Arbitrarily Fast CRR Schemes

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Abstract. We introduce a method for the approximation of a lognormal stock price process by a Cox, Ross and Rubinstein (CRR) type of binomial scheme, which allows to reach arbitrary speed of convergence of order $\mathcal{O}(n^{-\frac{N}{2}})$, for any integer $N > 0$.

1. Introduction and Setting

Let σ, r, T be the standard parameters in the Black-Scholes model, and consider a European call option with strike K, with K expressed in the form $K = S_0 \exp(\alpha\, r\, T)$ for some real number α, where S_0 is the spot price of the underlying asset. Also let $\left\{S^{(n)}\right\}_{n\in\mathbb{N}}$ denote risk neutral binomial schemes such that at every positive time t in $\frac{T}{n}\mathbb{N}$ the random walk $S^{(n)}$ has a probability $p\,(n)$ of jumping from its current state $S_t^{(n)}$ to the state $S_t^{(n)} u\,(n)$, and a probability $1-p\,(n)$ of jumping to the state $S_t^{(n)} d\,(n)$. Risk neutral binomial schemes are of the CRR type if $u\,(n) = \exp(\sigma\sqrt{\frac{T}{n}} + \lambda\,(n)\,\frac{T}{n})$ and $d\,(n) = \exp(-\sigma\sqrt{\frac{T}{n}} + \lambda\,(n)\,\frac{T}{n})$, for some bounded real valued function λ. Such schemes are also called *flexible* CRR binomial schemes.

Analyzing the convergence behavior of binomial schemes to calculate option prices has been a popular topic, in particular for the *European*, *American*, *Continuously Paying*, *Lookback*, *Digital*, *Game*, and *Barrier* option types. In the case of European options approximated by CRR-type binomial schemes, let us mention —*among others*— [6], [2], [3], [1], and [5]. Let $C\,(n) := C\,(\varphi, n)$ be the price of a European option with payoff φ under the CRR-type scheme and let $C_0 := C_0\,(\varphi)$ be the price of the same option in the Black-Scholes model. Considering a special case of flexible CRR scheme, Walsh [6] obtained an explicit formula $C_2\,(n) := C_2\,(\varphi, n)$ relating $C\,(n)$ and C_0:

$$C\,(\varphi, n) = C_0\,(\varphi) + \frac{C_2\,(\varphi, n)}{n} + \mathcal{O}(n^{-\frac{3}{2}}).$$

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Considering call options in [2] and digital options in [3], Diener and Diener showed how *coefficients* $C_\ell(n)$ can be explicitly calculated such that

$$C(n) = C_0 + \sum_{\ell=2}^{i_0} C_\ell(n)\, n^{-\frac{\ell}{2}} + \mathcal{O}(n^{-\frac{i_0+1}{2}}). \tag{1.1}$$

Chang and Palmer [1] showed how one can choose $\lambda := \lambda(n)$ in such a way that, for European Call and digital options,

$$C(n) = C_0 + \frac{\mathfrak{m}_0}{n} + o(n^{-1}).$$

Korn and Müller [5] showed how to choose $\lambda := \lambda(n)$ in order to minimize $\mathfrak{m}_0$ in absolute value. In the cases where $\mathfrak{m}_0 = 0$, this provides an acceleration of the convergence to order $o(n^{-1})$. Given any integer $N > 2$, we show in this paper how $\lambda := \lambda(n)$ can be chosen to obtain

$$C(n) = C_0 + \mathcal{O}(n^{-\frac{N}{2}}).$$

Such rate of convergence had been obtained for *special binomial trees* in Joshi [4] for n odd, and Xiao [7] extended the argument to n even. These *special* binomial trees differ from the *classical flexible CRR trees* among other things by the fact that exactly half of the values taken by $S_T^{(n)}$ are above the strike. Most of the commonly used binomial scheme are flexible CRR schemes. The method proposed in this paper is not only an alternative to the Joshi's trees, but it also shows how arbitrarily fast convergence can be achieved in a *quite straightforward manner* for *classical flexible CRR schemes*, simply by choosing the parameter λ appropriately. Furthermore, our method has the advantage to immediately extend, with nothing but trivial modifications, the digital options in [3], and in fact, to virtually any situation where an error formula of the form (1.1) exists. For the sake of simplicity we restrict our attention to flexible CRR schemes and to call options in the setting of [2], described below.

