Optimal consumption and investment in the economy with infinite number of consumption goods

Pawel Kliber

Poznan University of Economics

10 February 2014

OPTIMAL CONSUMPTION AND INVESTMENT IN THE ECONOMY WITH INFINITE NUMBER OF CONSUMPTION GOODS

PAWEŁ KLIBER

ABSTRACT. In the article we present some extension for the classical problem of dynamic investment optimization. We take the neoclassical model of growth with one product and many consumption goods. The number of consumption goods can be infinite and the consumption bundle is defined on some abstract, measurable space. The instantaneous social utility of consumption is measured as the integral of individual utilities of the consumption goods. The process of transforming product into consumption goods is described by another measure. The performance of the economy is measured by current value of the total utility in some planning horizon. We show that the problem of choosing optimal consumption paths for each good can be decomposed into 1) problem of choosing optimal aggregate consumption, which can be solved using standard methods of optimal control theory, 2) problem of distribution aggregate consumption into consumption of specific goods.

1. CLASSIC PROBLEM OF CONSUMPTION OPTIMIZATION

Let the total production in the economy be described by given by the function $f : \mathbb{R}_+ \to \mathbb{R}_+$. Outputs depend solely on the capital. For the inputs k the outputs are $f(k)$. We assume that f is increasing and concave, what implies that it is continuous and differentiable except for at most countably many points [1]. At each moment t output is devided into investments and consumption. The utility of consumption is described by the increasing and concave utility function $u : \mathbb{R}_+ \to \mathbb{R}_+$. Investments increase the capital stock and thus the production. Without investments the captal tends to shrink with constant depretiation rate δ. Let $c(t)$ be consumpiton at the moment t, the capital dynamics is then given by differential equation:

$$\dot{k}(t) = f(k(t)) - c(t) - \delta k(t). \tag{1}$$

Let ρ be the social discount rate. One unit of utility now equals $e^{-\rho t}$ units at time t. If the performance of the economy is measured by the sum of discounted utiliy in some period $[0, T]$ (with $0 < T \leqslant +\infty$), then we obtain the following optimal control problem.

$$\max \int_0^T e^{-\rho t} u\left(c\left(t\right)\right) dt, \tag{2}$$

subject to

$$\dot{k}(t) = f(k(t)) - c(t) - \delta k(t), \tag{3}$$

$$\forall t \in [0, T] \colon 0 \leqslant c(t) \leqslant f\left(k\left(t\right)\right), \tag{4}$$

where we consider all functions $c(t)$ which fulfill condition (4) and which are rightly continuous with at most finite number of jump discontinuities. These are standard assumpitnons about control variable path in the optimal control theory, when one uses Pontriagin theorem - see [1], [5], [6], [13]. Notice that exactly the same the same

[1] See [9], theorems 10.1.1 and 25.3.

problem (2)-(4) is obtained in the setting with capital and labor if the production function has constant returns to scale, labor grows with constant rate, and the performance is measured with the sum of discounted consumption *per capita* (see for example [11] chapter 1.)

The problem (2)-(4) was thoroughly analysed in the literature. Although the analitical solution for all possible utility functions u and production functions f cannot be given, one can however sketch the main features of such solution[2].

(i) If the planning horizon is limited ($T < \infty$), then there is a final consumption phase. In this phase all output is consumed, i.e. $c(t) = f(k(t))$.

(ii) If the planning horizon is long enough, then there is a turnpike phase in the optimal solution. In this phase the capital stock remains unchanged. The turnpike level of the capital fulfills the condition $f'(k) = \delta + \rho$ (if the function f is not differentiable in this point, then $\delta + \rho \in \partial f(k)$, where $\partial f(k)$ is subgradient of f). If $T = +\infty$, then the capital stock approches turnpike leves as $t \to +\infty$.

(iii) If the utility function is not strictly concave (i.e. there are elements of linearity), then there are strict phases in the optimal solution – investment phase, consumption phase and turnpike phase. The solution begins with the phase of reaching the turnpike. If the initial capital $k(0)$ is below turnpike level, then all the production should be invested until capital reaches the turnpike level. Otherwise, when initial capital is below turnpike level, the investments should be stopped and capital shrinks towards required level.

(iv) If the production function is strictly concave, then these phases are not so strict. The capital moves towards turnpike level along saddle path. The consumption on this path is given by

$$u'(c(t)) = \lambda(t),$$

where λ is the dual price (see [3], [11]).

