Incentive and normative analysis on sequencing problem

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INCENTIVE AND NORMATIVE ANALYSIS ON SEQUENCING PROBLEM

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ABSTRACT. We identify the complete class of transfer rules that guarantee strategyproofness of any non-increasing in completion time allocation rule for the sequencing problem. We then characterize the class of mechanisms satisfying efficiency of decision (or aggregate cost minimization), egalitarian equivalence and strategyproofness. There is no mechanism in this class that satisfies either feasibility or weak group strategyproofness. Finally we identify the restrictions under which egalitarian equivalence, efficiency of decision, identical preference lower bound and strategyproofness are compatible.

Keywords: Sequencing problems, Strategyproofness, Egalitarian equivalence, Identical preference lower bound

JEL Classification: C72, D63, D71, D82

1. INTRODUCTION

Consider a public decision making problem where (1) there are n agents and a single server, (2) the server can provide services of non-identical processing length but can process only one particular service at a time. (3) Jobs may not be identical across agents, so their processing time may differ, we assume that processing time is common knowledge. (4) Waiting for the service is costly, monetary transfers are given to the agents to compensate them. (5) Agents have quasi-linear preferences over position in queue and monetary transfers. The problem with the above mentioned structure is known as the sequencing problem. Typically the natural problems in this set-up are how to solve the problem of information asymmetry and what is a 'reasonable' the order in which to serve the agents.

A real life example of sequencing problem was given by Suijs[15]. He considered a large firm that has several divisions that need to have a service facility which is provided by maintenance and repairing unit of the firm. When a number of divisions ask for service for this facility, as the maintenance and repairing unit can only serve one division at a

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time, each division has to incur a downtime cost. In order to minimize the total downtime cost firm has to use a true cost revelation mechanism since cost are private information to the corresponding units. Apart from the above example we can have situations like a diagnostic centre, installed with a machine(due to space shortage) that can provide multiple services but can serve one agent at a time, where a certain number of enlisted patients visits for diagnosis or software installation problem to PCs of a set of agents. All these examples capture the structure of sequencing problem.

Assuming quasi-linear preferences, it is possible to design mechanisms that satisfy non-manipulability and efficiency of decision. This is a consequence of the result due to Hölmstrom's[17] on the uniqueness of the class of Vickrey-Clarke-Gorves (VCG) mechanisms.[1] From Suijs[15] and Mitra[4] we know that linearity of cost structure is a crucial assumption to ensure 'first best'. By 'first best' we mean one can find mechanisms that satisfy efficiency of decision, dominant strategy incentive compatibility and budget balancedness.

Sequencing problem has also been analyzed in a cooperative game set-up. In this respect we mention the pioneering work of Curiel, Pederzoli and Tijs[24] where they axiomatically characterizes a division rule (*Equal Gain Split*) to share cost savings among agents and shows the advantages of *Equal Gain Split* over *Equal Division Rule*. The other important work in this context is by Mishra [23] who characterized the Shapley value.

Our analysis on sequencing problem is from non-cooperative point of view. In sequencing context efficiency of decision is a well studied allocation rule. Queue efficiency implies minimal aggregate waiting cost. But there many situations where efficiency of decision may not be the primary objective. Think of situations where some well defined priority across the set of agents exists. For example, in an academic institute, faculty members may be given priority over research scholars and research scholars are given priority over graduate students in using printing or photocopying facilities. Mishra and Mitra[2] has given many such examples in the context of scheduling problems. Hence we first ask the following general question: What are the class of strategyproof (or non-manipulable) allocation rules and what is the class of transfers associated with such allocation rule? We identify the complete class of allocation rule and then associate the unique class of transfer rule associated with such rules. We show that allocation rule must be non-increasing in completion time and this is also sufficient with the associated unique class of transfers.

Next we try to further classify Vickrey-Clarke-Groves transfer scheme in such a way that the mechanism is egalitarian equivalent. Egalitarian equivalence is a well known equity concept that was introduced by

[1] See Vickery[25], Clarke[21], Groves[22].

Pazner and Schmeidler [14] and is based on the idea that all individuals should be placed in a situation which is Pareto-indifferent to a perfectly egalitarian allocation. In this respect our findings are similar to the work of Chun, Mitra and Mutuswami[1] in *Queueing*[2] context.

We then show that egalitarian equivalence is incompatible with first best situation since feasibility is not compatible with efficiency, strategyproofness and egalitarian equivalence. Here we try to focus a sharp difference in result found by Chun, Mitra and Mutuswamy[1]. While they had possibility result with feasibility in queuing problems we have impossibility result with feasibility in case of sequencing problems and we find the explanation is hidden in the heterogeneity of individual processing speed.

As we explore more stronger notion of non-manipulability, we must mention the findings of Mitra and Mutuswamy[9] that shows there does not exist any mechanism that satisfies queue-efficiency and strong group strategy-proofness, in a single machine queueing context. In our framework even pair-wise group strategyproofness is incompatible with efficiency of decision and egalitarian equivalence. Next we introduce another normative notion namely identical preference lower bound. It is a notion of individual rationality based on the idea that agent's utility is at least as much as that of consuming his equal share of resources. This concept was first introduced by Moulin [18]. We identify the class of mechanism that satisfies efficiency, strategyproofness, egalitarian equivalence and identical preference lower bound.