Let $i_0 \geq 2$ be an integer, and let $\lambda_1 = \sigma$ and $\overrightarrow{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_{i_0})$. Consider binomial schemes of the form

$$\begin{aligned}
\mathrm{u}(n,\ \overrightarrow{\lambda}) &= \exp\left(\sigma\sqrt{\frac{T}{n}} + \lambda_2\,\sigma^2\frac{T}{n} + \sum_{\ell=3}^{i_0}\lambda_\ell\frac{2\,\sigma}{T}\sqrt{\frac{T}{n}}^{\ell}\right),\\
\mathrm{d}(n,\ \overrightarrow{\lambda}) &= \exp\left(-\sigma\sqrt{\frac{T}{n}} + \lambda_2\,\sigma^2\frac{T}{n} + \sum_{\ell=3}^{i_0}\lambda_\ell\frac{2\,\sigma}{T}\sqrt{\frac{T}{n}}^{\ell}\right),\\
\mathrm{p}(n,\ \overrightarrow{\lambda}) &= \frac{\exp(\frac{rT}{n}) - \mathrm{d}(n,\ \overrightarrow{\lambda})}{\mathrm{u}(n,\ \overrightarrow{\lambda}) - \mathrm{d}(n,\ \overrightarrow{\lambda})},
\end{aligned}$$

and write

$$\begin{aligned}
\mathrm{a}(n,\ \overrightarrow{\lambda}) &= \frac{\ln\left(\frac{K}{S_0}\right) - n\ln\left(\mathrm{d}(n,\ \overrightarrow{\lambda})\right)}{\ln\left(\mathrm{u}(n,\ \overrightarrow{\lambda})\right) - \ln\left(\mathrm{d}(n,\ \overrightarrow{\lambda})\right)} \\
&= \frac{T}{2}\left(\frac{T}{n}\right)^{-1} - \frac{\lambda_2\sigma^2 T - \alpha r T}{2\sigma}\left(\frac{T}{n}\right)^{-\frac{1}{2}} - \sum_{\ell=3}^{i_0}\lambda_\ell\sqrt{\frac{T}{n}}^{\ell-3}, \\
\overline{\kappa}(n,\ \overrightarrow{\lambda}) &= frac\left(\mathrm{a}(n,\ \overrightarrow{\lambda})\right).
\end{aligned}$$

Diener and Diener [2] showed that

$$C(n,\ \overrightarrow{\lambda}) = C_0 + \sum_{\ell=2}^{i_0} C_\ell\left(\overrightarrow{\lambda},\ \overline{\kappa}(n,\ \overrightarrow{\lambda})\right) n^{-\frac{\ell}{2}} + \mathcal{O}\left(n^{-\frac{i_0+1}{2}}\right), \tag{1.2}$$

where, for $\ell = 2, ..., i_0$,

$$C_\ell\left(\overrightarrow{\lambda},\ \kappa\right) = \exp(-\frac{T\,(-2\,\alpha\,r + \sigma^2 + 2\,r)^2}{8\,\sigma^2})\,S_0\mathcal{P}_\ell(\overrightarrow{\lambda},\ \kappa),$$

and $\mathcal{P}_\ell$ is a multivariate polynomial in $(\lambda_2, \dots, \lambda_\ell, \kappa)$ (together with the parameters α, σ, r, T) which is of degree one in λ_ℓ. Moreover, the error $\mathcal{O}(n^{-\frac{i_0+1}{2}})$ is uniform in $(\lambda_2, \dots, \lambda_{i_0}) \in [-\mathcal{L}, \mathcal{L}]^{i_0-1}$, for any real number $\mathcal{L} \geq 0$. It is sometimes convenient to write $\mathcal{P}_\ell\,(\lambda_2, \dots, \lambda_\ell, \kappa) := \mathcal{P}_\ell(\overrightarrow{\lambda},\ \kappa)$ and we use a similar convention for C and the C_ℓ's.

