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VOLATILITY MODELLING OF FOREIGN EXCHANGE RATE :

DISCRETE GARCH FAMILY VERSUS CONTINUOUS GARCH

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Non-linearity is the general characteristic of financial series. Thus, common non-linear models such as GARCH, EGARCH and TGARCH are used to obtain the volatility of data. In addition , continuous time GARCH (COGARCH) model that is the extansion and analogue of the discrete time GARCH process, is the new approach for volatility and derivative pricing. COGARCH has a single source variability like GARCH, but also it is constructed on driving Levy Process since increments of Levy Process is replaced with the innovations in the discrete time. In this study, the proper model for the volatility is shown to represent foreign exchange rate of USD versus TRY for different period of time from January 2009 to December 2011.

Keywords : volatility, Levy Process, COGARCH

I. INTRODUCTION

The modelling and forecasting the foreign exchange rates are important subject for the international financial markets.The fluctations in supply and demand for the foreign currency, and the fluctations in interest rates are more effective on the foreign exchange rates.So, volatility modelling becomes one of the most important study for currency data. The volatility modelling of time series is highly utilised in predicting economic and business trends. The financial data are usually non-linear. Many forecasting methods have been developed for the non-linear data in the last few decades; such as discrete conditional variance models GARCH, EGARCH and TGARCH are well-known. In 2004, Klüppelberg and her co-workers introduce a new model continuous GARCH that is the analogue of the discerete conditional variance model GARCH (COGARCH), is constructed on driving Levy Process.

II.METHODOLOGY

In this study, after differencing data BDS test is done for testing the non-linearity of data that was first devised by W.A. Brock, W. Dechert and J. Scheinkman in 1987.According the test results the non-linear models could be used. The best candidate discrete GARCH model is choosen by comparing the Akaike Information Criterias, Bayessian Information Criterias and their maximum-likelihood values. Gaussian, Student-t, Generalised Error Distribution (GED), Normal Inverse Gaussian (NIG) and Double Exponential distributions are used as conditional distributions for the error terms. GARCH diagnostics are done by the Jarque-Berra and Shapiro-Wilk normality tests, Ljung-Box test for standardized residuals and squared standardized residuals , Langrange Multiplier test. The COGARCH process is constructed on NIG Levy process. The discrete GARCH family models and COGARCH model are compared according to their volatiliy plot and qq-plots.

II.I. GARCH MODEL

Bollerslev (1986) proposes a useful extension known as the generalized ARCH (GARCH) model. For a log return series r_t, we assume that the mean equation of the process can be adequatedly described by an ARMA model. Let $a_t = r_t - \mu_t$ be the mean-corrected log return. Then a_t follows a GARCH(m, s) model if

$$a_t = \sigma_t \epsilon_t, \qquad \sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i a_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2$$

where again $\{\epsilon_t\}$ is a sequence of iid random variables with mean 0 and variance 1.0, $\alpha_0 > 0$, $\alpha_i \geq 0$, $\beta_j \geq 0$, and $\sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i) < 1$. Here it is understood that $\alpha_i = 0$ for $i > m$ and $\beta_j = 0$ for $j > s$. The latter constraint on $\alpha_i + \beta_i$ implies that the unconditional variance of a_t is finite, whereas its conditional variance σ_t^2 evolves over time. (Tsay, 2002)

II.II. EXPONENTIAL GARCH MODEL

To overcome some weaknesses of the GARCH model in handling financial time series, Nelson (1991) proposes the exponential GARCH (EGARCH) model. In particular, to allow for asymmetric effects between positive and negative asset returns, he considers the weighted innovation

$$g(\epsilon_t) = \theta \epsilon_t + \gamma[|\epsilon_t| - E(|\epsilon_t|)]$$

where θ and γ are real constants. Both ϵ_t and $|\epsilon_t| - E(|\epsilon_t|)$ are zero-mean iid sequences with continuous distributions. Therefore, $E[g(\epsilon_t)] = 0$. The asymmetry of $g(\epsilon_t)$ can easily be seen by rewriting it as

$$g(\epsilon_t) = \begin{cases} (\theta + \gamma)\epsilon_t - \gamma E(|\epsilon_t|) & \text{if } \epsilon_t \geq 0, \\ (\theta - \gamma)\epsilon_t - \gamma E(|\epsilon_t|) & \text{if } \epsilon_t < 0. \end{cases}$$

