Multidimensional Black-Scholes options

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Abstract

In this article we propose an extension of the classical Black−Scholes option in a multidimensional setup. The underlying financial asset is a basket of equity stocks on which a general European type option pay−off is considered. Using the distributional Fourier transform, we derive a general formal solution and provide a sufficient condition to construct the former explicitly in a fairly rich set of functions. Finally, we develop two derivative options, which are given in closed−form: the first option can be expressed as a linear combination of the classical call/put options, while the second one is a new option with multidimensional underlying, nameley a χ^2−option.

Keywords: Black-Scholes model, pricing equation, linear constant coefficients PDE, distributional Fourier transform, plain vanilla option, χ^2−option.

*The opinions expressed here are mine and do not necessarily reflect the views and opinions of AIB. Any mistake contained in this work remains, of course, mine.

Contents

1 Introduction

In this article we adopt the arbitrage argument introduced in the seminal paper [1] with the aim to provide a general solution to the pricing equation of any European option written on a basket of stocks, whose dynamics is described by a multidimensional version of the original Black−Scholes model. The solution is obtained through the exponentiation of the operator defining the partial differential equation (PDE) involved in pricing, yielding the convolution operator G which produces the option price when applied to the pay−off function. We identify a very large family of functions, or regular distributions, that establishes the set of admissible pay−offs which grant numerical or closed form solutions to the problem of derivative pricing and the design of the replicating portfolio. We note that this family could be slightly enlarged considering singular distributions, but we do not further discuss this point in the present paper.
Depending on the form of the payout, the correlation structure of the random sources enters into the pricing formula affecting the value. We finally provide two closed form examples where, in the first one, the dimensionality of the problem might be disregarded and the solutions turn out to be the linear combination of individual plain vanilla options. In the second example we develop a novel European option, the χ^2−option, on a basket of equity stocks.
The work is organized as follows. In section 2 we recall the multidimensional Black−Scholes model and set the pricing equation for any derivative written on the equity basket as a consequence of the arbitrage argument. In section 3 we derive the general solution for the pricing equation in a suitable functional space. In section 4 we discuss two applications, through which closed form solutions are found. In section 5 we draw the conclusions.

2 Multidimensional Black-Scholes model

2.1 Model Setup

In this work we follow [1] in modeling the stock prices. The market price dynamics of a single equity stock value is modeled with the following stochastic differential equation (SDE)

$$\mathrm{d}S_t = S_t(\,\mu\,\mathrm{d}t + \sigma\,\mathrm{d}B_t\,), \tag{1}$$

where μ and σ are real constants representing the trend and the volatility of the stock price S. The stochastic process B_τ, $\tau \in [0,T]$, is the canonical Brownian motion observed from time 0 to time T. The equation (1), referred as the Black−Scholes model[1], is the *geometric Brownian motion* and in financial terms

[1]From a statistical perspective, the estimation of the coefficient μ for a dividend−stripped stock is very difficult, yielding non significantly different from zero parameters on high-frequency data sample of a large basket. Other remarks about (1) confined to short-term horizons, consist in the well documented volatility clustering phenomenon and a slight asymmetry in the empirical price distributions (cfr. [3], [2]), which are not reproduced by the conditional probability distributions generated by the SDE.

it states that the continuous time compounding law is subject to a Gaussian random noise which affects the growth rate.
When considering an equity stock basket as the derivative reference underlying, the equation (1) has to take into account the interaction among the elements of the set of investment opportunities. Therefore, we choose to model the market price dynamics of the stocks with the *multidimensional Black–Scholes model* that is defined as

$$\left\{ \mathrm{d}S_t^j = S_t^j (\mu_j \, \mathrm{d}t + \sigma_j \, \mathrm{d}W_t^j) \right\}_{j=1,\ldots,n}, \tag{2}$$

where W_j, $j = 1, \ldots, n$ are the correlated Brownian motions starting at $t = 0$, with $B_0^j = 0$ a.s. , $\forall j$. The correlated random sources are assumed to be generated by the equation $W_t = Q \cdot B_t$ with Q square, full rank and such that $Q = \left[\sqrt{a_{ii}}\right]^{-1} \cdot U$ and $A = U \cdot U'$, $A > 0$. By construction, $Q \cdot Q'$ is the correlation matrix of the growth rate of the basket components and $\mathrm{d}\left[W^i, W^j\right] = \rho^{ij} \mathrm{d}t$. The full rank assumption could be dropped without major modifications.