This paper has been arranged in the following way. In Section 2 we formally introduce the model and add necessary definitions. In Section 3 we state and prove our characterization results. Then in Section 4 we draw our conclusions.

2. The Model

We consider the set of agents $N = \{1, \ldots, n\}$ with a single machine. Each individual has a different kind of work to be executed by the machine. The machine can process one job at a time. Let $\forall i \in N$, $s_i \in \Re_{++}$ where s_i denotes the processing time of ith agent and we assume that not all s_i's are same. Each agent is identified with a waiting cost $\theta_i \in \Re_{++}$, the cost of waiting per unit of time. The profile of waiting costs of the set of all agents is typically denoted by $\theta = (\theta_1, \ldots, \theta_n) \in \Re^n_{++}$. For any $i \in N$, θ_{-i} denotes the profile $(\theta_1 \ldots \theta_{i-1}, \theta_{i+1}, \ldots \theta_n) \in \Re^{n-1}_{++}$. A sequencing game is denoted by $\Omega = \langle N, \Re^n_{++}, \Re^n_{++} \rangle$.

[2]Here processing speed is constant across agents and agent's own processing speed do not contribute to his own waiting time.

An allocation of n jobs can be done in many ways. An allocation rule is a mapping $\sigma : \Re^n_+ \to \Sigma(N)$ that specifies for each profile $\theta \in \Re^n_{++}$ an allocation(rank) vector $\sigma(\theta) \in \Sigma(N)$. Agent i's position is denoted by $\sigma_i(\theta)$ which is an input of the vector $\sigma(\theta)$. Let $\Sigma(N)$ denote the set of all possible sequence of agents in N. Given $\sigma \in \Sigma(N), \forall \in N, P_i(\sigma) = \{j \in N | \sigma_j(\theta) < \sigma_i(\theta)\}$ denotes the set of predecessors of i and similarly $P'_i(\sigma) = \{j \in N | \sigma_j(\theta) > \sigma_i(\theta)\}$ denotes the set of successors of i. Agent i's waiting time is denoted by $S_i(\sigma(\theta))$ and corresponding waiting cost is $S_i(\sigma(\theta))\theta_i$. A transfer rule is a mapping $t : \Re^n_+ \to \Re^n$ that specifies for each profile $\theta \in \Re^n_{++}$ a transfer vector $t(\theta) = (t_i(\theta), \dots, t_n(\theta)) \in \Re^n$. We assume that the utility function of each agent $i \in N$ is quasi-liner and is of the form $U_i(\sigma(\theta), t_i(\theta), \theta_i) = -S_i(\sigma(\theta)(\theta_i) + t_i(\theta)$, where $t_i(\theta)$ is the monetary transfer of agent to i.

Definition 1. $\forall \theta \in \Re^n_{++}$, a queue $\sigma \in \Sigma(N)$ is efficient if $\sigma \in \text{argmin}_{\sigma \in \Sigma(N)} \sum_{i=1}^n S_i(\sigma)\theta_i$.

The implication of efficiency is that agents are ranked according to the non-increasing order of their waiting costs (that is, if $\theta_i \geq \theta_j$ under a profile θ, then $S_i(\sigma(\theta)) \leq S_i(\sigma(\theta)))$. Moreover, there are profiles for which more than one rank vector is efficient. For example, if all agents have the same waiting cost, then all rank vectors are efficient. Therefore, we have an efficiency correspondence. In this paper we consider a particular efficient rule (that is, a single valued selection from the efficiency correspondence). For our efficient rule, we use the following *tie breaking rule*: if $i < j$ and $\theta_i = \theta_j$ then $S_i(\sigma(\theta)) < S_i(\sigma(\theta))$. This tie breaking rule guarantees that, given a profile $\theta \in \Re^n_{++}$, the efficient rule selects a single rank vector from $\Sigma(N)$.

A mechanism is (σ, θ) constitutes of an allocation rule σ and a transfer rule t. We are interested in strategy proof mechanism for the sequencing problem.

Definition 2. A mechanism(σ, t) is strategy-proof (SP) if $\forall i \in N, \forall \theta_i, \theta'_i \in \Re_{++}$ and $\forall \theta_{-i} \in \Re^{n-1}_{++}$ we have,
$-S_i(\sigma(\theta_i, \theta_{-i}))\theta_i + t_i(\theta_i, \theta_{-i}) \geq -S_i(\sigma(\theta'_i, \theta_{-i}))\theta_i + t_i(\theta'_i, \theta_{-i})$.

It means for any agent truthful reporting is weakly dominates false reporting irrespective of other players report.

Definition 3. A mechanism(σ, t) is efficient (EFF) if for all announced profile

$$\theta \in \Re^n_{++}, \sigma(\theta) \in \text{argmin}_{\sigma \in \Sigma(N)} \sum_{i=1}^n S_i(\sigma)\theta_i.$$

Efficiency here basically implies minimization of aggregate waiting cost.

Definition 4. A mechanism (σ, t) satisfies *egalitarian equivalence* (EE) if $\forall \theta \in \Re^n_{++}$ $\exists\, (\bar{S}(\theta), t(\theta)) \ni \forall i \in N$, $-S_i(\sigma(\theta))\theta_i + t_i(\theta) = -\bar{S}(\theta)\theta_i + t(\theta)$.