(v) There are cases in which the problem does not have a solution for $T = +\infty$. For example if u is linear and production function f is of CES-type (constant elasticity of substitution), then there are consumption paths that yields infinite total discounted consumption.

2. The economy with multiple of goods

In the previous model it was assumed that there is only one good in the economy and that it can serve for consumption as well as for investments. Now we assume that there is a multiple of goods which differs with respect to their utility and to the way they are produced. In the model goods are named with certain index θ. Let Θ be the set of all such indexes. Mathematically we assume that Θ is a measurable space, equipped with some finite measure ν. The measure μ measures the impact of the goods in different kinds on the total social welfare.

Each good has different differs with respect to the social utility it brings. The utility is now measured with the function $u : \Theta \times \mathbb{R}_+ \to \mathbb{R}_+$. The value of $u(\theta, y)$ denotes the utility from consumption of the θ-type good in the amount of y. We assume that $u(\theta, \cdot)$ is Lipschitz-continuous, increasing and concave with respect to y for all $\theta \in \Theta$. This means that consumers are nonsaturiated with respect for every good and that each good has decreasing mariginal utility. Let $c\colon \Theta \to \mathbb{R}_+$ be a measurable function that describes the consumption of goods of different types, i.e. the value $c(\theta)$ denotes the amount of good of the type θ, which is consumed.

[2]See [3], [7], [8], [11].

One can think of the function c as of a "consumption plan". Then the social utility at any specific moment is thus given by the integral

$$\int_{\Theta} u\left(\theta, c\left(\theta\right)\right) d\nu\left(\theta\right).$$

For this section we assume that all goods are produced in similar way. The total output can be transformed into different goods, although perheps to it may require different amounts of output to produce one unit of different goods. The coefficients that measure how much output is needed to produce one unit of specific good are given in the form of measure μ on Θ. We also assume that the measures μ and ν are equivalent, that is

$$\mu(A) = 0 \iff \nu(A) = 0$$

for every measurable set $A \subseteq \Theta$. For any "consumption plan" at any given moment t, $c(t, \cdot)\colon \Theta \to \mathbb{R}_+$, the total amount of output that is transformed into consumable goods is thus given by integral

$$c(t) = \int_{\Theta} c\left(t, \theta\right) d\mu\left(\theta\right). \tag{5}$$

The rest of the output is turned into investments and increases capital.

With this economics we can state the problem. We assume that the performance of the economy is measured by the sum of discounted utility in some period of time $[0, T]$ $(0 < T \leqslant +\infty)$. The output is described by increasing and concave production function $f(k)$. We assumptions from the last paragrafs lead to the following optimization problem:

$$\max \int_{\Theta} \int_0^T e^{-\rho t} u\left(\theta, c\left(\theta, t\right)\right) d\nu\left(\theta\right) dt, \tag{6}$$

subject to

$$\dot{k}(t) = f(k(t)) - \int_{\Theta} c\left(\theta, t\right) d\mu\left(\theta\right) - \delta k(t), \tag{7}$$

$$\forall t \in [0, T]\colon \int_{\Theta} c\left(\theta, t\right) d\mu\left(\theta\right) \leqslant f\left(k\left(t\right)\right), \tag{8}$$

where decisions are made about consumption plans $c(\theta, t)$ in all time points t of the planning horizon. Now we have to restrict somehow the class of feasible solutions $c\colon \Theta \times [0, T] \to \mathbb{R}_+$.

The measures μ and ν equivalent, so the Radon-Nikodyn derivative $\frac{d\nu}{d\mu}$ is positive almost everywhare. We assume that there exist constants $\varepsilon_1, \varepsilon_2 > 0$, such that

$$\varepsilon_1 < \frac{d\nu}{d\mu} < \varepsilon_2. \tag{9}$$

From (9) it follows that for any $p \geqslant 1$ we have[3] $\mathcal{L}_p(\nu) = \mathcal{L}_p(\mu)$. Indeed for any $x \in \mathcal{L}_p(\mu)$:

$$\int_{\Theta} |x(\theta)|^p d\nu(\theta) = \int_{\Theta} |x(\theta)|^p \frac{d\nu}{d\mu}(\theta) d\mu(\theta) < \varepsilon_2 \int_{\Theta} |x(\theta)|^p d\mu(\theta) < \infty.$$

Similarly for $x \in \mathcal{L}_p(\nu)$:

$$\int_{\Theta} |x(\theta)|^p d\mu(\theta) = \int_{\Theta} |x(\theta)|^p \frac{d\mu}{d\nu}(\theta) d\nu(\theta) < \frac{1}{\varepsilon_1} \int_{\Theta} |x(\theta)|^p d\mu(\theta) < \infty.$$

In the rest of the paper we will denote $\mathcal{L}_p(\mu) = \mathcal{L}_p(\nu)$ by $\mathcal{L}_p$.