Note that, specializing to $i_0 = 3$, C_2 and C_3 can be written as

$$\begin{aligned}
C_2(\lambda_2,\ \kappa) &= -\frac{1}{96}\,\frac{\sqrt{2}\,\exp(-\frac{T\,(-2\,\alpha\,r+\sigma^2+2\,r)^2}{8\,\sigma^2})\,S_0\,\sqrt{T}}{\sqrt{\pi}\,\sigma}\mathcal{P}_2(\lambda_2,\ \kappa), \\
C_3(\lambda_2,\ \lambda_3,\ \kappa) &= -\frac{1}{3}\,\frac{\sqrt{2}\,\exp(-\frac{T\,(-2\,\alpha\,r+\sigma^2+2\,r)^2}{8\,\sigma^2})\,S_0}{\sqrt{\pi}}\mathcal{P}_3(\lambda_2,\ \lambda_3,\ \kappa),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{P}_2(\lambda_2,\ \kappa) &= \sigma^4\,T^2\ -\ 32\,\lambda_2\,\sigma^2\,T^2\,r\ +\ 12\,T^2\,r^2\ +\ 4\,\alpha^2\,r^2\,T^2 \\
&\quad + 8\,\alpha\,r^2\,T^2\ +\ 12\ \sigma^2\,T -\ 96\,T\,\sigma^2\,\kappa\ +\ 24\,\lambda_2^2\,\sigma^4\,T^2 \\
&\quad + 96\,T\,\sigma^2\,\kappa^2\ -\ 16\,\alpha\,r\,T^{\ 2}\,\lambda_2\,\sigma^2, \\
\mathcal{P}_3(\lambda_2,\ \lambda_3,\ \kappa) &= -\ 4\ \kappa^3\,r\,T\ +\ 4\,\alpha\,r\,T\,\kappa^3\ +\ 6\,\kappa^2\,r\,T\ -\ \alpha\,r\,T\,\lambda_3 \\
&\quad + 2\,\alpha\,r\,T\,\kappa +\ 3\ \lambda_2\,\sigma^2\,T\,\lambda_3 -\ 2\,\kappa\,r\,T\ -\ 2\,\lambda_3\,r\,T \\
&\quad - 6\,\alpha\,r\,T\,\kappa^2.
\end{aligned}$$

2. The *Acceleration Method*

Given i_0, and n, we describe in this section a method allowing to map the parameters $(\lambda_2^{(n)}, \dots, \lambda_{i_0}^{(n)}) =: \overrightarrow{\lambda}_n$ of the random walk $S_t^{(n)}$ into the coefficients $C_\ell(\overrightarrow{\lambda}_n, \overline{\kappa}(n,\ \overrightarrow{\lambda}_n))$ of $n^{-\frac{\ell}{2}}$ in (1.2), in such a way that $\{\overrightarrow{\lambda}_n\}$

remains bounded and that, for every n, $C_\ell(\overrightarrow{\lambda}_n, \overline{\kappa}(n, \overrightarrow{\lambda}_n)) = 0$, for $\ell = 2, ..., i_0$. As a result, (1.2) reduces to $C(n, \overrightarrow{\lambda}_n) = C_0 + \mathcal{O}(n^{-\frac{i_0+1}{2}})$, and a convergence of order $\mathcal{O}(n^{-\frac{i_0+1}{2}})$ is achieved.

First, we consider the coefficient $C_2(\lambda_2, \kappa)$. In order for it to vanish, one must have $\mathcal{P}_2(\lambda_2, \kappa)$ vanishing. This is a quadratic equation in λ_2, yielding

$$\lambda_2 = \frac{8Tr + 4\alpha rT \pm \sqrt{D(\kappa)}}{12T\sigma^2},$$

where

$$D(\kappa) \stackrel{def}{=} -8T^2r^2(\alpha-1)^2 - 6\sigma^4T^2 - 72T\sigma^2 + 576T\sigma^2\kappa(1-\kappa).$$

We choose (arbitrarily) the "+" solution and define the function

$$\lambda_2^f(\kappa) \stackrel{def}{=} \frac{8Tr + 4\alpha rT + \sqrt{D(\kappa)}}{12T\sigma^2}.$$

Now in order to have $\mathcal{P}_3(\lambda_2, \lambda_3, \kappa)$ vanishing, it suffices have

$$\lambda_3 = \frac{-2\kappa r(2\kappa-1)(\kappa-1)(\alpha-1)}{3\lambda_2\sigma^2 - (2+\alpha)r},$$

and we define the function

$$\lambda_3^f(\kappa) \stackrel{def}{=} \frac{-2\kappa r(2\kappa-1)(\kappa-1)(\alpha-1)}{3\lambda_2^f(\kappa)\sigma^2 - (2+\alpha)r}.$$