An EGARCH(m, s) model can be written as

$$a_t = \sigma_t \epsilon_t, \quad \ln(\sigma_t^2) = \alpha_0 + \frac{1 + \beta_1 B + \cdots + \beta_s B^s}{1 - \alpha_1 B - \cdots - \alpha_m B^m} g(\epsilon_{t-1})$$

where α_0 is a constant, B is the back-shift (or lag) operator such that $Bg(\epsilon_t) = g(\epsilon_{t-1})$, and $1 + \beta_1 B + \cdots + \beta_s B^s$ and $1 - \alpha_1 B - \cdots - \alpha_m B^m$ are polynomials with zeros outside the unit circle and have no common factors. By outside the unit circle, we mean that absolute values of the zeros are greater than 1. The model differs from the GARCH model in several ways. First, it uses logged conditional variance to relax the positiveness constraint of model coefficients. Second, the use of $g(\epsilon_t)$ enables the model to respond asymmetrically to positive and negative lagged values of a_t .(Tsay, 2002)

II.III. THRESHOLD GARCH MODEL

The threshold GARCH (TGARCH) model proposed by Zakoian (1994) and GJR GARCH model studied by Glosten, Jagannathan, and Runkle (1993) define the conditional variance as a linear piecewise function. TGARCH is another GARCH variant that is capable of modeling leverage efects, threshold GARCH (TGARCH) model, which has the following form:

$$\sigma_t^2 = a_0 + \sum_{i=1}^{p} a_i \epsilon_{t-i}^2 + \sum_{i=1}^{p} \gamma_i S_{t-i} \epsilon_{t-i}^2 + \sum_{j=1}^{q} b_j \sigma_{t-j}^2 \quad \text{where} \quad S_{t-i} = \begin{cases} 1 & \text{if} \quad \epsilon_{t-i} < 0 \\ 0 & \text{if} \quad \epsilon_{t-i} \geq 0 \end{cases}$$

That is, depending on whether ϵ_{t-i} is above or below the threshold value of zero, ϵ_{t-i}^2 has diferent efects on the conditional variance σ_t^2 ; when ϵ_{t-i} is positive, the total efects are given by $a_i\epsilon_{t-i}^2$; when ϵ_{t-i} is negative, the total efects are given by $(a_i + \gamma_i)\epsilon_{t-i}^2$. So one would expect γ_i to be positive for bad news to have larger impacts. (Zivot, 2006)

II.IV. CONTINUOUS GARCH MODEL

II.IV.I. Normal inverse Gaussian process (NIG)

L = {Lt ; $t \geq 0$}is an infnitely divisible continuous time stochastic process, $L_t : \Omega \rightarrow \mathbb{R}$, with stationary and independent increments.Levy processes are more versatile than Gaussian A c_adl_ag, adapted, real valued stochastic process L = {Lt ; $t \geq 0$}with L0 = 0 a.s. is called a Levy process .

The *normal inverse Gaussian process* (NIG) is a Levy process $\{X(t)\}_{t\geq 0}$ that has normal inverse Gaussian distributed increments. Specically, $X(t)$ has a NIG $(\alpha, \beta, \delta t, \mu t)$ distribution with parameters $\alpha > 0,\ |\beta| < \alpha,\ \delta > 0$ and $\mu \in \mathbb{R}$.

The NIG$(\alpha, \beta, \delta t, \mu t)$ distribution has probability density function

$$f_{\text{NIG}}(x; \alpha, \beta, \delta, \mu) = \frac{\alpha\delta}{\pi} \frac{K_1\left(\alpha\sqrt{\delta^2 - (x-\mu)^2}\right)}{\sqrt{\delta^2 + (x-\mu)^2}} \exp\left\{\delta\sqrt{\alpha^2 - \beta^2} + \beta(x-\mu)\right\}$$

Where $$K_v(z) = \frac{1}{2}\int_0^{\infty} u^{v-1} \exp\left\{-\frac{z}{2}\left(u + \frac{1}{u}\right)\right\} du$$

is the modi¯ed Bessel function of the third kind, while the characteristic function is given by

$$\phi_{\mathrm{NIG}}(u;\alpha,\beta,\delta,\mu)=\exp\Big(-\delta\big(\sqrt{\alpha^2-(\beta+iu)^2}-\sqrt{\alpha^2-\beta^2}\big)\Big)\,\mathrm{e}^{i\mu u}$$

