A recursive method for solving a climate-economy model: value function iterations with logarithmic approximations

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Abstract

A recursive method for solving an integrated assessment model of climate and the economy is developed in this paper. The method approximates value function with a logarithmic basis function and searches for solutions on a set satisfying optimality conditions. These features make the method suitable for a highly nonlinear model with many state variables and various constraints, as usual in a climate-economy model.

Key words

Dynamic programming; recursive method; value function iteration; integrated assessment

JEL Classification

Q54; C61; C63

1 Introduction

This paper develops a numerical method for solving a climate economy model. Since an integrated assessment model (IAM) of climate and the economy is highly nonlinear and is subject to various constraints, it is not possible to solve the model analytically. Nonlinear programming has been usually applied for numerically solving IAMs. For instance the DICE 2007 model (Nordhaus, 2008) is solved with CONOPT (nonlinear programming) in GAMS modelling system. However the need for solving an IAM recursively (e.g., solving the Bellman equation) is growing because it helps investigate the effect of uncertainty and learning on policy and welfare (e.g., Kelly and Kolstad, 1999).

The method of this paper is suitable for this kind of numerical analysis because it is less prone to the number of state variables than the existing methods in the literature. The main advantage of the method of this paper is that it is simple and transparent because it obtains solutions from optimality conditions. The disadvantage is that one should specify the first order conditions analytically, which may require tedious calculations if the number of state variables and control variables becomes large.

In most dynamic programming literature solving an IAM, the problem is reformulated in a recursive way and the value function is approximated to a flexible basis function. Then the fixed-point theorem is applied to find solutions.[1] One of the main differences among existing papers is the approximation method. For instance, Kelly and Kolstad (1999) and Leach (2007) use neural networks approximations, Kelly and Tan (2013) apply spline approximations, and

[1] See Bellman and Dreyfus (1962), Stokey and Lucas (1989), Rust (1996), Judd (1998), and Miranda and Fackler (2004) for more on dynamic programming.

Cai et al. (2012b) and Lemoine and Traeger (2014) apply Chebyshev polynomials approximations. A basis function is useful in that it has an analytical functional form.

A dynamic climate-economy model is not generally time autonomous since it has many exogenous variables such as labor force and technology. To address this issue, Kelly and Kolstad (1999), Leach (2007), and Lemoine and Traeger (2014) add time as an argument for the value function. Cai et al. (2012b) let the coefficients of the basis function vary each time period. Kelly and Tan (2013) make the model time independent.

This paper presents a different method from the literature: logarithmic approximations. Exogenous variables can be added as arguments for the basis function in order to address the problem of time dependence, but whether or not exogenous variables are added does not affect the accuracy of the method. In addition, the solution method of this paper differs from the literature in that it searches for solutions on an ergodic set, whereas the other papers generally search for solutions on a carefully designed grid. A grid based method is generally prone to the 'curse of dimensionality' (for more discussion, see Judd et al., 2011). For instance, DICE has 2 control variables, 6 endogenous state variables, and 9 (time-dependent) exogenous variables. Thus, the number of the total grid points will be n^7, if we apply a grid based method with n grid points per each state variable and time is added as a state variable instead of exogenous variables. Thus an extension of the model to incorporate interesting topics such as uncertainty and learning is demanding because the total number of grid points grows fast. The method of this paper, however, searches for solutions on simulated data points, which satisfy optimality conditions. Therefore it is less prone to the curse of dimensionality.

This paper proceeds as follows. Section 2 presents the general method. As an application, a simple analytical economic growth model is solved in Section 3. The DICE model is solved in Section 4. Section 5 concludes.

2 The Method

The problem of a decision maker in a dynamic model can be reformulated as in Equation (1): the Bellman equation. The decision maker chooses the vector of control variables every time period so as to maximize the objective function, which is the discounted sum of expected utility.

$$W(\boldsymbol{s}_t;\ \boldsymbol{\theta}_t) = \max_{\boldsymbol{c}_t}[U(\boldsymbol{s}_t, \boldsymbol{c}_t; \boldsymbol{\theta}_t) + \beta\mathbb{E}_t W(\boldsymbol{s}_{t+1}; \boldsymbol{\theta}_{t+1})] \tag{1}$$

where $\mathbb{E}_t$ is the expectation operator given information at point in time t, W is the value function, $\boldsymbol{c}$ is the vector of control variables, $\boldsymbol{s}$ is the vector of state variables, $\boldsymbol{\theta}$ is the vector of uncertain variables, and β is the discount factor.

The logarithmic function as in Equation (2) is used to approximate the value function. The main criteria for the choice of the basis function are its simplicity, convenience for deriving the first order conditions, and its accuracy. Maliar and Maliar (2005) apply this functional form to a time-autonomous economic growth model. Hennlock (2009) applies this function to a theoretical model of climate and the economy.