Here $(\bar{S}(\theta), t(\theta))$ denotes the reference bundle, where $\bar{S}(\theta)$ is the reference waiting time and $t(\theta)$ his the reference transfer.

Egalitarian equivalence was introduced by Pazner and Schmeidler [14] and is based on the idea that all individuals should be placed in a situation which is Pareto-indifferent to a perfectly egalitarian allocation. In this sequencing problem's context $(\bar{S}(\theta), t(\theta))$ is such a reference bundle, where if the agent is placed remains indifferent to is original bundle that he receives under efficiency and strategyproofness.

Definition 5. A mechanism (σ, t) satisfies *budget balancedness* (BB) if $\forall \theta \in \Re^n_{++},\ \sum_{i\in 1}^n t_i(\theta) = 0$.

Definition 6. A mechanism (σ, t) satisfies *feasibility* (FSB) if $\forall \theta \in \Re^n_{++},\ \sum_{i\in 1}^n t_i(\theta) \leq 0$.

The profile θ and θ' are *S-variants* if $\forall i \in N \setminus S, \theta_i = \theta'_i$.

Definition 7. A mechanism (σ, t) is *weak group strategyproof* (WSP) if for all *S-variants* θ, θ' $U_i(\sigma(\theta), t_i(\theta), \theta_i) \geq U_i(\sigma(\theta'), t_i(\theta'), \theta_i)$ for at least one $i \in S$.

This implies as long as all the group member are not strictly better off by deviating from their true profile, such group will not be formed.

Definition 8. A mechanism (σ, t) is *pair-wise group strategyproof* (PWSP) if for all *S-variants* θ, θ' where $|S| = 2$,
$U_i(\sigma(\theta), t_i(\theta), \theta_i) \geq U_i(\sigma(\theta'), t_i(\theta'), \theta_i)$ for at least one $i \in S$.

This implies pair of agents deviates from their true profile by jointly misreporting if an only if they are both strictly better off from the situation when they truthfully reports.

Definition 9. A mechanism (σ, t) satisfies *identical preference lower bound* (IPLB) if $\forall \theta \in \Re^n_{++}$ $\forall i \in N$, $U_i(\sigma(\theta), t_i(\theta), \theta_i) \geq -(s_i + \frac{\sum_{j\neq i} s_j}{2})\theta_i$.

This concept was first introduced by Mouin[18] in 1990 and is based on the idea that an agent's welfare is atleast that of consuming his equal share of resources. In the context of sequencing problem agents are considered identical as long as their relative waiting cost's are same. That is if for all $i, j \in N$, $\frac{\theta_i}{s_i} = \frac{\theta_j}{s_j}$ then agents are considered to be identical. Identical preference lower bound implies that any agent's utility should be at least as that of average or expected utility of that agent when the agent perceives all the other agents identical to herself.

3. Results

3.1. **Strategy-Proof Mechanism.**

Proposition 1. The allocation rule σ is strategy-proof (*dsic*) if $\forall i \in N, S_i(\sigma(\theta))$ is non-increasing with θ_i.

Proof: We first prove the necessity, that is, non-increasingness of $S_i(\sigma(\theta))$ with θ_i is a necessary condition for allocation rule to be strategy-proof.

Let σ is strategy-proof (i.e.) Take an $i \in N$ and $\theta_{-i} \in \Re^{n-1}_{++}$, $\forall i \in N, \forall \theta_i, \theta_i' \in \Re_{++}$ we have,

$$-S_i(\sigma(\theta_i, \theta_{-i}))\theta_i + t_i(\theta_i, \theta_{-i}) \geq -S_i(\sigma(\theta_i', \theta_{-i}))\theta_i + t_i(\theta_i', \theta_{-i}). \tag{1}$$

$$-S_i(\sigma(\theta_i', \theta_{-i}))\theta_i' + t_i(\theta_i', \theta_{-i}) \geq -S_i(\sigma(\theta_i, \theta_{-i}))\theta_i' + t_i(\theta_i, \theta_{-i}). \tag{2}$$

From now on we will suppress $\theta_{-i} \in \Re^{n-1}_{++}$ as it is an obvious argument in σ and t i.e. $S_i(\sigma(\theta_i, \theta_{-i})$ is denoted as $S_i(\sigma(\theta_i))$ and $t(\theta_i, \theta_{-i})$ is denoted as $t(\theta_i)$.

Solving (1)&(2) we have,

$$\theta_i(S_i(\sigma(\theta_i')) - S_i(\sigma(\theta_i))) \geq t_i(\theta_i') - t_i(\theta_i) \geq \theta_i'(S_i(\sigma(\theta_i')) - S_i(\sigma(\theta_i))) \tag{3}$$

Further we get,

$$(\theta_i - \theta_i')(S_i(\sigma(\theta_i')) - S_i(\sigma(\theta_i))) \geq 0 \tag{4}$$

Equation (4) implies $S_i(\sigma(\theta_i))$ is non-increasing with θ_i.

