CVA, Wrong Way Risk, Hedging and Bermudan Swaption

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Abstract.[1] “Roughly two-thirds of credit counterparty losses were due to credit valuation adjustment losses and only one-third were due to actual defaults” according to the Basel Committee on Banking Supervision, highlighting the importance of counterparty credit risk management to the derivatives contracts. Today, managing counterparty credit risk has become an integrated part of many derivative trading desks’ day-to-day activities and the need of accurate pricing, efficient hedging strategies and practical proxies has become critical. As a result, banks have sharpened their CVA pricing and modeling infrastructure and most have a dedicated trading desk dynamically hedging their CVA.
However, if pricing techniques have become standard over the past few years, the expected positive exposure (EPE) modeling is usually not taking into account the embedded correlation between the counterparty and underlying market movements. This correlation known as wrong way risk can substantially affect the price and the related hedging strategy and is the main focus of this article.

1. Introduction.

“Roughly two-thirds of credit counterparty losses were due to credit valuation adjustment losses and only one-third were due to actual defaults” according to the Basel Committee on Banking Supervision, highlighting the importance of counterparty credit risk management to the derivatives contracts. Today, managing counterparty credit risk has become an integrated part of many derivative trading desks’ day-to-day activities and the need of accurate pricing, efficient hedging strategies and practical proxies has become critical. As a result, banks have sharpened their CVA pricing and modeling infrastructure and most have a dedicated trading desk dynamically hedging their CVA.

However, if pricing techniques have become standard over the past few years, the expected positive exposure (EPE) modeling is usually not taking into account the embedded correlation between the counterparty and underlying market movements. This correlation known as wrong way risk can substantially affect the price and the related hedging strategy and is the main focus of this article.

In the following we present a practical way of introducing the wrong way risk in standard credit valuation adjustment. We also present an alternative hedging strategy using Bermudan Swaption and infer counterparty characteristics in terms of spread and correlation levels for the over-hedge to be financially attractive.

2. CVA and Wrong Way Risk.

In this section, we present the general CVA pricing formula and introduce the wrong way risk and the adjustment function which allows to take into account wrong way risk without modifying the underlying dynamics. The adjustment function appears to be particularly suitable for a copula approach.

[1]Disclaimer: The views expressed in this paper are the authors’ own and may not necessarily reflect those of Grupo Santander. The authors would like to thank Norddine Bennani, Jon Gregory, Igor Smirnov and Rob Smith for valuable comments and suggestions.

2.1. General Pricing Formula.

The general pricing formula as stated in [1], [2] and [3] among others, assuming zero recovery is given by:

$$\mathrm{CVA}_0 := N_0\, \mathbb{E}_0^{\mathbb{Q}} \left[\frac{V_\tau^+}{N_\tau} \right] \tag{1}$$

where N is the cash account numeraire, $\mathbb{Q}$ the pricing measure associated to this numeraire and τ the counterparty default time. V_t is the mark-to-market position of the bank at time t.

The exact definition of the numeraire relies on the collateral's rate of return of the market hedges. For the reader interested in the matter we refer to the following articles [4], [5] and [6]. In a simple example where the market hedges are collateralized in cash with overnight rate of return r, the numeraire is simply:

$$N_t := N_0\, e^{\int_0^t r_s ds}$$

In the following we will refer to $\beta_t := e^{-\int_0^t r_s ds}$ as the path discount factor and $B(0,t) := N_0 \mathbb{E}_0^{\mathbb{Q}} \left[\frac{1}{N_t} \right] = \mathbb{E}_0^{\mathbb{Q}} \left[\beta_t \right]$ as the discount factor.

The other formula usually encountered in the literature is:

$$\mathrm{CVA}_0 = \int_0^T B(0,t) \mathrm{EPE}_t \, dp_t \tag{2}$$

where EPE is the expected positive exposure and p the default probability. In order to properly define the expected positive exposure and the default probability, we start from the definition.

$$\begin{aligned}
\mathrm{CVA}_0 &:= N_0 \mathbb{E}_0^{\mathbb{Q}} \left[\tfrac{V_\tau^+}{N_\tau} \right] \\
&= \int_0^T \mathbb{E}_0^{\mathbb{Q}} \left[\beta_t\, V_t^+ \delta(\tau - t) \right] dt \\
&= \int_0^T B(0,t) \mathbb{E}_0^{\mathbb{Q}} \left[\tfrac{\beta_t}{B(0,t)} V_t^+ \delta(\tau - t) \right] dt \\
&= \int_0^T B(0,t) \mathbb{E}_0^{\mathbb{Q}^t} \left[V_t^+ \delta(\tau - t) \right] dt \\
&= \int_0^T B(0,t) \mathbb{E}_0^{\mathbb{Q}^t} \left[V_t^+ \tfrac{\delta(\tau - t)}{\partial_t p_t} \right] dp_t \\
&= \int_0^T B(0,t) \mathbb{E}_0^{\mathbb{Q}^t} \left[V_t^+ | \tau = t \right] dp_t \\
&= \int_0^T B(0,t) \mathrm{EPE}_t dp_t
\end{aligned} \tag{3}$$

where δ is the Dirac delta function, $\mathbb{Q}^t$ the t-forward measure and p_t the default probability, defined implicitely by $\partial_t p_t := \mathbb{E}_0^{\mathbb{Q}^t} \left[\delta(\tau - t) \right]$. This definition is consistent with the pricing of the underlying credit default swap protection leg. Indeed if the exposure at default is a constant K then:

$$\mathrm{CVA}_0 = K \times \mathrm{EL}_T$$

where EL_T is the protection leg present value of the CDS contract of nominal 1 and maturity T, in particular market practice is to define EL_T as :

$$\mathrm{EL}_T = \int_0^T B(0,t) dp_t \tag{4}$$

where p_t is the market implied default probability, usually assumed to be $p_t := 1 - e^{-\int_0^t \lambda_s ds}$ where λ is the default intensity.