Without loss of generality, we assume that there are two control variables ($c_{1,t},\ c_{2,t}$), two endogenous state variables ($s_{1,t},\ s_{2,t}$), and one exogenous state variable ($s_{3,t}$). The endogenous variables depend on uncertain parameter $\boldsymbol{\theta}_{\mathrm{t}}$.

$$W(\boldsymbol{s}_t; \boldsymbol{b}, \boldsymbol{\theta}_t) \approx b_0 + b_1 \, ln\left(s_{1,t}\right) + b_2 \, ln\left(s_{2,t}\right) + b_3 \, ln\left(s_{3,t}\right) \tag{2}$$

The first order conditions for the Bellman equation are:

$$\frac{\partial U(\boldsymbol{s}_t, \boldsymbol{c}_t;\ \boldsymbol{\theta}_t)}{\partial \boldsymbol{c}_t} + \beta \mathbb{E}_t \frac{\partial \boldsymbol{g}(\boldsymbol{s}_t, \boldsymbol{c}_t;\ \boldsymbol{\theta}_t)}{\partial \boldsymbol{c}_t} \cdot \frac{\partial W(\boldsymbol{s}_{t+1}; \boldsymbol{b}, \boldsymbol{\theta}_{t+1})}{\partial \boldsymbol{s}_{t+1}} = \boldsymbol{0} \tag{3}$$

$$\frac{\partial W(\boldsymbol{s}_t; \boldsymbol{b}, \boldsymbol{\theta}_t)}{\partial \boldsymbol{s}_t} = \frac{\partial U(\boldsymbol{s}_t, \boldsymbol{c}_t; \boldsymbol{\theta}_t)}{\partial \boldsymbol{s}_t} + \beta \mathbb{E}_t \frac{\partial \boldsymbol{g}(\boldsymbol{s}_t, \boldsymbol{c}_t; \boldsymbol{\theta}_t)}{\partial \boldsymbol{s}_t} \cdot \frac{\partial W(\boldsymbol{s}_{t+1}; \boldsymbol{b}, \boldsymbol{\theta}_{t+1})}{\partial \boldsymbol{s}_{t+1}} \tag{4}$$

where $\boldsymbol{g}$ are the law of motions for the state variables.

Equations (3) and (4) give four equations for two unknown control variables at point in time t and two unknown state variables at point in time $t+1$. Therefore solutions are obtainable as long as the vector of coefficients of the basis function ($\boldsymbol{b}$) are chosen. An initial guess on $\boldsymbol{b}$ can be chosen from equilibrium conditions. If the model is highly nonlinear and subject to various constraints, as usual in a climate-economy model, numerical methods for finding solutions can be used (see Judd, 1998; Miranda and Fackler, 2004 for various methods). Then optimal policy rules at point in time t become functions of the given state variables at point in time t and the chosen coefficients of the basis function $\boldsymbol{b}$. Solving Equations (3) and (4) throughout the whole time periods, we are ready to calculate the left hand side (LHS) and the right hand side (RHS) of the Bellman equation (1). For the expectation operator, numerical integration such as Monte Carlo integration or Gauss–Hermite quadrature (GH) can be used (Judd, 1998).

By the fixed point theorem, optimal solutions equate LHS and RHS of the Bellman equation (Stokey and Lucas, 1989). Since our initial value for $\boldsymbol{b}$ is chosen by a guess, more iterations may be required. To this end the stopping rule is specified as in Equation (5).

$$max \left| \frac{W(\boldsymbol{s}_t, \boldsymbol{\theta}_t)^{(p+1)} - W(\boldsymbol{s}_t, \boldsymbol{\theta}_t)^{(p)}}{W(\boldsymbol{s}_t, \boldsymbol{\theta}_t)^{(p)}} \right| \leq \omega \tag{5}$$

where ω is the tolerance level and p refers to the p^{th} iteration. An arbitrarily high value for $W(\boldsymbol{s}_t, \boldsymbol{\theta}_t)^{(0)}$ is used to initiate iterations.

If p^{th} iteration does not satisfy the stopping rule, a new $\boldsymbol{b}$ should be chosen. To this end, the updating rule for $\boldsymbol{b}$ is specified as in Equation (6).

$$\boldsymbol{b}^{(p+1)} = (1 - \vartheta)\boldsymbol{b}^{(p)} + \lambda\widehat{\boldsymbol{b}} \tag{6}$$

where $\widehat{\boldsymbol{b}}$ denotes the estimator minimizing approximation errors between LHS and RHS of Equation (1), and ϑ is a parameter ($0< \vartheta<1$). Technically, in order to avoid the problem of ill-conditioning, the least-square method using singular value decomposition (SVD) can be applied (Judd et al., 2011).

The above procedure continues until the stopping rule is satisfied. If $\boldsymbol{b}^*$ satisfy the stopping rule, then the resulting policy rules are optimal solutions in the sense that they are the fixed point of the Bellman equation (Stokey and Lucas, 1989).

3 An Application: A Simple Economic Growth Model

The procedure for solving a simple economic growth model is shown below. The model is useful for an illustration of the solution method since it is analytically solvable without tedious calculations.