Next we try to find the appropriate transfer rule that will ensure strategy-proofness in the above mentioned set-up. Consider an agent $i \in N$, for a given $\theta_i \in \Re^n_{++}$ agent i can face maximum 2^{n-1}(i.e.$\sum_{j=0}^{n-1}(n-1)_{C_j}$) different processing speed. But the number different processing speed is faced by i is actually depends on the concerned allocation rule. So depending on the allocation rule the agent can face same(i.e. single) processing time[3] (i.e. constant allocation rule) or $R+1$ different processing time where $R \in \{1, 2, 3, \ldots, 2^{n-1} - 1\}$. We assume $0 < \theta_i^{1*} < \theta_i^{2*} < \ldots < \theta_i^{R*} < \infty$ such that $S_i(\sigma(\theta_i^{1*})) > S_i(\sigma(\theta_i^{2*})) > \ldots > S_i(\sigma(\theta_i^{R*}))$ and $\forall \theta_i \in (\theta_i^{R*}, \infty)$, $S_i(\sigma(\theta_i^{R*})) > S_i(\sigma(\theta_i)) > 0$ [4]. So for agent $i \in N$, $\forall k \in \{1, 2, 3, \ldots, R\}$, $\theta_i^{k*} (k \neq 0)$ acts as level-mark waiting cost i.e. if for agent i, $\theta_i \in (\theta_i^{k-1}, \theta_i^k]$ then corresponding waiting time of agent i is $S_i(\sigma(\theta_i^k))$. We assume $\theta_i^{0*} \equiv 0$. If $\theta_i \in (0, \theta_i^{1*}]$ then the corresponding waiting time is $S_i(\sigma(\theta_i^{1*}))$ and if $\theta \in (\theta_i^{R*}, \infty)$

[3]For this constant allocation rule we can use equation (3) to find the appropriate transfer rule. It is trivial that constant transfer rule will be the appropriate transfer scheme.

[4]Note that by Proposition (1), $S_i(\sigma(\theta))$ can not be increasing in θ_i

then the corresponding waiting time is $S_i(\sigma(\theta_i))$ i.e. the *minimum allowable waiting time* (specified by the allocation rule concerned)[5]. Let the transfer rule for agent i with waiting cost θ_i be

$$(5)\quad \begin{array}{l} \forall\theta_i \in (0, \theta_i^{1*}] \\ t_i(\theta_i) = h_i(\theta_{-i}) \\ \forall k \in \{1, 2, \dots, R-1\}, \forall\theta_i \in (\theta_i^{k*}, \theta_i^{(k+1)*}] \\ t_i(\theta_i) = h_i(\theta_{-i}) - \sum_{j=1}^{k}\{S_i(\sigma(\theta_i^{j*})) - S_i(\sigma(\theta_i^{(j+1)*}))\}\theta_i^{j*} \\ \forall\theta_i \in (\theta_i^{R*}, \infty) \\ t_i(\theta_i) = t_i(\theta_i^{R*}) - \theta_i^{R*}[S_i(\sigma(\theta_i^{R*})) - S_i(\sigma(\theta_i))] \end{array}$$

Theorem 1. An allocation rule σ is strategy-proof if and only if the transfer of each agent $i \in N$ is given by (7).

Proof: Fix $\theta_{-i} \in \Re_{++}^{n-1}$. Form equation (1)&(2) we have,

$$(6)\quad \forall i \in N, \forall\theta_i, \theta_i' \in \Re_{++} \qquad S_i(\sigma(\theta_i)) = S_i(\sigma(\theta_i')) \Rightarrow t_i(\theta_i) = t_i(\theta_i')$$

For an agent $i \in N$, $\forall\theta_i, \theta_i' \in (0, \theta_i^{1*}]$, $S_i(\sigma(\theta_i^{1*}))$ denotes the waiting time. So by equation (6) we have, $t_i(\theta_i) = t_i(\theta_i') = h_i(\theta_{-i})$(as the expression is θ_i independent). So

$$(7)\quad t_i(\theta_i^{1*}) = h_i(\theta_{-i})$$

Consider any pair $\{\theta_i, \theta_i'\} \in (\theta_i^{1*}, \theta_i^{2*}] \times (0, \theta_i^{1*}]$. By applying equation (3)&(6) and the fact that condition (3) must hold for all $\{\theta_i, \theta_i'\} \in (\theta_i^{1*}, \theta_i^{2*}] \times (0, \theta_i^{1*}]$ we get,

$$t_i(\theta_i) = t_i(\theta_i^{1*}) - \theta_i^{1*}\{S_i(\sigma(\theta_i^{1*})) - S_i(\sigma(\theta_i^{2*}))\}$$

i.e. the expression reduces to,

$$t_i(\theta_i) = h_i(\theta_{-i}) - \theta_i^{1*}\{S_i(\sigma(\theta_i^{1*})) - S_i(\sigma(\theta_i^{2*}))\}$$

Hence $$t_i(\theta_i^{2*}) = h_i(\theta_{-i}) - \theta_i^{1*}\{S_i(\sigma(\theta_i^{1*})) - S_i(\sigma(\theta_i^{2*}))\}$$

Now in general for any pair $\{\theta_i, \theta_i'\} \in (\theta_i^{k*}, \theta_i^{(k+1)*}] \times (\theta_i^{(k-1)*}, \theta_i^{k*}]$ where $k \in \{1, 2, \dots, R-1\}$ (note $\theta_i^{0*} \equiv 0$) using equation (3)&(6) and by the
similar fashion we get,

$$(8)\quad t_i(\theta_i^{(k+1)*}) = t_i(\theta_i^{k*}) - \theta_i^{k*}\{S_i(\sigma(\theta_i^{k*})) - S_i(\sigma(\theta_i^{(k+1)*}))\}$$

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[5]There is a potential difference between *minimum possible waiting time*(MPWT)and *minimum allowable waiting time*(MAWT). Depending on the allocation rule MPWT i.e. s_i may or may not be equal to MAWT. If MAWT is s_i then so is MPWT but the reverse is not always true.