2.2 Replicating portfolio

In order to price the European option on the joint value of the basket components described in (2), which matures at time $T \in \mathbb{R}$ and $\mathbf{1}_{\mathbb{R}} \sim 1$ year, we exploit the arbitrage argument as it has been presented in [1] and later formalized in [5], [6]. This approach allows to write explicitly the partial differential equation whose solution yields the price of the sought financial derivative as a parameterized function of the equity prices.
We find useful to perform the change of variable $X_t^j = \log\left(\frac{S_t^j}{S_0^j}\right)$, and to express the derivative's price in terms of the growth factors of the basket components $f(X^1, \ldots, X^n)$ rather then their price. Eventually the pricing formula can be converted into the S dependent function $h = f \circ X$.
Let Π be the portfolio made of a short position in the derivative f and exposed to the n stocks $\left\{S^j\right\}_{j=1,\ldots,n}$

$$\Pi = \sum_{j=1}^{n} \Delta_j S_j - f \;\Rightarrow\; \mathrm{d}\Pi = \textstyle\sum_j \Delta_j \mathrm{d}S_j - \mathrm{d}f,$$

where $\Delta_j S_j$ is the cash amount invested in the equity stock j at time t. Recursively applying Ito's rule and setting $\Delta_j = \frac{\partial_j f}{S_j}$ we are able to generate an asset that is free from stochastic variations. In a frictionless world this artificial asset would grow at the risk−free rate $r \geq 0$. Therefore, set $\mathrm{d}\Pi = r\Pi \mathrm{d}t$ and let ∂_j be the partial derivative with respect to x_j and ∂_t be the partial derivative with respect to time, we obtain the evolutionary partial differential equation (PDE)

$$\partial_t f = rf - \textstyle\sum_j \left(r - \sigma_j^2/2\right) \partial_j f - \frac{1}{2} \sum_j \sum_k \rho_{jk} \sigma_j \sigma_k \partial_{jk}^2 f,$$

which is called the *pricing equation*.
Finally, we reverse the time axes and set $\bar{t} = T - t$, actually turning the terminal

pay−off condition into the Cauchy problem. We write the pricing equation in the following compact form

$$\begin{cases} \partial_{\bar{t}} f = -D\left(\partial_X\right) \cdot f \\ \\ f|_{\bar{t}=0} = g. \end{cases} \tag{3}$$

where we define the polynomial operator $D\left(\partial_X\right) = r - \sum_j \alpha_j \partial_j - {}^1\!/\!_2 \sum_j \sum_k \rho_{jk}\sigma_j\sigma_k\partial^2_{jk}$, with $\alpha_j = (r - \sigma_j^2/2)$. The function $g\colon X \in \mathbb{R}^n \mapsto g(X) \in \mathbb{R}$ represents the prescribed *pay−off* profile.

3 Deriving the general solution

The objective of this section is to construct a general solution of (3) that allows us to price and hedge any basket derivative of the European type, whose underlying dynamics is represented by (2). The solution of the pricing equation depends on the specific *initial value problem*, which in financial terms is represented by the final (in the reversed time) pay−off of the basket option. As a result, the option formula is the image of the terminal pay−off through the bounded operator (6). Conditions upon the set of admissible terminal value functions arise naturally during the process of solving the main equation, thus for the sake of developing applications we will take the set of admissible initial conditions as large as possible. As previously remarked, it is possible to enlarge this set introducing the class of *pseudo−functions*.

We set up our study in the context of the theory of distributions, exploiting the properties of these mathematical objects to identify and characterize the space of solutions. Hence, when not explicitly noted, the definitions and results must be taken in the distributional sense. In this work, the following notation is used, cfr. [10]. We will refer to the distributional Fourier transform $\mathscr{F}(u) = \check{u}$ and the functional spaces $\mathcal{D}^*$ the space of distributions, $\mathcal{S}^*$ the space of tempered distributions, $\mathcal{E}^*$ the space of distributions with compact support and $\mathcal{Z}^* := \mathscr{F}(\mathcal{D}^*)$. We will also refer to the space of functions of rapid descent $\mathcal{S}$, the space of infinitely smooth functions $\mathcal{E} := \mathcal{C}^\infty$ and the space of locally summable and absolutely summable functions, respectively $\mathscr{L}^1_{\mathbf{loc}}$ and $\mathscr{L}^1$. An index t is conventionally appended to the symbol of the space to indicate that the support of its elements belongs to $\mathbb{R}^n \times \mathbb{T}$, where $\mathbb{T} := [0, \infty)$. When the $f \in \mathscr{G}$ does not depend on time, the index t is dropped and $\operatorname{supp} f \subseteq \mathbb{R}^n$.

