Induced Uncertainty, Market Price of Risk, and the Dynamics of Consumption and Wealth

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Abstract

In this paper we examine implications of model uncertainty due to robustness (RB) for consumption and saving and the market price of uncertainty under limited information-processing capacity (rational inattention or RI). We first solve the robust permanent income models with inattentive consumers and show that RI by itself creates an additional demand for robustness that leads to higher "induced uncertainty" facing consumers. Second, we explore how the induced uncertainty composed of (i) model uncertainty due to RB and (ii) state uncertainty due to RI, affects consumption-saving decisions and the market price of uncertainty. We find that induced uncertainty can better explain the observed market price of uncertainty – low attention increases the effect of model misspecification. We also show the observational equivalence between RB and risk-sensitivity (RS) in environment.

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1 Introduction

Hansen and Sargent (1995) first introduced robustness (RB, a concern for model misspecification) into linear-quadratic (LQ) economic models.[1] In robust control problems, agents do not know the true data-generating process and are concerned about the possibility that their model (denoted the approximating model) is misspecified; consequently, they choose optimal decisions as if the subjective distribution over shocks was chosen by an evil nature in order to minimize their expected utility.[2] Robustness (RB) models produce precautionary savings but remain within the class of LQ models, which leads to analytical simplicity. A second class of models that produces precautionary savings but remains within the class of LQ models is the risk-sensitive (RS) model of Hansen and Sargent (1995) and Hansen, Sargent, and Tallarini (1999). In the RS model agents effectively compute expectations through a distorted lens, increasing their effective risk aversion by overweighting negative outcomes. The resulting decision rules depend explicitly on the variance of the shocks, producing precautionary savings, but the value functions are still quadratic functions of the states. As shown in Hansen and Sargent (2007), the risk-sensitivity preference can be used to interpret the desire for robustness as they lead to the same consumption-saving decisions, and similar asset pricing implications.[3]

Sims (2003) first introduce rational inattention into economics and argued that it is a plausible method for introducing sluggishness, randomness, and delay into economic models. In his formulation agents have finite Shannon channel capacity, limiting their ability to process signals about the true state of the world. As a result, an impulse to the economy induces only gradual responses by individuals, as their limited capacity requires many periods to discover just how much the state has moved. Since RI introduces additional uncertainty, the endogenous noise due to finite capacity, into economic models, RI by itself creates an additional demand for robustness. In addition, agents with finite capacity need to use a filter to update their perceived state upon receiving noisy signals, which may lead to another demand for robustness, the robust Kalman

[1] See Hansen and Sargent (2007) for a textbook treatment on robustness and Hansen and Sargent (2010) for a recent survey. For decision-theoretic foundations of the robustness preference, see Maccheroni, Marinacci, and Rustichini (2006) and Strzalecki (2011) for detailed discussions.

[2] The solution to a robust decision-maker's problem is the equilibrium of a max-min game between the decision-maker and nature.

[3] It is worth noting that although both RB (or RS) and CARA preferences (i.e., Caballero 1990 and Wang 2003) increase the precautionary savings premium via the intercept terms in the consumption functions, they have distinct implications for the marginal propensity to consume out of permanent income (MPC). Specifically, CARA preferences do not alter the MPC relative to the LQ case, whereas RB or RS increases the MPC. That is, under RB, in response to a negative wealth shock, the consumer would choose to reduce consumption more than that predicted in the CARA model (i.e., save more to protect themselves against the negative shock).

filter.[4]

In this paper we first construct a discrete-time robust permanent income model with inattentive consumers who have concerns about two types of model misspecification: (i) the disturbances to the perceived permanent income (the disturbances here include both the fundamental shock and the RI-induced noise shock) and (ii) the Kalman gain.[5] For ease of presentation, we will refer to the first type of model misspecification as Type I and the second as Type II.[6] After solving the model explicitly, we first examine how the preference of robustness affects optimal consumption and precautionary savings via interacting with finite capacity. Specifically, we show that given finite capacity, concerns about the two types of model misspecification have opposing impacts on the marginal propensity to consume out of perceived permanent income (MPC) and precautionary savings. In the case with only Type I model misspecification, since agents with low capacity are very concerned about the confluence of low permanent income and high consumption (meaning they believe their permanent income is high so they consume a lot and then their new signal indicates that in fact their permanent income was low), they take actions which reduce the probability of this bad event – they save more.[7] As for Type II misspecification, an increase in the strength of the preference for robustness increases the Kalman gain, which leads to lower total uncertainty about the true level of permanent income and then lower precautionary savings. In addition, the strength of the precautionary effect is positively related to the amount of this uncertainty that always increases as finite capacity gets smaller. Using the explicit expression for consumption dynamics, we also show that increasing RB increases the robust Kalman filter gain and thus leads to less relative volatility of consumption to income (smoother consumption process) when we only consider Type II misspecification. In contrast, RB increases the relative volatility of consumption by increasing the MPC out of changes in permanent income when we only consider Type I misspecification.[8]

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[4] The key assumption in Luo and Young (2010) is that agents with finite capacity distrust their budget constraint, but still use an ordinary Kalman filter to estimate the true state; in this case, a distortion to the mean of permanent income is introduced to represent possible model misspecification. However, this case ignores the effect of the RI-induced noise on the demand for robustness.

[5] Anderson, Hansen, and Sargent (2003) provide a general framework to study and quantify robustness in continuous-time. See Cagetti *et al.* (2002) and Maenhout (2004) for the applications of robustness in pricing, growth, and portfolio choice in continuous-time.

[6] When modeling Type II misspecification, we assume that the agent faces the commitment on the part of the minimizing agent to previous distortions.

[7] Luo, Nie, and Young (2012) apply Type I RB in the SOE-RBC model proposed in Aguiar and Gopinath (2007) and show that this type of RB can help generate realistic relative volatility of consumption to income and the current account dynamics observed in emerging and developed small-open economies.

[8] This mechanism is similar to that examined in Luo and Young (2010).

Furthermore, we compare the implications of RS and RB for consumption and savings when considering both control and filtering decisions of inattentive consumers. In the risk-sensitive permanent income model with imperfect-state-observation due to RI, the classical Kalman filter that extremizes the expected value of a certain quadratic objective function is still optimal. After solving the RB and RS models with filtering, we establish the observational equivalence (OE) conditions between RB and RS. We find that the simple and linear OE between RB and RS established in Hansen and Sargent (2007) and Luo and Young (2010) no longer holds, we instead have a complicated and nonlinear OE between RB and RS under RI.

We next investigate the asset pricing implications of RB and RI within the PIH setting.[9] Following Hansen (1987) and Hansen, Sargent, and Tallarini (1999), we interpret the consumption-saving decisions in terms of a social planning problem and these decisions are equilibrium allocations for a competitive equilibrium. We can then deduce asset prices as in the consumption-based asset pricing literature by finding the shadow prices that clear security markets. Since these asset prices include information about the agent's intertemporal preferences, they measure the risk and uncertainty aversion of the agent. Given the explicit solutions for consumption and saving decisions, we can explicitly solve for the market prices of induced uncertainty under RB and RI.[10] We find that induced uncertainty due to RB and RI significantly increases the market price of risk. The mechanism is straightforward to describe. Under RB, the market price of uncertainty is related to the norm of the worst-case shock (that is, the size of the pessimistic distortion to the underlying stochastic process for income); adding rational inattention increases the size of these distortions and therefore amplifies the effect on asset prices. We find that our model, under plausible calibrations of the fear of model misspecification based on detection error probabilities (as in Hansen and Sargent 2007), produces stochastic discount factors that satisfy the Hansen-Jagannathan bounds.

The remainder of the paper is organized as follows. Section 2 presents the robustness versions of the RI permanent income model with filtering, and examines how the preference for robustness affects individual consumption and saving decisions. Section 3 explores how these two informational frictions affect individual consumption and saving dynamics. Section 4 compares the risk-sensitive and robust filtering problem under RI. Section 5 computes how induced uncertainty due to RB and RI affects the market prices of risk. Section 6 concludes.

[9] See Epstein and Wang (1994), Chen and Epstein (2002), and Ju and Miao (2012) for ambiguity, risk aversion, and asset returns.

[10] To explore how induced uncertainty due to RB and RI affects market prices of uncertainty, we follow the procedure adopted in Epstein and Wang (1994) and Hansen, Sargent, and Tallarini (1999).

2 Robust Control and Filtering under Rational Inattention

2.1 A Rational Inattention Version of the Standard Permanent Income Model

In this section we consider a rational inattention (RI) version of the standard permanent income model. In the standard permanent income model (Hall 1978, Sargent 1978, Flavin 1981), households solve the dynamic consumption-savings problem

$$v(s_0) = \max_{\{c_t\}} E_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

subject to

$$s_{t+1} = Rs_t - c_t + \zeta_{t+1}, \tag{1}$$

where $u(c_t) = -\frac{1}{2}(\overline{c} - c_t)^2$ is the period utility function, $\overline{c} > 0$ is the bliss point, c_t is consumption,

$$s_t = b_t + \frac{1}{R} \sum_{j=0}^{\infty} R^{-j} E_t [y_{t+j}] \tag{2}$$

is permanent income, i.e., the expected present value of lifetime resources, consisting of financial wealth (b_t) plus human wealth (i.e., the discounted expected present value of current and future labor income: $\frac{1}{R}\sum_{j=0}^{\infty} R^{-j} E_t [y_{t+j}]$),

$$\zeta_{t+1} \equiv \frac{1}{R} \sum_{j=t+1}^{\infty} \left(\frac{1}{R}\right)^{j-(t+1)} (E_{t+1} - E_t)[y_j], \tag{3}$$

is the time $(t+1)$ innovation to permanent income, b_t is financial wealth (or cash-on-hand), y_t is a labor income process with Gaussian white noise innovations, β is the discount factor, and $R > 1$ is the constant gross interest rate at which the consumer can borrow and lend freely.[11] In this paper, we assume that income y_t takes the following AR(1) process with the persistence coefficient $\rho \in [0, 1]$,

$$y_{t+1} = \rho y_t + x_t + \varepsilon_{t+1}, \tag{4}$$

where $x_t \equiv (g - \rho) g^t y_0$ is the growth component, g is the constant gross growth rate of income, y_0 is defined as the initial level of income, and ε_{t+1} is iid with mean 0 and variance ω^2.[12] Given this income specification, we have $s_t \equiv b_t + \frac{1}{R-\rho} y_t + \frac{1}{(R-g)(R-\rho)} x_t$ and $\zeta_{t+1} = \varepsilon_{t+1} / (R - \rho)$, where and $\omega_\zeta^2 \equiv \text{var}(\zeta_{t+1}) = \omega^2 / (R - \rho)^2$.[13] Finally, financial wealth (b) follows the process

$$b_{t+1} = Rb_t + y_t - c_t. \tag{5}$$

[11] We only require that y_t and R are such that permanent income is finite.

[12] Note that when $g = 1$, this specification reduces to $y_{t+1} = \rho y_t + (1 - \rho) y_0 + \varepsilon_{t+1}$, one of the most popular income specifications in the consumption literature.

[13] For the rest of the paper we will restrict attention to points where $c_t < \overline{c}$, so that utility is increasing and concave.

This specification follows that in Hall (1978) and Flavin (1981) and implies that optimal consumption is determined by permanent income:

$$c_t = \left(R - \frac{1}{\beta R}\right) s_t - \frac{1}{R-1}\left(1 - \frac{1}{\beta R}\right) \bar{c}. \tag{6}$$

We assume for the remainder of this section that $\beta R = 1$, since this setting is the only one that implies zero drift in consumption under rational expectations. Under this assumption the model leads to the well-known random walk result of Hall (1978):

$$\Delta c_{t+1} = (R-1)\,\zeta_{t+1}; \tag{7}$$

the change in consumption depends neither on the past history of labor income nor on anticipated changes in labor income. We also point out the well-known result that the standard PIH model with quadratic utility implies the certainty equivalence property holds: uncertainty has no effect on consumption, so that there is no precautionary saving.

To motivate what follows, we now remind readers why (7) is inadequate as an empirical representation of consumption. There are many routes we could take here; we choose to follow Campbell and Deaton (1989) and run a simple bivariate VAR of (demeaned) labor income growth and the (demeaned) saving rate out of labor income, obtaining

$$\begin{bmatrix} \Delta \log(y_t) - \mu_y \\ \frac{s_t}{y_t} - \mu_s \end{bmatrix} = \begin{bmatrix} 0.0357 & 0.02 \\ -0.072 & 0.9947 \end{bmatrix} \begin{bmatrix} \Delta \log(y_{t-1}) - \mu_y \\ \frac{s_{t-1}}{y_{t-1}} - \mu_s \end{bmatrix} + \begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix}.$$

The data we use is quarterly NIPA data on total compensation of employees and gross saving deflated using the PCE deflator (defined as labor income minus consumption of nondurables and services), from 1947Q1-2014Q1. Campbell and Deaton (1989) show that the PIH implies the autocorrelation matrix should take the form

$$\begin{bmatrix} \delta & 0 \\ \delta & \tau^{-1} \end{bmatrix} \tag{8}$$

where δ is a unrestricted coefficient and $\tau \neq 0$ is the effective discount rate for future cash flows (the growth-adjusted interest rate); this matrix embodies both a test of "excess sensitivity" (that consumption responds to predictable changes in income) and a test of "excess smoothness" (consumption growth does not vary enough with income growth).[14] Our estimated matrix does not satisfy the restrictions embodied in (8) – we can reject the equality of the elements in the first column and our estimate of τ is economically nonsensical (including durables makes these

[14] In fact, Campbell and Deaton (1989) show that the two excesses are the same – if consumption responds excessively to anticipated income changes it necessarily must respond insufficiently to unanticipated ones.

rejections more pronounced). One method for attacking these rejections is to assume that the information set of the agents differs systematically from that of the econometrician. It is common to assume that agents have more information than the econometrician about their individual situation; rational inattention implies that, for some variables, agents may actually have less information.

To this end we follow Sims (2003, 2010) and incorporate rational inattention (RI) due to finite information-processing capacity into the model. Under RI, consumers have only finite Shannon channel capacity to observe the state of the world. Specifically, we use the concept of entropy from information theory to characterize the uncertainty about a random variable; the reduction in entropy is thus a natural measure of information flow.[15] With finite capacity $\kappa \in (0, \infty)$, a variable s following a continuous distribution cannot be observed without error and thus the information set at time $t+1$, $\mathcal{I}_{t+1}$, is generated by the entire history of noisy signals $\left\{ s_j^* \right\}_{j=0}^{t+1}$. Following the literature, we assume the noisy signal takes the additive form $s_{t+1}^* = s_{t+1} + \xi_{t+1}$, where ξ_{t+1} is the endogenous noise caused by finite capacity. We further assume that ξ_{t+1} is an iid idiosyncratic shock and is independent of the fundamental shock. Agents with finite capacity will choose a new signal $s_{t+1}^* \in \mathcal{I}_{t+1} = \left\{ s_1^*, s_2^*, \cdots, s_{t+1}^* \right\}$ that reduces their uncertainty about the state variable s_{t+1} as much as possible. Formally, this idea can be described by the information constraint

$$\mathcal{H}\left(s_{t+1} | \mathcal{I}_t\right) - \mathcal{H}\left(s_{t+1} | \mathcal{I}_{t+1}\right) \leq \kappa, \tag{9}$$

where κ is the consumer's information channel capacity, $\mathcal{H}\left(s_{t+1} | \, \mathcal{I}_t\right)$ denotes the entropy of the state prior to observing the new signal at $t+1$, and $\mathcal{H}\left(s_{t+1} | \, \mathcal{I}_{t+1}\right)$ is the entropy after observing the new signal. κ imposes an upper bound on the amount of information – that is, the change in the entropy – that can be transmitted in any given period. Finally, following the literature, we suppose that the prior distribution of s_{t+1} is Gaussian.