Thus using equation (7) and solving equation (8) recursively we get,

$$\theta_i \in (\theta_i^{k*}, \theta_i^{(k+1)*}] \ni k \in \{1, 2, \ldots, R-1\}$$

$$t_i(\theta_i) = h_i(\theta_{-i}) - \textstyle\sum_{j=1}^{k} \theta_i^{j*}\{S_i(\sigma(\theta_i^{j*})) - S_i(\sigma(\theta_i^{(j+1)*}))\} \tag{9}$$

Solving $t_i(\theta_i^{R*})$ form equation (9) and using equation (3) we get,

$$\forall \in (\theta_i^{R*}, \infty)$$

$$t_i(\theta_i) = t_i(\theta_i^{R*}) - \theta_i^{R*}\{S_i(\sigma(\theta_i^{R*})) - S_i(\sigma(\theta_i))\} \tag{10}$$

Thus we have proved the necessity of transfer rule given by equation (5) for ensuring strategy-proofness. The proof of sufficiency is quite easy, hence omitted.

So the mechanism (σ, t) that ensures strategy-proofness in this set-up must have an allocation rule such that $\forall i \in N, S_i(\sigma(\theta))$ is non-increasing with θ_i and the corresponding transfer rule $t_i(\theta_i)$ is given by equation (5).

3.2. **Strategy-Proof, Efficient and Egalitarian Equivalence Mechanism.**

Here we examine the implication of egalitarian equivalence on a strategy-proof mechanism with a specific allocation criteria called efficient allocation rule. In this section we will use a slightly different notation to refer an agent. Here we refer the agent at the i-th position of the queue as agent (i). So the true waiting cost profile is $\theta = \{\theta_{(1)}, \theta_{(2)}, \ldots, \theta_{(n)}\}$ is such that $\lambda_{(1)} > \lambda_{(2)} > \ldots > \lambda_{(n)} > 0$ where $\forall i \in N,\ \lambda_{(i)} = \frac{\theta_{(i)}}{s_{(i)}}$. Hence $\forall \theta_{(N)} = (\theta_{(i)}, \theta_{-(i)}) \in \Re^n_{++}, \forall i \in N, s_i \neq s_{(i)}$.

The crucial fact behind the idea of egalitarian equivalent allocation where everyone consumes the same "reference bundle" and derives same utility as they get with initial allocation is trivially egalitarian. In case of queueing problem Chun, Mitra and Mutuswamy[1] have completely characterized EFF, SP and EE mechanisms. In queueing problem each element of the set $\{1, 2, \ldots, n\}$ can be a potential reference position because jobs are homogeneous hence permutation of agents nothing to do with agents job completion time. Unlike queuing problems, in sequencing problems individuals can only think of the "last position" of the queue where for everyone the job completion time will necessarily be same that is $\bar{S}(\theta) = \bar{S} = \sum_{i \in N} s_{(i)}$. So we need to find the class of VCG transfer rule that respects EE at the only available reference position $\bar{S}$.

Theorem 2. A mechanism(σ, t) satisfies EE,SP and EFF if and only if the reference bundle for the profile $\theta \in \Re^n_{++}$ is of the form $(\bar{S}, t(\theta))$ where $\bar{S} =$ and $t(\theta) = \sum_{i \in N}\{\bar{S} - S_{(i)}(\sigma(\theta))\}\theta_{(i)} + \bar{k}$ $=\sum_{i \in N}\{\bar{S} - S_i(\sigma(\theta))\}\theta_i + \bar{k}$.

Proof: Let us consider an announcement profile $\theta = (\theta_{(1)}, \theta_{(2)}, \ldots, \theta_{(n)}) \in \Re^n_{++}$. So with EFF & tie breaking rule we can arrange agents uniquely i.e. $\sigma_{(i)}(\theta) = i$.

Since the domain of preference is $\Re_{++}$ is convex, it follows from Hölmstrom's result on efficient and strategy-proof mechanisms that (σ, t) must be a VCG mechanism. This implies that the transfer is given by

$$\forall i \in N : t_{(i)}\theta) = -\sum_{j \neq i} \theta_{(j)} S_{(j)}(\sigma(\theta)) + h_{(i)}(\theta_{-(i)}) \tag{11}$$

If we set $h_{(i)}(\theta_{-(i)}) = \sum_{j \neq i} S_{(j)} \theta_{(j)}(\sigma(\theta_{-(i)})) + g_{(i)}(\theta_{-(i)})$ in equation (11) we get

$$\forall i \in N : t_{(i)}(\theta) = -s_{(i)} \sum_{j \in P'_{(i)}(\sigma(\theta))} \theta_{(j)} + g_{(i)}(\theta_{-(i)}) \tag{12}$$