3.1 Solving the PDE

We search for a weak solution in the space of distributions $\mathcal{D}^*_{\bar{t}}$, the dual of $\mathcal{D}_{\bar{t}}$. The solution strategy consists in constructing a cut−off of the pay−off function and then extending the solution in the limit. While doing so, we must consider the peculiarity of the problem with respect to the time parameter dependency. Therefore we will distinguish the two cases, where $\bar{t} \in (0, T]$

and $\bar{t} = 0$. The situation when $T \to +\infty$ is not particularly interesting in the financial perspective because we always fix a (finite) settlement date for the option. We notice that in order for the equation to have a solution, the derivative's profile at maturity as a function $g\colon \mathbb{R}^n \to \mathbb{R}$ must lie in a suitable subspace of $\mathcal{D}^*$. Let $\mathscr{K} = \left\{ \mathrm{e}^{-\|x+\tilde{\alpha}\|^2} \right\}_{\tilde{\alpha}\in\mathbb{R}}$ be a parametric family of functions of rapid descent, that is $\mathscr{K} \subset \mathcal{S}$.
We assume that the initial value function g is in $\mathscr{L}^1_{\mathbf{loc}}$ and is such that

$$gh \in \mathscr{L}^1,\ \forall h \in \mathscr{K}.$$

A condition for the last statement to be verified, is that g has an appropriate behavior at infinity. Let $\mathscr{A}$ be

$$\mathscr{A} = \left\{ g \in \mathscr{L}^1_{\mathbf{loc}} \colon g \in O\left(\|x\|^{-\alpha} \mathrm{e}^{\|x\|^2} \right),\ \alpha > n \right\}.$$

It is understood that if $g \in \mathscr{A} \Rightarrow gh \in \mathscr{L}^1$, $\forall h \in \mathscr{K}$ by Lebesgue's dominated convergence. Therefore, within the described setup, we are in the position to solve the prototype equation.

Proposition 3.1. *Let $g \in \mathscr{A}$ and $P(\xi) = \delta + \alpha \cdot \xi + {}^1\!/_2\, \xi \cdot \mathbf{\Omega} \cdot \xi$, with $\xi \in \mathbb{C}^n$ a polynomial such that $\mathbf{\Omega} > 0$ and $V = \sqrt{\mathbf{\Omega}}$. The solution $f \in \mathcal{D}^*_{\bar{t}}$ of the equation*

$$\begin{cases} \partial_{\bar{t}} f = P\left(\partial_X\right) \cdot f \\ \\ f\,|_{\bar{t}=0} = g \end{cases} \tag{4}$$

is the f such that

$$f = \left(G \star g\right) \mathrm{e}^{\delta \bar{t}} \tag{5}$$

where G is the Gaussian function

$$G = \tfrac{|V|^{-1}}{\sqrt{2\pi\bar{t}}^{\,n}} \mathrm{e}^{-\frac{1}{2\bar{t}}\|\widetilde{X}\|^2} \tag{6}$$

and $\widetilde{X} = (V')^{-1} \cdot [X + \bar{t}\alpha]$.

Proof. See the appendix. ☐

We further notice that equation (4) admits a steady state behavior when $\delta \leq 0$ implying f to converge to the zero distribution when we let $\bar{t} \to +\infty$.

3.2 Derivative Pricing

Finally, we are in the position to solve the pricing equation (3) for the derivative $h(T-t, S)$. The main step of the procedure consists in converting the initial pay−off, whenever expressed in price levels, into the performance dependent functions $g\left(X\right)$, by means of the transformation $S = S_0 \mathrm{e}^X$. In order to apply 3.1 and obtain a derivative price, which is computationally more handy, we need further to reverse the time direction and set $t = T - \bar{t}$. We can easily formulate the solution to the pricing problem, considering the symbolic transformation $f = h \circ t \circ S$. Therefore, let $g \in \mathscr{A}$

Corollary 3.1. *The price of the derivative* $f(\bar{t}, X)$ *written on the performance of the basket* $\{S_j\}_{j=1,\ldots,n}$ *which pays out* $g(X)$ *at maturity* T*, that is* $\bar{t}=0$*, is the* f *such that*

$$f = (G \star g)\ \mathrm{e}^{-r\bar{t}} \tag{7}$$

Proof. This is a straightforward application of the proposition (3.1). □

The corollary 3.1 allows to construct numerical and/or closed−form solutions for any derivative written on the reference basket. Provided that the statistical measure is given by the SDE (2), the price of any derivative on the performance of the basket components can be seen as a functional of the terminal pay−off $g(X_T)$, by the formula (7). If necessary, the pricing equation might be converted back to the $h = f \circ \bar{t} \circ X$.