Under the linear-quadratic-Gaussian (LQG) setting, as has been shown in Sims (2003, 2010), the true state under RI also follows a normal distribution $s_t | \mathcal{I}_t \sim N\left(E\left[s_t | \mathcal{I}_t\right], \Sigma_t\right)$, where $\Sigma_t = E_t\left[\left(s_t - \widehat{s}_t\right)^2\right]$.[16] In addition, given that the noisy signal takes the additive form $s_{t+1}^* = s_{t+1} + \xi_{t+1}$ and the noise ξ_{t+1} is an iid Gaussian variable, the posterior variance of s_{t+1} is also Gaussian within

[15]Formally, entropy is defined as the expectation of the negative of the (natural) log of the density function, $-E\left[\ln\left(f\left(s\right)\right)\right]$. For example, the entropy of a discrete distribution with equal weight on two points is simply $E\left[\ln_2\left(f\left(s\right)\right)\right] = -0.5 \ln\left(0.5\right) - 0.5 \ln\left(0.5\right) = 0.69$, and the unit of information contained in this distribution is 0.69 "nats". In this case, an agent can remove all uncertainty about s if the capacity devoted to monitoring s is $\kappa = 0.69$ nats.

[16]Shafieepoorfard and Raginsky (2013) derive the result formally, as opposed to the heuristic approach from Sims (2003).

the LQG structure.[17] The information-processing constraint, (9), can then be reduced to

$$\ln\left(R^2\Sigma_t+\omega_\zeta^2\right)-\ln\left(\Sigma_{t+1}\right)\leq 2\kappa; \tag{10}$$

Since this constraint is always binding, we can compute the value of the steady state conditional variance Σ: $\Sigma=\omega_\zeta^2/\left(\exp\left(2\kappa\right)-R^2\right)$. Given this Σ, we can use the usual formula for updating the conditional variance of a Gaussian distribution Σ to recover the variance of the endogenous noise (Λ):

$$\Lambda=\left(\Sigma^{-1}-\Psi^{-1}\right)^{-1}, \tag{11}$$

where $\Psi=R^2\Sigma+\omega_\zeta^2$ is the posterior variance of the state. Finally, $\widehat{s}_t$ is governed by the following Kalman filtering equation:

$$\widehat{s}_{t+1}=(1-\theta)\left(R\widehat{s}_t-c_t\right)+\theta\left(s_{t+1}+\xi_{t+1}\right), \tag{12}$$

given $s_0\sim N\left(\widehat{s}_0,\Sigma\right)$, where $\theta=\Sigma\Lambda^{-1}=1-\exp\left(-2\kappa\right)$ is the Kalman gain.[18]

Note that after substituting (1) into (12), we have an alternative expression of the regular Kalman filter:

$$\widehat{s}_{t+1}=R\widehat{s}_t-c_t+\eta_{t+1}, \tag{13}$$

where

$$\eta_{t+1}=\theta R\left(s_t-\widehat{s}_t\right)+\theta\left(\zeta_{t+1}+\xi_{t+1}\right) \tag{14}$$

is the innovation to the mean of the distribution of perceived permanent income,

$$s_t-\widehat{s}_t=\frac{(1-\theta)\,\zeta_t}{1-(1-\theta)R\cdot L}-\frac{\theta\xi_t}{1-(1-\theta)R\cdot L}, \tag{15}$$

and $E_t\left[\eta_{t+1}\right]=0$ because the expectation is conditional on the perceived signals and inattentive agents cannot perceive the lagged shocks perfectly.[19] The variance of the innovation to the perceived state is:

$$\omega_\eta^2=\mathrm{var}\left(\eta_{t+1}\right)=\frac{\theta}{1-(1-\theta)\,R^2}\omega_\zeta^2, \tag{16}$$

[17] This result is often assumed as a matter of convenience in signal extraction models with exogenous noises, and RI can rationalize this assumption.

[18] As argued in Sims (2010), instead of using fixed finite channel capacity to model limited information-processing ability, one could assume that the marginal cost of information processing (i.e., the shadow price of information-processing ability) is constant. That is, the Lagrange multiplier on (10) is constant. Luo and Young (2014) show that in the univariate case these two RI modeling strategies are observationally equivalent in the sense that they lead to the same conditional variance. Under the observational equivalence, we can construct a mapping between fixed information-processing cost and fixed channel capacity.

[19] In order that the variance of η be finite we need $\kappa>\ln(R)\approx R-1$. For short time periods this requirement is obviously not very restrictive. Since $R>1$, some minimum level of capacity is needed to control the conditional mean of permanent income.

which means that ω_η^2 reflects two sources of uncertainty facing the consumer: (i) fundamental uncertainty, ω_ζ^2 and (ii) induced uncertainty, i.e., state uncertainty due to RI, $\left[\frac{\theta}{1-(1-\theta)R^2}-1\right]\omega_\zeta^2$. Therefore, as κ decreases, the relative importance of induced uncertainty to fundamental uncertainty increases.

In the next section, we will discuss alternative ways to robustify this RI-PIH model and their different implications for consumption, precautionary savings, and the welfare costs of uncertainty. The RB-RI model proposed here encompasses the hidden state (HS) model discussed in Hansen, Sargent, and Wang (2002, henceforth, HSW) and Hansen and Sargent (2005); the main difference is that agents in the RB-RI model cannot observe the entire state vector perfectly, whereas agents in the RB-hidden state model can observe some part of the state vector (in particular, the part they control).

2.2 Concerns about the Fundamental Shock and the Noise Shock

As shown in Hansen and Sargent (2007), we can robustify the permanent income model by assuming agents with finite capacity distrust their model of the data-generating process (i.e., their income process), but still use an ordinary Kalman filter to estimate the true state. Note that without the concern for model misspecification, the consumer has no doubts about the probability model used to form the conditional expectation of permanent income (s). It is clear that the Kalman filter under RI, (13), is not only affected by the fundamental shock (ζ_{t+1}), but also affected by the endogenous noise (ξ_{t+1}) induced by finite capacity; these noise shocks could be another source of the demand for robustness. We therefore need to consider this demand for robustness in the RB-RI model. By adding the additional concern for robustness developed here, we are able to strengthen the effects of robustness on decisions.[20] Specifically, we assume that the agent thinks that (13) is the approximating model.

A simple version of robust optimal control considers the question of how to make decisions when the agent does not know the probability model that generates the data. Specifically, an agent with a preference for robustness considers a range of models surrounding the given approximating model, (13):

$$\widehat{s}_{t+1} = R\widehat{s}_t - c_t + \omega_\eta w_t + \eta_{t+1}. \tag{17}$$

where w_t distorts the mean of the innovation, and makes decisions that maximize lifetime expected utility given this worst possible model (i.e., the distorted model).[21] To make that model (13) is a

[20] Luo, Nie, and Young (2012) use this approach to study the joint dynamics of consumption, income, and the current account in emerging and developed countries.

[21] Formally, this setup is a game between the decision-maker and a malevolent nature that chooses the distortion process w_t.

good approximation when (17) generates the data, we constrain the approximation errors by an upper bound ψ_0:

$$E_0\left[\sum_{t=0}^{\infty}\beta^{t+1}w_t^2\right]\leq\psi_0, \tag{18}$$

where $E_0[\cdot]$ denotes conditional expectations evaluated with model, and the left side of this inequality is a statistical measure of the discrepancy between the distorted and approximating models. Note that the standard full-information RE case corresponds to $\psi_0=0$. In the general case in which $\psi_0>0$, the evil agent is given an intertemporal entropy budget $\psi_0>0$ which defines the set of models that the agent is considering. Following Hansen and Sargent (2007), we compute robust decision rules by solving the following two-player zero-sum game: a minimizing decision maker chooses the optimal consumption process $\{c_t\}$ and a maximizing evil agent chooses the model distortion process $\{w_t\}$.

Following Hansen and Sargent (2007a), a simple robustness version of the PIH model proposed above can be written as

$$v\left(\widehat{s}_t\right)=\max_{c_t}\min_{w_t}\left\{-\frac{1}{2}\left(\overline{c}-c_t\right)^2+\beta\left(\frac{1}{2}\vartheta w_t^2+E_t\left[v\left(\widehat{s}_{t+1}\right)\right]\right)\right\} \tag{19}$$

subject to the distorted transition equation (i.e., the worst-case model), (17), where $\vartheta>0$ is the Lagrange multiplier on the constraint specified in (18) and controls how bad the error can be. (19) is a standard dynamic programming problem and can be easily solved using the standard procedure.[22] The following proposition summarizes the solution to the RB-RI model.

Proposition 1 *Given ϑ and κ, the consumption function under RB and RI is*

$$c_t=\frac{R-1}{1-\Pi}\widehat{s}_t-\frac{\Pi\overline{c}}{1-\Pi} \tag{20}$$

with $\Pi<1$, the mean of the worst-case shock is

$$\omega_\eta w_t=\frac{(R-1)\Pi}{1-\Pi}\widehat{s}_t-\frac{\Pi\overline{c}}{1-\Pi}, \tag{21}$$

and $\widehat{s}_t$ is governed by

$$\widehat{s}_{t+1}=\rho_s\widehat{s}_t+\frac{\Pi\overline{c}}{1-\Pi}+\eta_{t+1} \tag{22}$$

under the approximation model, where $\rho_s=\frac{1-R\Pi}{1-\Pi}\in(0,1)$,

$$\Pi=\frac{R\omega_\eta^2}{\vartheta}\in(0,1), \tag{23}$$

η_{t+1} and ω_η^2 are defined in (14) and (14), and $\theta=1-1/\exp(2\kappa)$.

[22] There is a one-to-one correspondence between ψ_0 in (18) and ϑ in (19).

Proof. See Appendix 7.1. $\Pi < 1$ can be obtained because the second-order condition for the optimization problem is

$$\frac{R(R-1)}{2\left(1 - R\omega_\eta^2/\vartheta\right)} > 0, \text{ i.e., } \Pi < 1.$$

■

It is worth noting that (20) can also be obtained using multiplier preferences to represent a fear of model misspecification:

$$\widehat{v}(\widehat{s}_t) = \max_{c_t} \left\{ -\frac{1}{2}(c_t - \overline{c})^2 + \beta \min_{m_{t+1}} E_t\left[m_{t+1}\widehat{v}(\widehat{s}_{t+1}) + \vartheta m_{t+1} \ln(m_{t+1})\right] \right\}, \tag{24}$$

where m_{t+1} is the likelihood ratio, $E_t[m_{t+1}\ln(m_{t+1})]$ is defined as the relative entropy of the distribution of the distorted model with respect to that of the approximating model, and $\vartheta > 0$ is the shadow price of capacity that can reduce the distance between the two distributions, i.e., the Lagrange multiplier on the constraint:

$$E_t[m_{t+1}\ln(m_{t+1})] \leq \eta,$$

where $\eta \geq 0$ defines an entropy ball of the distribution of the distorted model with respect to that of the approximating model. Following the same procedure adopted in Hansen and Sargent (2007), we can also obtain the corresponding value function:

$$\widehat{v}(\widehat{s}_t) = \Omega\left(\widehat{s}_t - \frac{\overline{c}}{R-1}\right)^2 + \Omega_0, \tag{25}$$

where $\Omega = -\frac{R(R-1)}{2(1-\Pi)}$ and $\Omega_0 = \frac{\vartheta}{2(R-1)}\ln\left(1 - \frac{(R-1)\Pi}{1-\Pi}\right)$. Although the two-player minmax game and multiplier preferences lead to the same consumption-saving decisions, they have different asset pricing implications. (See Section 5 for a detailed discussion.)

Equations (20) and (23) determine the effects of model uncertainty due to RB and state uncertainty due to RI on the marginal propensity to consume out of perceived permanent income ($MPC_\eta \equiv \frac{R-1}{1-\Pi}$) and the constant precautionary saving premium ($PS \equiv \frac{\Pi\overline{c}}{1-\Pi}$). These two expressions show that Π governs how RB and RI interact and then affect the consumption function and precautionary savings. Since Π is increasing with the degrees of both RB (smaller ϑ) and RI (smaller κ and θ), it is straightforward to show that either RB or RI leads to more constant precautionary savings and higher marginal propensity to consume, holding other factors constant and given that $\Pi < 1$:

$$\frac{\partial(MPC_\eta)}{\partial\vartheta} < 0 \text{ and } \frac{\partial(PS)}{\partial\vartheta} < 0.$$

We now present the intuition about the effects of robustness (ϑ) on precautionary savings. Since agents with low capacity are very concerned about the confluence of low permanent income and high consumption (meaning they believe their permanent income is high so they consume a lot and then their new signal indicates that in fact their permanent income was low), they take actions which reduce the probability of this bad event – they save more. The strength of the precautionary effect is positively related to the amount of uncertainty regarding the true level of permanent income, and this uncertainty increases as θ gets smaller.

However, RB and RI affect consumption and precautionary savings through distinct channels. RI affects Π by increasing the variance of the innovation to the perceived state, ω_η^2, whereas RB affects Π via changing the structure of the response of consumption to income shocks. Furthermore, if we consider the marginal propensity to consume out of true permanent income,

$$MPC_\zeta \equiv \frac{R-1}{1-R\theta/\left[\vartheta\left(1-(1-\theta)R^2\right)\right]\omega_\zeta^2}\theta, \tag{26}$$

we can immediately see that

$$\frac{\partial\left(MPC_\zeta\right)}{\partial\vartheta} < 0, \frac{\partial\left(MPC_\zeta\right)}{\partial\theta} > 0.$$

That is, both an increase in the demand for robustness and an increase in inattention increases the marginal propensity to consume out of true (but unobserved) permanent income.