II.IV.II. Continuous GARCH Model

Nelson introduce COGARCH model that includes two independent Brownian motions B(1) and B(2)

$$dG_t=\sigma_t dB_t^{(1)}, \qquad t\geq 0$$

$\sigma_t^2=(\beta-\eta\sigma_t^2)dt+\varphi\sigma_t^2 dB_t^{(2)}$, $t\geq 0$ where $\beta>0$, $\eta\geq 0$, and $\phi\geq 0$ are constants.

Since empirical work indicates upwards jumps in the volatility, a model driven by a L´evy process seems a natural approach. In Kluppelberg et al. (2004, 2006) such amodel was suggested. Klüppelberg shows that COGARCH model is analogue of the discrete time GARCH model, based on a single background driving Lévy process. COGARCH model has the basic properties of discrete time GARCH process. They iterated the volatility equation to get

$$\begin{aligned}
\sigma_i^2 &= \omega_0\sum_{k=0}^{i-1}\prod_{j=k+1}^{i-1}(\delta+\lambda\epsilon_j^2)+\sigma_0^2\prod_{j=0}^{i-1}(\delta+\lambda\epsilon_j^2)\\
&= \omega_0\int_0^i \exp\left\{\sum_{j=\lfloor u\rfloor+1}^{i-1}\log(\delta+\lambda\epsilon_j^2)\right\}du+\sigma_0^2\exp\left\{\sum_{j=0}^{i-1}\log(\delta+\lambda\epsilon_j^2)\right\}
\end{aligned}$$

$$\begin{aligned}
&= \left[\omega_0\int_0^i \exp\left\{-\sum_{j=0}^{\lfloor u\rfloor}\log(\delta+\lambda\epsilon_j^2)\right\}du+\sigma_0^2\right]\exp\left\{\sum_{j=0}^{i-1}\log(\delta+\lambda\epsilon_j^2)\right\}\\
&= \left[\omega_0\int_0^i \exp\left\{\eta(\lfloor u\rfloor+1)-\sum_{j=0}^{\lfloor u\rfloor}\log(1+\varphi\epsilon_j^2)\right\}du+\sigma_0^2\right]\\
&\qquad\times\exp\left\{-\eta i+\sum_{j=0}^{i-1}\log(1+\varphi\epsilon_j^2)\right\},
\end{aligned}$$

where ⌊u⌋ denotes the integer part of u ∈ R, η := −log(δ) and ϕ := λ/δ, then suggested to replace the noise variables ϙi by the jumps _Lt = Lt − Lt− of a Levy process L = (Lt)t≥0, which allowed them to define the volatility process σ2 for all t ≥ 0 by

$$\sigma_t^2:=\left(\omega_0\int_0^t e^{X_s}ds+\sigma_0^2\right)e^{-X_{t-}}$$

where

$$X_t := \eta t - \sum_{0<s<t} \log(1 + \varphi(\Delta L_s)^2)$$

The parameter space is given by ω0, η > 0 and ϕ > 0 and the process X will be referred to as the auxiliary process. The COGARCH(1, 1) process G = (Gt)t≥0 is defined as the solution to the stochastic differential equation (SDE) The COGARCH process $(G_t)_{t\geq 0}$ is defined in terms of its stochastic differential dG, such that

$dG_t = \sigma_t dL_t$ t≥0 $d\sigma_t^2 = (\beta - \eta\sigma_{t-}^2)dt + \phi\sigma_{t-}^2 d[L,L]_t$, $t>0$ where $\beta>0$, $\eta\geq 0$, and $\phi\geq 0$ are constants. The solution for the stochastic equation is

$$\sigma_i^2 = \sigma_{i-1}^2 - \beta + \int_0^t \sigma_s^2 ds + \varphi \sum_{0<s\leq t} \sigma_s^2 (\Delta L_t)^2 + \sigma_0^2$$