4 Applications

In the following sections we develop two applications which provide closed−form solutions. In the first one, it is evident how the pay−off function might impact the formula in terms of the joint performance of the basket components, eventually clearing off the correlation structure of the underlying items. The second application is a novel multidimensional option.

4.1 Classical basket option

In the first application, the specific form of p will allow to partition the solution of the equation (4) into particularly simple integrals that can be solved with the *method of sections*, cfr. [9]. In this case, the multivariate Black−Scholes formula turns out to be a linear combination of classic equity options.

Let us assume that the option pays out a linear combination of the excess return on a unitary value dependent on a coefficient a, that is

$$p = \sum_j \phi_j \begin{cases} S^j - S_0^j \mathrm{e}^a, & S^j > S_0^j \mathrm{e}^a \\ 0, & S_0^j \mathrm{e}^{-a} < S^j \leq S_0^j \mathrm{e}^a \\ S_0^j \mathrm{e}^{-a} - S^j, & S^j \leq S_0^j \mathrm{e}^{-a} \end{cases}, \tag{8}$$

where $\phi_j \in \mathbb{R}$ are multipliers. Equation (8) implies that on each single terminal condition, the option pays out the absolute excess variation if the underlying price exceeds $\pm a$ log−return on each settled initial value. Seen from another point of view, on each underlying the option pays−out the absolute variation with respect to the boundary strikes $K_j^{\pm} = S_0^j \mathrm{e}^{\pm a}$.
Applying the operator (6) to the initial condition $g = p \circ S$, we formally obtain the option price. Then, each convolution can be decomposed into integrals of flats orthogonal to each j^{th} axes, that is

$$\begin{aligned} f = \mathrm{e}^{-r\bar{t}} \textstyle\sum_j \phi_j S_0^j \Big\{ & \textstyle\int_{\Omega_j^+} (\mathrm{e}^{x_j} - \mathrm{e}^a)\, G_{\bar{t}} + \\ & + \textstyle\int_{\Omega_j^0} G_{\bar{t}} + \int_{\Omega_j^-} (\mathrm{e}^{-a} - \mathrm{e}^{x_j})\, G_{\bar{t}} \Big\} \end{aligned} \tag{9}$$

where Ω_j^+, Ω_j^0, Ω_j^- are, respectively, the sectors of the Euclidean space $e^j \cdot X > a$, $-a < e^j \cdot X \leq a$, $e^j \cdot X \leq -a$ and e^j is the j^{th} canonical vector.
After some algebra, the following closed form solution is obtained

$$\begin{aligned} f = \textstyle\sum_j \phi_j S_0^j \big\{ & \mathrm{e}^{x_j} F(\alpha_{1j}) - \mathrm{e}^{-r\bar{t}+a} F(\omega_{1j}) + \\ & + \mathrm{e}^{-r\bar{t}-a} F(\omega_{2j}) - \mathrm{e}^{x_j} F(\alpha_{2j}) \big\} \end{aligned} \tag{10}$$

the functions $F(\cdot)$ being the cumulative distribution function of a standard normal variable. The parameters α and ω are, respectively

$$\alpha_{1j} = \frac{-a+X_j+\left(r+\sigma_j^2/2\right)\bar{t}}{\sigma_j\sqrt{\bar{t}}} \qquad \omega_{1j} = \frac{a+X_j+\left(r+\sigma_j^2/2\right)\bar{t}}{\sigma_j\sqrt{\bar{t}}}$$
$$\alpha_{2j} = \frac{a-X_j-\left(r-\sigma_j^2/2\right)\bar{t}}{\sigma_j\sqrt{\bar{t}}} \qquad \omega_{2j} = \frac{-a-X_j-\left(r-\sigma_j^2/2\right)\bar{t}}{\sigma_j\sqrt{\bar{t}}}$$