We finally define the expected positive exposure (EPE), in line with the EPE^{mod} in [2]:

$$\mathrm{EPE}_t := \mathbb{E}_0^{\mathbb{Q}^t} \left[V_t^+ | \tau = t \right] \tag{5}$$

Remark: the EPE defined previously is specific to CVA computation and can differ slightly from the expected exposure defined in the Basel regulatory framework.

2.2. Wrong Way Risk.

2.2.1. Definition.

The wrong way risk represents the correlation between the exposure at default V^+ and the counterparty default time. As such it is more an intuitive concept than a true mathematical definition. It occurs when the mtm of the underlying contract appreciates at the time of default producing an important jump-to-default. The right way risk (RWR) describes the opposite behavior, it is in the right way as the value of the underlying derivative tends to zero as the counterparty goes to default, reducing the jump-to-default.

The literature tackling wrong way risk modeling essentially models the dependency through the intensity process like in [7]. The approach followed here is derived from [2] and starts from the alternative formulation which exhibits the dependency function, starting from (3):

$$\begin{aligned} \text{CVA}_0 &= \int_0^T B(0,t)\mathbb{E}_0^{\mathbb{Q}^t}\left[V_t^+ \tfrac{\delta(\tau-t)}{\partial_t p_t}\right] dp_t \\ &= \int_0^T B(0,t)\mathbb{E}_0^{\mathbb{Q}^t}\left[V_t^+ \mathbb{E}_0^{\mathbb{Q}^t}\left[\tfrac{\delta(\tau-t)}{\partial_t p_t}|V_t\right]\right] dp_t \\ &= \int_0^T B(0,t)\mathbb{E}_0^{\mathbb{Q}^t}\left[V_t^+ \psi_t^\tau\right] dp_t \end{aligned} \tag{6}$$

where ψ^τ is defined as follows:

$$\psi_t^\tau = \psi^\tau(t, V_t) = \mathbb{E}_0^{\mathbb{Q}^t}\left[\frac{\delta(\tau-t)}{\partial_t p_t}|V_t\right] \tag{7}$$

As $\psi_t^\tau \geq 0$ and $\mathbb{E}_0^{\mathbb{Q}^t}\left[\psi_t^\tau\right] = 1$, this multiplicative adjustment can be seen as a density of measure, enabling to pass from a space of probability where counterparty and underlying are independent to a space where they are correlated.

Also note that the strong dependency between the underlying mtm and the default can be relaxed slightly assuming shallow dependency between the default and the underlying market variable. In this case, assuming $V_t = v(t, X_t)$, we alternatively define:

$$\psi_t^\tau = \psi^\tau(t, X_t) = \mathbb{E}_0^{\mathbb{Q}^t}\left[\frac{\delta(\tau-t)}{\partial_t p_t}|X_t\right]$$

In practice, wrong way risk occurs when the adjustment function over-weights high mtms, right way risk when the adjustment function over-weights low mtms. The uncorrelated case is represented by the constant 1. The graph 1 illustrates the three typical situations.

2.2.2. Monte Carlo.

One of the main advantage of this formulation is its tractability in a Monte Carlo simulation framework. Let us start with the uncorrelated case. The simulation process can be decomposed in four steps:

1. Simulate path discount factors $(\beta_t(k))_{1\leq k\leq M}$ and the mtms of each deal j composing the counterparty's portfolio $\left(V_t^j(k)\right)_{1\leq k\leq M}$ under the risk neutral measure $\mathbb{Q}$.
2. Apply netting agreements, collateral, threshold to obtain the exposure $(V_t(k)^+)_{1\leq k\leq M}$
3. Compute the expected positive exposure:

$$\text{EPE}_t = \frac{1}{M}\sum_{k=1}^{M}\frac{\beta_t(k)}{B(0,t)}V_t^+(k)$$

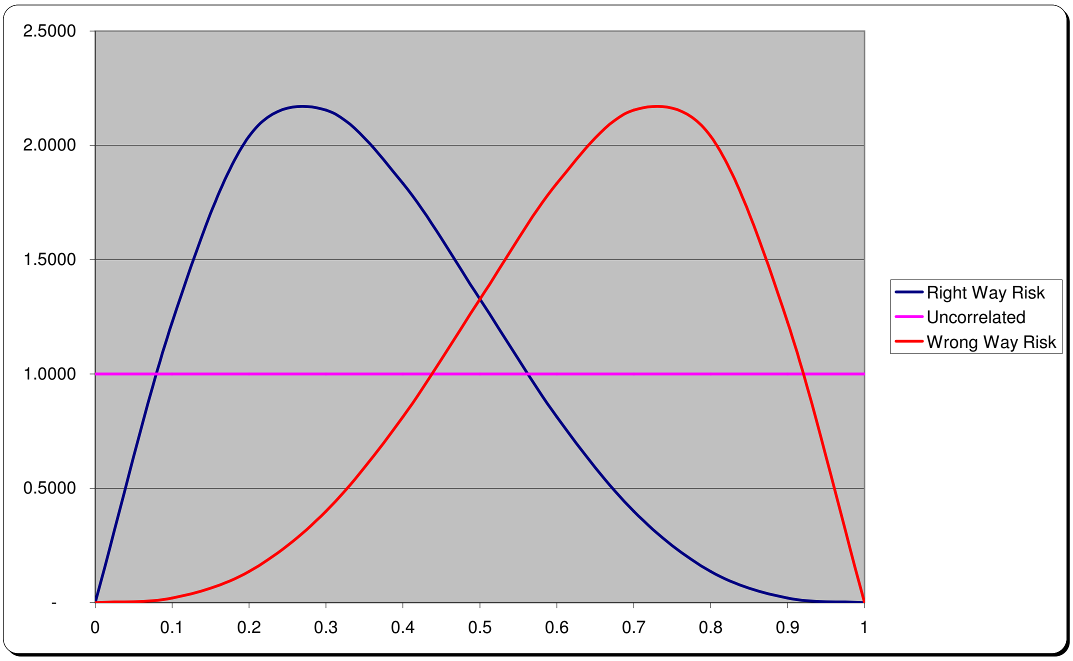

Figure 1: The adjustment function ψ^τ as a function of the normalized underlying variable.

4. Compute the CVA using formula (2).

$$\mathrm{CVA}_0 = \int_0^T B(0,t)\mathrm{EPE}_t dp_t$$

Adding wrong way risk through the adjustment function can be easily performed by adding one extra step between the steps 2. and 3. :

2.b Replace the exposure by:

$$V_t^+(k) \leftarrow V_t^+(k)\psi^\tau(t, V_t(k)) = V_t^+(k)\psi^\tau(t, V_t^+(k))$$

The main advantages of this method are:

a. The underlying market variables can be simulated independently of the counterparty. This is particularly interesting for standard products which only need to be simulated once for all counterparties.

b. The adjustment function is applied at the very end of the computation chain. In particular once the risk-neutral simulation cube is saved, multiple scenarios of wrong way risk can be applied with little computation efforts.

2.2.3. Copula Approach.

In the previous section, we have seen that the wrong way risk can be modeled using the adjustment function ψ^τ. Most of the time this function is unknown and need to be parameterized. One of the interesting property of the adjustment function is the following : for any arbitrary choice of copula function C we can define:

$$\psi^\tau(t,v) = \partial^2_{ab} C\,(a,b)\big|_{a=p_t, b=F^V(t,v)} = c(p_t, F^V(t,v)) \tag{8}$$

where $F^V(t,v) = \mathbb{Q}^t(V_t \le v)$ is the cumulative distribution function of V and if not known analytically can be computed using the alternative formulation:

$$F^V(t,v) = \mathbb{E}_0^{\mathbb{Q}}\left[\frac{\beta_t}{B(0,t)}\mathbf{1}_{V_t \le v}\right]$$

In a Monte Carlo framework we use the empirical cumulative distribution estimator:

$$\hat{F}^V(t,v) = \frac{1}{M}\sum_{k=1}^{M}\frac{\beta_t(k)}{B(0,t)}\mathbf{1}_{V_t(k)\le v}$$

The proof of (8) relies on the following. Let us consider a reference probability measure $\mathbb{P}$ and two continuous random variable A and B with marginal cumulative distribution functions F^A and F^B respectively. Let us also consider the copula function C such that:

$$C(F^A(a), F^B(b)) := \mathbb{P}(A \le a, B \le b)$$

Then we have:

$$\begin{aligned}\mathbb{E}\left[\tfrac{\delta(A-a)}{\mathbb{E}[\delta(A-a)]}|B=b\right] &= \tfrac{\mathbb{E}[\delta(A-a)\delta(B-b)]}{\mathbb{E}[\delta(A-a)]\mathbb{E}[\delta(B-b)]} \\ &= \partial^2_{uv}C(F^A(a), F^B(b))\end{aligned}$$

Remark 1: the copula function can be made time dependent. However, the copula structure remains a static representation of the dependency of the underlying variables given a time horizon. In particular we do not make any particular assumption on the joint dynamic of (τ, V) other than their joint density function.

Remark 2: Coupling asset pricing and credit risk models with copula functions was initially proposed in [8].

2.2.4. A Simple Adjustment Function.

Let us assume a gaussian copula structure:

$$C(u,v) = \Phi_2\left(\Phi^{-1}(u), \Phi^{-1}(v), r\right)$$

where Φ_2 is the bivariate cumulative normal, Φ the cumulative normal and r the correlation. We therefore have:

$$c(u,v) = \phi_2(u,v,r) = \left(1-r^2\right)^{-\frac{1}{2}}\exp\left(-\frac{\left(r\Phi^{-1}(u)\right)^2 - 2r\Phi^{-1}(u)\Phi^{-1}(v) + \left(r\Phi^{-1}(v)\right)^2}{2(1-r^2)}\right)$$

Figure 2 shows the Gaussian copula density for a correlation of 80%.

We define the adjustment function as:

$$\psi^\tau_\rho(t,v) = \phi_2\left(p_t, F^V(t,v), \rho_t\right) \tag{9}$$

with $\rho_t = -\rho$ if $p_t \le 0.5$ and $+\rho$ otherwise. This choice of ρ_t ensures positive values of ρ generate wrong way risk and negative values of ρ generate right way risk.

2.3. A Few Words on Hedging.

In case the underlying mtm does not depend on the default time then the hedging strategy is directly given by the pricing formula. First assume no wrong way risk.