Continuing this way, that is isolating λ_ℓ in the equation $\mathcal{P}_\ell(\lambda_2, \ldots, \lambda_\ell, \kappa) = 0$, and substituting λ_j by $\lambda_j^f(\kappa)$, for $j = 2, ..., \ell-1$, one defines functions $\lambda_\ell^f(\kappa)$, for $\ell = 2, ..., i_0$. This is easily done since $\mathcal{P}_\ell$ is linear in λ_ℓ. By induction, it is clear, that all $\lambda_\ell^f(\kappa)$ have the form

$$\lambda_\ell^f(\kappa) = \frac{P_\ell\left(\kappa, \sqrt{D(\kappa)}\right)}{Q_\ell\left(\kappa, \sqrt{D(\kappa)}\right)},$$

for some polynomials $P_\ell(x, y)$ and $Q_\ell(x, y)$. Assume that $D(\kappa) > 0$ on some subinterval I of $(0, 1)$. Obviously, $Q_\ell(\kappa, \sqrt{D(\kappa)}) = 0$, only for finitely many values of κ in I. Staying in between two such points, one can pick a close bounded subinterval I_0 of I such that the functions $\lambda_\ell^f(\kappa)$ are all real-valued and bounded on I_0, for $\ell = 2, ..., i_0$.

Recall that

$$\overline{\kappa}(n, \lambda_2, ..., \lambda_{i_0}) = frac\left(\frac{n}{2} - \frac{\lambda_2\sigma^2T - \alpha rT}{2\sigma\sqrt{T}}\sqrt{n} - \sum_{\ell=3}^{i_0}\lambda_\ell\sqrt{\frac{T}{n}}^{\ell-3}\right) \tag{2.1}$$

and define the function

$$\overline{\kappa}^f(n, \kappa) \stackrel{def}{=} \overline{\kappa}^f\left(n, \lambda_2^f(\kappa), ..., \lambda_{i_0}^f(\kappa)\right).$$

If, for all n sufficiently large, we can solve the equation

$$\kappa_n = \overline{\kappa}^f(n, \kappa_n),$$

for $\kappa_n \in I_0$, then, setting

$$\lambda_\ell^{(n)} \stackrel{def}{=} \lambda_\ell^f(\kappa_n),$$

for $\ell = 2, ..., n$, and defining

$$\overrightarrow{\lambda}_n \stackrel{def}{=} \left(\sigma, \lambda_2^{(n)}, ..., \lambda_{i_0}^{(n)}\right)$$

one gets $\kappa_n = \overline{\kappa}(n, \overrightarrow{\lambda}_n)$ and $C_\ell(\overrightarrow{\lambda}_n, \overline{\kappa}(n, \overrightarrow{\lambda}_n)) = 0$, for $\ell = 2, ..., i_0$, so that

$$C(n, \overrightarrow{\lambda}_n) = C_0 + \mathcal{O}(n^{-\frac{i_0+1}{2}}),$$

as wanted.

A glimpse at (2.1) reveals that solving $\kappa = \overline{\kappa}^f(n, \kappa)$, is the same as solving $\mathring{\kappa}^f(n, \kappa) \in \mathbb{N}$, where

$$\mathring{\kappa}^f(n, \kappa) \stackrel{def}{=} \frac{n}{2} - \frac{\lambda_2^f(\kappa)\,\sigma^2 T - \alpha r T}{2\sigma\sqrt{T}}\sqrt{n} - \sum_{\ell=3}^{i_0} \lambda_\ell^f(\kappa)\sqrt{\frac{T}{n}}^{\ell-3} - \kappa.$$

Note that for sufficiently large values of n, $\mathring{\kappa}_f(n, \kappa)$ behaves (as a function of $\kappa \in I_0$) as $(\lambda_2^f(\kappa)\,\sigma^2 T - \alpha r T)\sqrt{n}/(2\sigma\sqrt{T})$, and it is obvious that, as n tends to infinity, the number of solutions κ_n to $\mathring{\kappa}_f(n, \kappa_n) \in \mathbb{N}$ tends to infinity. It is trivial to find such solutions numerically in a logarithmic time by exploiting the mean value theorem.