and, in the usual Black–Scholes notation

$$X_j = \log \frac{S_j}{S_0^j}$$
$$S_0^j \mathrm{e}^{\pm a} = K_j^{\pm}$$

The formula (10) is a combination of call and put single-stock options, each of which has been written on the j^{th} stock and paying out unitary excess returns with respect to the strikes $S_0^j\mathrm{e}^{a}$ and $S_0^j\mathrm{e}^{-a}$.
It can be noticed that the equation (10), because of the particular form of g, is independent of the correlation structure of the stochastic processes that describe the dynamics of the underlings. The final formula results in a linear combination of straddle–like options each of which is paying unitary value with respect to the *in–the–money* interval that is determined by the a coefficient. Inside the hypercube that encloses the origin in the X space, the option pay–out is 0, whereas outside this neighborhood the option pays out a combination of linearly increasing returns in any direction.
In figure 1 we plot the shape of the function (10) in the X space, with 2 stocks, $a = 0.03$, $\sigma = 0.2$, $r = 0.02$ and $T - t = 1.5$, 1, 0.5 and 0.0001.

4.2 The χ^2–option

With the second application we develop a new option on an equity basket whose dynamics is described by the multivariate Black–Scholes model. Exploiting a result based on the geometrical properties of the underlying probability distributions and the peculiar design of the pay–off function, we are able to provide a closed–form solution for the pricing equation. To simplify the procedure we fix the derivative price with respect to the performance of the basket components.

Let $A = P\Lambda P'$ be the time unitary covariance matrix of the process W_t, where P and Λ contain respectively the eigenvectors and eigenvalues of the symmetric positive definite matrix A. Define the elliptic neighborhood of the point x^* as

$$\mathscr{E} = \left\{ x \in \mathbb{R}^n :\ \|z\|^2 \leq 1,\ z = \left[\sqrt{\Lambda\tau}\right]^{-1} P'(x - x^*) \right\}.$$

In figure 2 we show the set $\mathscr{E}$ with $n = 2$, $\tau = 0.5$, the point $x^* = (+10\%, +10\%)$ and the covariance matrix

$$A = \begin{bmatrix} 0.04 & 0.02 \\ 0.02 & 0.04 \end{bmatrix}$$

which corresponds to the correlation coefficient $\rho = 0.5$ and yearly volatilities $\sigma_j = 20\%$, $j = 1, 2$.
Now, we search for the price in t of the derivative $\mathscr{P}^*$ paying out the value $g(X)$ if at the expiration time T the stochastic growth vector $X_t = \log\left(S_t/S_0\right)$ lies in $\mathscr{E}$. The pay−off profile function $g\left(X\right)$ is defined such that

$$g(x) = \begin{cases} K\left[\mathrm{e}^{\phi\left(1-\|z\|^2\right)} - 1\right], & x \in \mathscr{E} \\ \\ 0, \quad x \in \mathbb{R}^n \setminus \mathscr{E} \end{cases} \tag{11}$$

where K is the option notional value and the coefficient $\phi > 0$. In fig. 3 we see the pay−off option value, with $\phi = 0.4$ and $K = 1$. The sought option delivers a progressively increasing payment as much as the basket components performance is closer to x^* at maturity. The set of points where this payment is not null coincides with the elliptic neighborhood of x^*, oriented by the angles of the eigenvectors frame and scaled by the eigenvalues parameters. At maturity, outside the set $\mathscr{E}$ the option is worth nothing.

The price $f\left(\bar{t}, X\right)$ of the derivative $\mathscr{P}^*$ follows from the application of the operator $\left(G_{\bar{t}}\,\mathrm{e}^{-r\bar{t}}\right)(\star)$ to the pay−off profile function g. Hence, we can finally state

Proposition 4.1. *The price of the derivative $\mathscr{P}^*$ written on the performance of the basket $\left\{S_j\right\}_{j=1,\ldots,n}$ which pays out $g\left(X\right)$ at maturity T, is the f such that*

$$f = \left[\alpha^{\frac{n}{2}}\mathrm{e}^{\phi(1-\alpha\kappa_0)}F_{\chi^2_2}\left(\theta_2\right) - F_{\chi^2_1}(\theta_1)\right]K\mathrm{e}^{-r(T-t)} \tag{12}$$