The problem of the decision maker is to choose the level of consumption each time period so as to maximize social welfare defined as in Equation (7) subject to Equation (8).

$$\max_{C_t} \sum_{t=0}^{\infty} \beta^t L_t U\,(C_t/L_t) = \sum_{t=0}^{\infty} \beta^t L_t \frac{(C_t/L_t)^{1-\alpha}}{1-\alpha} \tag{7}$$

$$K_{t+1} = (1-\delta_k)K_t + Q_t - C_t \tag{8}$$

where U is the utility function, L is labor force (exogenous), C is consumption, K is the capital stock, $Q = F(A, K, L)$ is gross output, A is the total factor productivity (exogenous), δ_k is the depreciation rate of the capital stock, α is the elasticity of marginal utility.

The Bellman equation and the basis function for the problem are:

$$W(K_t, L_t, A_t; \boldsymbol{b}) = \max_{C_t}[L_t U(C_t/L_t) + \beta W(K_{t+1}, L_{t+1}, A_{t+1}; \boldsymbol{b})] \tag{9}$$

$$W(K_t, L_t, A_t; \boldsymbol{b}) \approx b_0 + b_1 \ln(K_t) + b_2 \ln(L_t) + b_3 \ln(A_t) \tag{10}$$

The first order conditions are:

$$(C_t/L_t)^{-\alpha} - \beta b_1/K_{t+1} = 0 \tag{11}$$

$$b_1/K_t = \beta(1 - \delta_k + \partial Q_t/\partial K_t) b_1/K_{t+1} \tag{12}$$

Since there are two unknowns (C_t, K_{t+1}) and we have two equations, solutions are obtainable as follows.

$$C_t = L_t \left\{ \frac{b_1}{K_t(1 - \delta_k + \partial Q_t/\partial K_t)} \right\}^{-1/\alpha} \tag{13}$$

$$K_{t+1} = \beta b_1 \left\{ \frac{b_1}{K_t(1 - \delta_k + \partial Q_t/\partial K_t)} \right\}^{\alpha} \tag{14}$$

If δ_k=1, α=1, and the production function is Cobb-Douglas, Equations (7) and (8) are analytically solvable (see Stokey and Lucas, 1989: Exercise 2.2). The solution for a finite time horizon problem is:

$$k_{t+1} = \beta\gamma \left\{ \frac{1 - (\beta\gamma)^{T-t+1}}{1 - (\beta\gamma)^{T-t+2}} \right\} A_t {k_t}^{\gamma} \tag{15}$$

where $k_t = K_t/L_t$, T is the time horizon.

The left panel of Figure 1 shows the rate of saving for the problem of Equations (7) and (8), calculated from Equation (15). As expected, longer time horizon increases the rate of saving. The optimal rate of saving for the infinite time horizon problem ($T \to \infty$) is 0.295567. As shown in the right panel of Figure 1, our dynamic programming method with the logarithmic approximations produces exact solution (up to 16th decimal place), which is more precise than nonlinear programming with finite time horizon. The inclusion of the other exogenous variables into the basis function does not affect the results (not shown).

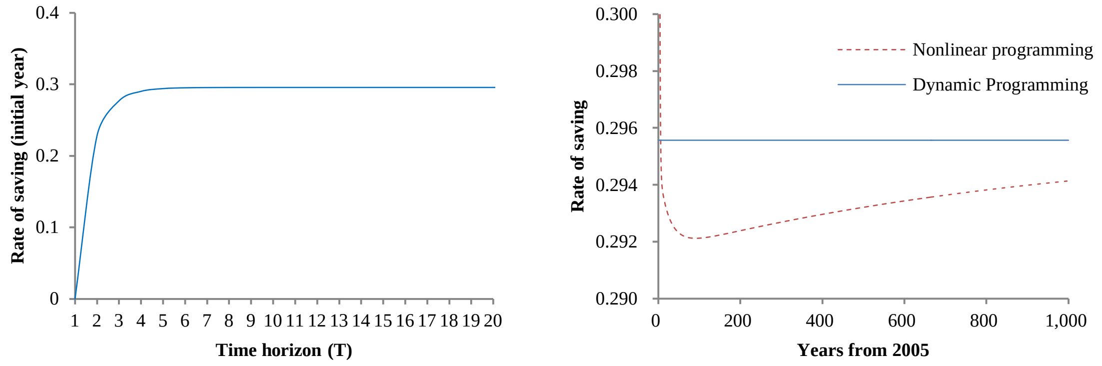

Figure 1 The rate of saving (Left): analytical solutions **(Right):** numerical solutions. Dynamic programming refers to the solutions obtained from the method of this paper. Only a constant and the capital stock are included as arguments for the value function. The maximum tolerance level and the simulation length are set at 10^{-6} and 1,000, respectively. Nonlinear programming refers to the solutions obtained from CONOPT (nonlinear programming) in GAMS (time horizon 1,000 years). For numerical simulations, the initial value of the capital stock and the evolutions of exogenous variables are drawn from DICE 2007.

Applying δ_k=0.1 and α=2, the rate of saving is higher for the dynamic programming than for the nonlinear programming with finite time horizon. Put differently, optimal investment is higher for the dynamic programming (see the top left panel of Figure 2). One of the reasons is that the dynamic programming solves the infinite time horizon problem, whereas the nonlinear programming solves the finite time horizon model.