On one hand:

$$\begin{aligned}\text{CVA}_0 &= \textstyle\int_0^T N_0\mathbb{E}_0^{\mathbb{Q}}\left[\frac{V_t^+}{N_t}\right]dp_t \\ &= \textstyle\sum_k \Delta_k p \times \pi_{t_k}\end{aligned}$$

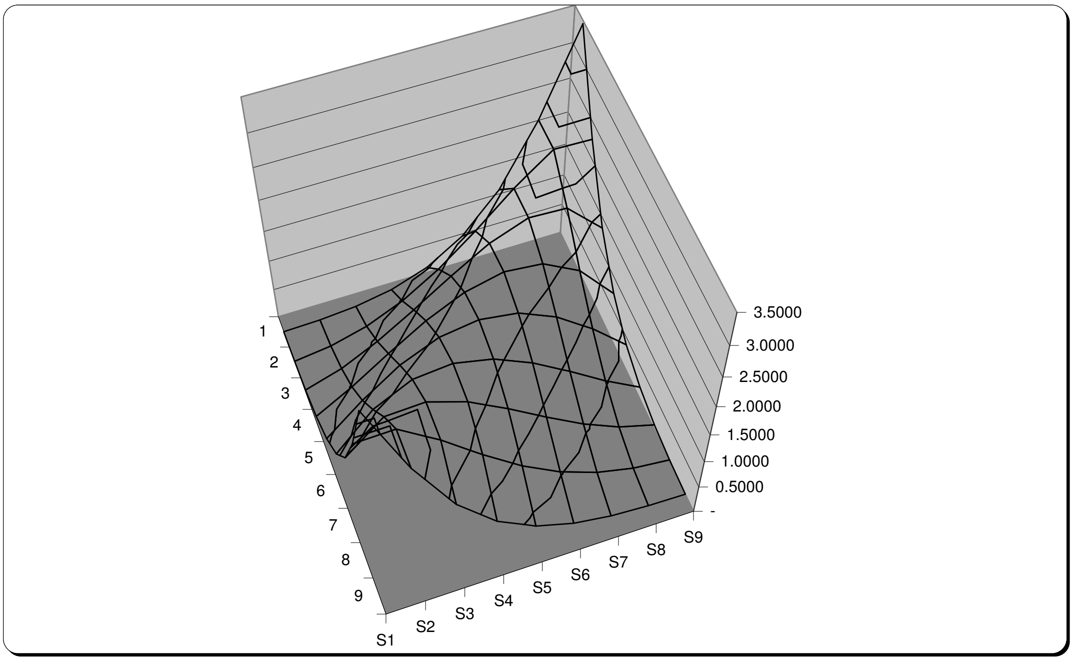

Figure 2: Gaussian copula density for a correlation of 80% as a function of two normalized variables

where $\pi_t = N_0 \mathbb{E}_0^{\mathbb{Q}} \left[\frac{V_t^+}{N_t} \right]$ is the price of an option.
On the other hand, performing an integration by part:

$$\begin{aligned} \mathrm{CVA}_0 &= \int_0^T \mathrm{EPE}_t \times (B(0,t) dp_t) \\ &= \int_0^T \mathrm{EPE}_t d\mathrm{EL}_t \\ &= \mathrm{EPE}_T \mathrm{EL}_T - \int_0^T \mathrm{EL}_t d\mathrm{EPE}_t \\ &= \mathrm{EPE}_{t_N} \times \mathrm{EL}_{t_N} - \sum_k \Delta_k \mathrm{EPE} \times \mathrm{EL}_{t_k} \end{aligned}$$

where EL_t is the protection leg value of a credit default swap of nominal 1 and maturity t.

The hedging strategy therefore consists in a basket of traded options and CDSs, the hedge notionals are given in the following table:

Tenor	Option	CDS
t_k	$\Delta_k p$	$-\Delta_k \mathrm{EPE}$
t_N	$\Delta_N p$	EPE_{t_N}

In presence of wrong way risk, the hedging strategy remains the same for the CDS hedge notionals. Most of the time, the option is unfortunately not a standard product and needs to be managed as an hybrid with correlation risk limits and reserves.

Remark: in the simple case of a swap, as described in [9], the hedge portfolio consists in a basket of co-terminal swaptions weighted by the counterpart's default probability.

2.4. Case Study : Vanilla Swap. Let us consider for simplicity a 10Y EUR 6M payer swap. Nominal is set to 1 and swap is supposed to be at par. We use a gaussian copula to model the wrong way risk as described previously. The following table shows the CVA upfront in percentage as a function of the correlation and the counterparty spread.