To examine the relative importance of the two informational frictions in determining the consumption function and precautionary savings, we compare the effects from proportionate shifts in ϑ governing RB and κ governing RI. Specifically, the marginal effects on Π from an increase in ϑ and κ are given by

$$\frac{\partial\Pi}{\partial\kappa} = \frac{R\left(1-R^2\right)\exp\left(-2\kappa\right)}{\vartheta\left[1-\exp\left(-2\kappa\right)R^2\right]^2}\omega_\zeta^2,\ \frac{\partial\Pi}{\partial\vartheta} = -\frac{R\omega_\eta^2}{\vartheta^2},$$

respectively. Therefore, the marginal rate of transformation between proportionate changes in ϑ and changes in κ can be written as

$$\text{MRT} = -\frac{\partial\Pi/\partial\kappa}{\left(\partial\Pi/\partial\vartheta\right)\vartheta} = \frac{2\left(R^2-1\right)\exp\left(-2\kappa\right)}{\left(1-\exp\left(-2\kappa\right)\right)\left(1-\exp\left(-2\kappa\right)R^2\right)} > 0. \tag{27}$$

This expression gives the proportionate reduction in ϑ (i.e., a stronger preference for RB) that compensates, at the margin, for a decrease in κ (i.e., more inattentive) — in the sense of preserving the same effect on the consumption function for a given $\widehat{s}_t$. Equation (27) shows that this compensating change depends on the interest rate (R) and the degree of inattention (κ). Figure 1 clearly shows that MRT is decreasing with κ for any given R. Since $\partial\left(\text{MRT}\right)/\partial\kappa < 0$, consumers with lower capacity will ask for higher compensation in an proportionate increase in model uncertainty facing them for an increase in capacity. For example, when $R = 1.03$, MRT $= 0.256$ when

$\kappa = 0.5$ bits, while MRT $= 0.054$ when $\kappa = 1$ bit. In other words, to maintain the same effect on the consumption function, a decrease in κ by 50 percent (from 1 bit to 0.5 bits) matches up approximately with a proportional decline in ϑ of 2.7 percent. We will show later that there is a model-independent procedure for estimating ϑ; the tradeoff here could in principle be used to discipline the choice for κ.[23]

It is also instructive to examine exactly what agents "fear" – that is, what are the dynamics of permanent income under the worst-case model? Substituting (20) and (21) into (17) yields the law of motion for $\widehat{s}_t$ under the worst-case model:

$$\widehat{s}_{t+1} = \widehat{s}_t + \eta_{t+1} = (1 - \theta R)\, \widehat{s}_t + \theta R s_t + \theta\, (\zeta_{t+1} + \xi_{t+1})$$

as compared to the actual process

$$\widehat{s}_{t+1} = \left(\frac{1 - R^2 \omega_\eta^2 / \vartheta}{1 - R\omega_\eta^2/\vartheta} - \theta R \right) \widehat{s}_t + \frac{\left(R\omega_\eta^2/\vartheta\right) \bar{c}}{1 - R\omega_\eta^2/\vartheta} + \theta R s_t + \theta\, (\zeta_{t+1} + \xi_{t+1}) \,. \tag{28}$$

The key difference between the two processes is the autocorrelation parameter; since $\frac{1-R^2\omega_\eta^2/\vartheta}{1-R\omega_\eta^2/\vartheta} < 1$, the worst case model is more persistent than the true process. As noted in Kasa (2006), the most destructive distortions are low-frequency ones, so naturally the agents in the model design their decision rules to be robust against precisely those kinds of processes. ϑ does not appear in (**??**), as it only determines the size of the distortion process $\{w_t\}$ needed to achieve the worst-case model.[24]

2.3 Robust Kalman Filter Gain

Another source of robustness could arise from the Kalman filter gain. In Section 2.2, we assumed that the agent distrusts the innovation to the perceived state but trusts the regular Kalman filter gain. Following Hansen and Sargent (2005) and Hansen and Sargent (Chapter 17, 2007), in this section we consider a situation in which the agent pursues a robust Kalman gain and faces the commitment on the part of the minimizing agent to previous distortions. Specifically, assume that at t the agent observes the noisy signal

$$s_t^* = s_t + \xi_t, \tag{29}$$

[23] κ (or θ) are difficult to estimate outside the model; the literature on processing information provides estimates of the total ability of humans, but little guidance on how much of that ability would be dedicated to monitoring economic data. Obviously it would not be feasible to model all the competing demands for attention.

[24] If $\theta = 1$ (so that $\widehat{s}_t = s_t$) then the worst-case model is a random walk.

where s_t is the true state and ξ_t is the iid endogenous noise. The variance of the noise term, $\Lambda = \text{var}(\xi_t)$, is determined by:

$$\Lambda = \frac{\left(\omega_\zeta^2 + R^2\Sigma\right)\Sigma}{\omega_\zeta^2 + \left(R^2 - 1\right)\Sigma},$$

and $\Sigma = \omega_\zeta^2 / \left(\exp(2\kappa) - R^2\right)$ is the steady state conditional variance. Given the budget constraint,

$$s_{t+1} = Rs_t - c_t + \zeta_{t+1}, \tag{30}$$

we consider the following time-invariant robust Kalman filter equation,

$$\widehat{s}_{t+1} = (1-\theta)\left(R\widehat{s}_t - c_t\right) + \theta\left(s_{t+1} + \xi_{t+1}\right), \tag{31}$$

where $\widehat{s}_{t+1}$ is the estimate of the state using the history of the noisy signals, $\left\{s_j^*\right\}_{j=0}^{t+1}$. We want θ to be robust to unstructured misspecifications of Equations (29) and (30). To obtain a robust Kalman filter gain, the agent considers the following distorted model:

$$s_{t+1} = Rs_t - c_t + \zeta_{t+1} + \omega\nu_{1,t+1}, \tag{32}$$

$$s_{t+1}^* = s_{t+1} + \xi_{t+1} + \varrho\nu_{2,t+1}, \tag{33}$$

where $\omega = \omega_\zeta$, $\varrho = \sqrt{\Lambda}$, and $\nu_{1,t+1}$ and $\nu_{2,t+1}$ are distortions to the conditional means of the two shocks, ζ_{t+1} and ξ_{t+1}, respectively.

Combining (30), (31), (32) with (33) gives the following dynamic equation for the estimation error:

$$e_{t+1} = (1-\theta)Re_t + (1-\theta)\zeta_{t+1} - \theta\xi_{t+1} + (1-\theta)\omega\nu_{1,t+1} - \theta\varrho\nu_{2,t+1}, \tag{34}$$

where $e_t = s_t - \widehat{s}_t$.[25] We can then solve for the robust Kalman filter gain corresponding to this problem by solving the following deterministic optimal linear regulator problem:

$$e_0^T P e_0 = \max_{\{\nu_{t+1}\}} \sum_{t=0}^{\infty} \left(e_t^T e_t - \vartheta\nu_{t+1}^T\nu_{t+1}\right), \tag{35}$$

subject to

$$e_{t+1} = (1-\theta)Re_t + D\nu_{t+1}, \tag{36}$$

where $D = \left[\begin{array}{cc} (1-\theta)\omega & -\theta\varrho \end{array}\right]$ and $\nu_{t+1} = \left[\begin{array}{cc} \nu_{1,t+1} & \nu_{2,t+1} \end{array}\right]^T$. We can compute the worst-case shock by solving the corresponding Bellman equation and obtain

$$\nu_{t+1}^* = Qe_t, \tag{37}$$

[25] Note that control variable, c, does not affect the estimation error equation.

where

$$Q = \left(\vartheta I - D^T P D\right)^{-1} D^T P \left(1-\theta\right) R. \tag{38}$$

Here I is the identity matrix, P is the value function matrix, and Q depends on robustness (ϑ) and channel capacity (κ).

For arbitrary Kalman filter gain θ and (37), the error in reconstructing the state s can be written as

$$e_{t+1} = \left\{\left(1-\theta\right) R + \left[\left(1-\theta\right)\omega - \theta\varrho\right] Q\right\} e_t + \left(1-\theta\right)\zeta_{t+1} - \theta\xi_{t+1}. \tag{39}$$

Taking unconditional mean on both sides of (39) gives

$$\Sigma_{t+1} = \left\{\left(1-\theta\right) R + \left[\left(1-\theta\right)\omega - \theta\varrho\right] Q\right\} \Sigma_t + \left(1-\theta\right)^2 \omega_\zeta^2 + \theta^2\omega_\xi^2, \tag{40}$$

where $\Sigma_{t+1} = E\left[e_{t+1}^2\right]$. From (40), it follows directly that in the steady state

$$\Sigma\left(\theta; Q\right) = \frac{\left(1-\theta\right)^2 \omega_\zeta^2 + \theta^2\omega_\xi^2}{1-\chi^2},$$

where $\chi = \left(1-\theta\right) R + \left[\left(1-\theta\right)\omega - \theta\varrho\right] Q$, and the robust Kalman filter gain $\theta\left(\vartheta,\kappa\right)$ minimizes the variance of e_t, $\Sigma\left(\theta; Q\right)$:

$$\theta\left(\vartheta,\kappa\right) = \arg\min\left(\Sigma\left(\theta; Q\left(\vartheta,\kappa\right)\right)\right). \tag{41}$$

Figure 2 illustrates how robustness (measured by ϑ) and inattention (measured by κ) affect the robust Kalman gain when $R = 1.02$ and $\omega_\zeta = 1$.[26] It clearly shows that holding the degree of attention (i.e., channel capacity κ) fixed, increasing robustness (reducing ϑ) increases the Kalman gain (θ). In addition, for given robustness (ϑ), the Kalman gain is increasing with capacity. For example, when $\log\left(\vartheta\right) = 3$, the robust Kalman gain will increase from 60.17 percent to 77.35 percent when capacity κ increases from 0.6 bits to 1 bit; when $\kappa = 0.6$ bits, the robust Kalman gain will increase from 58.31 percent to 60.17 percent if ϑ falls from $\log\left(\vartheta\right) = 4$ to 3.[27]

After obtaining the robust Kalman gain $\theta\left(\vartheta,\kappa\right)$, we can solve the Bellman equation proposed in Section 2.2 using the Kalman filtering equation with robust θ. The following proposition summarizes the solution to this problem:

Proposition 2 *Given ϑ and κ, the consumption function is*

$$c_t = \frac{R-1}{1-\Pi}\widehat{s}_t - \frac{\Pi\overline{c}}{1-\Pi}, \tag{42}$$

[26] We use the program `rfilter.m` provided in Hansen and Sargent (2007) to compute the robust Kalman filter gain $\theta\left(\vartheta,\kappa\right)$.

[27] This result is consistent with that obtained in a continuous-time filtering problem discussed in Kasa (2006).

where $\Pi = \frac{R\omega_\eta^2}{\vartheta} \in (0,1)$,

$$\omega_\eta^2 = \text{var}(\eta_{t+1}) = \frac{\theta(\vartheta,\kappa)}{1-(1-\theta(\vartheta,\kappa))R^2}\omega_\zeta^2, \tag{43}$$

and $\widehat{s}_t$ *is governed by*

$$\widehat{s}_{t+1} = \rho_s \widehat{s}_t + \eta_{t+1}, \tag{44}$$

where $\rho_s = \frac{1-R\Pi}{1-\Pi} \in (0,1)$.

Proof. The proof is the similar to that provided in Appendix 7.1. Here we just need to replace $\theta(\kappa) = 1-\exp(-2\kappa)$ with $\theta(\vartheta,\kappa)$. ■

Note that here θ is a function of both ϑ (concerns about Kalman gain) and κ (channel capacity), rather than simply $1 - 1/\exp(2\kappa)$ as obtained in Section 2.2. In this case the agent is not only concerned about disturbances to the perceived permanent income, but also concerned about the Kalman gain. It is clear from (42) and (43) that the preference for robustness has opposing effects on both the marginal propensity to consume out of permanent income, i.e., the responsiveness of c_t to $\widehat{s}_t$ $\left(MPC_\eta = \frac{R-1}{1-\Pi}\right)$ and precautionary savings, i.e., the intercept of the consumption profile $\left(PS = \frac{\Pi\bar{c}}{1-\Pi}\right)$.[28] Specifically, if we temporarily shut down the concern about disturbances to perceived permanent income, we can see from (42) that the smaller the value of ϑ the lower the MPC and the smaller the precautionary saving increment

$$\frac{\partial(MPC_\eta)}{\partial\vartheta} > 0 \text{ and } \frac{\partial(PS)}{\partial\vartheta} > 0$$

because $\frac{\partial\omega_\eta^2}{\partial\vartheta} > 0$, $\frac{\partial\omega_\eta^2}{\partial\theta} < 0$, $\frac{\partial\omega_\eta^2}{\partial\kappa} < 0$, and $\frac{\partial\theta}{\partial\vartheta} < 0$. From (42), we can see that the precautionary savings increment in the RB-RI model is determined by the interaction of three factors: labor income uncertainty, preferences for robustness (RB), and finite information-processing capacity (RI). Figure 2 also illustrates how robustness (ϑ) and channel capacity (κ) affect ω_η^2. We now provide some intuition about the effects of robustness (ϑ) on precautionary savings in this case. An increase in robustness (a reduction in ϑ) will increase the Kalman gain θ, which leads to lower ω_η^2 and then low precautionary savings. We can see that under certain conditions a greater reaction to the shock can either be interpreted as an increased concern for robustness in the presence of model misspecification, or an increase in information-processing ability when agents only have finite channel capacity.

We now discuss the concern about disturbances to perceived permanent income. Figure 3 illustrates Π as functions of ϑ for different values of κ in this case, and shows that increasing the

[28]Note that given the consumption function Π has the same effect on the marginal propensity to consume and precautionary savings.

robustness preference for the shock to the perceived state (decreasing ϑ) increases Π and thus the effects of robustness on consumption and precautionary savings. In addition, we can also see from the figure that the effect of RB (ϑ) on Π dominates the effect of RI (κ). The main reason for this result is that RB has a strong direct effect on Π via the ratio $R\omega_\eta^2/\vartheta$.

3 Consumption and Wealth Dynamics

3.1 Sensitivity and Smoothness of Consumption Process

We will now discuss the effect of RI-RB on the dynamics of consumption, in particular the excess smoothness and sensitivity puzzles noted earlier. Since the deterministic growth component does not affect the stochastic properties of the model, for simplicity, here we assume that there is no growth component ($g = 1$). Combining (42) with (44) yields an expression for individual consumption in the RI-RB economy:

$$c_t = \frac{1 - R\Pi}{1 - \Pi} c_{t-1} + \frac{(R-1)\Pi}{1 - \Pi}\bar{c} + \frac{R-1}{1-\Pi}\left[\frac{\theta\zeta_t}{1-(1-\theta)R\cdot L} + \theta\left(\xi_t - \frac{\theta R\xi_{t-1}}{1-(1-\theta)R\cdot L}\right)\right], \quad (45)$$

where L is the lag operator and we assume that $(1-\theta)R < 1$. This expression implies that consumption growth is a weighted average of all past permanent income and noise shocks. In addition, it is also clear from (45) that the propagation mechanism of the model is determined by the robust Kalman filter gain, $\theta(\vartheta, \kappa)$. Figure 4 illustrates that consumption in the RB-RI model reacts gradually to income shocks, with monotone adjustments to the corresponding RB asymptote. Note that when $\log(\vartheta) = 2$, the robust Kalman gain is $\theta = 0.6688$. This case is illustrated by the dash-dotted line in Figure 4. Similarly, the dotted line corresponds to the case in which $\log(\vartheta) = 5$ ($\theta = 0.5824$). With a stronger preference for robustness, the precautionary savings increment is larger and thus an income shock that is initially undetected would have larger impacts on consumption during the adjustment process.[29]

Using (45), we can obtain the expression for the relative volatility of consumption growth relative to income growth.[30] The following proposition provides the expression of this relative volatility.

[29] Estimating the process (45) on quarterly real nondurable and service consumption, logged and linearly detrended with $R = 1.01$, produces a nonstationary process (the autoregressive roots are 1.3038 and -0.3027), so we do not pursue this direction further. We note in passing that the estimate for θ is close to 0 and the estimate for Π is close to 1. Lower values for R move these values closer to 0 and 1.

[30] Here we follow the consumption literature and use $\frac{\mathrm{sd}(\Delta c_t)}{\mathrm{sd}(\Delta y_t)}$ instead of $\frac{\mathrm{sd}(c_t)}{\mathrm{sd}(y_t)}$ to measure the relative volatility of the consumption process. Note when $\rho = 1$ in the income process or $\vartheta = \infty$ (no RB), $\mathrm{sd}(y_t)$ or $\mathrm{sd}(c_t)$ are not well defined.