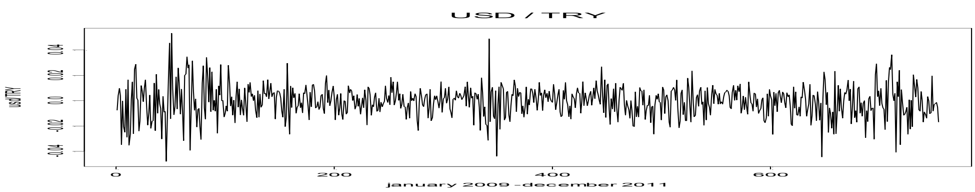
where the Levy process is constructed by using NIG process

III. RESULTS

The data is taken from web site of Turkish Central Bankwhich shows daily foreign exchange rate of USD versus TRY for different period of time from January 2009 to December 2011.

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
1.39510	1.50050	1.54740	1.58038	1.61615	1.91570

Dickey-Fuller and Phillips-Perron unit root tests Show that tha data is not stationary .By taking the first difference it becomes stationary.



Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
-0.0479000000	-0.0079000000	0.0001000000	-0.0005127321	0.0070000000	0.0532000000

BDS Test for Independence and Identical Distribution

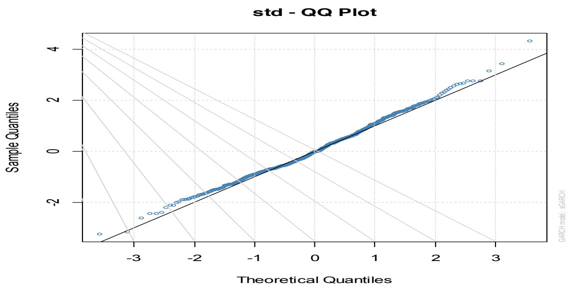
Test Statistics =				p-value =				
[0.01]	[0.01]	[0.02]	[0.02]		[0.01]	[0.01]	[0.02]	[0.02]
5.3947	5.2091	5.0984	4.7197	[2]	0	0	0	0
6.4430	6.9627	7.0882	6.9264	[3]	0	0	0	0
7.0720	8.4045	8.4096	8.3259	[4]	0	0	0	0
7.4289	9.5891	9.4309	9.3674	[5]	0	0	0	0

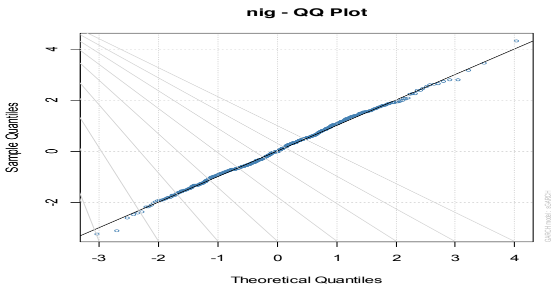
BDS tests results give that the data is not linear or independently and identically distributed. Although the data isnot linear conditional mena model ARMA(0,1) was construct for difference data and the coefficients of the mean model are not statistically significant. But all GARCH models are made up with include mean coefficient. Test for ARCH Effects is done by LM Test which has a null hypothesis; ther is no no ARCH effects . The results statistics are t-Statistics:82.0833 and p.value 0.0000. So, GARCH models can be used to model the data. First , it is started by GARCH model. Note that only the best candidate models' coefficients and graphs will be given in this study.

Comparing GARCH Models

	Gaussian	Student-t	Double Exponential	NIG	GED
AIC	-4597	*-4599*	-4534	*-4598*	-4597
BIC	-4578	*-4576*	-4515	*-4579*	-4573
Likelihood	2302	*2305*	2271	*2304*	2303

There are two improper GARCH model according the AIC, BIC and maximum likelihood value. They are student-t GARCH and NIG GARCH. But the qq-plot of two model points out the best one. NIG GARCH satisfies better fit on qq-line.





NIG GARCH parameters also satisfies the stationary condition before likely said the estimation of mean parameter is not statistically significant. NIG GARCH diagnostics show that there is no autocorellation between residuals.