Note that $\overrightarrow{\lambda}_n$ exists if and only if there is a subinterval I_1 of $(0, 1)$ on which $\lambda_f(\kappa)$ is real valued, that is for which $D(\kappa) > 0$. Clearly this is the case if $D(\kappa)$ has some roots, which occurs when

$$72\sigma^2 - 8Tr^2 + 16\alpha r^2 T - 8\alpha^2 r^2 T - 6\sigma^4 T > 0. \tag{2.2}$$

This condition is at least satisfied for small values of T. In the important case of $\alpha = 1$ (i.e. the strike is chosen such that the stock is at the money) then condition (2.2) is valid for $\sigma^2 T < 12$ which should always be satisfied in practical applications.

3. Numerical Illustration

To demonstrate the performance of our *acceleration method* we considered the case of $i_0 = 4$ Given a strike K —recall incidentally that $K = S_0 \exp(\alpha\, r\, T)$— we define the error $Err_T^n(K)$ as

$$Err_T^n(K) \stackrel{def}{=} C(n, \lambda_2^{(n)}, \lambda_3^{(n)}, \lambda_4^{(n)}) - C_0.$$

Using $\sigma = 0.5$, $T = 1$, $r = 0.05$, $S_0 = 1$ and $\alpha = 1.5$, we computed the quantity $n^{\frac{5}{2}} Err_T^n(K)$ which oscillates heavily, but remains bounded, illustrating numerically that the convergence is of order $\mathcal{O}(n^{-\frac{5}{2}})$ (see Figure 1).

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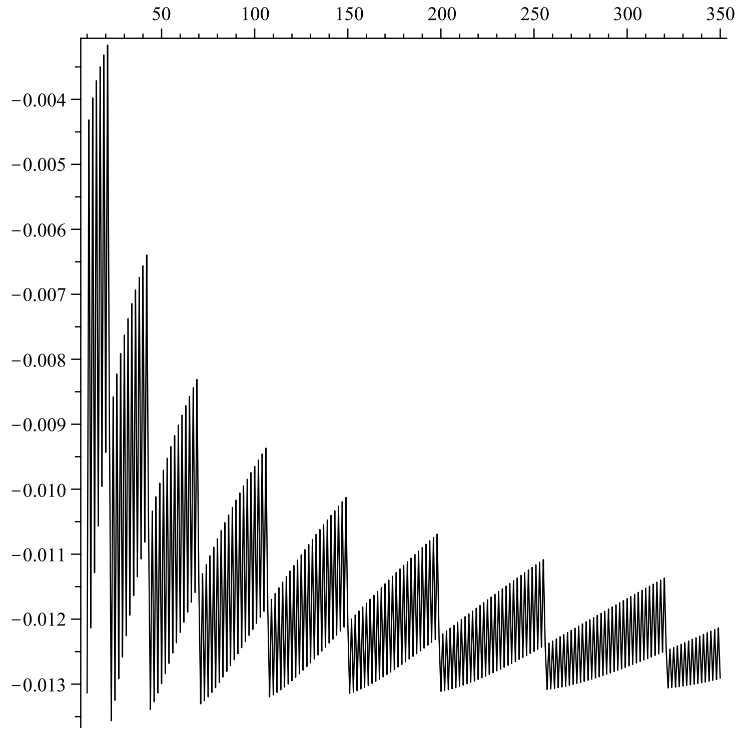

FIGURE 1. The quantity $n^{\frac{5}{2}} Err_T^n(K)$ remains bounded.

REFERENCES

[1] Chang, L.B. and Palmer, K., 2007. *Smooth convergence in the binomial model*, Finance and Stochastics **11** no. 1, 91–105.

[2] Diener, F. and Diener, M., 2004. *Asymptotics of the price oscillations of a European call option in a tree model*, Mathematical finance **14** no. 2, 271–293.

[3] Diener, F. and Diener, M., 2005. *Higher-order terms for the de Moivre-Laplace theorem*, Contemporary Mathematics **373** 191–206.

[4] Joshi, M.S., 2010. *Achieving higher order convergence for the prices of European options in binomial trees*, Mathematical Finance **20** no. 1, 89–103.

[5] R. Korn and S. Müller, 2012. *The optimal-drift model: an accelerated binomial scheme*, Finance and Stochastics, 1–26.

[6] Walsh, J.B., 2003. *The rate of convergence of the binomial tree scheme*, Finance and Stochastics **7** no. 3, 337–361.

[7] Xiao, X., 2010. *Improving speed of convergence for the prices of European options in binomial trees with even numbers of steps*, Applied Mathematics and Computation **216**, no. 9, 2659–2670.

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