Proposition 3 *The relative volatility of consumption growth relative to income growth is*

$$\mu \equiv \frac{\mathrm{sd}(\Delta c_t)}{\mathrm{sd}(\Delta y_t)} = \frac{\theta}{1-\Pi}\sqrt{\sum_{j=0}^{\infty}\Upsilon_j^2 + \frac{\rho_\theta}{\theta R(1-\rho_\theta R)}\sum_{j=0}^{\infty}(\Upsilon_j - R\Upsilon_{j-1})^2}, \tag{46}$$

where we use the fact that $\omega_\xi^2 = \mathrm{var}(\xi_t) = \frac{1-\theta}{\theta(1-(1-\theta)R^2)}\omega_\zeta^2$, $\rho_s = \frac{1-R\Pi}{1-\Pi} \in (0,1)$, $\rho_\theta = (1-\theta)R \in (0,1)$, *and* $\Upsilon_j = \sum_{k=0}^{j}\left(\rho_s^{j-k}\rho_\theta^k\right) - \sum_{k=0}^{j-1}\left(\rho_s^{j-1-k}\rho_\theta^k\right)$, *for* $j \geq 1$, *and* $\Upsilon_0 = 1$.

Proof. See Online Appendix. ■

The upper panel of Figure 5 illustrates how RB (ϑ) affects the relative volatility of consumption growth to income growth for different values of κ. It clearly shows that for any given value of ϑ, the relative volatility μ is decreasing with κ. The intuition is that less capacity leads to higher total induced uncertainty $\left(\omega_\eta^2\right)$ for given values of RB. Furthermore, when κ is relatively low (e.g., 0.3 bits), the relative volatility μ is increasing with ϑ. The reason is that reducing ϑ has opposing effects on μ (or Π) for the two types of model misspecification: μ (Π) is decreasing (increasing) with ϑ for Type I misspecification and is increasing (decreasing) with ϑ for Type II misspecification. Note that here the effect due to Type I misspecification is direct and the effect due to the impact of ϑ on the robust Kalman gain θ and ω_η^2 is indirect. When κ is relatively low, the indirect channel dominates the direct channel because low κ amplifies the effect of ϑ on θ and thus μ. In contrast, when κ is relatively high (e.g., 0.5 bits), the relative volatility μ is decreasing with ϑ. The reason for this result is that in this case the direct effect dominates the indirect effect. To explore the intuition behind this result, we consider the perfect-state-observation case in which $\kappa = \infty$. In this case, the relative volatility of consumption growth to income growth reduces to

$$\mu = \frac{1}{1-\Pi}\sqrt{\frac{2}{1+\rho_s}}, \tag{47}$$

which clearly shows that ϑ increases the relative volatility via two channels. First, a higher ϑ increases the marginal propensity to consume out of permanent income $\left(\frac{R-1}{1-\Pi}\right)$, and second, it increases consumption volatility by reducing the persistence of permanent income measured by ρ_s: $\frac{\partial\rho_s}{\partial\Pi} < 0$.

In the presence of robustness, rational inattention measured by κ affects consumption volatility via two channels: (i) the gradual and smooth responses to income shocks (i.e., the $1-\rho_\theta \cdot L$ term in (45) and (ii) the RI-induced noises (ξ_t). Specifically, a reduction in capacity κ decreases the Kalman gain θ, which strengthens the smooth responses to income shock and increases the volatility of the RI-induced noise. Luo (2008) shows that the noise effect dominates the smooth response effect, and the volatility of consumption growth decreases with κ. The lower panel of Figure 5 illustrates how ϑ affects the relative volatility of consumption growth to income growth for different values of κ and there is no noise term. In this case, $\Delta c_t = \frac{(1-R)\Pi}{1-\Pi}(c_{t-1}-\overline{c}) + \frac{R-1}{1-\Pi}\frac{\theta\zeta_t}{1-\rho_\theta \cdot L}$. The reason is that reducing ϑ increases the robust Kalman gain θ, which leads to a more volatile consumption process because the smooth response effect completely dominates the noise effect.

When $\rho \neq 1$, we can also compute the relative volatility of consumption to income:

$$\mu_0 \equiv \frac{\mathrm{sd}(c_t)}{\mathrm{sd}(y_t)} = \frac{\theta(R-1)\sqrt{1-\rho^2}}{(1-\Pi)(R-\rho)}\sqrt{\frac{(1+\rho_s\rho_\theta)}{(1-\rho_s\rho_\theta)\left[(1+\rho_s\rho_\theta)^2-(\rho_s+\rho_\theta)^2\right]}\left[1+\frac{\rho_\theta(1+R^2)}{\theta R(1-\rho_\theta R)}\right]}, \tag{48}$$

which implies that $\frac{\partial\mu_0}{\partial\kappa} < 0$ and $\frac{\partial\mu_0}{\partial\vartheta} > 0$. That is, the relative volatility of consumption to income is increasing with the degree of inattention holding ϑ fixed because RI introduces endogenous noises. This result is the same as that for the relative volatility of consumption growth to income growth. In contrast, the relative volatility of consumption to income is increasing with the degree of RB holding κ fixed. The main reason is that robustness reduces the autocorrelation of consumption; for example, under FI-RE consumption is a random walk but RB makes it a stationary AR(1).

It is worth noting that from the expressions of μ or μ_0, we can obtain an interesting testable implication of our RB-RI model. Specifically, if we allow capacity to elastic as in Sims (2010), κ and θ are increasing functions of the fundamental uncertainty $\left(\omega_\zeta^2\right)$.[31] In this case, for given ϑ, consumption will be smoother when income becomes more volatile because greater income uncertainty makes consumers devote more attention to monitoring the state. On the other hand, if we use the detection error probabilities (DEP) as the deep RB parameter as in Hansen and Sargent (2007), ϑ will adjust in response to changes in income uncertainty to maintain the same level of DEP.[32] In this case, for given values of κ, greater income uncertainty leads to less values of ϑ to maintain the same value of DEP. (Here we follow Hansen and Sargent 2007 and assume that DEP is stable over time and across regions.) These theoretical results might provide a potential explanation for the empirical evidence documented in Blundell, Pistaferri, and Preston (2008) that income and consumption inequality diverged over the sampling period they study.[33]

3.2 Saving Process

Combining the original budget constraint, $b_{t+1} = Rb_t + y_t - c_t$, with the consumption function (42), we can obtain the following expression for individual saving d_t:

$$d_t \equiv b_{t+1} - b_t = \frac{-\Pi(R-1)}{1-\Pi}\left(b_t - \overline{b}\right) + \left(1 - \frac{R-1}{(1-\Pi)(R-\rho)}\right)(y_t - \overline{y}) + \varsigma_{t+1}, \tag{49}$$

[31] See Luo and Young (2014) for a proof.

[32] See Section 5 for a detailed discussion on the DEP calibration procedure.

[33] They create a new panel series of consumption that combines information from PSID and CEX, focusing on the period when some of the largest changes in income inequality occurred.

where the evolution of individual financial wealth (b_t) follows

$$b_{t+1} = \rho_s b_t + \frac{\Pi}{1-\Pi}\left(\overline{c}-\overline{y}\right) + \left(1 - \frac{R-1}{\left(1-\Pi\right)\left(R-\rho\right)}\right)\left(y_t - \overline{y}\right) + \varsigma_{t+1}, \tag{50}$$

where $\varsigma_{t+1} = \frac{R-1}{1-\Pi}\left(s_t - \widehat{s}_t\right)$ is determined by the estimation error, $s_t - \widehat{s}_t = \frac{(1-\theta)\zeta_t}{1-(1-\theta)R\cdot L} - \frac{\theta\xi_t}{1-(1-\theta)R\cdot L}$ and $\overline{b} = \frac{\overline{c}-\overline{y}}{R-1}$ is the steady state value of b_t. From (49), we have the following proposition for unconditional mean of individual saving:

Proposition 4 $E\left[d_t\right] = 0$. *That is, induced uncertainty due to the interaction of RB and RI does not affect the amount of individual saving on average.*

Proof. The proof is straightforward. ■

Furthermore, using (49), we can compute the relative volatility of individual savings to income. The following proposition provides the expression of this ratio.

Proposition 5 *The relative volatility of individual savings is*

$$\mu_d \equiv \frac{\mathrm{sd}\left(\Delta b\right)}{\mathrm{sd}\left(\Delta y\right)} = \sqrt{\frac{\left(1+\rho\right)\left(R-\rho\right)^2}{2}}\sqrt{\begin{array}{c}\frac{1-\rho}{1+\rho} + \frac{\Gamma^2}{1-\rho_s^2} + \left(\frac{R-1}{1-\Pi}\right)^2 \frac{1-\theta}{1-R^2(1-\theta)} \\ +\frac{2(1-\rho)\Gamma}{1-\rho\rho_s} + \left(\frac{R-1}{1-\Pi}\right)\frac{2(1-\rho)(1-\theta)}{1-\rho\rho_\theta} + \left(\frac{R-1}{1-\Pi}\right)\frac{2\Gamma(1-\theta)}{1-\rho_s\rho_\theta}\end{array}}, \tag{51}$$

where $\Gamma = -\frac{(R-1)\Pi}{1-\Pi} < 0$, $\rho_s = \frac{1-R\Pi}{1-\Pi}$, *and* $\Pi = \frac{\theta(\vartheta,\kappa)}{1-(1-\theta(\vartheta,\kappa))R^2} R\omega_\zeta^2/\vartheta$.

Proof. See Online Appendix. ■

The complexity of this expression prevents us from obtaining clear results about how RI and RB affect the variance of individual savings. Figure 6 illustrates the effects of RB (ϑ) on the relative volatility for different values of κ, when $\omega_\zeta^2 = 1$ and $\rho = 0.9$. We can see from the figure that μ_d is increasing with channel capacity (κ) and is decreasing with robustness (ϑ). The main reason for this result is that consumers are more information-constrained or are less concerned about model specification, and saving is treated as a residual in the consumption-saving problem with a given income process.[34] If we now consider an aggregate economy with a continuum of ex ante identical inattentive consumers with the same preference for robustness and each of them has the consumption function (42), then the total saving demand in the economy is equal to zero. The intuition is simple. The saving function can be expressed as a combination of different types of income and noise shocks: ς, ε or ζ, and ξ, and all of these shocks are idiosyncratic. These idiosyncratic shocks cancel out after aggregating across consumers and therefore have no effect on aggregate savings.

[34] Note that from (48), we have $\frac{\partial\mu_0}{\partial\kappa} < 0$ and $\frac{\partial\mu_0}{\partial\vartheta} > 0$.

4 Comparison with Risk-sensitive Control and Filtering

Risk-sensitivity (RS) was first introduced into the LQ-Gaussian framework by Jacobson (1973) and extended by Whittle (1990). Exploiting the recursive utility framework of Epstein and Zin (1989), Hansen and Sargent (1995) introduce discounting into the RS specification and show that the resulting decision rules are time-invariant. In the RS model agents effectively compute expectations through a distorted lens, increasing their effective risk aversion by overweighting negative outcomes. The resulting decision rules depend explicitly on the variance of the shocks, producing precautionary savings, but the value functions are still quadratic functions of the states.[35] Hansen, Sargent, and Tallarini (1999, henceforth, HST) and Hansen and Sargent (2007) interpret RS preferences in terms of a concern about model uncertainty (robustness or RB) and argue that RS introduces precautionary savings because RS consumers want to protect themselves against model specification errors. In the corresponding risk-sensitive filtering LQ problem, the problem is that when the state cannot be observed perfectly, is the classical Kalman filter that minimizes the expected loss function still optimal? In our LQ-PIH model setting, we can easily see that the regular Kalman filter is still optimal given the quadratic forms of the utility function and the value function.[36] In this section we will explore how the RS filtering affects consumption dynamics and precautionary savings and show that the OE between RB and RS is no longer linear, but takes a more complicated non-linear form. The RI version of risk-sensitive control based on recursive preferences with an exponential certainty equivalence function can be formulated as

$$\widehat{v}\left(\widehat{s}_t\right) = \max_{c_t}\left\{-\frac{1}{2}\left(c_t - \overline{c}\right)^2 + \beta\mathcal{R}_t\left[\widehat{v}\left(\widehat{s}_{t+1}\right)\right]\right\} \tag{52}$$

subject to the Kalman filter equation (13).[37] The distorted expectation operator is now given by

$$\mathcal{R}_t\left[\widehat{v}\left(\widehat{s}_{t+1}\right)\right] = -\frac{1}{\alpha}\log E_t\left[\exp\left(-\alpha\widehat{v}\left(\widehat{s}_{t+1}\right)\right)\right],$$

[35] Formally, one can view risk-sensitive agents as ones who have non-state-separable preferences, as in Epstein and Zin (1989), but with a value for the intertemporal elasticity of substitution equal to one.

[36] As shown in Moore, Elliott, and Dey (1997), even if the agent has risk-sensitive preferences when filtering,

$$\min \ln E_t\left\{\exp\left[-\vartheta\left(s_t - \widehat{s}_t^{RS}\right)^2\right]\right\},$$

the risk-sensitive estimate $\widehat{s}_t^{RS}$ is identical to the minimum variance estimate $\widehat{s}$ obtained from solving

$$\min E_t\left[\left(s_t - \widehat{s}_t\right)^2\right].$$

[37] Given the quadratic form of the value function, introducing risk-sensitivity does not change the optimality of the *ex post* Gaussianity of the true state and the induced noise; see Luo and Young (2010) for more discussion.

where $s_0 | \mathcal{I}_0 \sim N\left(\widehat{s}_0, \overline{\sigma}^2\right)$, $\widehat{s}_t = E_t[s_t]$ is the perceived state variable, θ is the optimal weight on the new observation of the state, and ξ_{t+1} is the endogenous noise. The optimal choice of the weight θ is given by $\theta(\kappa) = 1 - 1/\exp(2\kappa) \in [0,1]$. It is worth noting that given that the value function in the RS model is quadratic, the regular Kalman filter is still optimal because the objective function in the filtering problem is the square of the estimation error.

Following the same procedure used in Hansen and Sargent (1995) and Luo and Young (2010), we can solve this risk-sensitive control problem explicitly. The following proposition summarizes the solution to the RI-RS model when $\beta R = 1$:

Proposition 6 *Given finite channel capacity κ and the degree of risk-sensitivity α, the consumption function of a risk-sensitive consumer under RI is*

$$c_t = \frac{R-1}{1-\Pi}\widehat{s}_t - \frac{\Pi\overline{c}}{1-\Pi}, \tag{53}$$

where

$$\Pi = R\alpha\omega_\eta^2 \in (0,1), \tag{54}$$

ω_η^2 is defined in (16), and $\theta(\kappa) = 1 - 1/\exp(2\kappa)$.