GARCH Conditional Distribution: Normal Inverse Gaussian

Estimated coefficients:	Value	Std.Error	t value	Pr(>|t|)
C	0.000301	0.000388	0.77655	0.437422
A	0.000004	0.000002	2.21845	0.026524
ARCH(1)	0.100122	0.024791	4.03860	0.000054
GARCH(1)	0.869387	0.032246	26.96072	0.000000

Normal Inverse Gaussian GARCH Diagnostics

Normality Test:	Ljung-Box test for standardized residuals:	Ljung-Box test for squared standardized residuals:	Lagrange multiplier test:	Test for ARCH Effects:
Jarque-Bera P-value 17.22 0.0001952 Shapiro-Wilk P-value 0.9868 0.6231	Statistic P-value 8.483 0.746	Statistic P-value 7.771 0.7962	TR^2 P-value F-stat P-value 6.684 0.88 0.609 0.9178	LM Test Null Hypothesis: no ARCH effects Test Stat 6.6679 p.value 0.8811

The improper EGARCH whose conditional error distribution is gaussian, satisfies the conditionality . The followings are the results of Gaussian EGARCH. All the coefficients are statistically significant except estimated mean coefficient.

Comparing EGARCH Models

	Gaussian	Student-t	Double Exponential	GED
AIC	*-4592*	-4591	-4528	-4526
BIC	*-4569*	-4568	-4510	-4503
Likelihood	*2301*	2300	2268	2268

EGARCH Conditional Distribution: Gaussian

Estimated Coefficients:	Value	Std.Error	t value	Pr(>|t|)
C	-0.0003988	0.0003979	-1.002	3.165e-001
A	-0.6248296	0.1787430	-3.496	5.007e-004
ARCH(1)	0.2020199	0.0440169	4.590	5.206e-006
GARCH(1)	0.9479995	0.0174055	54.466	0.000e+000
LEV(1)	-0.3414637	0.1071940	-3.185	1.505e-003

Gaussian EGARCH Diagnostics

Normality Test:	Ljung-Box test for standardized residuals:	Ljung-Box test for squared standardized residuals:	Lagrange multiplier test:	Test for ARCH Effects:
Jarque-Bera P-value 11.86 0.002656 Shapiro-Wilk P-value 0.9887 0.8465	Statistic P-value 9.08 0.6961	Statistic P-value 10.88 0.5394	TR^2 P-value F-stat P-value 9.299 0.6772 0.8561 0.6945	LM Test Null Hypothesis: no ARCH effects Test Stat 9.2987 p.value 0.6772

The TGARCH comparison indicate that the Gaussian TGARCH superior model from the other conditional distributions. In addition, the main parameters are significant again except the mean and the residuals have no autocorrelation is understood from TGARC diagnostics.

Comparing TGARCH Models

	Gaussian	Student-t	Double Exponential	GED
AIC	*-4602*	-4601	-4534	-4532
BIC	*-4578*	-4574	-4511	-4504
Likelihood	*2307*	2306	2272	2272

TGARCH Conditional Distribution: Gaussian

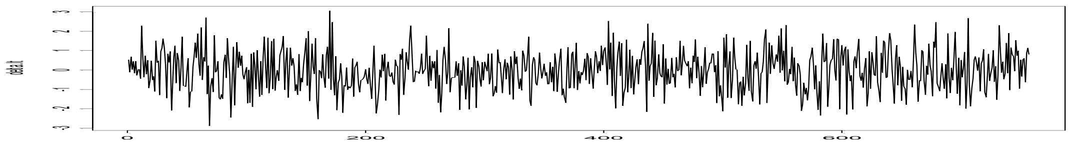
Estimated Coefficients:	Value	Std.Error	t value	Pr(>|t|)
C	-3.869e-004	4.041e-004	-0.9574	0.338696
A	4.797e-006	1.812e-006	2.6467	0.008299
ARCH(1)	5.225e-002	2.131e-002	2.4520	0.014435
GARCH(1)	8.727e-001	3.019e-002	28.9020	0.000000
GAMMA(1)	7.727e-002	2.823e-002	2.7370	0.006347