The rate of saving should satisfy the following equation at equilibrium: $s_\infty = (n + g + \delta) K_\infty / Q_\infty$, where ∞ denotes the variables at equilibrium, n and g are the growth rates of labor force and the total factor productivity, respectively (for more on this, see Romer, 2006). The top right panel of Figure 2 shows that our dynamic programming finds the optimal path that satisfies the relation at equilibrium.

The decision maker consumes less in the near future for the dynamic programming than for the nonlinear programming. Such decisions produce more consumption in the future (more specifically, after 2028). The evolutions of the other variables including gross output, and the capital stock depend on the behavior of the rate of saving (see the bottom panel of Figure 2).

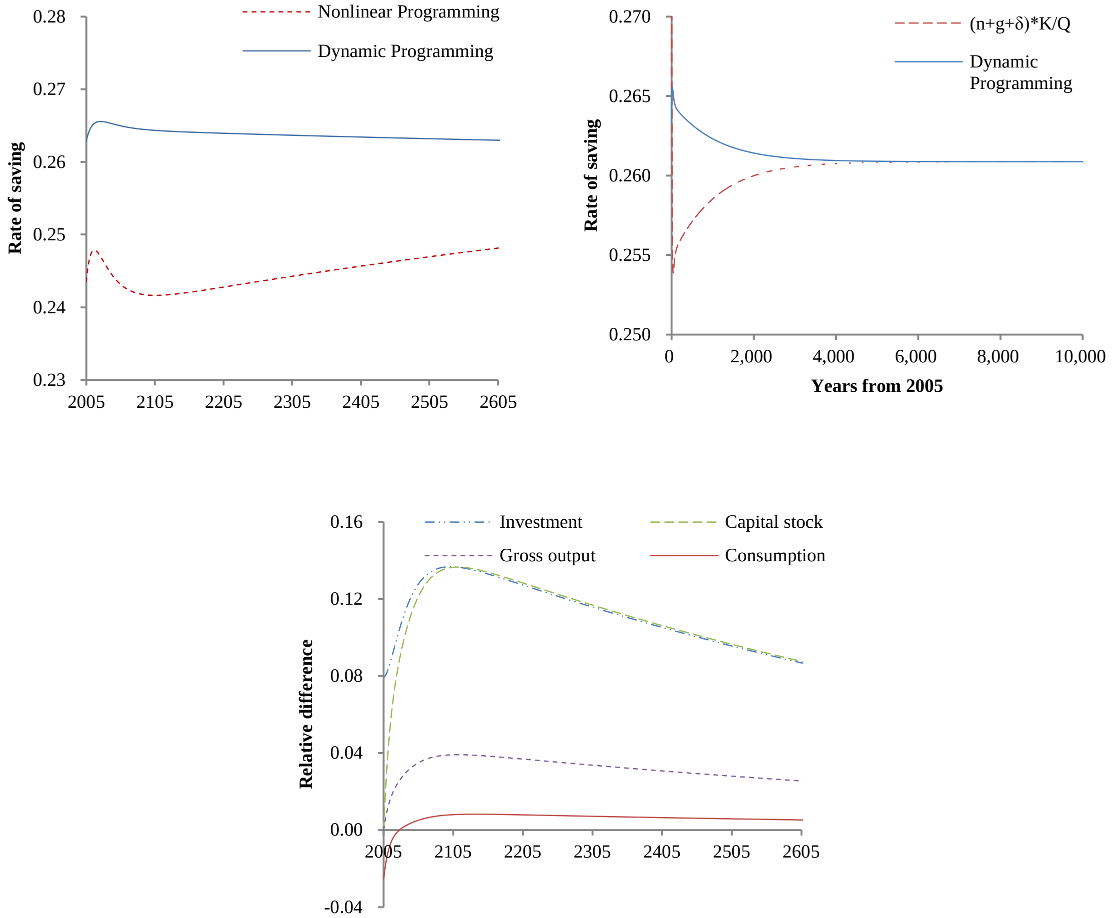

Figure 2 The optimal solutions (the simple growth model) (Top): The rate of saving. **(Bottom):** The relative difference between the dynamic programming and the nonlinear programming calculated as follows: (the results of DP – the results of NP) / the results of DP. For dynamic programming, the maximum tolerance level and simulation length are set at 10^{-6} and 1,000 years, respectively. For nonlinear programming time horizon is set at 1,000 years.

4 An Application: The DICE Model

Our solution method is applicable to more complex models such as the DICE model. The full model is given in Appendix A. The Bellman equation and the basis function for this problem are:

$$W(\boldsymbol{s_t};\boldsymbol{b}) = \max_{C_t,\mu_t}[L_t U + \beta W(\boldsymbol{s_{t+1}};\boldsymbol{b})] \tag{16}$$

$$W \approx b_0 + \sum_{i=1} b_i ln(s_{i,t}) \tag{17}$$

where μ_t is the rate of GHG emissions control. Note that there are two control variables (C_t, μ_t) and six endogenous state variables $(K_{t+1},\ M_{AT_{t+1}},\ M_{U_{t+1}},\ M_{L_{t+1}},\ T_{AT_{t+1}},\ T_{L_{t+1}})$, where $M_{AT_{t+1}}$, $M_{U_{t+1}}$, $M_{L_{t+1}}$ are the carbon stock in the atmosphere, the upper ocean, and the lower ocean, respectively, $T_{AT_{t+1}}$ and $T_{L_{t+1}}$ are air temperature changes and ocean temperature changes (from 1900), respectively. Applying the first order conditions we get eight equations. Arranging the first order conditions results in Equations (18) and (19) for optimal policy rules:

$$B_{1,t}\mu_t^{\theta_2} + B_{2,t}\mu_t^{\theta_2-1} + B_{3,t} = 0 \tag{18}$$

where

$$B_{1,t} = -\theta_{1,t}\frac{b_1}{K_t}\frac{\zeta_3}{M_{AT_t}}\frac{(1-\xi_4)Q_t\Omega_t'}{(1-\xi_4)(\zeta_1 f + \zeta_2) - \xi_4\zeta_4} + \theta_{1,t}(\theta_2 - 1)Q_t\Omega_t'\left\{\frac{b_2}{M_{AT_t}} - P_t\right\} \tag{18-1}$$

$$B_{2,t} = -\theta_{1,t}(\theta_2 - 1)Q_t'\Omega_t\left\{\frac{b_2}{M_{AT_t}} - P_t\right\} + \theta_{1,t}\theta_2\frac{b_1}{K_t}\frac{\Omega_t}{\sigma_t}P_0 \tag{18-2}$$

$$B_{3,t} = \frac{b_1}{K_t}\frac{\zeta_3}{M_{AT_t}}\frac{(1-\xi_4)Q_t\Omega_t'}{(1-\xi_4)(\zeta_1 f+\zeta_2)-\xi_4\zeta_4} - \{(1-\delta_k)+Q_t'\Omega_t\}\left\{\frac{b_2}{M_{AT_t}} - P_t\right\} \qquad (18\text{-}3)$$

$$P_t = \frac{\delta_{UA}}{\delta_{UL}\delta_{LU}-\delta_{LL}\delta_{UU}}\left\{\frac{\delta_{UL}b_4}{M_{L_t}} - \frac{\delta_{LL}b_3}{M_{U_t}}\right\} + \frac{\zeta_3}{M_{AT_t}}\frac{\left\{\frac{(1-\xi_4)b_5}{T_{AT_t}} - \frac{\xi_4 b_6}{T_{L_t}}\right\}}{(1-\xi_4)(\zeta_1 f+\zeta_2)-\xi_4\zeta_4} \qquad (18\text{-}4)$$

$$(C_t/L_t)^{-\alpha} = b_1/K_t\{(1-\delta_k)+Q_t'\Omega_t(1-\Lambda_t)-(1-\mu_t)Q_t'\Omega_t\Lambda_t'\}^{-1} \qquad (19)$$

where Ω_t is the damage function, Λ_t is the abatement cost function, $\zeta_1 = \xi_1\eta/\lambda_0$, $\zeta_2 = 1-\zeta_1$, $\zeta_3 = \xi_1\eta/\ln(2)$, $f = 1-\lambda_0/\lambda$, $Q_t' = \partial Q_t/\partial K_t$, $\Omega_t' = \partial\Omega_t/\partial T_{AT\,t}$, $\Lambda_t' = \partial\Lambda_t/\partial\mu_t$, and $P_0 = \delta_{AA} + \delta_{UA}\delta_{AU}\delta_{LL}/(\delta_{UL}\delta_{LU}-\delta_{LL}\delta_{UU})$. See Appendix A for notations and the parameter values.

Since the model is highly nonlinear and subject to irreversibility constraint ($0 \le \mu_t \le 1$), numerical methods for finding solutions are applied. More precisely, Newton's method with Fisher's function for the root-finding problem is applied (Judd, 1998; Miranda and Fackler, 2004).

As shown in Figure 3, our solution method finds equilibrium far in the future. This is because of the evolutions of the exogenous variables of the DICE model as shown in Figure 4. With various experiments it is found that the simulation length larger than 1,000 does not affect solutions. The tolerance level was set at 10^{-6}.