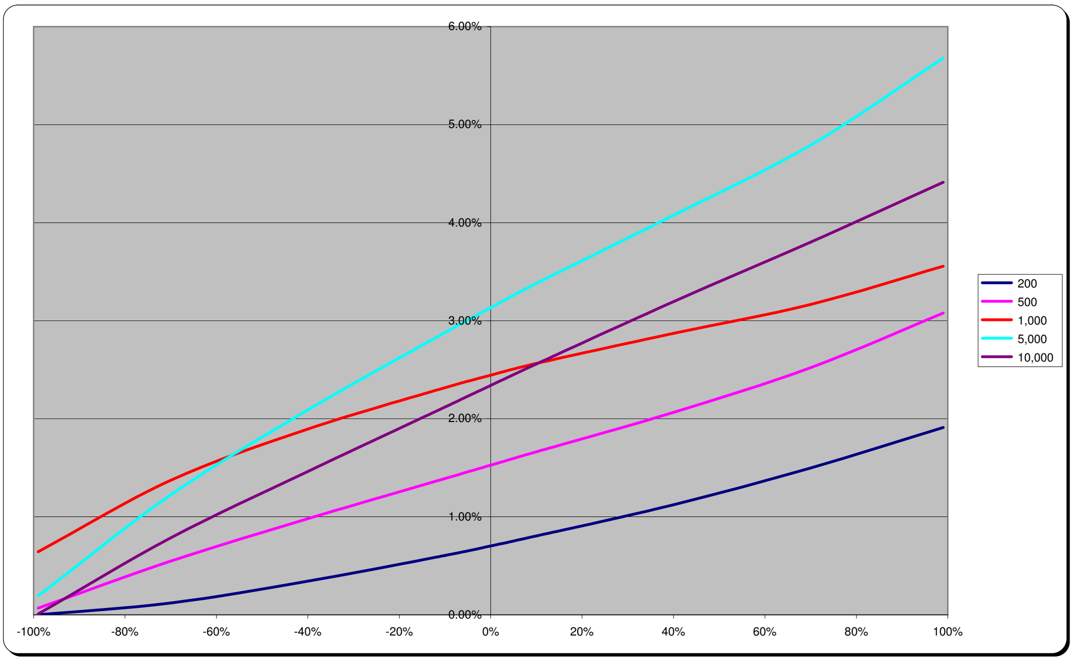

Figure 3: CVA upfront as a function of the correlation for different spread levels.

Spread\Correlation	−99%	−70%	−40%	−10%	0	10%	40%	70%	99%
200	0.00	0.12	0.34	0.61	0.70	0.80	1.12	1.50	1.91
500	0.07	0.55	0.98	1.39	1.53	1.66	2.07	2.52	3.08
1000	0.64	1.37	1.90	2.32	2.44	2.56	2.87	3.17	3.56
5000	0.19	1.22	2.09	2.88	3.13	3.38	4.08	4.79	5.68
10000	0.01	0.79	1.46	2.12	2.34	2.56	3.20	3.80	4.41

We can make a few remarks from these results. First, depending on the correlation value, the spectrum of CVA values varies from 0 to a number closed to twice the uncorrelated number. At a fixed spread, the CVA upfront is essentially linear as a function of the correlation:

$$\mathrm{CVA}_0(s,\rho) \simeq (1+\rho)\mathrm{CVA}_0(s,0)$$

Remark: The study of the CVA upper bound is one of the objective of the next section. In the swap example the upper bound is 7.05%.

We also observe that fixing the correlation, the CVA has a bell shape. This is coming from two opposite effects, on one hand as the default probability is increasing, the counterparty is more likely to default, hence the CVA increases, but on the other hand, as the swap is at par, as the counterparty is very likely to default in a short period of time, the mtm of the swap does not have the time to take extreme values. To model mtm default jumps we could alternatively use extreme-value copulas.

3. Bermudan Swaption.

In the following we assume the underlying asset is a swap but most of the techniques can be extended to more exotic payoffs. We proceed as follows : after having introduced the general pricing formula and its dynamic programming dual, we prove that the Bermudan Swaption is a suitable over-hedge for a risky swap. In order to estimate which market conditions make this over-hedge a practical hedge, we assume the optimal exercise time is a counterparty default time and provide a description of this stopping time as a spread level and wrong way risk adjustment function.

3.1. Pricing Formula.

A Bermudan Swaption is an option to enter into a swap at some given dates in the future, it is the discrete time equivalent of the American Swaption which initial value is given by:

$$\theta_0 = N_0 \text{ ess sup}_{\tau \in \tau_{[0,T]}} \mathbb{E}_0^{\mathbb{Q}} \left[\frac{V_\tau}{N_\tau} \right]$$

where $\tau_{[0,T]}$ is the set of stopping times with values in $[0, T]$.

The classical way of solving this problem is to solve the dual dynamic programming formulation.

$$\begin{cases} Z_{t_N} & = & 0 \\ Z_{t_k} & = & \max \left(V_{t_k} \, ; \, \mathrm{N}_{t_k} \mathbb{E}_{t_k}^{\mathbb{Q}} \left[\frac{Z_{t_{k+1}}}{\mathrm{N}_{t_{k+1}}} \right] \right) \end{cases}$$

Remark: previous formulas are unaffected by the change of V to V^+, in particular:

$$\theta_0 = \mathrm{N}_0 \text{ess sup}_{\tau \in \tau_{[0,T]}} \mathbb{E}_0^{\mathbb{Q}} \left[\frac{V_\tau^+}{\mathrm{N}_\tau} \right]$$

3.2. CVA Price Triangle and Over-Hedge.

Using previous results one can infer the following inequalities holding for the CVA upfront value:

$$0 \leq \mathrm{CVA}_0 \leq \theta_0$$

This is a direct consequence of the definition of the Bermudan Swaption pricing formula as an optimal exercise time. In particular these inequalities hold independently of the spread level or the adjustment function. Moreover the Bermudan Swaption can be exercised suboptimally at the default time, meaning the Bermudan Swaption is not only a price upper bound but an over-hedge.

In the following, for practical implementation details we assume the numerical method is a Monte Carlo (MC).

3.3. Implied Default Time.

The dynamic programming problem provides a description of the optimal exercise time τ^*. In particular, in MC, τ^* can be inferred using an algorithm like the one proposed in [11].

In the following, we consider this optimal exercise time as a counterparty default time. By doing so we ensure:

$$\mathrm{CVA}_0 = N_0 \mathbb{E}_0^{\mathbb{Q}} \left[\frac{V_{\tau^*}^+}{N_{\tau^*}} \right] = \theta_0$$

The study of this stopping time therefore provides a good indicator of actual spread levels and correlations required for the upper bound to be reached. Unless stated otherwise, the case study is the swap of the previous section.

