Generating Functions for and

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Generating Functions for $P(n, p,*)$ and $P(n,*, p)$

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Abstract

This paper shows how to prove the Theorem $P(n, p,*) = P(n,*, p)$, i.e., the number of partitions of n into p-parts is equal to the number of partitions of n having largest part p.

Key Words: Irrelevant, decreasing order, *p*-parts

1. Introduction

We give some definitions of a partition, $P(n, p,\le q)$, $P(n,\le q, p)$, $P(n, p,*)$ and $P(n,*, p)$. We generate the generating functions for $P(n, p,\le q)$, $P(n,\le q, p)$, $P(n, p,*)$ and $P(n,*, p)$ and prove the theorem $P(n, p,*) = P(n,*, p)$ by graphically. Finally we give a numerical example when *n* = 8.

2. Definitions

Partition: A partition of a number is a representation of *n* as the sum of any number of positive integral parts. Thus, 5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1. The order of the parts is irrelevant, so that parts to be arranged in decreasing order of magnitude, we denote by $P(n)$, the number of partitions of *n*. Thus, $P(5) = 7$.

$P(n, p,\le q)$: The number of partitions of *n* into *p*-parts, none of which exceeds *q*.

$P(n,\le q, p)$: The number of partitions of *n* into *p* or any smaller number of parts, the greatest of which is *p*.

$P(n, p,*)$**:** The number of partitions of *n* into *p*-parts.

$P(n,*, p)$: The number of partitions of *n* having largest part *p*.

3. Generating Functions for $P(n, p,\le q)$

The Generating functions for $P(n, p,\le q)$ is of the form [1]:

$$\frac{1}{(1-zx)(1-zx^2)\ldots(1-zx^q)}$$

$$= 1 + \sum_{n=1}^{\infty} x^n \left\{ \sum_{p=1}^{n} z^p P(n, p,\le q) \right\}. \qquad (1)$$

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It is convenient to define $P(n, p, \le q) = 0$ if $n < p$. The coefficient $P(n, p, \le q)$ is the number of partitions of *n* into *p*-parts, none of which exceeds *q*.

Again the generating function for $P(n, \le q, p)$ is of the form;

$$\frac{1}{(1-zx)(1-zx^2)\ldots(1-zx^q)}$$
$$= 1 + xz + x^2(z+z^2) + x^3(z+z^2+z^3) + x^4(z+2z^2+z^3+z^4) + \ldots\infty$$
$$= 1 + \sum_{n=1}^{\infty} x^n \left\{ \sum_{p=1}^{n} z^p P(n, \le q, p) \right\}. \qquad (2)$$

The proof of the Theorem $P(n, p, \le q)$ = $P(n, \le q, p)$ is given in Hardy and Wright [2]. If $q \to \infty$, in (1), such as $\underset{q\to\infty}{Lim}\, x^q = 0$ when $|x| < 0$, then (1) becomes;

$$\frac{1}{(1-zx)(1-zx^2)(1-zx^3)\ldots\infty}$$
$$= 1 + xz + x^2(z+z^2) + x^3(z+z^2+z^3) + x^4(z+2z^2+z^3+z^4) + \ldots\infty$$
$$= 1 + \sum_{n=1}^{\infty} x^n \left\{ \sum_{p=1}^{n} z^p P(n, p, *) \right\} \qquad (3)$$

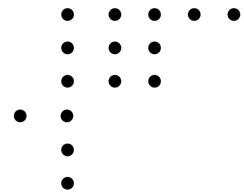
where the coefficient $P(n, p, *)$ is the number of partitions of *n* into *p*-parts. Again (2) becomes;

$$\frac{1}{(1-zx)(1-zx^2)(1-zx^3)\ldots\infty}$$
$$= 1 + xz + x^2(z+z^2) + x^3(z+z^2+z^3) + x^4(z+2z^2+z^3+z^4) + \ldots\infty$$
$$= 1 + \sum_{n=1}^{\infty} x^n \left\{ \sum_{p=1}^{n} z^p P(n, *, p) \right\} \qquad (4)$$

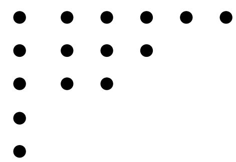
where the coefficient $P(n, *, p)$ is the number of partitions of *n* having largest part *p*.

Now we can consider a Theorem as follows:

Theorem: $P(n, p, *) = P(n, *, p)$ i.e., the number of partitions of *n* into *p*-parts is equal to the number of partitions of *n* having largest part *p*.

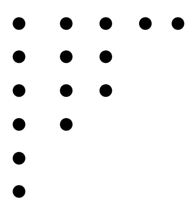
Proof: We establish a one-to-one correspondence between the partitions enumerated by $P(n, p, *)$ and those enumerated by $P(n, *, p)$. Let $n = a_1 + a_2 + \ldots + a_p$ be a partition of *n* into *p*-parts. We transfer this into a partition of *n* having largest part *p* and can represent a partition of 15 graphically by an array of dots or nodes such as, *A* 

The dots in a column correspond to a part. Thus *A* represents the partition 6+4+3+1+1 of 15. We can also represent *A* by transposing rows and columns in which case it would represent the partition graphically as conjugate of *A*.

B = Conjugate of *A* 

The dots in a column correspond to a part, so that it represents the partition 5+3+3+2+1+1 of 15. Such pair of partitions are said to be conjugate. The number of parts at 1^{st} one portion is equal to the largest part of 2^{nd} one partition, so that our corresponding is one-to-one.

Conversely, we can represent the partition *B* = conjugate of *A*, by transposing rows and columns, in which case it would represent the same partition like *A*, so we can say that the largest part of the partition is equal to the number of parts of the partition, then our corresponding is onto, i.e., the number of partitions of *n* into *p*-parts is equal to the number of partitions of *n* having largest part *p*. Consequently,



$P(n, p, *) = P(n, *, p)$.

Hence the Theorem.

4. A Numerical Example When n = 8

The list of partitions of 8 into 4 parts is given as follows:

5+1+1+1 = 4+2+1+1 = 3+3+1+1 = 3+2+2+1= 2+2+2+2. The number of such partitions is 5 i.e., $P(8,4,*) = 5$.

Again the list of partitions of 8 having largest part 4 is given by;

4+4 = 4+3+1 = 4+2+1+1 = 4+1+1+1+1 = 4+2+2.

So the number of such partitions is 5, i.e., $P(8,*,4) = 5$. Here 4+4, 4+3+1, 4+2+1+1, 4+1+1+1+1 and 4+2+2 are the conjugate partitions of 2+2+2+2+, 3+2+2+1, 4+2+1+1, 5+1+1+1 and 3+3+1+1 respectively. Thus the number of partitions of 8 into parts, the largest of which is 4 i.e., $P(8,4,*) = P(8,*,4)$.

5. Conclusion

For any positive integer of *n*, we can verify the Theorem $P(n, p, *) = P(n, *, p)$. We have already satisfied the Theorem when $n = 8$.

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