with the parameters $\alpha = \left(1 + \frac{2\phi}{\theta_1}\right)^{-1}$, $\theta_1 = \frac{\tau}{T-t}$, $\theta_2 = \frac{\theta_1}{\alpha}$, $\kappa_0 = \|\eta\|^2$ and the densities $\chi^2_1 = \chi^2\left(n, \kappa_1\right)$, $\chi^2_2 = \chi^2\left(n, \kappa_2\right)$, with $\kappa_1 = \theta\kappa_0$ and $\kappa_2 = \alpha\kappa_1$. The vector η is a function of the relative portfolio performance

$$\eta = \left[\sqrt{\Lambda\tau}\right]^{-1} P'\left[X - x^* + (T-t)\left(r\iota - \tfrac{1}{2}\,\mathrm{diag}A\right)\right]$$

Proof. See the appendix. □

where F is the cumulative distribution function of the non−central $\chi^2\left(n,\, \kappa\right)$ with n degrees of freedom and non centrality parameter κ. In fig. 4 we plot the shape of the function (12) in the X space, with 2 stocks, $a = 0.03$, $\sigma = 0.2$, $r = 0.02$ and $T - t = 0.5$, 0.3, 0.1 and 0.001. We remark that the option return at maturity depends on the possibility that the performance vector point would fall within the elliptic neighborhood of x^*. As any other option, the price of the χ^2−option is related to the probability that the reference event happens in $T - t$ years. This probability can be easily calculated. In fact, the probability of X_t belonging to the $\mathscr{E}$−neighborhood of x^* is

$$\mathbb{P}\{X_t \in \mathscr{E} | X_0\} = \int\limits_{\left\|u-\sqrt{\theta}\zeta\right\|^2 \leq \theta} \mathrm{d}u \, \frac{\mathrm{e}^{-\frac{1}{2}\|u\|^2}}{\sqrt{2\pi}^n} = F_{\chi^2}(\theta), \tag{13}$$

with $\chi^2\left(n,\, \theta\|\zeta\|^2\right)$, $\theta = \tau/t$ and the point $\zeta = \left[\sqrt{\Lambda\tau}\right]^{-1} P'\left(x^* + \frac{1}{2}\, t\, \mathrm{diag} A\right)$ resulting from the change of variable in (13). If we set, for instance, $t = 1$ then $F = 0.179$ is the probability of the option $\mathscr{P}^*$ being in−the−money in one year. Eventually, the option function transform into monetary value the probability content of the Euclidean space mapping the performance of the array of the basket components.

5 Conclusions

In this paper we have described an approach to construct the solution to the problem of pricing an European option in the multidimensional Black−Scholes model. The option price is not only a local solution, but it is extended to the complete price / performance space $\mathbb{R}^n$. The solution takes the form of a functional on a range of functions with an appropriate behavior at the infinity. This space of function is the set of admissible pay−offs of the option at the maturity time. We provide a sufficient condition to identify this set $\mathscr{A}$. We argue that this space can be enlarged considering the Cauchy principal value. This completion has been discarded in this work, whose intention has been the development of a method of solution for a specific kind of financial applications. This method provides a week solution, which justifies the exploitation of numerical techniques to construct any price function that cannot be obtained in closed form.

In the final section, we have constructed two applications, which deliver a closed−form output. In the first exercise, we are prompted to a case where the correlation structure did not enter the pricing function, where the final basket option formula is the result of the linear combination of plain vanilla call / put options on each single underlying. This result is consequent upon the specific formulation of the pay−out function. In the second example, the typical situation arises where the return profile of the basket option at the expiration time requires the treatment of the option pricing as a joint problem on the set of the underlying. While elaborating the exercise, we are able to obtain the pricing formula using a result concerning the probability content of regions of $\mathbb{R}^n$ under normal distribution. The pricing function is indicated as the χ^2−option because it involves the non−central χ^2 cumulative distribution function in the determination of the option value. Under this paradigm, several other options that engage probability content of regions of the Euclidean space might be easily obtained.