3.3.1. Adjustment Function.

Recall the adjustment function as defined in (1):

$$\psi^\tau(t, v) = \mathbb{E}_0^{\mathbb{Q}^t} \left[\frac{\delta(\tau - t)}{\partial_t p_t} | V_t = v \right]$$

Equivalently we have:

$$\psi^\tau(t,v) = \frac{\mathbb{E}_0^{\mathbb{Q}^t}\left[\delta(\tau-t)\delta(V_t-v)\right]}{\mathbb{E}_0^{\mathbb{Q}^t}\left[\delta(\tau-t)\right]\mathbb{E}_0^{\mathbb{Q}^t}\left[\delta(V_t-v)\right]} \tag{10}$$

In order to determine the marginal and the joint densities we use a smoothing kernel K_h, see Appendix for more details. This leads to the following estimator under the pricing measure:

$$\hat{\psi}^\tau(t,v) = \frac{\frac{1}{M}\sum_{k=1}^{M}\frac{\beta_t(k)}{B(0,t)}K_{h_{\tau^*}}(\tau^*(k)-t)K_{h_{V_t}}(V_t(k)-v)}{\left(\frac{1}{M}\sum_{k=1}^{M}\frac{\beta_t(k)}{B(0,t)}K_{h_{\tau^*}}(\tau^*(k)-t)\right)\left(\frac{1}{M}\sum_{k=1}^{M}\frac{\beta_t(k)}{B(0,t)}K_{h_{V_t}}(V_t(k)-v)\right)}$$

For simplicity we use a gaussian kernel:

$$K_h(x-y) = \frac{1}{h}\varphi\left(\frac{x-y}{h}\right)$$

And we define the bandwidth using the Silverman's rule of thumb as described in [12]:

$$h_{V_t} = 1.06\,\hat{\sigma}_{V_t}\,M^{-\frac{1}{5}}$$

where M is the number of paths and $\hat{\sigma}_{V_t}$ the estimated standard deviation of the sample under the $\mathbb{Q}^t$ measure estimated as follows:

$$\hat{m}_{V_t} := \frac{1}{M}\sum_{k=1}^{M}\frac{\beta_t(k)}{B(0,t)}V_t(k)$$

$$\hat{\sigma}_{V_t} = \frac{1}{M-1}\sum_{k=1}^{M}\frac{\beta_t(k)}{B(0,t)}\left(V_t(k)-\hat{m}_{V_t}\right)^2$$

Remark: note that because of the change of measure, the estimator is biased. For the purpose of defining an approximate bandwidth the bias is ignored.

In the following graph we observe, as expected, a strong dependency between the implied default density and the underlying mtm. As time goes, the center of the distribution displaces itself to lower values. This is essentially due to possible values taken by the swap for large times. As in most extreme cases, the option is exercised earlier, in order to maximize the exposure at default, the adjustment function over-weights high values of the mtm for the short term and low values of the mtm for the long term.

3.3.2. Implied Gaussian Correlation.

Assuming a gaussian copula structure, we wish to infer the termstructure of correlation implied by the optimal stopping time. We therefore define:

$$\rho_t = \text{argmin}_{\rho\in[0,1]}\mathbb{E}_0^{\mathbb{Q}^t}\left[\left(\psi_\rho^{\tau^*}(t,V_t)-\psi_t^{\tau^*}\right)^2\right]$$

where $\psi_t^{\tau^*}$ is the implied adjustment function and $\psi_\rho^\tau(t,V_t)$ the gaussian adjustment function as defined in (9). To find the optimal correlation, we use a stochastic approximation method like the one proposed by Kiefer and Wolfowitz, see Appendix and [13] for more details:

$$\begin{cases} \rho_t^0 & = & 0 \\ \rho_t^{k+1} & = & \rho_t^k - \frac{1}{(k+1)^{\frac{2}{3}}}\left(\Delta_t\left(k+1,\rho_t^k+\frac{1}{(k+1)^{\frac{1}{3}}}\right)-\Delta_t\left(k+1,\rho_t^k-\frac{1}{(k+1)^{\frac{1}{3}}}\right)\right)\end{cases}$$

where the implied adjustment function is estimated using (10), hence:

$$\Delta_t(k,\rho) = \frac{\beta_t(k)}{B(0,t)}\left(\psi_\rho^{\tau^*}(t,V_t(k))-\hat{\psi}^{\tau^*}(t,V_t(k))\right)^2$$