[3]($\mathcal{L}_p(\nu)$ is the set of functions $x\colon \Theta \to \mathbb{R}$ that fulfills the condition $\int_{\Theta} |x(\theta)|^p d\nu(\theta) < \infty$ and $\mathcal{L}_p(\mu)$ is a similar set for the measure μ. See eg. [4] Ch. 12, or [12], Ch.1.

Let $\mathcal{L}$ be the set of all nonnegative functions from $\mathcal{L}_p(\mu)$ for some $p > 1$ (if $\mathcal{L}_1(\mu) = \mathcal{L}_\infty(\mu)$, then one can take also $p = 1$). We assume that the set of feasible solutions for c is such that for every moment t $c(\cdot, t) \in \mathcal{L}$ and that the function $\int_\Theta c(\theta, \cdot)d\mu$ is right countinuous with at most finite number of jump discontinuities. Moreover for each $t \in [0, T]$, $\|c\|_p \leqslant A$ for some positive constant A.

Example 1. Consider n different types of goods. The social utility of good i is u_i, ν is the counting measure and $\mu(\{i\}) = a_i$, where a_i is the amount of output that is needed to be transformed in one unit of good i. The problem (6)-(8) takes the form:

$$\max \sum_{i=1}^{n} \int_0^T e^{-\rho t} u_i\left(c_i\left(t\right)\right) dt,$$

subject to

$$\begin{aligned}
&\dot{k}(t) = f(k(t)) - \sum_{i=1}^{n} a_i c_i\left(t\right) - \delta k(t), \\
&\forall t \in [0, T] \colon 0 \leqslant c_i(t) \text{ dla kadego } i = 1, \ldots, n, \\
&\forall t \in [0, T] \colon \sum_{i=1}^{n} a_i c_i\left(t\right) \leqslant f\left(k\left(t\right)\right),
\end{aligned}$$

where $c_i(t)$ is the consumption of good i at the moment t.

Example 2. Suppose that goods can differ with respect to some their characteristics, which can be described quantitavely by some vector $\mathbf{x} \in \mathbb{R}^n$. We also assume that all posible combination of characteristics lie in some compact set $\Theta \in \mathbb{R}^n$. Let the function $h(\mathbf{x})$ describe how much output is needed to produce an unit of good with the characteristic $\mathbf{x}$. Utility frm all possible kinds of goods has the same impact on total utility. The problem (6)-(8) can be than restated as:

$$\max \int_\Theta \int_0^T e^{-\rho t} u\left(\mathbf{x}, c\left(\mathbf{x}, t\right)\right) d\mathbf{x} dt,$$

subject to

$$\begin{aligned}
&\dot{k}(t) = f(k(t)) - \int_\Theta c\left(\mathbf{x}, t\right) h\left(\mathbf{x}\right) d\mathbf{x} - \delta k(t), \\
&\forall t \in [0, T] \colon 0 \leqslant c(\mathbf{x}, t), \\
&\forall t \in [0, T] \colon \int_\Theta c\left(\mathbf{x}, t\right) g\left(\mathbf{x}\right) d\mathbf{x} \leqslant f\left(k\left(t\right)\right).
\end{aligned}$$

We are now ready to state the main result of the paper.

Theorem 1. Define the social utility function $U : \mathbb{R}_+ \to \mathbb{R}$ as follows:
(10)

$$U^A\left(y\right) = \sup\left\{\int_\Theta u\left(\theta, c\left(\theta\right)\right) d\nu\left(\theta\right) : c \in \mathcal{L} \wedge \int_\Theta c\left(\theta\right) d\mu\left(\theta\right) = y \wedge \|c\|_p \leqslant A\right\}.$$

Then the problem (6)-(8) is equivalent with the problem (2)-(4) with the social utility function U^A. Any of these both problems has a solution if only if there is a solution for the other problem. The paths of the aggregate consumptions (respectively: $c(t)$ and $\int_\Theta c(\theta, t)d\mu(\theta)$) and of the capital are the same in the optimal solutions to these problems.

Proof. Notice first that U^A in is increasing and concave, thus can serve as an utility function in the problem (2)-(4).