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Appendix

Proof of Proposition 3.1. Let $f \in \mathcal{D}_t^*$, we have $\mathscr{F}_X(\partial_t f) = \partial_t \check{f} = P(\mathrm{i}\xi)\,\check{f}$, possibly $\check{f}_t \in \mathcal{Z}^*$. The multidimensional polynomial $P \in \mathcal{E}$, hence the partial $\mathscr{F}-$transform of the (4) is a parameterized ordinary differential equation, whose solution is the product of the initial value times the exponential $F = \mathrm{e}^{P(\mathrm{i}\xi)t}$. We will consider the f as the limit of the sequence of local solutions obtained substituting the initial condition $g \in \mathscr{A}$ with its cut$-$off $g^K = \chi_K g$, where χ_K is the indicator function of the open ball $B_0(s) =: K \subset \mathbb{R}^n$. Clearly, $g^K \in \mathscr{L}^1_{\mathbf{loc}}$ and $g^K \to g$ as $s \to \infty$. By the Cauchy$-$Schwarz inequality we get also, $\forall K$

$$\left|\langle g^K,\, a\rangle\right| \le \|\, g\,\|_{1,K} \|\, a\,\|_{\infty,K} < +\infty,\ \forall a \in \left\{\mathrm{e}^{-i\xi\cdot x}\right\}_{\xi\in\mathbb{R}^n}$$

which implies that the function $\mathcal{E} \ni \check{g}^K : \mathbb{R}^n \to \mathbb{C}$ is of bounded variation, i.e. $\check{g}^K \in \mathcal{E} \cap \mathcal{S}^*$. Hence, plugging the $\mathscr{F}-$transform of the initial condition into the solution of the parameterized ODE, we get

$$\check{f}^K = F\check{g}^K \tag{14}$$

Notice that $F \in \mathcal{E}_t$ and $F_t \in \mathcal{E} \cap \mathcal{S}^*$, $\forall t \in \mathbb{T}$. More specifically, if we split the interval $\mathbb{T}$ into two subsets, we see that as $\Omega > 0$ then $F \in \mathcal{S}$, $\forall t \in (0, T]$ and taking the $\lim_{t\to 0^+} F = 1 \in \mathcal{E} \cap \mathcal{S}^*$.
Because $\check{g}^K$ is a function of bounded variation and $F|_t$ is, generically, in $\mathcal{E} \cap \mathcal{S}^*$, $\forall t \in \mathbb{T}$, we see that their product $F\check{g}^K = \check{f}^K \in \mathcal{E} \subset \mathcal{D}^*$. Taking the inverse transform of both sides, we get $f^K = \mathscr{F}_X^{-1}\left(F\check{g}^K\right) = \left(G \star g^K\right)\mathrm{e}^{\delta t}$. The latter convolution is well defined, because $g^K \in \mathscr{L}^1_{\mathbf{loc}} \subset \mathcal{E}^*$ and $G \in \mathcal{S}_t^*$. As we let $s \to \infty$ and $g^K \to g$ we obtain $f^K \to f$ and the convergence is in $\mathscr{L}^1$, $\forall t \in (0, T]$, because $G \in O\left(\mathrm{e}^{-\|x\|^2}\right)$ and $g \in \mathscr{A}$. Whereas $t \to 0^+$ we have G converging to δ_X, the identity element with respect to the convolution operation. This completes the proof.

□

Proof of Proposition 4.1. The solution is obtained applying the operator (6) to the final pay$-$off (11) and then manipulating the equations in order to achive integrals as in (13). These type of integrals can be solved with a geometrical argument or through the application of the method of sections, cfr. [9]. □

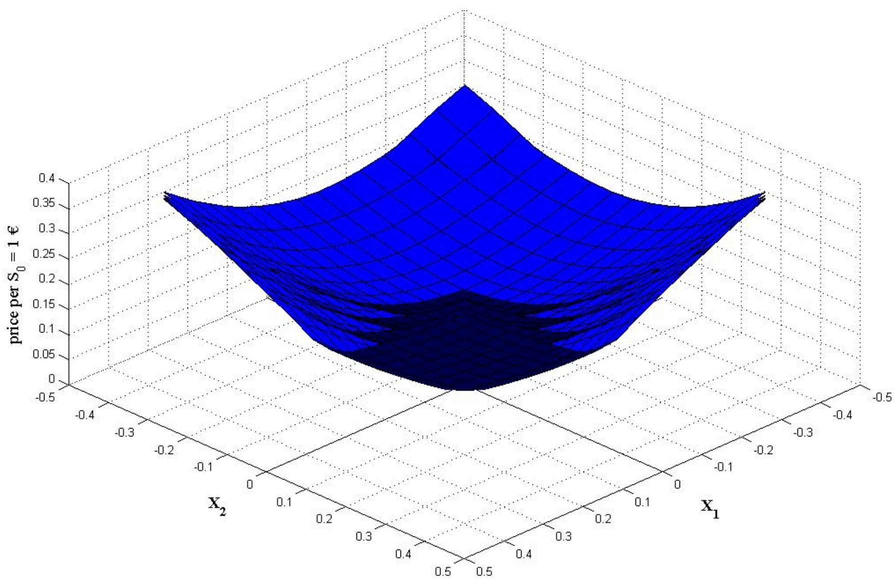

Figure 1: $2D$ classic Black-Scholes option price with time to maturity $\bar{t} = 1.5$, 1, 0.5 and 0.0001.