Gaussian TGARCH Diagnostics

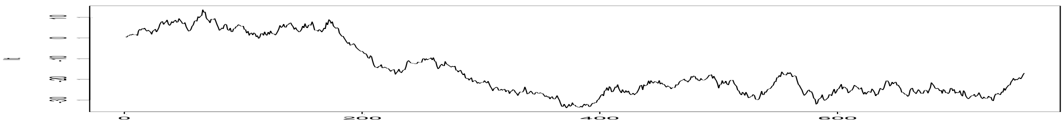
Normality Test:	Ljung-Box test for standardized residuals:	Ljung-Box test for squared standardized residuals:	Lagrange multiplier test:	Test for ARCH Effects:
Jarque-Bera P-value 11.88 0.002637 Shapiro-Wilk P-value 0.9882 0.8024	Statistic P-value 9.407 0.6679	Statistic P-value 8.418 0.7517	TR^2 P-value F-stat P-value 7.072 0.8528 0.6491 0.8878	LM Test Null Hypothesis: no ARCH effects Test Stat 7.0718 p.value 0.8528

Continuous Volatility Modeling

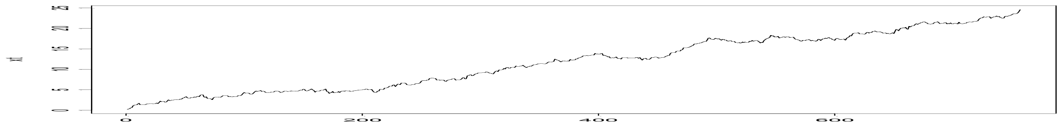
The parameters of COGARCH model is obtained from the discrete GARCH model's parameters $\beta = \beta$, $\eta = \ln \delta$, $\phi = \lambda / \delta$ where β is the constant of GARCH model, δ is the coefficient of GARCH term and $\lambda\ \delta$ is the coefficient of ARCH term. The numerical solutions for dG_t and $d\sigma_t^2$ is done by using Lévy process driven by NIG process. The exact solution of stochastic differential equation of $d\sigma_t^2$ brings the results of the COGARCH process.