For any $c \in \mathcal{L}$ let $C(c) = \int_\Theta c(\theta) d\mu(\theta)$. C is a continuous linear mapping $C : \mathcal{L} \to \mathbb{R}$, thus the set of all possible $c \in \mathcal{L}$ in (10) is closed as an inverse image of the set $\{y\}$ with respect to the mapping C. As $\|c\|_p \leqslant A$, it is also norm bounded. The space $\mathcal{L}_p$ is reflexive and according to Alaoglu theorem (see eg. [4]) all closed and norm bounded sets are compact in the weak topology and thus $B = \left\{c\colon c \in \mathcal{L} \wedge \int_\Theta c(\theta)\, d\mu(\theta) = y \wedge \|c\|_p \leqslant A\right\}$ is weak compact. Consider the mapping $\mathcal{U}\colon \mathcal{L} \to \mathcal{L}$ defined as follows: $\mathcal{U}(x) = \int_\Theta u(\theta, c(\theta)) d\nu(\theta)$. The mapping is concave and thus (see [2], proposition 1.8.3) is upper-semicontiuous in weak topology. In every compact set (in particularly in B) it reaches its maximal value. There exists such c, for which the RHS obtains its supremum.

Suppose that $u(\theta, t)$ is a feasible solution to (6)-(8). Then $c(t) = \int_\Theta c(\theta, t)$ is a feasible solution to (2)-(4). Subsequently if $c(t)$ is a feasible solution to (2)-(4) then for each moment t there exists a function $c_t(\theta)$ for which the suppremum in (10) is reached. The process $c_t(\theta)$ has a modification which is cadlag, e.a. there is such a function $c(\theta, t)$ that $c(\theta) = c(\theta, t)$ almost everywhere. (This follows from the Doob's Theorem, see [10], Ch. II.61-67). There exists one to one correspondence between feasible solutions in (2)-(4) and in (6)-(8). The objective functions in the both problems are the same (by Fubini Theorem). Hence there is a correspondence between optimal solutions to the both problems. □

The assumption that $\|c\|_p \leqslant A$ is necessary only to make the set of feasible solutions in 10 norm bounded, because form $\int_\Theta c(\theta) d\mu(\theta) = y$ does not imply generally that c is bounded in $\mathcal{L}_p$ for $p > 1$. If $\mathcal{L}_1 = \mathcal{L}_\infty$ (as for example in 1) then $\mathcal{L}_1$ is reflexive and the condition $\int_\Theta c(\theta) d\mu(\theta) = y$ implies boundedness (as $\|c\|_1 = y$. Thus the following result follows:

Theorem 2. If $\mathcal{L}_1 = \mathcal{L}_\infty$, then (6)-(8) with $A = \infty$ is equivalent with (2)-(4) with social utility function U^∞. In particular this is the case when Θ is a finite set.

Notice that if the mariginal utlity of $u(\theta, \cdot)$ declines sufficiently rapidly for each θ so that $\|c\|_1 = y$ implies uniform boundedness of $\|u(c)\|_p$, then the result from Thm. 1 is valid also for $A = \infty$.

The economics in the theorems 1 and 2 concerns the problems of aggregation. The problem of optimal consumption in the multi-good economy (6)-(8) is mathematically complicated problem in which one concerns *how much* to consume as well as *what* to consume. The theorems shows that this problem can be deaggregated into two separate ones. First one should choose the optimal consumption path, solving (2)-(4) with appriopriate social utility function U. The general solution to this problem has the same properties as if it were an economy with single good. The optimal consumption path in the one-good economy are the same as optimal aggregate consumption paths in multi-good economy. Having chosen the aggregate consumption path, one should choose the optimal structure of consumption at every moment t separetely, by solving the optimisation problem in (10).

3. Optimal consumption paths with finite number of goods

To investigate problem more thoroughly we consider the economy with finite number of goods. Let the total number of differend kinds of good be n. The goods are numbered with $i = 1, \ldots, n$. The measure ν in the utility function can than be a counting measure. As for the measure μ, let the measure of the good i be μ_i. The coefficient μ_i denotes how much output is needed to produce one unit of the good i. The optimal control problem is

$$max \int_0^T e^{-\rho t} \sum_{i=1}^n u_i\left(c_i(t)\right), \tag{11}$$

subject to

$$\dot{k}(t) = f\left(k\left(t\right)\right) - \sum_{i=1}^{n} a_i c_i(t), \tag{12}$$

$$\sum_{i=1}^{n} a_i c_i(t) \leq f\left(k\left(t\right)\right), \qquad \forall t \in [0, T], \tag{13}$$

where u_i is the utility form consumption of good i and $c_i(t)$ is the consumption of the good i at the moment t.