As the mechanism (σ, t) satisfies EE, SP and EFF the following condition must hold

$$\forall i \in N : -\theta_{(i)} S_{(i)}(\sigma(\theta)) + t_{(i)}(\theta) = -\theta_{(i)} \bar{S}(\theta) + t(\theta)$$

Where the left side of the above equation is the utility from a VCG mechanism and the right hand side is the utility from EE requirement. The above expression can alternatively be written as $(\bar{S(\theta)} = \bar{S})$

$$t(\theta) = -\theta_{(i)} S_{(i)}(\sigma(\theta)) - s_{(i)} \sum_{j \in P'_{(i)}(\sigma(\theta))} \theta_{(j)} + g_{(i)}(\theta_{-(i)}) + \theta_{(i)} \bar{S} \tag{13}$$

Putting $i = 1$ into equation (13) we get

$$t(\theta) = -\theta_{(1)} S_{(1)}(\sigma(\theta)) - s_{(1)} \sum_{j \in P'_{(1)}(\sigma(\theta))} \theta_{(j)} + g_{(1)}(\theta_{-(1)}) + \theta_{(1)} \bar{S}$$

Similarly for $i = 2$ we have

$$t(\theta) = -\theta_{(2)} S_{(2)}(\sigma(\theta)) - s_{(2)} \sum_{j \in P'_{(2)}(\sigma(\theta))} \theta_{(j)} + g_{(2)}(\theta_{-(2)}) + \theta_{(2)} \bar{S}$$

Equating the expressions for $t(\theta)$ we get,

$$-s_{(1)}\theta_{(1)} - s_{(1)}\theta_{(2)} + g_{(1)}(\theta_{-(1)}) = -\theta_{(2)}(s_{(1)} + s_{(2)})$$
$$+(s_{(1)} - s_{(2)}) \sum_{j \in P'_{(2)}(\sigma(\theta))} \theta_{(j)} + g_{(2)}(\theta_{-(2)}) - \bar{S}(\theta_{(1)} - \theta_{(2)})$$

Since $g_{(1)}(\theta_{-(1)})$is independent of $\theta_{(1)}$ and $g_{(2)}(\theta_{-(2)})$ is independent of $\theta_{(2)}$ we get

$$g_{(1)}(\theta_{-(1)}) = (\bar{S} - s_{(2)})\theta_{(2)} + f_{(1)}(\theta_{(N)\setminus\{1,2\}})$$

and

$$g_{(2)}(\theta_{-(2)}) = (\bar{S} - s_{(1)})\theta_{(1)} + f_{(2)}(\theta_{(N)\setminus\{1,2\}})$$

Now comparing the expression for $t(\theta)$ for $i = 1$ and $i = 3$ and using the expression of $g_{(1)}(\theta_{-(1)})$ we have

$$(\bar{S} - s_{(1)})\theta_{(1)} + (\bar{S} - s_{(2)})\theta_{(2)} + f_{(1)}(\theta_{(N)\setminus\{1,2\}}) = -\theta_{(3)}(s_{(1)} + s_{(2)} + s_{(3)})$$
$$+s_{(1)}\theta_{(2)} + s_{(1)}\theta_{(3)} + (s_{(1)} - s_{(2)}) \sum_{j \in P'_{(3)}(\sigma(\theta))} \theta_{(j)} \quad + g_{(3)}(\theta_{-(3)}) + \bar{S}\theta_{(3)}$$

Comparing the expressions on both sides in the similar fashion we get

$g_{(1)}(\theta_{-(1)}) = (\bar{S} - s_{(2)})\theta_{(2)} + \{\bar{S} - (s_{(2)} + s_{(3)})\}\theta_{(3)} + f'_{(1)}(\theta_{(N)\setminus\{1,2,3\}})$

and

$g_{(3)}(\theta_{-(3)}) = (\bar{S} - s_{(1)})\theta_{(1)} + \{\bar{S} - (s_{(1)} + s_{(2)})\}\theta_{(2)} + f_{(3)}(\theta_{(N)\setminus\{1,2,3\}})$

By using the same argument recursively we get

$$g_{(1)}(\theta_{-(1)}) = \sum_{j \neq 1}^{n} \{\bar{S} - S_{(j)}(\sigma(\theta_{(N)\setminus\{1\}}))\}\theta_{(j)} + k_{(1)}$$

In fact the above expression hold not only for $i = 1$ but for all $i \in N$ i.e.

$$g_{(i)}(\theta_{-(i)}) = \sum_{j \neq i}^{n} \{\bar{S} - S_{(j)}(\sigma(\theta_{(N)\setminus\{i\}}))\}\theta_{(j)} + k_{(i)}$$

Now we further get $\forall i, j \in N, \quad k_{(i)} = k_{(j)} = \bar{k}$ by using the above expression of $g_{(i)}(\theta_{(i)})$ into $t(\theta)$ in equation (13) and equating them. Hence

$$\forall i \in N: \quad g_{(i)}(\theta_{-(i)}) = \sum_{j \neq i}^{n} \{\bar{S} - S_{(j)}(\sigma(\theta_{(N)\setminus\{i\}}))\}\theta_{(j)} + \bar{k}$$

Using the above expression of $g_{(i)}(\theta_{-(i)})$ in equation (13) we have,

$$t(\theta) = \textstyle\sum_{i \in N} \{\bar{S} - S_{(i)}(\sigma(\theta))\}\theta_{(i)} + \bar{k}$$

Since $\sum_{i \in N} \{\bar{S} - S_{(i)}(\sigma(\theta))\}\theta_{(i)} = \sum_{i \in N} \{\bar{S} - S_i(\sigma(\theta))\}\theta_i$ the expression of reference transfer rule can also be written as follows: $t(\theta) = \sum_{i \in N} \{\bar{S} - S_i(\sigma(\theta))\}\theta_i + \bar{k}$.