Proof. See Online Appendix. ■

Comparing (20) obtained from the model with only concerns about the innovation to the perceived state (i.e., without robust Kalman filtering) in Section 2.2 and (53), it is straightforward to show that RB and RS under RI are indistinguishable using only consumption-savings decisions if

$$\alpha = \frac{1}{\vartheta}. \tag{55}$$

Note that (55) is exactly the same as the observational equivalence condition obtained in the full-information RE model (see Backus, Routledge, and Zin 2004). That is, under the assumption that the agent trusts the Kalman filter equation, the OE result obtained under full-information RE still holds under RI.[38]

Hansen, Sargent, and Tallarini (1999) show that as far as the quantity observations on consumption and savings are concerned, the robustness version ($\vartheta > 0$ or $\alpha > 0, \widetilde{\beta}$) of the PIH model is observationally equivalent to the standard version ($\vartheta = \infty$ or $\alpha = 0, \beta = 1/R$) of the PIH model for a unique pair of discount factors.[39] The intuition is that introducing a preference for risk-sensitivity (RS) or a concern about robustness (RB) increases savings in the same way as

[38]The states are different, of course – s_t vs. $\widehat{s}_t$.

[39]HST (1999) derive the observational equivalence result by fixing all parameters, including R, except for the pair (α, β).

increasing the discount factor, so that the discount factor can be changed to offset the effect of a change in RS or RB on consumption and investment.[40] Alternatively, holding all parameters constant except the pair (α, β), the RI version of the PIH model with RB consumers ($\vartheta > 0$ and $\beta R = 1$) is observationally equivalent to the standard RI version of the model ($\vartheta_0 = \infty$ and $\widetilde{\beta} > 1/R$). To do so, we compare the consumption function obtained from the RI model ($\vartheta = \infty$ and $\widetilde{\beta} > 1/R$), $c_t = \left(R - \frac{1}{\beta R}\right) \widehat{s}_t - \frac{1}{R-1}\left(1 - \frac{1}{\beta R}\right) \overline{c}$, with (42) and (53), and find that when $\widetilde{\beta} = \frac{1}{R} \frac{1-R\omega_\eta^2/\vartheta}{1-R^2\omega_\eta^2/\vartheta} = \frac{1}{R} \frac{1-R\alpha\omega_\eta^2}{1-R^2\alpha\omega_\eta^2} > \frac{1}{R}$, consumption and savings are identical in the RI, RB-RI, and RS-RI models.

However, if we compare (42) obtained from the model with both concerns about the innovation to the perceived state and concerns about Kalman gain with (53), it is obvious that the observational equivalence between RB and RS under RI, (55), no longer holds. Given the same value of κ, the Kalman gain only depends on κ in the RS model, whereas it depends on both κ and ϑ (the preference for robust Kalman gain) in the RB model. The two Kalman gains are therefore different for any finite value of ϑ. If we allow for different values of κ, the models are observationally equivalent when $\alpha = \vartheta^{-1}$ and

$$\theta\left(\vartheta, \kappa_{RB}\right) = 1 - \frac{1}{\exp(2\kappa_{RS})}.$$

Figure 7 illustrates how κ_{RS} varies with ϑ and κ_{RB} when the OE between RB and RS holds under RI. It clearly shows that given the level of ϑ, κ_{RS} is increasing with κ_{RB}.

5 Market Price of Induced Uncertainty

The PIH model presented in Section 2.2 is usually regarded as a partial equilibrium model. However, as noted in Hansen (1987) and HST (1999), it can be interpreted as a general equilibrium model with a linear production technology and an exogenous income process. Given the expression of optimal consumption in terms of the state variables derived from the robust version of the PIH model with inattentive agents, we can price assets by treating the process of aggregate consumption that solves the model as though it were an endowment process. In this setup, equilibrium prices are shadow prices that leave the agent content with that endowment process. HST (1999) study how robustness and risk-sensitivity affect the predicted market price of risk within a PIH model with shocks to both labor income and preferences, and find that RB or RS significantly alter the model's predictions on the market price of risk and thus provides an alternative explanation for the equity premium puzzle.

[40] As shown in HST (1999), the two models have different implications for asset prices because continuation valuations change as one varies (α, β) within the observationally-equivalent set of parameters.

In this section, using the optimal consumption and saving decisions derived in the previous sections, we will explore how induced uncertainty due to the interactions of RB and RI with income shocks affect the market price of uncertainty. We first consider the single-period asset pricing case. In this case, we assume that the agent purchases a security at period t at a price q_t, holds it for one period, and then sells it at $t+1$ for a total payoff ϕ_{t+1} in terms of the consumption good after collecting the dividend. Under this assumption, the following Euler equation holds:

$$q_t = \widetilde{E}_t \left[\left(\beta \frac{u'(\widehat{s}_{t+1})}{u'(\widehat{s}_t)} \right) \phi_{t+1} \right], \tag{56}$$

where $\beta \frac{u'(\widehat{s}_{t+1})}{u'(\widehat{s}_t)}$ is the stochastic discount factor (SDF) and $\widetilde{E}_t [\cdot]$ is the distorted conditional expectations operator. Note that here SDF depends on the perceived states because optimal consumption is a linear function of perceived permanent income; because $\widehat{s}_{t+1}$ is a function of the true state s_{t+1}, s_t will also affect the SDF. The corresponding formula for q_t in terms of the original conditional expectations operator can be written as

$$q_t = E_t [m_{t,t+1} \phi_{t+1}], \tag{57}$$

where $m_{t,t+1}$ depends not only on the usual SDF but also on robustness. As has been shown in Hansen, Sargent, and Tallarini (1999), RB or RS are reflected in the usual measure of the SDF being scaled by a random variable with conditional mean 1. They also show that this multiplicative adjustment to the SDF increases the volatility of the SDF and thus drives up the risk premium. To explore the effects of induced uncertainty on the market price of uncertainty, we write (57) as

$$q_t = E_t [\phi_{t+1}] E_t [m_{t,t+1}] + \mathrm{cov}_t (m_{t,t+1}, \phi_{t+1}),$$

which leads to the following price bound:

$$q_t \geq E_t [\phi_{t+1}] E_t [m_{t,t+1}] - \mathrm{sd}_t (m_{t,t+1}) \, \mathrm{sd}_t (\phi_{t+1}),$$

where $\mathrm{sd}_t(\cdot)$ denotes the conditional standard deviation. If we define the market price of uncertainty (MPU) as $\mathrm{MPU} \equiv \frac{\mathrm{sd}_t(m_{t,t+1})}{E_t[m_{t,t+1}]}$, the Hansen-Jagannathan (HJ) bound can be rewritten as

$$\mathrm{MPU} \geq \frac{E_t [\phi_{t+1}/q_t]}{\mathrm{sd}_t (\phi_{t+1}/q_t)},$$

where the RHS is the Sharpe ratio and is above 0.2 for most industrial countries.[41] In the standard full-information state- and time-separable utility model, the value of MPU is an order

[41] In the U.S. data presented in Campbell (2002), the Sharpe ratio is about 0.52 (annualized) during $1947 - 1998$. Using a longer annual U.S. time series put together by Shiller yields a similar value of the Sharpe ratio.

of magnitude lower than what is required for this inequality to be satisfied, which is just another manifestation of the equity premium puzzle: consumption growth is smooth, uncorrelated with returns, and has near zero autocorrelation, leading to a small cost of bearing uncertainty.[42]

5.1 MPU under RB and RI

The SDF, $m_{t,t+1}$, can be decomposed into

$$m_{t,t+1} = m_{t,t+1}^{f} m_{t,t+1}^{rb},$$

where $m_{t,t+1}^{f} \equiv \beta \frac{u'(\widehat{s}_{t+1})}{u'(\widehat{s}_t)}$ is the "familiar" stochastic discount factor ($\vartheta = \infty$) and $m_{t,t+1}^{rb}$ is the Radon-Nikodym derivative, or the likelihood ratio of the distorted conditional probability of $\widehat{s}_{t+1}$ with respect to the approximating conditional probability. Under the two-player game specification of RB, (19), asset prices are computed using the pessimistic view of the next period's shock: $\widetilde{\eta}_{t+1} = \omega_\eta \widetilde{\epsilon}_{t+1} = \omega_\eta (\epsilon_{t+1} + w_t)$, where $\widetilde{\epsilon}_{t+1}$ is a normally distributed variable with mean $\omega_\eta w_t$ and variance ω_η^2. In this case, the Radon-Nikodym derivative can be written as

$$m_{t,t+1}^{rb} \equiv \frac{\exp\left(-\left(\widetilde{\epsilon}_{t+1} - w_t\right)^2/2\right)}{\exp\left(-\widetilde{\epsilon}_{t+1}^2/2\right)} = \exp\left(\widetilde{\epsilon}_{t+1} w_t - w_t^2/2\right).$$

By construction, we obtain $E_t\left[m_{t,t+1}^{rb}\right] = 1$. By straightforward calculations, we obtain the following conditional second moment of $m_{t,t+1}^{rb}$ as a means for computing its conditional variance:

$$E_t\left[\left(m_{t,t+1}^{rb}\right)^2\right] = \exp\left(w_t^2\right). \tag{58}$$

The following proposition summarizes the result on how induced uncertainty affects the market price of uncertainty.

Proposition 7 *The expression for the market price of induced uncertainty is*

$$\mathrm{sd}_t\left(m_{t,t+1}^{rb}\right) = \sqrt{\exp\left(w_t^2\right) - 1} \cong |w_t| \tag{59}$$

for small distortions, where w_t is the mean of the worse-case shock:

$$w_t = \Theta\left[(R-1)\widehat{s}_t - \overline{c}\right] \tag{60}$$

and $\Theta = \frac{\Pi/\omega_\eta}{1-\Pi}$.

[42] The standard deviation of aggregate consumption growth is 0.84 percent, the correlation with real returns on the S&P500 Index is 0.22, and the autocorrelation is 0.08, using nondurables and services deflated by the PCE deflator.

Proof. Given $E_t\left[m_{t,t+1}^{rb}\right] = 1$, we can obtain (59) using (58). It is also straightfoward to derive (60) using (21). ■

Expression (59) clearly shows that the amount of market price of uncertainty contributed by the Radon-Nikodym derivative is approximately equal to the norm of the mean of the worse-case shock (w).[43] Note that Θ can be used to measure the importance of RB and RI in determining the market price of uncertainty for given $\widehat{s}_t$ and $\partial\Theta/\partial\Pi > 0$. Specifically, both ϑ and κ influence $m_{t,t+1}^{rb}$ through their effects on Π.[44] Lowering ϑ strengthens the preference for robustness and then drive $m_{t,t+1}^{rb}$ away from 1 by increasing Π. Lowering κ reduces the Kalman gain, and then increases ω_η and Π. However, since the evolution of $\widehat{s}_t$ is also affected by ω_η and Π, we have to take both the Θ term and the $(R-1)\,\widehat{s}_t - \overline{c}$ term in (60) into account when evaluating how the interaction of RB and RI affects the market price of uncertainty.

To fully explore how induced uncertainty due to RB and RI affects the market price of uncertainty, we adopt the calibration procedure outlined in HSW (2002), AHS (2003), and Hansen and Sargent (Chapter 9, 2007) to calibrate the value of Π that summarizes the interaction between RB and RI. Specifically, we calibrate Π by using the notion of a model detection error probability that is based on a statistical theory of model selection. We can then infer what values of the RB parameter imply reasonable fears of model misspecification for empirically-plausible approximating models. The model detection error probability denoted by p is a measure of how far the distorted model can deviate from the approximating model without being discarded; low values for this probability mean that agents are unwilling to discard very many models, implying that the cloud of models surrounding the approximating model is large (since agents want errors to be rare, they push the two models very far apart). The value of p is determined by the following procedure. Let model A denote the approximating model, (13), and model B be the distorted model, (17). Define p_A as

$$p_A = \text{Prob}\left(\ln\left(\frac{L_A}{L_B}\right) < 0 \,\middle|\, A\right), \tag{61}$$

where $\ln\left(\frac{L_A}{L_B}\right)$ is the log-likelihood ratio. When model A generates the data, p_A measures the probability that a likelihood ratio test selects model B. In this case, we call p_A the probability of the model detection error. Similarly, when model B generates the data, we can define p_B as

$$p_B = \text{Prob}\left(\log\left(\frac{L_A}{L_B}\right) > 0 \,\middle|\, B\right). \tag{62}$$

[43] In other words, $|w_t|$ is an upper bound on the approximate enhancement to the market price of uncertainty caused by the interaction of RB and RI.

[44] When $\kappa = \infty$, i.e., no RI, (60) reduces to $w_t = \frac{\Pi/\omega_\eta}{1-\Pi}\left[(R-1)\,s_t - \overline{c}\right]$. Without RB, $w_t = 0$.

The detection error probability, p, is defined as the average of p_A and p_B:

$$p(\vartheta;\Pi) = \frac{1}{2}(p_A + p_B), \tag{63}$$

where ϑ is the robustness parameter used to generate model B. Given this definition, we can see that $1-p$ measures the probability that econometricians can distinguish the approximating model from the distorted model. Now we show how to compute the model detection error probability in the RB model. The general idea of the calibration exercise is to find a value of (or Π) such that $p(\vartheta_0;\Pi)$ equals a given value (for example, 5 percent or 10 percent) after simulating model A, (13), and model B, (17).[45]

Following the consumption and saving literature, we set $R = 1.02$, $\omega/y_0 = 0.15$, $\rho = 0.8$, and $\bar{c} = 3.5y_0$. Using these parameter values, Figures 8 shows how p affects the mean of MPU under RB and RI.[46] Using either the data set documented in Campbell (2002) or that provided by Shiller, the estimated Sharpe ratio for the postwar U.S. time series is greater than 50 percent a year. Figure 9 plots the Hansen-Jagannathan (HJ) bound under RB and RI when $\theta = 0.1$ using the Shiller data set. It is clear from this figure that the model's predicted MPU can enter the HJ bound when both the preference for RB and the degree of RI are strong, i.e., when $p = 0.05$ or $p = 0.1$ and $\theta = 0.1$. If we consider the international developed-country data set in Campbell (2002), the Sharpe ratio is between 15 percent and 20 percent for Australia and Italy, between 20 percent and 30 percent for Canada and Japan, and above 30 percent for all the other countries. In the long-run annual data sets the lower bound on the standard deviation exceeds 30% for all three countries. From Figure 9, we can see that the theoretical MPU satisfies the HJ bound for higher values of p. Figure 10 clearly shows that our results about how the interaction of RB and RI with income uncertainty affects the market price of uncertainty are robust to the changes in the values of R, ω/y_0, ρ, and $\bar{c}$.

Since θ is a critical parameter, we show explicitly in Figure 11 how the predictions for each value of p vary with θ. It is clear that a low value of θ is important for satisfying the HJ bounds; as we reduce θ we reduce the required level of p, so that the worst-case model can move closer to the approximating model (that is, we can reduce the required fear of model misspecification).

[45] The number of periods used in the simulation, T, is set to be the actual length of the data we study. For example, if we consider the post-war U.S. annual time series data provided by Shiller from $1946-2010$, $T = 65$.

[46] Because the effects of RB and RI on the mean and median of MPU are quite similar, we focus on the mean of MPU in our subsequent analysis.