The current-value Hamiltonian for the problem (11)-(13) is

$$H = \sum_{i=1}^{n} u_i\left(c_i(t)\right) + \lambda \left[f\left(k\left(t\right)\right) - \sum_{i=1}^{n} a_i c_i(t) \right], \tag{14}$$

where λ is the dual variable or the shadow price of the capital. According to Pontriagin Maximum Principle the optimal solution $c_i(t)$ $(i = 1, /ldots, n)$ should maximizes Hamiltonian H at every moment $t \in [0, T]$. The paths of capital k and its shadow price λ should be the solution to the following system of differential equations

$$\dot{k}(t) = \frac{\partial H}{\partial \lambda} = f\left(k\left(t\right)\right) - \sum_{i=1}^{n} a_i c_i(t), \tag{15}$$

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial k} = \lambda \left(\rho + \delta + f'\left(k\left(t\right)\right)\right), \tag{16}$$

with initial condition $k(0) = k_0$ (where k_0 is initial stock of capital) and transversality condition $\lambda(T) = 0$ (or $lim_{t\to 0} \lambda(t) e^{-rhot} = 0$ if $T = \infty$).

As for the aggregate problem (2)-(4) from the section 1 it is difficult to give detailed analitical solution without specific analitical forms for the functions f and u_i, $i = 1, /ldots, n$. It is however possible to sketch some main features of optimal solution. Notice that the first-order conditions for maximizing Hamiltonian are:

$$u_i'(c_i) = \lambda a_i \qquad i = 1, \ldots, n. \tag{17}$$

If there is an internal solution for maximisation problem, then the conditions (17) holds for all goods. Dividing equations (17) for any two different goods, we obtain that in the optimal solution the mariginal ratas of substitution of any two goods should equal to the ratio of their "costs" a_j/a_j.

4. Conclusions

The theorem 1 shows that it is possible to separate the decision concerning long-run investment patterns from the decissions concerning optimization the consumption bundle. The problem of optimizing consumption path in the economy with many different consumption goods can be solved in two stages. In the first stage one choose optimal path of aggregate consumption. This can be formulated as a classic problem of optimal control and one can solve it using standard methods, such as a Pontriagin maximum principle. The path of optimal aggregate consumption path will have properties of the solutions for similar problems for the models with one good, which are described in the literature. The second stage consists of dividing aggregate consumption into consumption of particular goods. The appropriate problem can be stated and solved for each moment of time independently. We have shown that there exists a solution for this problem. In general this is a problem of optimizing a functional on some abstract measurable space. In some cases the problem is less complicated. For example if the number of goods is finite, it is a problem of maximizing convex function with linear restrictions and can be solved using standard methods, like Kuhn-Tucker conditions.

References

[1] M. Athans and P. Falb. *Sterowanie optymalne: wstp do teorii i jej zastosowania.* Wydawnictwo Naukowo-Techniczne, Warszawa, 1969.

[2] A.V. Balakrishnan. *Analiza funkcjonalna stosowana.* Wydawnictwo Naukowe PWN, Warszawa, 1992.

[3] R. J. Barro and X. Sala-i Martin. *Economic Growth.* McGraw-Hill, New York, 1995.

[4] D. A. Charalambos and K. C. Border. *Infinite dimensional analysis.* Springer-Verlag, Nowy Jork, 1999.

[5] W. H. Fleming and R. W. Rishel. *Deterministic and stochastic optimal control.* Springer-Verlag, Nowy Jork, 1975.

[6] D. Leonard and N. Van Long. *Optimal Control Theory and Static Opimization.* Cambridge University Press, Cambridge, 1992.

[7] E. Panek. *Ekonomia matematyczna.* Wydawnictwo Akademii Ekonomicznej w Poznaniu, Pozna, 2000.

[8] F.P. Ramsey. A mathematical theory of saving. *Economic Journal*, 38, 1928.

[9] R.T. Rockafellar. *Convex Analysis.* Princeton University Press, Princeton, 1970.

[10] L. C. G. Rogers and D. Williams. *Diffusions, Markov processes and martingales*, volume 1. Cambrige University Press, Cambridge, 2000.

[11] D. Romer. *Makroekonomia dla zaawansowanych.* Wydawnictwo Naukowe PWN, Warszawa, 2000.

[12] W. Rudin. *Analiza funkcjonalna.* Wydawnictwo Naukowe PWN, Warszawa, 2001.

[13] A. Takayama. *Mathematical economics.* Cambridge University Press, Cambridge, 1985.