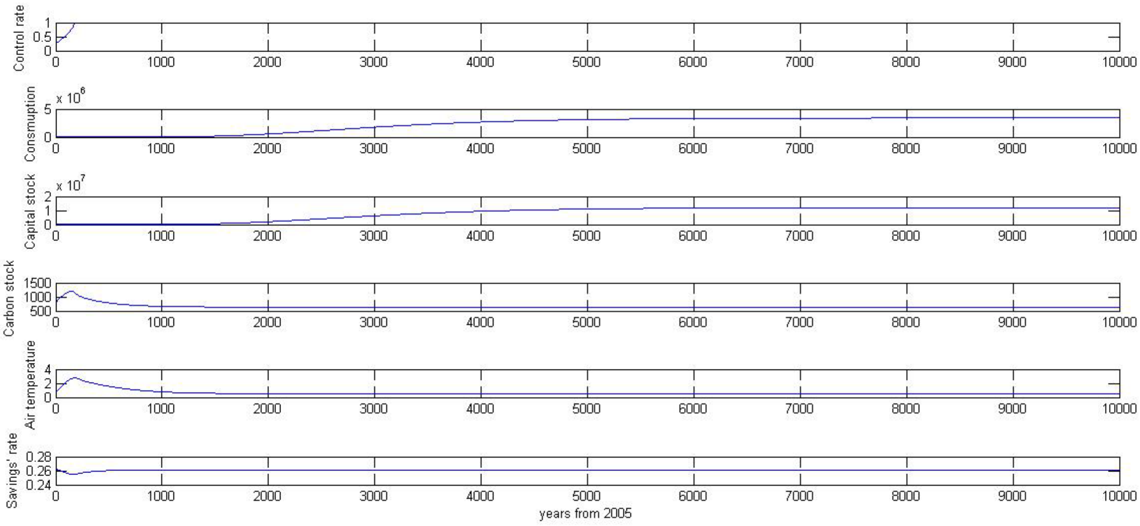

Figure 3 The optimal solutions (the DICE model) The units for consumption and the capital stock are 1,000US$/person. The units for the carbon stock and air temperature are GtC and °C, respectively.

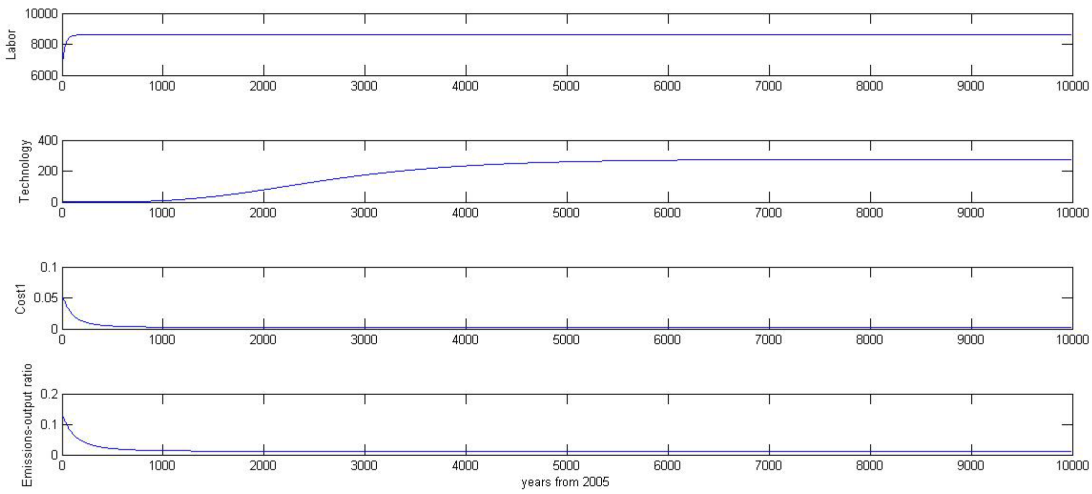

Figure 4 The evolutions of the exogenous variables (the DICE model) Labor, technology, cost1, and emissions-output ratio refer to $\boldsymbol{L_t}$, $\mathbf{A}_{\boldsymbol{t}}$, $\boldsymbol{\theta_{1,t}}$, and $\boldsymbol{\sigma_t}$, respectively.

Similar to the simple growth model, the rate of saving (in turn, investment and gross output) is higher for the dynamic programming method than for the nonlinear programming method.

As a result, (business as usual) greenhouse gas emissions are higher for the dynamic programming than for the nonlinear programming. This raises the rate of emissions control for the dynamic programming compared to the nonlinear programming (in turn, lower optimal temperature increases). Except for near future (until 2037), consumption is higher for the dynamic programming than for the nonlinear programming.