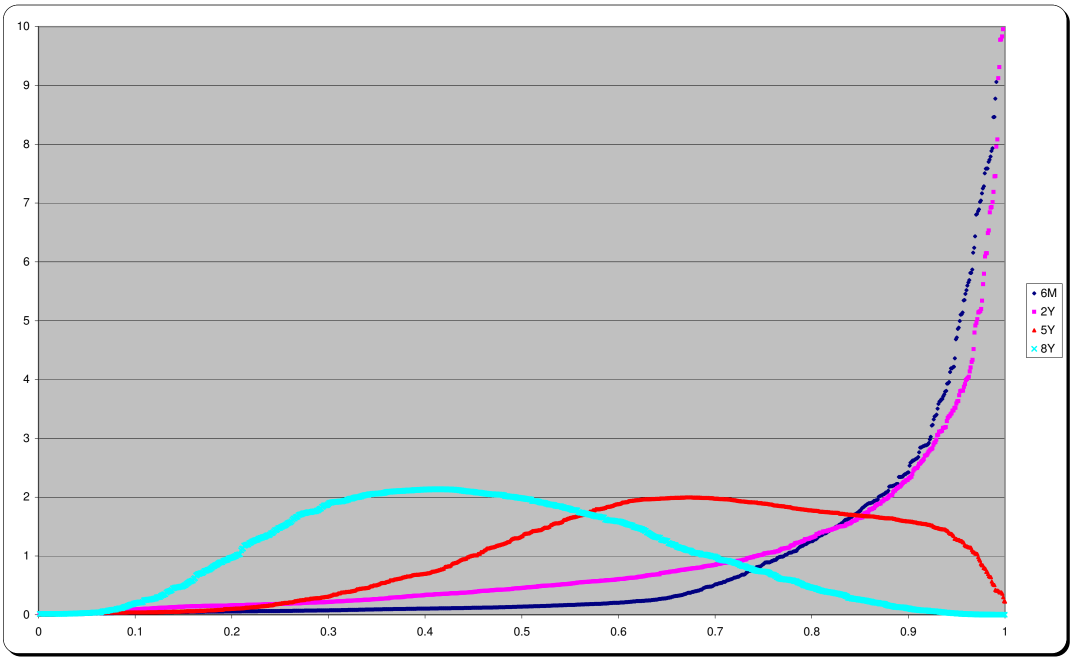

Figure 4: Implied adjustment function for some observable dates as a function of $u = F^V(t, v)$

Finally notice that the gaussian adjustment requires the cumulative distribution of V_t under $\mathbb{Q}^t$ which we estimate with the empirical cumulative distribution function as described in the previous section.

Numerical results indicate that the gaussian copula structure fails to capture the high level of implied correlation in the short term of the curve while producing a stable level of 70% in the mid term. Large times are not represented graphically as very few scenarios are actually relevant in this region.

3.3.3. Spread Level.

In order to determine the implied spread level, we proceed as follows:

1. Compute the expected loss.
2. Infer the default probability.
3. Infer the spread.

As the default probability is not an observable quantity, we start with the expected loss which, according to (4) is given by:

$$\mathrm{EL}_t = \int_0^t B(0,s)dp_s = \mathbb{E}_0^{\mathbb{Q}}\left[\beta_{\tau^*}\mathbf{1}_{\tau^*\le t}\right]$$

The expectation can be computed using the following MC estimator:

$$\hat{\mathrm{EL}}_{t_j} = \frac{1}{M}\sum_{k=1}^{M}\beta_{\tau^*(k)}(k)\mathbf{1}_{\tau^*(k)\le t_j}$$

Then infer the default probability using a time grid. As:

$$\mathrm{EL}_{t_{k+1}} - \mathrm{EL}_{t_k} = \left(\frac{B(0,t_k)+B(0,t_{k+1})}{2}\right)(p_{t_{k+1}} - p_{t_k})$$

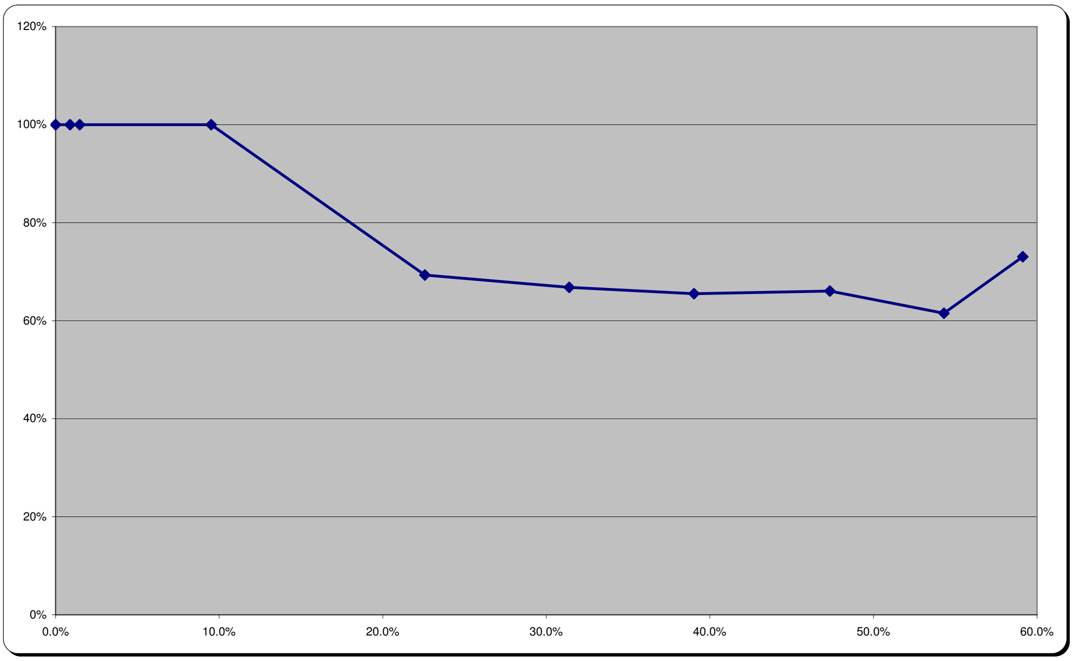

Figure 5: Implied gaussian correlation as a function of the default probability

The default probability is given by:

$$\begin{cases} p_{t_0} & = & 0 \\ p_{t_{k+1}} & = & 2\left(\frac{\mathrm{EL}_{t_{k+1}}-\mathrm{EL}_{t_k}}{B(0,t_k)+B(0,t_{k+1})}\right) \end{cases}$$

To finally infer the spread level, we use the approximation:

$$p_t \simeq 1 - e^{-s_t t}$$

Hence:

$$s_t = -\frac{1}{t}\ln(1-p_t)$$

The implied spreads, as shown in the following graph, are driven by two factors. On the short end of the curve, the initial spread relies on the present value of the swap. On the long end, as the default probability tends to one - the option is necessarily exercised at maturity, the spread increases until reaching its maximum value. In most cases, the spread curve is increasing from almost zero to a few thousands. When mtm is high enough the spread curve shape is inverted and the implied short term spread is close to a few hundreds of thousands. These spread levels are obviously unrealistic for most counterparties.