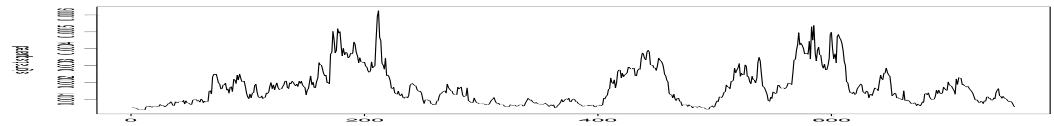
Plot of Delta Lt



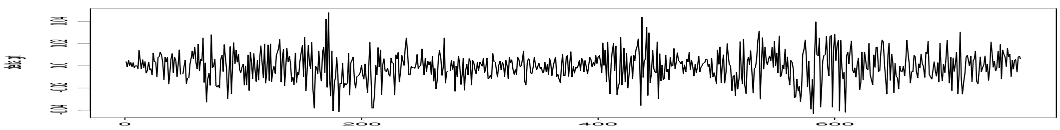
Plot of Lt



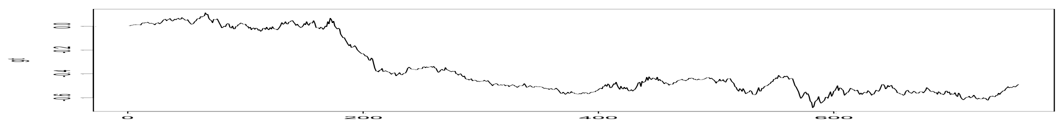
Plot of Xt auxiliary process



Plot of Sigma-t Squared

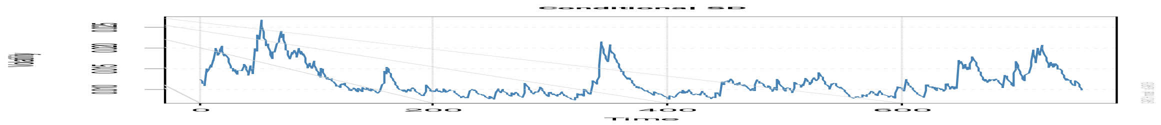


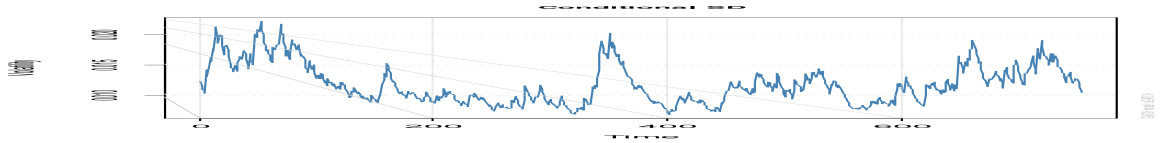
Plot of delta Gt

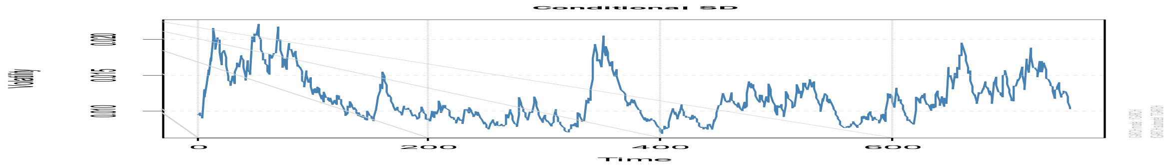


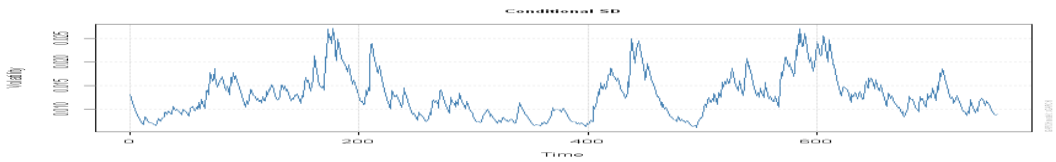
Plot of Gt

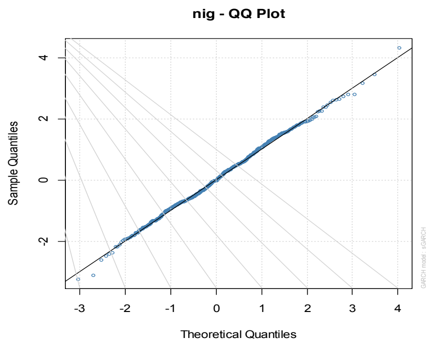
The discrete GARCH models can be compared with NIG Levy Driven COGARCH model according to their conditional variance plots and qq-plot.



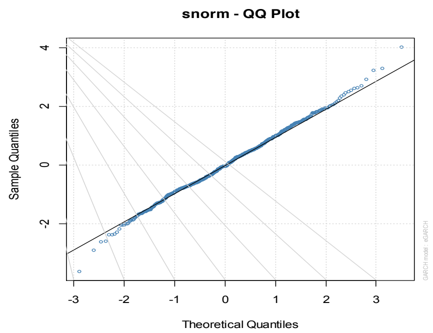




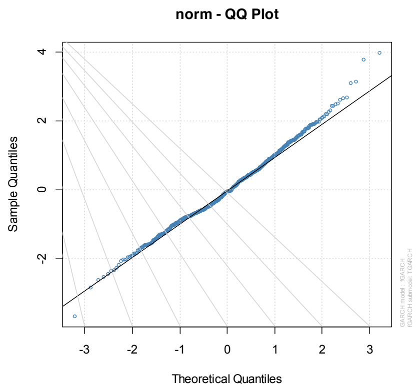




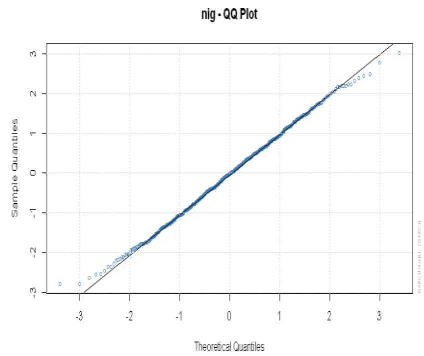
qq-plot of NIG GARCH



qq-plot of gaussian EGARCH



qq-plot of gaussian TGARCH



qq-plot of NIG COGARCH

As it is seen in the above figures discrete GARCH models capture the shocks and jumps better than continuous GARCH model, although COGARCH captures the jumps in the volatility at the right time, it almost figures out the same pattern with the other models.

CONCLUSION

The methodology to compare discrete models with a continuous model could not be the right way. The forecasting performance and the news impact curves of the models could be the way of opposing the models. The exponential COGARCH (Haug,2006) which is the extension of the COGARCH model, might be compared with discerete models. But, firstly the exact conditions for opposing the discrete models and their analogue model would be studied

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