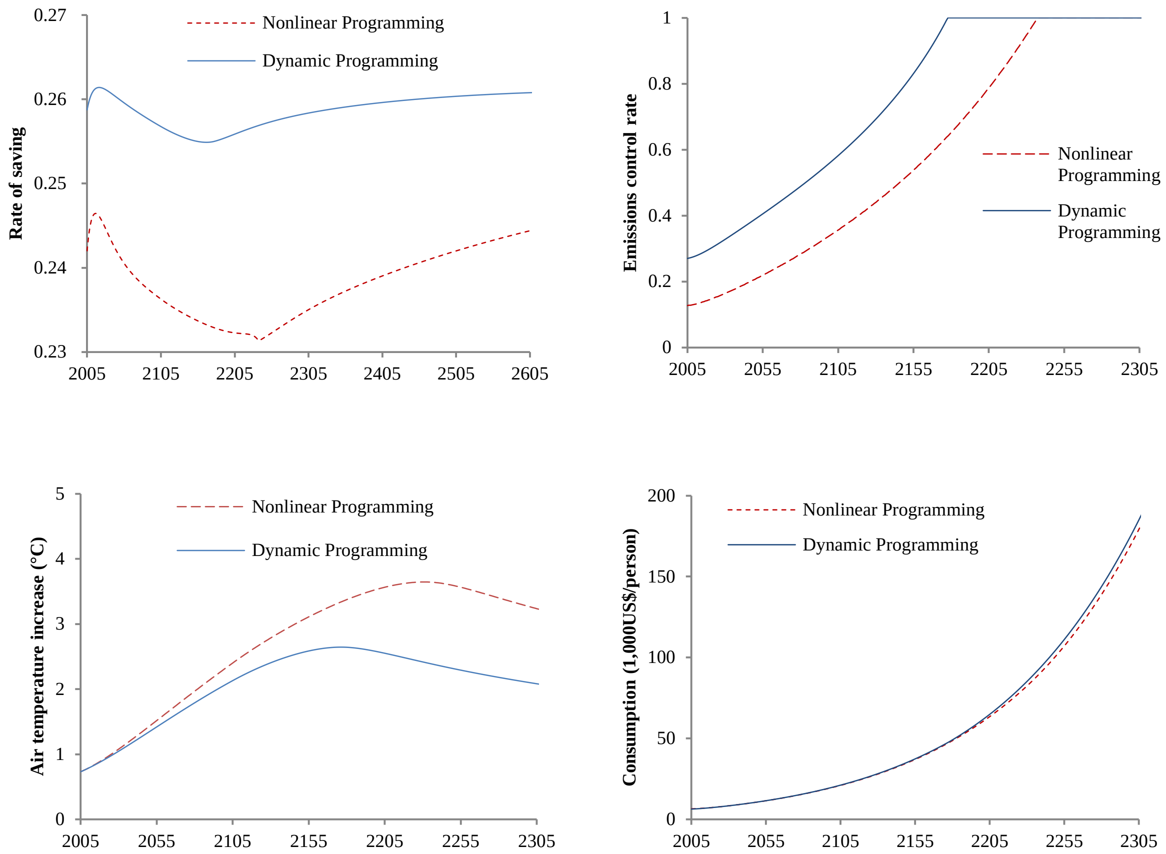

Figure 5 The optimal solutions (the DICE model) (Top Left): The rate of saving **(Top Right):** The rate of emissions control **(Bottom Left):** .Atmospheric temperature increases (from 1900) **(Bottom Right):** Consumption

As shown in Figure 3, the rate of saving does not change much over time. In addition, the rate of saving does not change much over time over a various plausible range of the climate sensitivity. For instance, the rate of saving (defined as the gross investment divided by the net production) changes in the range of 0.240 and 0.247 for the first 600 years in the DICE-CJL model (Cai et al., 2012a), which is a modified version of DICE with an annual time step. For instance, if the savings' rate is fixed at 0.245 all variables including the optimal carbon tax deviate only less than 3% from the original results. This holds even if the true value of the climate sensitivity is set at 25°C/2xCO_2.

Fixing the savings' rate as in the model of Solow (1956) helps reduce computational burden since the number of control variables is reduced. Furthermore fixing the savings' rate does not affect the optimal solutions much. For instance, the solutions of the reduced DICE 2007 model (fixing the savings' rate at constant) obtained from dynamic programming are compared with the results obtained from nonlinear programming in GAMS (i.e., using the original programming code made available by William Nordhaus) in Figure 6, which shows that our solution method produces almost the same results as the results from nonlinear programming.

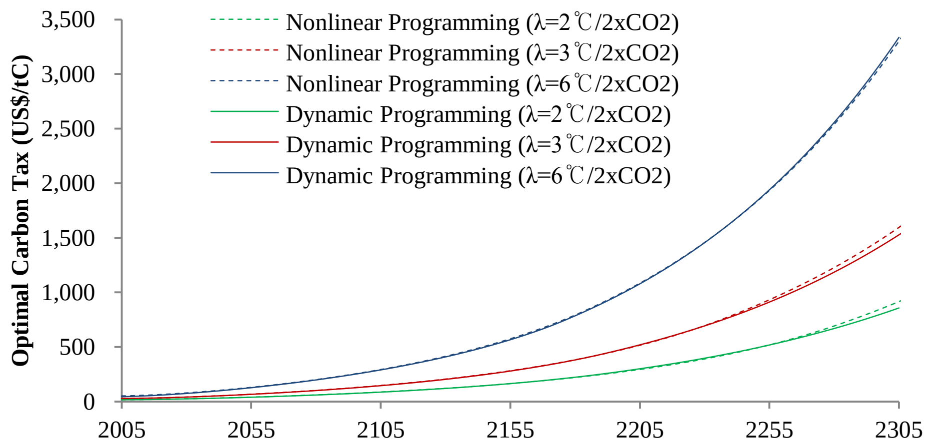

Figure 6 The optimal carbon tax **λ** refers to the equilibrium climate sensitivity.

The accuracy of the dynamic programming method is tested as follows. The maximum welfare over a grid of the control variables is calculated for every time period. More specifically, the model is simulated with a fixed emissions control rate (1,000 grid points from 0 to 1) and then the rate of emissions control which results in maximum welfare is chosen for every time period. The emissions control rate obtained above and the emissions control rate obtained from the dynamic programming method are compared. The result is that the maximum difference between the two values over the whole time periods is about 10^{-4}.

5 Concluding Remarks

This paper develops a numerical method for solving a climate economy model. Our method produces exact solutions to an (analytical) economic growth model and is useful for solving more demanding models such as DICE. Only the Bellman equation, arguments of the value function, and the first order conditions should be changed according to models. For instance, Hwang et al., (2013, 2014) solve uncertainty and learning models on climate change having up to 9 endogenous state variables with the method of this paper.