5.2 MPU under RS and RI

Under the risk-sensitivity specification of RB, (24), we can also compute the corresponding market price of uncertainty. The intertemporal marginal rate of substitution between t and $t+1$ can be written as

$$m_{t,t+1} = m_{t,t+1}^{f} m_{t,t+1}^{rs},$$

where $m_{t,t+1}^{f} \equiv \beta \frac{u'(\widehat{s}_{t+1})}{u'(\widehat{s}_t)}$ and $m_{t,t+1}^{rs} \equiv \frac{\exp(-v_{t+1}/\vartheta)}{E_t[\exp(-v_{t+1}/\vartheta)]} = \frac{\exp\left(-\left(\Omega \widehat{s}_{t+1}^2+\rho\right)/\vartheta\right)}{\exp\left(-\left(\widetilde{\Omega} \widehat{s}_t^2+\widetilde{\rho}\right)/\vartheta\right)}$. Using a formula found in Jacobson (1973) and used in HST (1999), we have $E_t\left[\left(m_{t,t+1}^{rs}\right)^2\right] = \exp\left(-2\left[\left(\widetilde{\Omega}-\widehat{\Omega}\right)\widehat{s}_t^2 + \left(\widetilde{\rho}-\widehat{\rho}\right)\right]/\vartheta\right)$, because $E_t\left[\exp\left(-2\left(\Omega \widehat{s}_{t+1}^2+\rho\right)/\vartheta\right)\right] = \exp\left(-2\left(\widetilde{\Omega}\widehat{s}_t^2+\widetilde{\rho}\right)/\vartheta\right)$. The following proposition summarizes the result on how induced uncertainty affects the amount of market price of uncertainty under the RS specification of RB.

Proposition 8 *Under RS-RI, the Radon-Nikodym derivative is*

$$E_t\left[\left(m_{t,t+1}^{rs}\right)^2\right] = \exp\left(\Xi^{rs}\left[(R-1)\,\widehat{s}_t - \overline{c}\right]^2\right)\Upsilon, \tag{64}$$

where $\Xi^{rs} \equiv \frac{R\Pi(1-R\Pi)}{\vartheta_0(1-\Pi)^2[1-(2R-1)\Pi]}$ *and* $\Upsilon \equiv \frac{1-(R-1)\Pi/(1-\Pi)}{\sqrt{1-2(R-1)\Pi/(1-\Pi)}}$. *The market price of induced uncertainty is*

$$\mathrm{sd}_t\left(m_{t,t+1}^{rs}\right) = \sqrt{\exp\left(\Xi^{rs}\left((R-1)\,\widehat{s}_t - \overline{c}\right)^2\right)\Upsilon - 1} \cong \left|\sqrt{\Xi^{rs}}\left((R-1)\,\widehat{s}_t - \overline{c}\right)\right| \tag{65}$$

Proof. See Appendix 7.2. ■

Denote $\Xi^{rb} = \Theta^2$, we have

$$\Delta \equiv \sqrt{\frac{\Xi^{rs}}{\Xi^{rb}}} = \sqrt{\frac{1-R\Pi}{1-(2R-1)\,\Pi}}, \tag{66}$$

which is close 1 when R is close 1 and Π is not too large. Using (66), it is straightforward to show that $\frac{\partial\Delta}{\partial\Pi} > 0$ and $\frac{\partial\Delta}{\partial\theta} < 0$, i.e., the stronger the degree of RI, the larger the difference of the market price of uncertainty under the RB and RS specifications. Figure 12 illustrates how Δ varies with Π for given values of R. It is clear that theoretically the difference of the market price of uncertainty between RB and RS under RI can be very significant. For example, when $R = 1.02$, $\Delta = 1.25$ when $\Pi = 0.93$. In other words, the MPU under the RS-RI specification is 25 percent higher than that under the RB-RI specification when the two models are observationally equivalent. However, after calibrating empirically-plausible ϑ (and Π) using the DEP, $\mathrm{sd}_t\left(m_{t,t+1}^{rb}\right)$ and $\mathrm{sd}_t\left(m_{t,t+1}^{rs}\right)$ are very close because R is close to 1 and the calibrated values of Π are between 0.1 and 0.2 for $p = 0.05$.[47] Figure 13 clearly shows that the two specifications have similar effects

[47] The calibrated values of Π are lower for higher values of p.

on the mean of for MPU different values of p and θ. In addition, Figure 14 plots the HJ bound under RS and RI when $\theta = 0.1$ using the same data set as in the above RB-RI specification, and it is clear that this HJ bound is similar to that shown in Figure 9.

6 Conclusion

This paper has provided a characterization of the consumption-savings behavior of agents who have a preference for robustness (worries about model misspecification) and limited information-processing ability. After obtaining the optimal individual decisions, we explore how two types of induced uncertainty, state uncertainty due to RI and model uncertainty due to RB, affect consumption and saving decisions as well as the market prices of uncertainty. Specifically, we show that concerns about different types of model misspecification – (i) disturbances to the perceived permanent income and (ii) the Kalman gain – can have opposite effects on consumption, savings, and asset prices via interacting with finite capacity in the control and filtering problems. In addition, we show that once allowing RB consumers to use the robust Kalman filter to update the perceived state, the simple observational equivalence (OE) between RB and RS obtained in HST (1999) no longer holds; instead, we find a more complicated OE between RB and RS. Finally, we explore how the two types of informational frictions affect the market price of risk in general equilibrium.

7 Appendix (For Online Publication)

7.1 Solving the Two-Player Game Version of the Robust Model

To solve the Bellman equation (19) subject to (17), $\widehat{s}_{t+1} = R\widehat{s}_t - c_t + \omega_\eta w_t + \eta_{t+1}$, we conjecture that

$$v(\widehat{s}_t) = -C - B\widehat{s}_t - A\widehat{s}_t^2, \tag{67}$$

where A, B, and C are constants to be determined. Substituting this guessed value function into the Bellman equation (19) gives

$$-C - B\widehat{s}_t - A\widehat{s}_t^2 = \max_{c_t} \min_{w_t} \left\{ -\frac{1}{2}(c_t - \overline{c})^2 + \beta E_t \left[\widetilde{\vartheta}_0 w_t^2 - C - B\widehat{s}_{t+1} - A\widehat{s}_{t+1}^2 \right] \right\}, \tag{68}$$

where $\widetilde{\vartheta}_0 = \vartheta_0/2$. We can do the min and max operations in any order, so we choose to do the minimization first. The first-order condition for w_t is

$$2\vartheta\nu_t - 2AE_t(\omega_\eta w_t + R\widehat{s}_t - c_t)\omega_\eta - B\omega_\eta = 0,$$

which means that

$$w_t = \frac{B + 2A\left(R\widehat{s}_t - c_t\right)}{2\left(\widetilde{\vartheta}_0 - A\omega_\eta^2\right)}\omega_\eta. \tag{69}$$

Substituting (69) back into (68) gives

$$-A\widehat{s}_t^2 - B\widehat{s}_t - C = \max_{c_t}\left\{-\frac{1}{2}\left(\overline{c} - c_t\right)^2 + \beta E_t\left[\widetilde{\vartheta}_0\left[\frac{B + 2A\left(Rs_t - c_t\right)}{2\left(\vartheta - A\omega_\eta^2\right)}\omega_\eta\right]^2 - As_{t+1}^2 - Bs_{t+1} - C\right]\right\},$$

where $\widehat{s}_{t+1} = R\widehat{s}_t - c_t + \omega_\eta w_t + \eta_{t+1}$. The first-order condition for c_t is

$$\left(\overline{c} - c_t\right) - 2\beta\widetilde{\vartheta}_0\frac{A\omega_\eta}{\vartheta - A\omega_\eta^2}w_t + 2\beta A\left(1 + \frac{A\omega_\eta^2}{\widetilde{\vartheta}_0 - A\omega_\eta^2}\right)\left(R\widehat{s}_t - c_t + \omega_\eta w_t\right) + \beta B\left(1 + \frac{A\omega_\eta^2}{\widetilde{\vartheta}_0 - A\omega_\eta^2}\right) = 0.$$

Using the solution for ν_t the solution for consumption is

$$c_t = \frac{2A\beta R}{1 - A\omega_\eta^2/\widetilde{\vartheta}_0 + 2\beta A}\widehat{s}_t + \frac{\overline{c}\left(1 - A\omega_\eta^2/\widetilde{\vartheta}_0\right) + \beta B}{1 - A\omega_\eta^2/\widetilde{\vartheta}_0 + 2\beta A}.$$

Substituting the above expressions into the Bellman equation gives

$$\begin{aligned}
&- A\widehat{s}_t^2 - B\widehat{s}_t - C \\
&= -\frac{1}{2}\left(\frac{2A\beta R}{1 - A\omega_\eta^2/\widetilde{\vartheta}_0 + 2\beta A}\widehat{s}_t + \frac{-2\beta A\overline{c} + \beta B}{1 - A\omega_\eta^2/\widetilde{\vartheta}_0 + 2\beta A}\right)^2 \\
&+ \frac{\beta\widetilde{\vartheta}_0\omega_\eta^2}{\left(2\left(\widetilde{\vartheta}_0 - A\omega_\eta^2\right)\right)^2}\left[\frac{2AR\left(1 - A\omega_\eta^2/\widetilde{\vartheta}_0\right)}{1 - A\omega_\eta^2/\widetilde{\vartheta}_0 + 2\beta A}\widehat{s}_t + B - \frac{2\overline{c}\left(1 - A\omega_\eta^2/\widetilde{\vartheta}_0\right)A + 2\beta AB}{1 - A\omega_\eta^2/\widetilde{\vartheta}_0 + 2\beta A}\right]^2 \\
&- \beta A\left\{\left[\frac{R}{1 - A\omega_\eta^2/\widetilde{\vartheta}_0 + 2\beta A}\widehat{s}_t - \frac{-B\omega_\eta^2/\widetilde{\vartheta}_0 + 2c + 2B\beta}{2\left(1 - A\omega_\eta^2/\widetilde{\vartheta}_0 + 2\beta A\right)}\right]^2 + \omega_\eta^2\right\} \\
&- \beta B\left[\frac{R}{1 - A\omega_\eta^2/\widetilde{\vartheta}_0 + 2\beta A}\widehat{s}_t - \frac{-B\omega_\eta^2/\widetilde{\vartheta}_0 + 2c + 2B\beta}{2\left(1 - A\omega_\eta^2/\widetilde{\vartheta}_0 + 2\beta A\right)}\right] - \beta C.
\end{aligned}$$

Collecting and matching terms, the constant coefficients turn out to be

$$A = \frac{\beta R^2 - 1}{2\beta - \omega_\eta^2/\widetilde{\vartheta}_0}, B = \frac{\left(\beta R^2 - 1\right)\overline{c}}{(R-1)\left(\omega_\eta^2/\left(\widetilde{\vartheta}_0\right) - \beta\right)}, C = \frac{R\left(\beta R^2 - 1\right)}{2\left(\beta R - R\omega_\eta^2/\widetilde{\vartheta}_0\right)(R-1)^2}\left((R-1)\,\omega_\eta^2 + \overline{c}^2\right),$$

where $\widetilde{\vartheta}_0 = \vartheta_0/2$. When $\beta R = 1$, we obtain the consumption function (20) in the text.

7.2 Computing the Market Price of Uncertainty

Given the value function we obtained in Section (2.2),

$$v(\widehat{s}_t) = \Omega \left(\widehat{s}_t - \frac{\overline{c}}{R-1} \right)^2 + \rho, \tag{70}$$

where $\Omega = -\frac{R(R-1)}{2(1-\Pi)}$ and $\rho = \frac{\vartheta_0}{2(R-1)} \ln \left(1 - \frac{(R-1)\Pi}{1-\Pi}\right)$, it follows from Jacobson (1973) and HST (1999) that the risk-sensitivity operator can be written as

$$\mathcal{R}_t \left[\widehat{v} \left(\widehat{s}_{t+1} \right) \right] = -\vartheta_0 \log E_t \left[\exp \left(-\widehat{v} \left(\widehat{s}_{t+1} \right) / \vartheta_0 \right) \right] = \widehat{\Omega} \left(\widehat{s}_t - \frac{\overline{c}}{R-1} \right)^2 + \widehat{\rho},$$

where

$$\widehat{\Omega} = \rho_s^2 \Omega \left(1 - \frac{2}{\vartheta_0} \Omega \omega_\eta^2 \left(1 + \frac{2}{\vartheta_0} \Omega \omega_\eta^2 \right)^{-1} \right) = -\frac{R(R-1)(1-R\Pi)}{2(1-\Pi)^2}$$

and

$$\widehat{\rho} = \rho + \frac{\vartheta_0}{2} \ln \left(1 + \frac{2}{\vartheta_0} \Omega \omega_\eta^2 \right) = \frac{\vartheta_0}{2(R-1)} \ln \left(1 - \frac{2(R-1)\Pi}{1-\Pi} \right) + \frac{\vartheta_0}{2} \ln \left(1 - \frac{(R-1)\Pi}{(1-\Pi)} \right),$$

where we assume that $\frac{2(R-1)\Pi}{1-\Pi} < 1$.

Given that

$$m_{t,t+1}^{rs} \equiv \frac{\exp(-v_{t+1}/\vartheta_0)}{E_t \left[\exp(-v_{t+1}/\vartheta_0) \right]} = \frac{\exp \left(-\left(\Omega \widehat{s}_{t+1}^2 + \rho \right) / \vartheta_0 \right)}{\exp \left(-\left(\widehat{\Omega} \widehat{s}_t^2 + \widehat{\rho} \right) / \vartheta_0 \right)},$$

we have

$$\left(m_{t,t+1}^{rs} \right)^2 = \frac{\exp \left(-2 \left(\Omega \widehat{s}_{t+1}^2 + \rho \right) / \vartheta_0 \right)}{\exp \left(-2 \left(\widehat{\Omega} \widehat{s}_t^2 + \widehat{\rho} \right) / \vartheta_0 \right)}.$$

Multiplying the numerator and denominator by the time t conditional mean of the exponential term in the numerator, $E_t \left[\exp \left(-2 \left(\Omega \widehat{s}_{t+1}^2 + \rho \right) / \vartheta_0 \right) \right]$, gives

$$\left(m_{t,t+1}^{rs} \right)^2 = \frac{E_t \left[\exp \left(-2 \left(\Omega \widehat{s}_{t+1}^2 + \rho \right) / \vartheta_0 \right) \right]}{\exp \left(-2 \left(\widehat{\Omega} \widehat{s}_t^2 + \widehat{\rho} \right) / \vartheta_0 \right)} \frac{\exp \left(-2 \left(\Omega \widehat{s}_{t+1}^2 + \rho \right) / \vartheta_0 \right)}{E_t \left[\exp \left(-2 \left(\Omega \widehat{s}_{t+1}^2 + \rho \right) / \vartheta_0 \right) \right]},$$

where the exponential term, $E_t \left[\exp \left(\sigma \left(\Omega \widehat{s}_{t+1}^2 + \rho \right) \right) \right]$, can be computed using a formula found in Jacobson (1973):

$$E_t \left[\exp \left(-2 \left(\Omega \widehat{s}_{t+1}^2 + \rho \right) / \vartheta_0 \right) \right] = \exp \left(-2 \left(\widetilde{\Omega} \widehat{s}_t^2 + \widetilde{\rho} \right) / \vartheta_0 \right),$$

where $\widetilde{\Omega} = \rho_s^2 \Omega \left(1 - \frac{4}{\vartheta_0} \Omega \omega_\eta^2 \left(1 + \frac{4}{\vartheta_0} \Omega \omega_\eta^2 \right)^{-1} \right) = \frac{R \rho_s^2 \Omega}{R + 4\Omega\Pi}$ and $\widetilde{\rho} = \rho + \frac{\vartheta_0}{4} \ln \left(1 + \frac{1}{\vartheta_0} \Omega \omega_\eta^2 \right) = \rho + \frac{\vartheta_0}{4} \ln \left(1 - \frac{2(R-1)\Pi}{(1-\Pi)} \right)$. Therefore, we obtain

$$E_t \left[\left(m_{t,t+1}^{rs} \right)^2 \right] = \exp \left(-2 \left(\left(\widetilde{\Omega} - \widehat{\Omega} \right) \widehat{s}_t^2 + \left(\widetilde{\rho} - \widehat{\rho} \right) \right) / \vartheta_0 \right),$$

which yields (64) in the main text.