Therefore it follows that a mechanism satisfies EE,SP and EFF only if the reference bundle for the profile θ i.e. $(\bar{S}(\theta), t(\theta))$ is of the form $t(\theta) = \sum_{i \in N} \{\bar{S} - S_i(\sigma(\theta))\}\theta_i + \bar{k}$ where $\bar{S} = \sum_{i \in N} s_{(i)}$.

Sufficiency is fairly obvious, hence omitted.

3.3. **Feasibility, Pair Wise Weak Group Strategyproofness and Identical Preference Lower-bound: Some possibilities and impossibilities.**

Proposition 2. *In a sequencing problem no mechanism satisfies EFF,SP,EE and FSB.*

Proof. We have $\forall\theta \in \Re^n_{++}\ \ \forall i \in N, t_{(i)}(\theta) = \sum_{j\neq i}(\bar{S} - S_{(j)})\theta_{(j)} + \bar{k}$. If FSB holds then $\forall\theta \in \Re^n_{++}\ \ \forall i \in N,\ \sum_{i\in N} t_{(i)}(\theta) \leq 0$ i.e. we have,

$$\sum_{i\in N}\{S_{(i)}(\sigma(\theta)) - \bar{S}\}\theta_{(i)} \geq \frac{n\bar{k}}{(n-1)}$$

If $\bar{k} \geq 0$, consider the profile $\theta = (\theta_n, \theta_{-n})$ where $\theta_{(j)\neq n} = 1$. Then we have,

$$\sum_{i\in N}\{S_{(i)}(\sigma(\theta)) - \bar{S}\}\theta_i < \frac{n\bar{k}}{(n-1)}$$

If $\bar{k} < 0$, consider the profile $\theta = (\theta_n, \theta_{-n})$ where $\theta_{(j)\neq 1} = 1$ and $\theta_{(1)} = \{\frac{2\bar{k}}{(S_{(1)}-\bar{S})} + 1\}$. Then we have,

$$\sum_{i\in N}\{S_{(i)}(\sigma(\theta)) - \bar{S}\}\theta_i < \frac{n\bar{k}}{(n-1)}$$

. Hence FSB is violated.

□

Remark 1. *The consequence of the Proposition(2) is, in a sequencing problem no mechanism satisfies EFF,SP,EE and BB.*

Proposition 3. *In a sequencing problem no mechanism satisfies EFF,PWSP,EE.*

Proof. If a mechanism (σ, t) satisfies EE,SP and EFF then $\forall\theta \in \Re^n_{++}$ the allocation of an agent (i) such that $i \in N$ is given by $(\sigma(\theta), t_{(i)}(\theta) = \sum_{(j)\in N\setminus(i)}\{\bar{S} - S_{(i)}(\sigma(\theta))\}\theta_{(j)})$.

Suppose the true waiting cost profile is $\theta = \{\theta_{(1)}, \theta_{(2)}, \ldots, \theta_{(n)}\}$ is such that $\lambda_{(1)} > \lambda_{(2)} > \ldots > \lambda_{(n)} > 0$ where $\forall i \in N,\ \lambda_{(i)} = \frac{\theta_{(i)}}{s_{(i)}}$.

Consider $\forall i \in N,\ \theta'_{(i)} = \theta_{(i)} + \epsilon_1$ such that

$$\epsilon_1 = \frac{\min(s_{(j)}\lambda_{(j-1)} - \theta_{(j)})}{2},\ \ j \in \{2, 3, \ldots, n\}.$$

Let agents (2) and (3) jointly misreports as $\theta'_{(2)} = \theta_{(2)} + \epsilon_1$ and $\theta'_{(3)} = \theta_{(3)} + \epsilon_1$. The basic idea is to construct a new profile so that under this new misreported profile relative queue positions remains unaltered. So under this new profile $\theta^* = (\theta_{(1)}, \theta'_{(2)}, \theta'_{(3)}, \theta_{(4)}, \ldots, \theta_{(n)}), t_{(2)}(\theta^*) > t_{(2)}(\theta)$ and $t_{(3)}(\theta^*) > t_{(3)}(\theta)$ since $\epsilon_1 > 0$ by construction. We find profitable group deviation exists for agents (2) and (3). Therefore PWSP impossible along with EFF and EE.

□

Proposition 4. *In case of two agents consider a mechanism that satisfies EFF,SP,EE then it also satisfies IPLB if and only if* $\bar{k} \geq -\frac{s_{(2)}\theta_{(1)}}{2}$

.

Proof. In two agent case a typical profile is $(\theta_{(1)}, \theta_{(2)}) \in \Re^2_{++}$. In the efficient allocation $\frac{\theta_{(1)}}{s_{(1)}} \geq \frac{\theta_{(2)}}{s_{(2)}}$ i.e. $s_{(2)}\theta_{(1)} \geq s_{(1)}\theta_{(2)}$.