4. Conclusion.

In this document we have provided a tractable formulation of the wrong way risk through an adjustment function. This adjustment function can be either inferred from a specific copula structure or implied. In the case of the copula structure, wrong way risk can be introduced with little extra complexity and can be applied independently of the simulation phase.

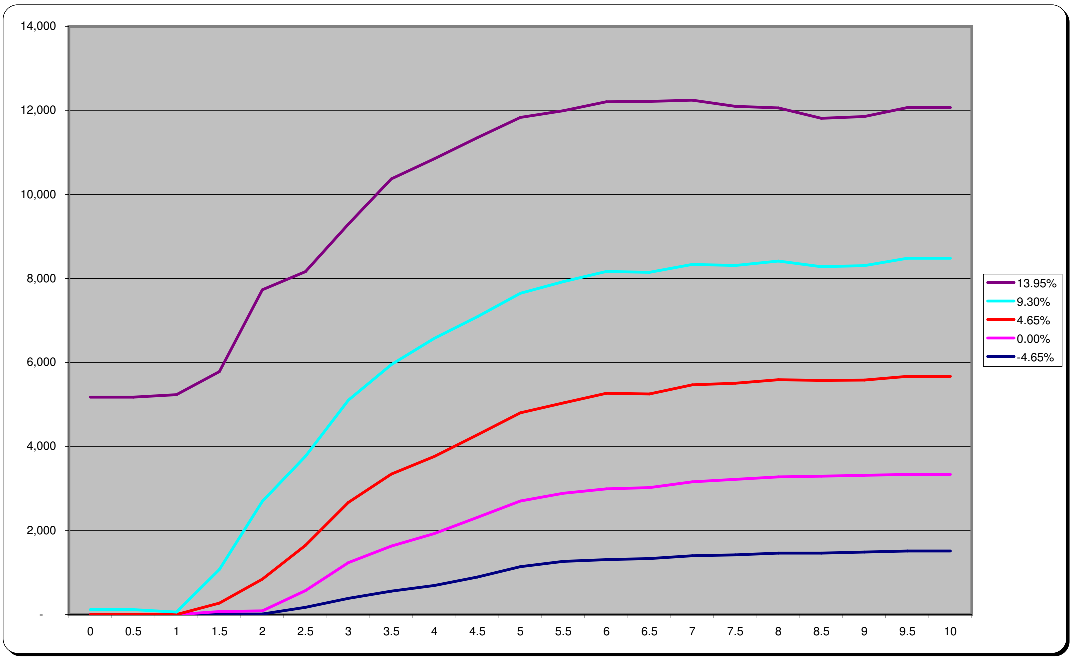

Figure 6: Implied spread for various initial swap mtm

In a second part of the document we have considered a Bermudan Swaption as an over-hedge and inferred the equivalent spread and adjustment function behavior implied by the optimal stopping time strategy. This allowed us to define a set of scenarios under which the price difference between the perfect hedge and the over-hedge can justify the deal. In practice, in the simple case of a swap, extreme spread behaviors implied by the Bermudan are incompatible with observable counterparty default spreads, for more reasonable spread levels, the hedge might be justified as long as it can be repacked as a contigent credit default swap (C-CDS) hence reducing the capital requirements imposed by the Basel rules.

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5. Appendix.

5.1. Smoothing Kernel.

In statistic, a kernel density estimation, or smoothing Kernel, aims at estimating the density function of a random variable. It approximates the density function by a sum of weighted kernel functions. Assuming $(X_k)_k$ is an iid sample drawn from an unknown density f, its kernel density estimator is:

$$\hat{f}(x) = \frac{1}{N}\sum_{k=1}^{N}\frac{1}{h}K\left(\frac{x - X_k}{h}\right)$$

where K is the kernel and h the smoothing parameter called bandwidth.

The gaussian kernel K_g is defined by:

$$K_g(x) = \varphi(x)$$

where φ is the gaussian density. The determine the bandwidth, we use the Silverman's rule of thumb as described in [9]:

$$h = 1.06\,\hat{\sigma}\,N^{-\frac{1}{5}}$$

where $\hat{\sigma}$ is the standard deviation of the samples.

5.2. Kiefer-Wolfowitz.

The Kiefer-Wolfowitz algorithm, see [13], provides a method which stochastically estimates the maximum of a function. Assume we are interested in the the following optimization problem:

$$\max_{\theta\in\Theta} f(\theta) = \mathbb{E}\left[F(\theta,\xi)\right]$$

The algorithm follows a gradient-like method:

$$\theta_{k+1} = \theta_k + a_k\left(\frac{F(\theta_k + c_k, \xi_k) - F(\theta_k - c_k, \xi_k)}{c_k}\right)$$

The sequence of $(\theta_k)_k$ converges to the optimal point θ^* provided that:

- f has a unique maximum and is strong concave.
- The sequences $(a_k)_k$ and $(c_k)_k$ satisfy:
  a. $c_k \to 0$, $a_k \to 0$ as $k \to +\infty$
  b. $\sum_{k=1}^{+\infty} a_k = \infty$, $\sum_{k=1}^{+\infty}\left(\frac{a_k}{c_k}\right)^2 < \infty$

A natural choice is $a_k = k^{-1}$ and $c_k = k^{-\frac{1}{3}}$.