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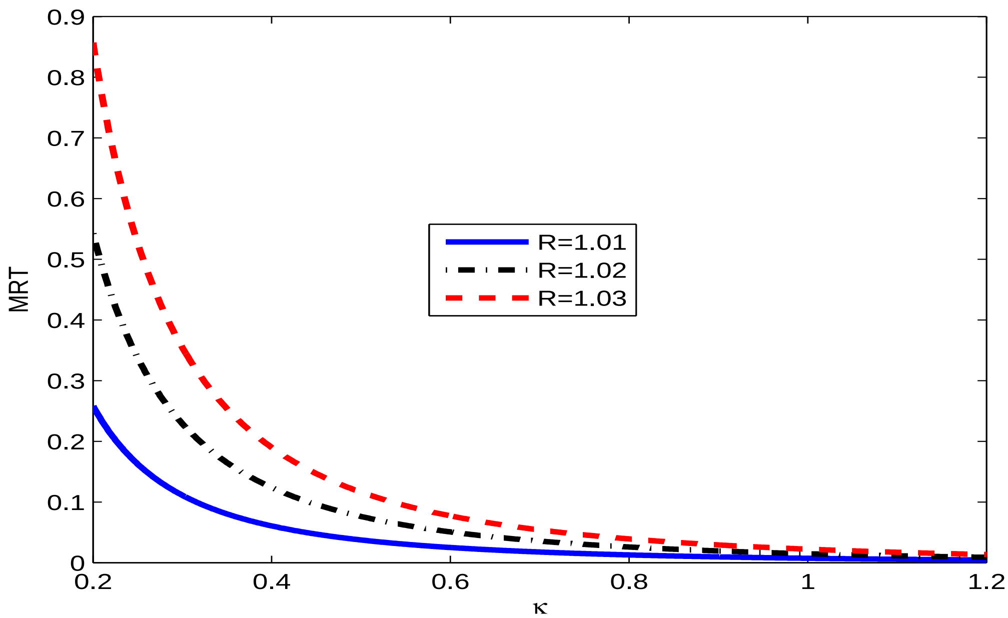