From Theorem(2) it is clear that for two agent case $\bar{S} = s_{(1)} + s_{(2)}$ is the reference position.

Now IPLB is compatible with a mechanism that satisfies EFF,SP,EE if $\forall i \in N, \forall \theta \in \Re_{++} \quad U_{(i)}(\sigma(\theta), t_{(i)}(\theta), \theta_{(i)}) \geq C_{(i)}(\theta)$ i.e.

$$-\bar{S}\theta_{(i)} + t(\theta) \geq -(s_{(i)} + \sum_{j \neq i} \frac{s_{(j)}}{2})\theta_{(i)} \tag{14}$$

Consider $i = 1$. Then following equation(14) we have,

$$-(s_{(1)} + s_{(2)})\theta_{(1)} + (\bar{S} - S_{(1)})\theta_{(1)} + (\bar{S} - S_{(2)})\theta_{(2)} + \bar{k} \geq -(s_{(1)} + \frac{s_{(2)}}{2})\theta_{(1)}$$

Solving the above equation we get, $\bar{k} \geq -\frac{s_{(2)}\theta_{(1)}}{2}$...(i).

Similarly for $i = 2$, following equation (14) we have,

$$-(s_{(1)} + s_{(2)})\theta_{(2)} + (\bar{S} - S_{(1)})\theta_{(1)} + (\bar{S} - S_{(2)})\theta_{(2)} + \bar{k} \geq -(s_{(2)} + \frac{s_{(1)}}{2})\theta_{(2)}$$

Solving this we get, $s_{(2)}\theta_{(1)} + \bar{k} \geq \frac{s_{(1)}\theta_{(2)}}{2}$ ...(ii).

If (i) holds then (ii) trivially holds. So condition (i) is necessary and sufficient for IPLB.

Hence the necessary and sufficient condition for a mechanism with two agents to be EFF, SP, EE and IPLB is $\bar{S} = s_{(1)} + s_{(2)}$ and $\bar{k} \geq -\frac{s_{(2)}\theta_{(1)}}{2}$.[6]

□

Proposition 5. *Consider a Mechanism* (σ, t) *that satisfies EFF, SP, EE, if* $\bar{k} \geq 0$ *then it satisfies IPLB.*

[6]It is the average negative externalities imposed by second agent on the first agent.

Proof. If a Mechanism (σ, t) satisfies EE, EFF, SP and IPLB then $\forall i \in N$, $\forall \theta \in \Re^n_{++}$ we have,

$$U_{(i)}(\sigma(\theta)) - C_{(i)}(\theta) = \left(\sum_{q \in P'_{(i)}(\sigma(\theta))} s_q - \sum_{r \in P_{(i)}(\sigma(\theta))} s_r \right) \theta_{(i)} + \sum_{j \neq i} (\bar{S} - S_{(j)})\theta_{(j)} + \bar{k} \geq 0 \tag{15}$$

Note that,
$\sum_{j \neq i}(\bar{S} - S_{(j)})\theta_{(j)} = \sum_{r \in P_{(i)}(\sigma(\theta)}(\bar{S} - S_r)\theta_r + \sum_{q \in P'_{(i)}(\sigma(\theta)}(\bar{S} - S_q)\theta_q$.
Again,
$\sum_{r \in P_{(i)}(\sigma(\theta)}(\bar{S} - S_r)\theta_r = s_{(i)} \sum_{r \in P_{(i)}(\sigma(\theta)} \theta_r + \sum_{r \in P_{(i)}(\sigma(\theta)} \left(\sum_{q \in P'_{(i)}(\sigma(\theta)} s_q \right) \theta_r$
$+ \sum_{r \in P_{(i)}(\sigma(\theta)} \left(\sum_{m=r+1}^{(i)-1} s_m \right) \theta_r$
But, $s_{(i)} \left(\sum_{r \in P_{(i)}(\sigma(\theta)} \theta_r \right) - \left(\sum_{r \in P_{(i)}(\sigma(\theta))} s_r \right) \theta_{(i)} \geq 0$, because agents with higher $\lambda_{(\cdot)}$ are placed in the earlier positions of the queue (EFF allocation).
Hence,
$\left(\sum_{q \in P'_{(i)}(\sigma(\theta))} s_q - \sum_{r \in P_{(i)}(\sigma(\theta))} s_r \right) \theta_{(i)} + \sum_{j \neq i}(\bar{S} - S_{(j)})\theta_{(j)} > 0$. Since $\bar{k} \geq 0$ therefore $U_{(i)}(\sigma(\theta)) - C_{(i)}(\theta) > 0$. So IPLB holds. □

4. Conclusion

In this paper we have analysed sequencing problem from both incentive and normative approaches. We have completely characterized the class of incentive compatible mechanisms. We have identified unique class of VCG mechanisms that ensures egalitarian equivalence and we also have shown the possibility result with identical preference lower bound in that unique class of VCG mechanisms. Sequencing game impose a tougher restriction on the possible set of "reference position", compared to queueing game and that in turn results into the failure of having a feasible VCG mechanism along with egalitarian equivalence.

Lastly, though we have found the necessary and sufficient condition for the above mentioned unique class of VCG mechanism to satisfy identical preference lower bound when the number of participating agents is two, necessary condition for the same when the number of participating agent is more than two remains an open question.

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