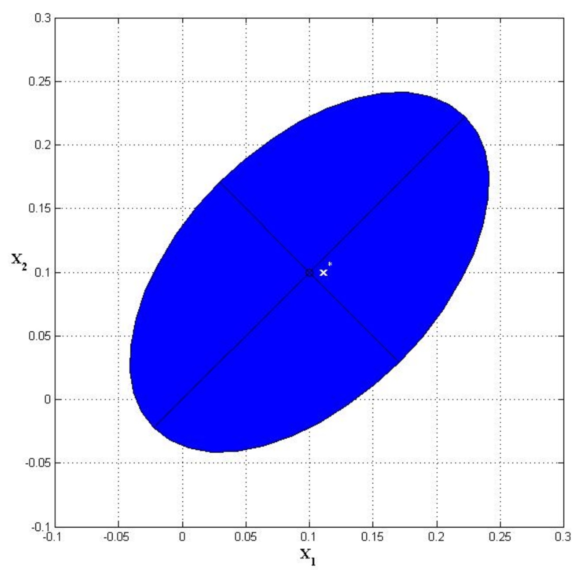

Figure 2: $\tau-$neighborhood of x^* with $\tau = 0.5$ and covariance matrix A and $n = 2$.

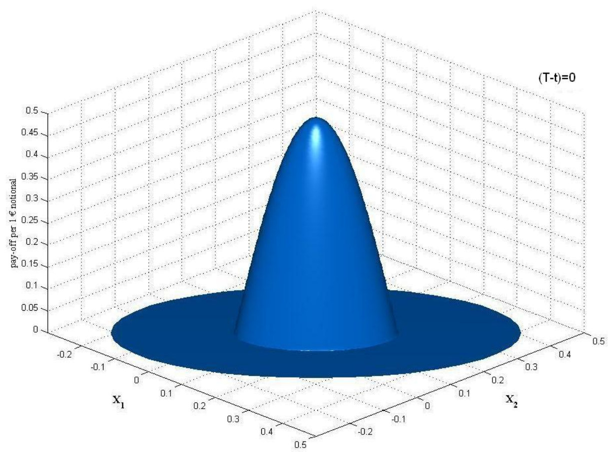

Figure 3: $2D$ χ^2-option terminal pay$-$off.

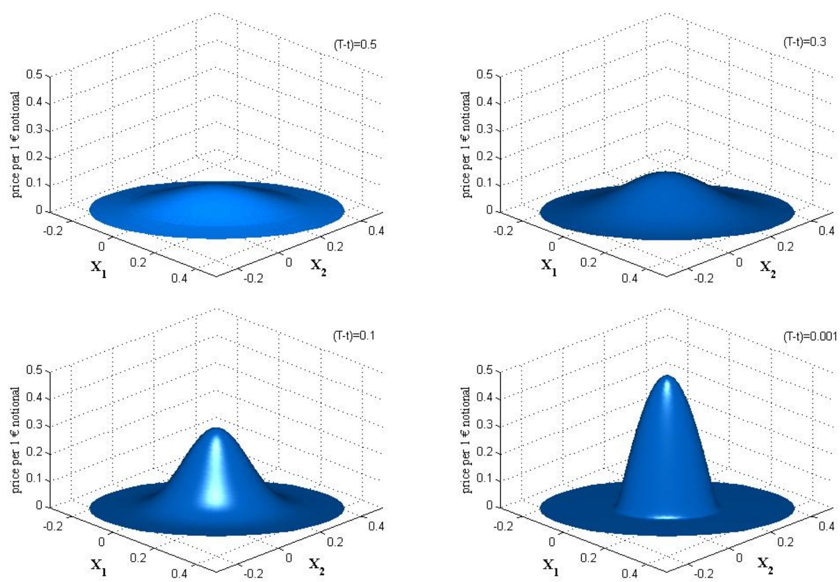

Figure 4: χ^2-option prices at different time to maturity.