Figure 1: MRT between RB and RI

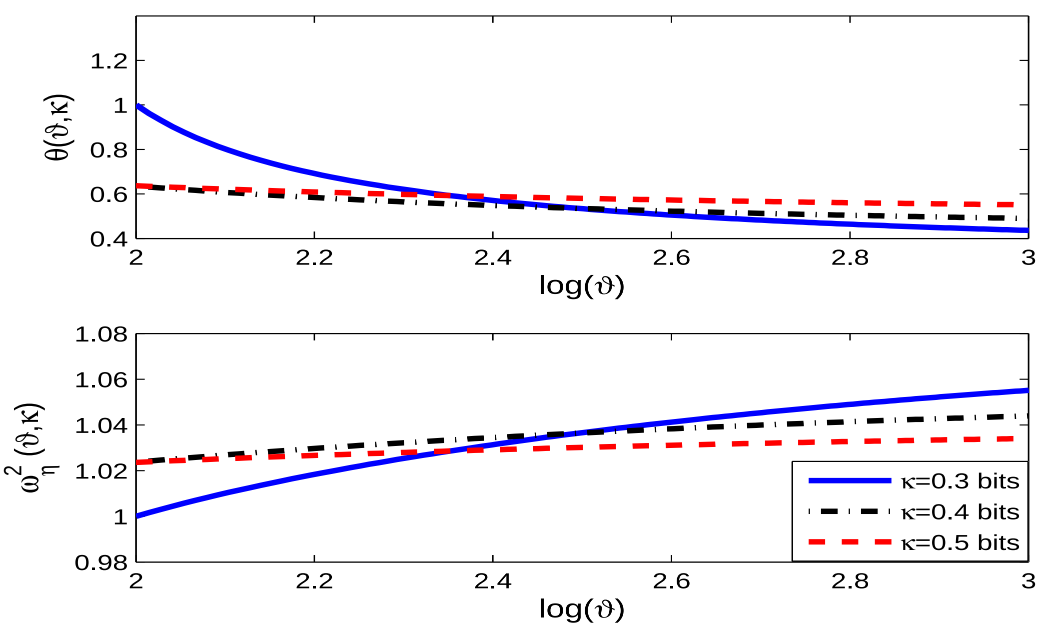

Figure 2: Effects of RB and RI on Robust Kalman Gain θ and ω_η^2

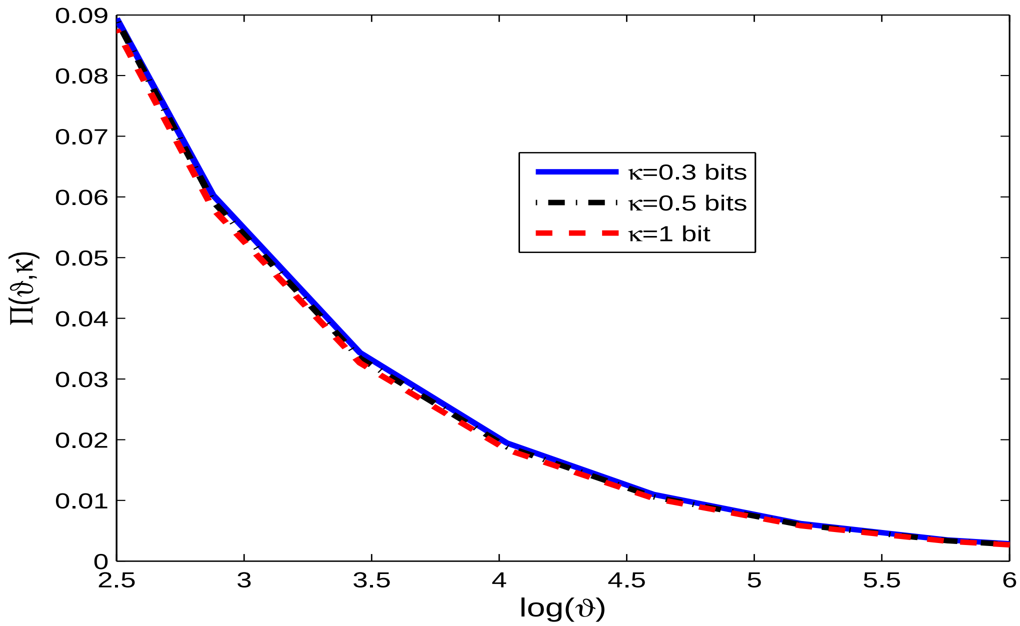

Figure 3: Effects of RB and RI on Π

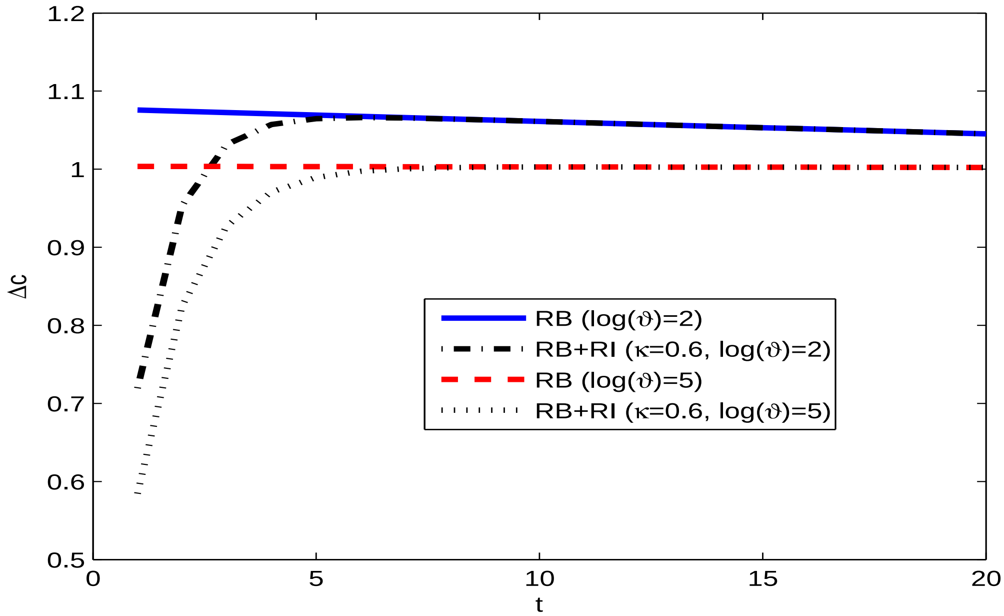

Figure 4: Impulse Responses of Consumption to Income Shock

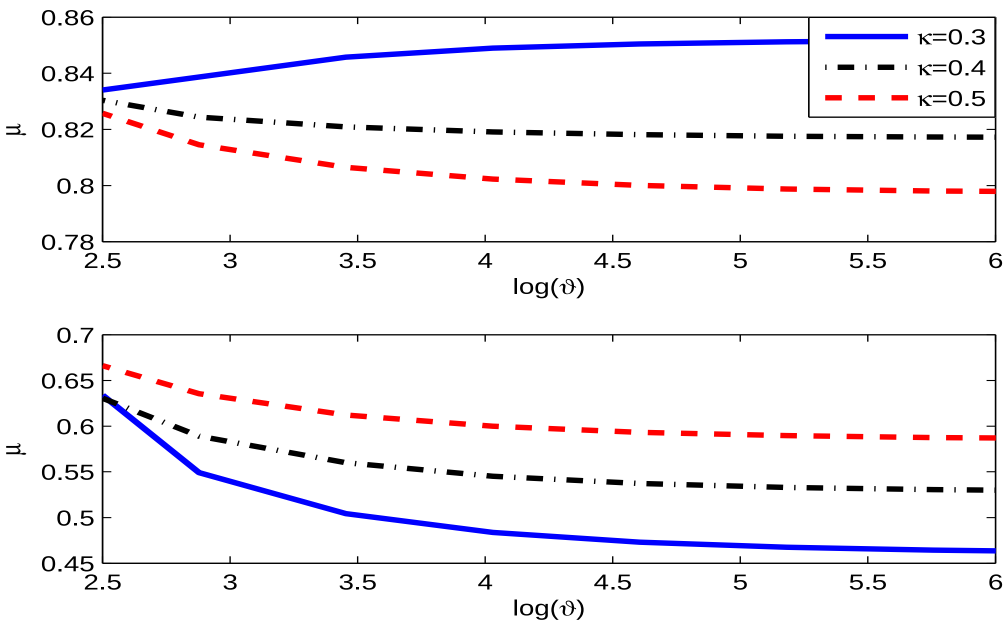

Figure 5: Relative Volatility of Consumption to Income under RB-RI

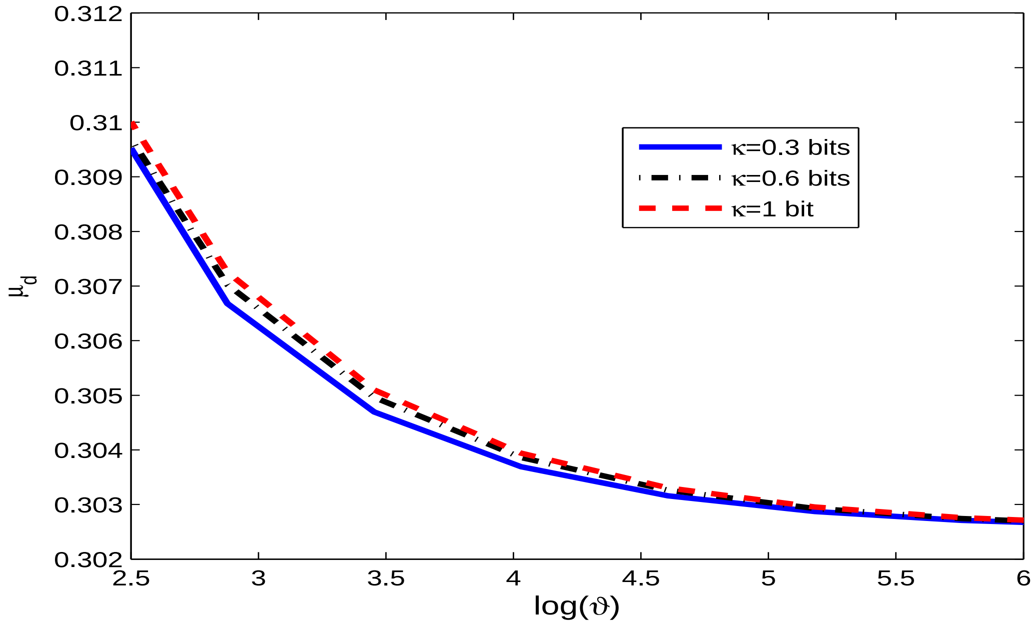

Figure 6: Relative Volatility of Individual Saving

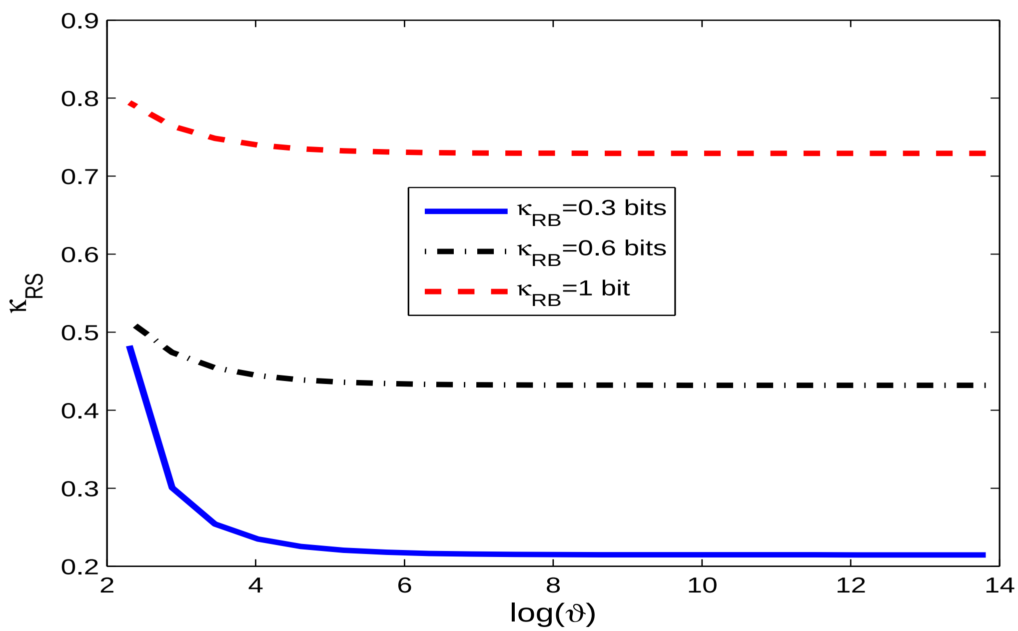

Figure 7: The OE between RB and RS

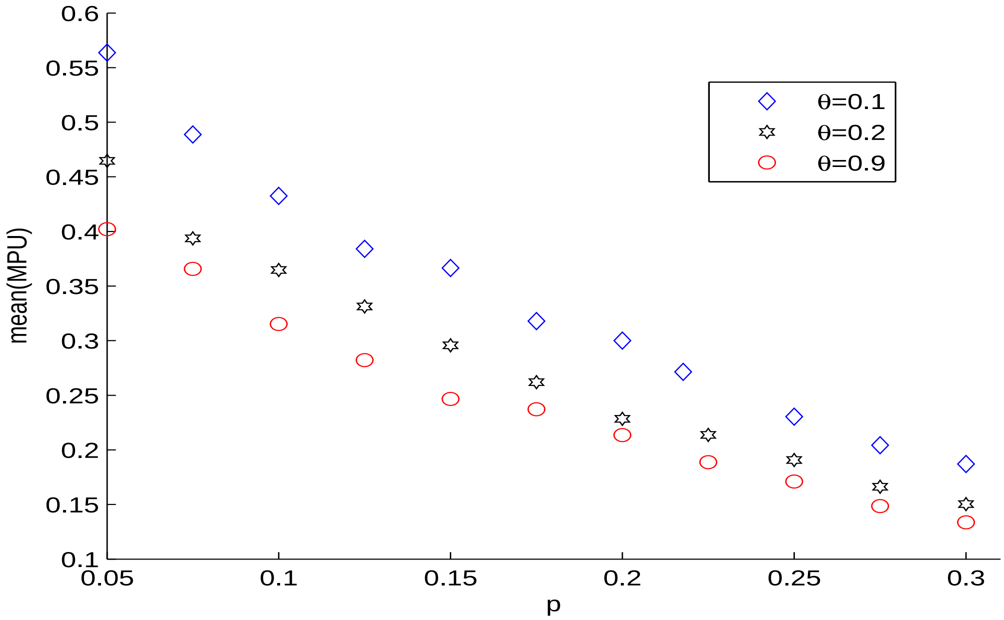

Figure 8: Effects of p on the mean of MPU for Different θ

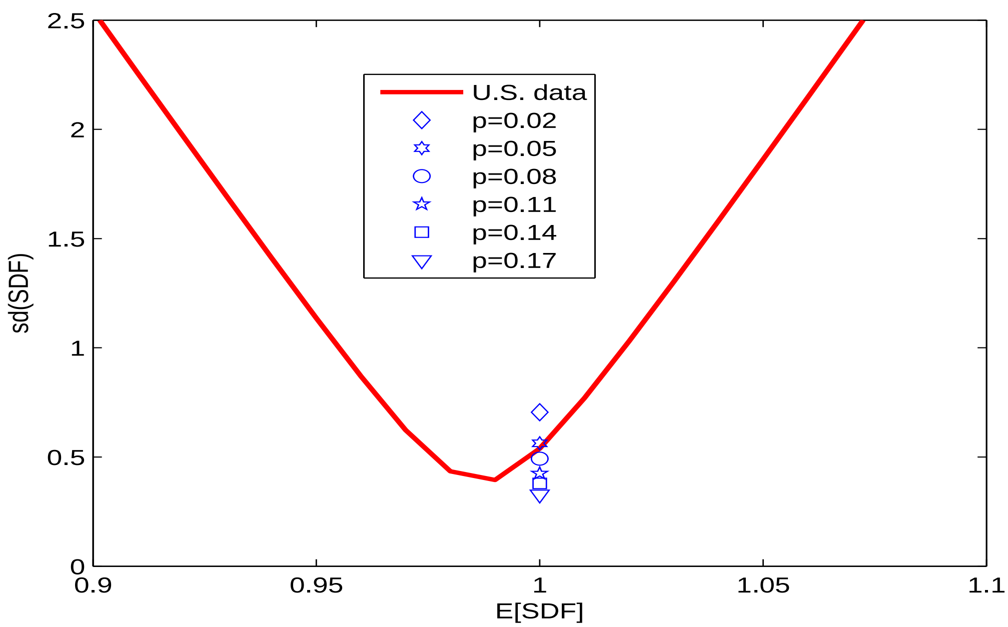

Figure 9: HJ Bound under RB and RI

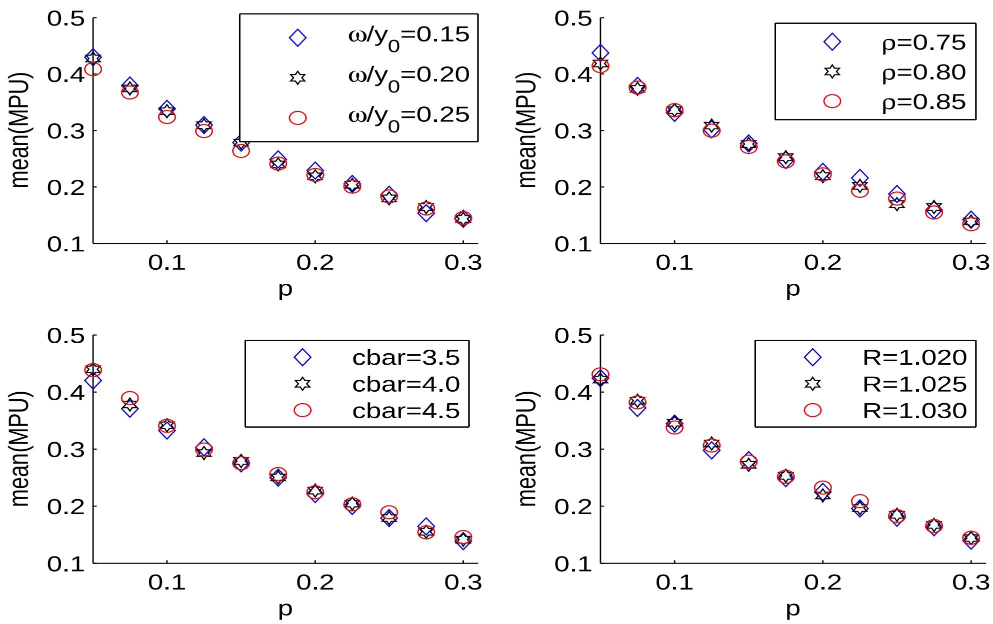

Figure 10: Sensitivity Analysis of the Effects of RB on MPU ($\theta = 0.3$)

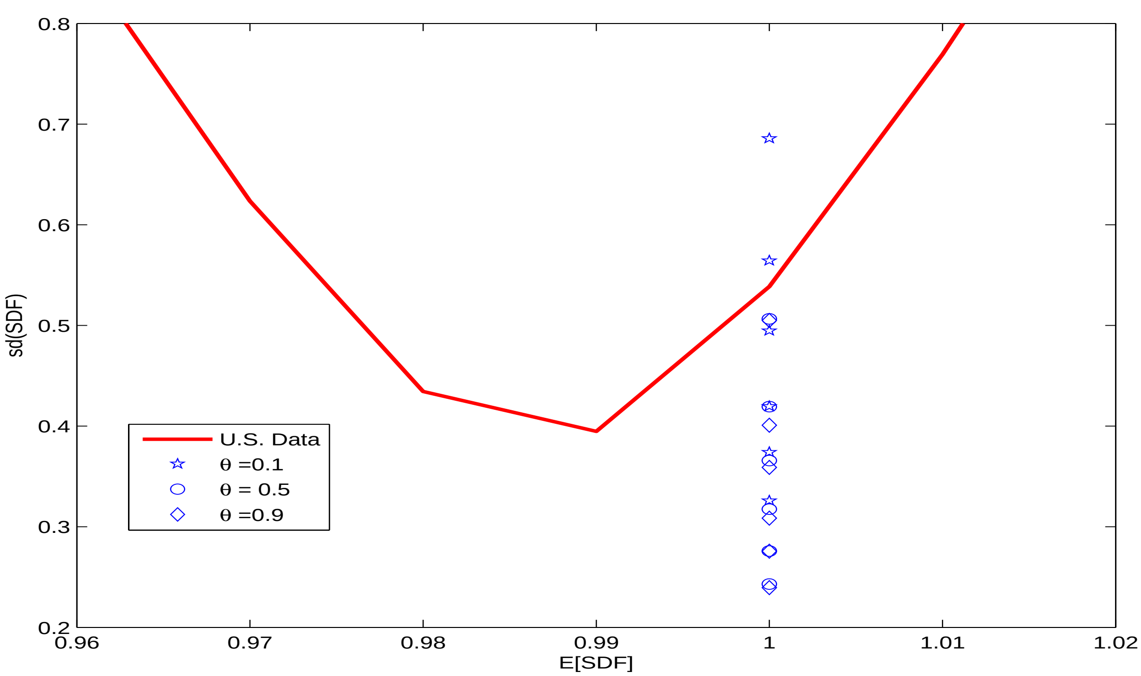

Figure 11: Sensitivity Analysis of the Effects of RB and RI on MPU

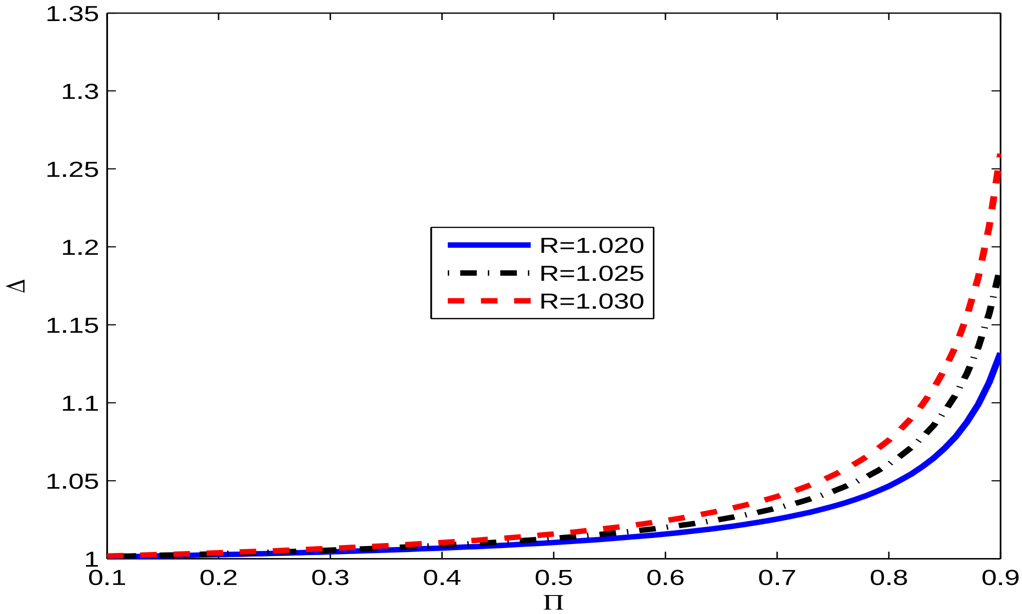

Figure 12: Difference of MPU under RB+RI and RS+RI

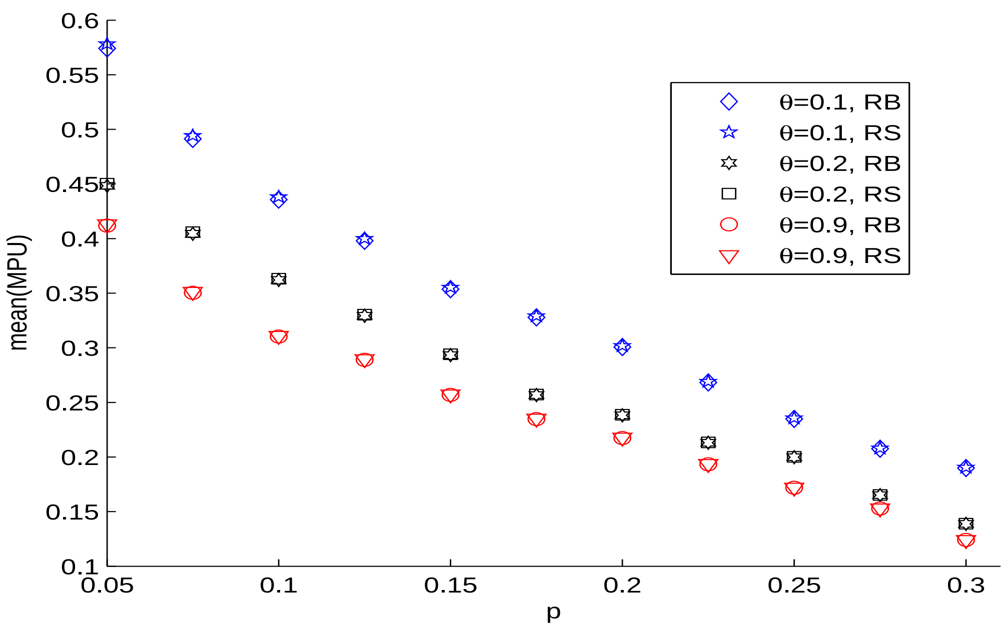

Figure 13: Comparison of the Effects of p on MPU under RB+RI and RS+RI

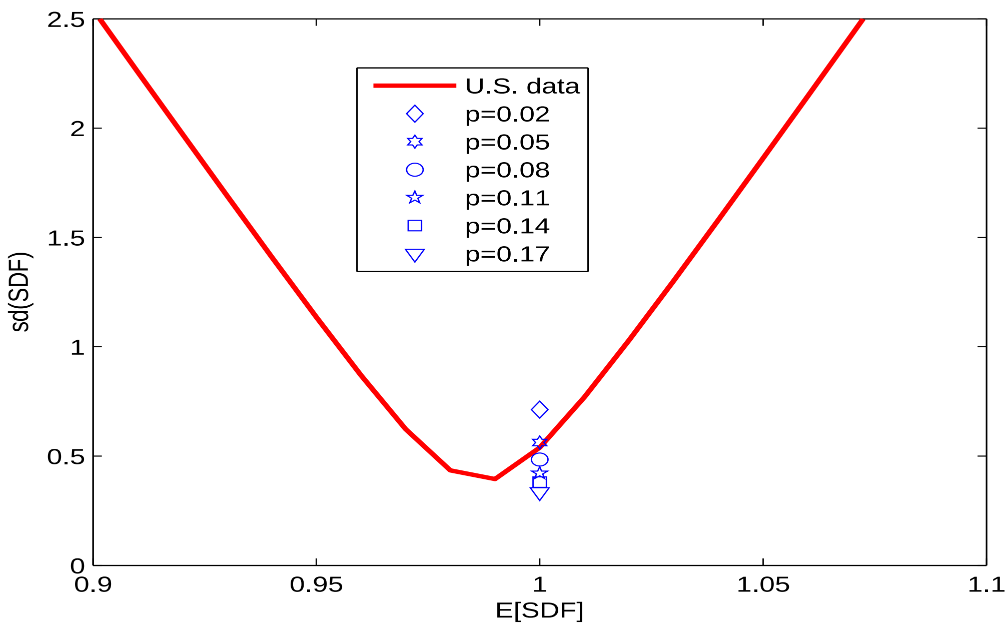

Figure 14: HJ Bound under RS and RI