From the applications of our method, we find that optimal investment is calculated to be slightly higher for the dynamic programming than for the nonlinear programming with finite time horizon. Such decisions induce slightly lower near future consumption but higher consumption in the future (after about 20-30 years) for the dynamic programming than for the nonlinear programming. The optimal rate of emissions control (in turn, the optimal level of temperature increases) is affected by the investment decision. More specifically, the decision maker increases the rate of emissions control compared to the nonlinear programming and thus temperature increases are lower for the dynamic programming.

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Appendix A: The Full Model

$$\max_{\mu_t, I_t} \mathbb{E} \sum_{t=1}^{\infty} L_t \beta^t U(C_t, L_t) = \sum_{t=1}^{\infty} L_t \beta^t U\left[\left\{\left(1 - \theta_{1,t} \mu_t^{\theta_2}\right)\Omega_t Q_t - I_t\right\} / L_t\right] \quad \text{(A.1)}$$

$$K_{t+1} = (1 - \delta_k)K_t + I_t \quad \text{(A.2)}$$

$$M_{AT\,t+1} = (1 - \mu_t)\sigma_t Q_t + E_{LAND_t} + \delta_{AA} M_{AT\,t} + \delta_{UA} M_{U\,t} \quad \text{(A.3)}$$

$$M_{U\,t+1} = \delta_{AU} M_{AT\,t} + \delta_{UU} M_{U\,t} \quad \text{(A.4)}$$

$$M_{L\,t+1} = \delta_{UL} M_{U\,t} + \delta_{LL} M_{L\,t} \quad \text{(A.5)}$$

$$T_{AT_{t+1}} = T_{AT_t} + \xi_1 \left\{\eta \, \frac{\ln(M_t/M_b)}{\ln(2)} + RF_{N,t} - \frac{\eta}{\lambda} T_{AT_t} - \xi_3 (T_{AT_t} - T_{LO_t})\right\} \quad \text{(A.6)}$$

$$T_{LO_{t+1}} = T_{LO_t} + \xi_4 \left\{T_{AT_t} - T_{LO_t}\right\} \quad \text{(A.7)}$$

where $\mathbb{E}$ is the expectation operator, t is time (annual). Notations, initial values, and parameter values are given in Tables (A.1) and (A.2).

Table A.1 Variables

Variables		Notes
U	Utility function	$=(C_t/L_t)^{1-\alpha}/(1-\alpha)$
C_t	Consumption	$=\left(1 - \theta_{1,t} \mu_t^{\theta_2}\right)\Omega_t Q_t - I_t$
μ_t	Emissions control rate	Control variable
I_t	Investment in general	Control variable
K_t	Capital stock	K_0=$137 trillion
$M_{AT\,t}$	Carbon stocks in the atmosphere	$M_{AT\,0}$=808.9GtC

M_{U_t}	Carbon stocks in the upper ocean	M_{U_0}=18,365GtC
M_{L_t}	Carbon stocks in the lower ocean	M_{L_0}=1,255GtC
T_{AT_t}	Atmospheric temperature deviations	T_{AT_0}=0.7307°C
T_{LO_t}	Ocean temperature deviations	T_{LO_0}=0.0068°C
Ω_t	Damage function	$=1/(1+\kappa_1 T_{AT_t}+\kappa_2 T_{AT_t}^2)$
Q_t	Gross output	$=A_t K_t^{\gamma} L_t^{1-\gamma}$
A_t	Total factor productivity	Exogenous
L_t	Labor force	Exogenous
σ_t	Emission-output ratio	Exogenous
$RF_{N,t}$	Radiative forcing from non-CO_2 gases	Exogenous
E_{LAND_t}	GHG emissions from non-energy consumption	Exogenous
$\theta_{1,t}$	Abatement cost function parameter	Exogenous

Note: The initial values for the state variables and the evolutions of the exogenous variables follow Nordhaus (2008).

Table A.2 Parameters

Parameters		Values
λ	Equilibrium climate sensitivity	$=\lambda_0/(1-f)$
f	Total feedback factors	0.6
λ_0	Reference climate sensitivity	1.2°C/2xCO_2
α	Elasticity of marginal utility	2
β	Discount factor	$=1/(1+\rho)$
ρ	Pure rate of time preference	0.015
γ	Elasticity of output with respect to capital	0.3
δ_k	Depreciation rate of the capital stock	0.1
κ_1, κ_2	Damage function parameters	κ_1=0, κ_2=0.0028388
θ_2	Abatement cost function parameter	θ_2=2.8
δ_{AA}, δ_{UA}, δ_{AU},	Climate parameters	δ_{AA}=0. 9810712, δ_{UA}=0.0189288,

δ_{UU}, δ_{UL}, δ_{LL}, ξ_1, ξ_3, ξ_4, η		δ_{AU}=0.0097213, δ_{UU}=0.005, δ_{UL}=0.0003119, δ_{LL}=0.9996881, ξ_1=0.022, ξ_3=0.3, ξ_4=0.005, η=3.8
M_b	Pre-industrial carbon stock	596.4GtC

Note: The parameter values are from Nordhaus (2008) and Cai et al. (2012a) except that λ_0 follow Roe and Baker (2007).