A proxy approach to dealing with the infeasibility problem in super-efficiency data envelopment analysis

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June 2012

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Abstract

Super-efficiency data envelopment analysis (SE-DEA) models are expressions of the traditional DEA models featuring the exclusion of the unit under evaluation from the reference set. The SE-DEA models have been applied in various cases such as sensitivity and stability analysis, measurement of productivity changes , outliers' identification , and classification and ranking of decision making units (DMUs). A major deficiency in the SE-DEA models is their infeasibility in determining super-efficiency scores for some efficient DMUs when variable, non-increasing and non-decreasing returns to scale (VRS, NIRS, NDRS) prevail. The scope of this study is the development of an oriented proxy approach for SE-DEA models in order to tackle the infeasibility problem. The proxy introduced to the SE-DEA models replaces the original infeasible DMU in the sample and guarantees a feasible optimal solution. The proxy approach yields the same scores as the traditional SE-DEA models to the feasible DMUs.

Keywords: Data envelopment analysis (DEA); Super-efficiency (SE); Infeasibility; Orientation

1. Introduction

Data Envelopment Analysis (DEA) is a comparative efficiency measurement methodology put forth by Charnes et al. (1978) that serves as a quantitative benchmarking technique. DEA draws on linear programming for distinguishing the relatively efficient from the inefficient operational units of a particular sample. Nevertheless, the distinction between efficient and inefficient units is not the only present as there are further dissimilarities in the production process of the efficient units. These dissimilarities are not detected by traditional DEA models.

Super-efficiency DEA (SE-DEA) models, initially developed by Banker et al. (1989), and Andersen and Petersen (1993), are appropriate for identifying premium efficiency among efficient units and ranking efficient DMUs. In the SE-DEA, the unit under evaluation is excluded from the reference set, so that its efficiency may be greater than 100%.

A major drawback of the SE-DEA models is their infeasibility in defining super-efficiency scores for some efficient DMUs under VRS technology. Several scholars (Dula & Hickman, 1997; Seiford & Zhu, 1999; Xue & Harker, 2002) discussed the conditions for infeasibility in SE-DEA models under VRS. Dula and Hickman (1997) and Seiford and Zhu (1999) proved the necessary and sufficient conditions for infeasibility in the VRS SE-DEA model. Taking into account these conditions, a number of methods have been developed to solve the infeasibility problem(Chen et al., 2011; Chen, 2005; Cook et al., 2009; Lee et al., 2011; Lovell & Rouse, 2003; Ray, 2008).

In this paper, we propose a new proxy approach which successfully overcomes the infeasibility problem. The novelty of the new approach is that it completely holds the original orientation of the SE-DEA model (input-orientation or output-orientation) by identifying a virtual proxy unit in the frontier. The proxy unit is located at the nearest point to the original infeasible efficient unit and it has a feasible super-efficiency score.

The paper is organized as follows. Section 2 describes the infeasibility problem in input- and output-oriented models. Section 3 presents existing VRS super-efficiency models and discusses both the procedure applied for overcoming the infeasibility problem and the appropriateness of the results of these models, in order to provide a basis for comparison between the existing models and the new approach presented in this paper. Section 4 analyses the proposed approach. Section 5 compares alternative approaches for tackling the infeasibility problem through a numerical example. The numerical example is based on a real-world dataset found in Bal et al. (2010). Conclusions are presented in the final section of the paper.

2. Infeasibility problem for SE-VRS model

2.1 Infeasibility for input-oriented SE-VRS model

The input-oriented VRS model for the evaluated DMU_k can be formulated as (Banker et al., 1984):

$$\begin{aligned}
&\min \theta \\
&\text{s.t.} \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{ik}, \quad i = 1, 2, \ldots, m \\
&\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rk}, \quad r = 1, 2, \ldots, s \\
&\sum_{j=1}^{n} \lambda_j = 1 \\
&\lambda_j \geq 0, \quad j = 1, 2, \ldots, n
\end{aligned} \tag{1}$$

For an efficient DMU_k, the SE-VRS model becomes (Andersen & Petersen, 1993):

$$\begin{aligned}
&\min \theta \\
&\text{s.t.} \sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j x_{ij} \leq \theta x_{ik}, \quad i = 1, 2, \ldots, m \\
&\sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j y_{rj} \geq y_{rk}, \quad r = 1, 2, \ldots, s \\
&\sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j = 1 \\
&\lambda_j \geq 0, \quad j = 1, 2, \ldots, n \quad (j \neq k)
\end{aligned} \tag{2}$$

The necessary and sufficient condition for infeasibility in the input-oriented VRS SE-DEA is that the evaluated DMU has at least one output greater than the convex combination formed by all the other DMUs. In such a condition, the efficient DMU_k cannot reach the frontier formed by the remaining DMUs because the constraint for outputs in (2) is infeasible, i.e. $\sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j y_{rj} \geq y_{rk}$ is infeasible.

A sufficient condition for infeasibility in the input-oriented VRS SE-DEA is that the evaluated DMU has at least one output greater than the corresponding output for all the other DMUs.

2.2 Infeasibility for output-oriented SE-VRS model

The output-oriented VRS model can be formulated as:

$$\max \varphi$$

$$s.t. \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{ik}, \quad i = 1, 2, \ldots, m$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} \geq \varphi y_{rk}, \quad r = 1, 2, \ldots, s$$

$$\sum_{j=1}^{n} \lambda_j = 1$$

$$\lambda_j \geq 0, \quad j = 1, 2, \ldots, n \qquad (3)$$

For an efficient DMU_k, the SE-VRS model is:

$$\max \varphi$$

$$s.t. \sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j x_{ij} \leq x_{ik}, \quad i = 1, 2, \ldots, m$$

$$\sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j y_{rj} \geq \varphi y_{rk}, \quad r = 1, 2, \ldots, s$$

$$\sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j = 1$$

$$\lambda_j \geq 0, \quad j = 1, 2, \ldots, n \ (j \neq k) \qquad (4)$$

The necessary and sufficient condition for infeasibility in the output-oriented VRS SE-DEA is that the evaluated DMU has at least one input less than the convex combination formed by all the other DMUs. In such a condition, the efficient DMU_k cannot reach the frontier formed by the rest of the DMUs because the constraint for inputs in (4) is infeasible, i.e. $\sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j x_{ij} \leq x_{ik}$ is infeasible.

A sufficient condition for infeasibility in the output-oriented VRS SE-DEA is that the evaluated DMU has at least one input less than the corresponding input for all the other DMUs.

3. Modified SE-DEA models dealing with infeasibility

Lovell and Rouse (2003) proposed an oriented method for tackling the infeasibility problem of traditional SE-DEA models. This method draws on a scaling procedure applied either to the inputs (input orientation) or the outputs (output orientation) of the efficient units for which the calculation of a super-efficiency score, based on traditional SE-DEA models, is infeasible. For the scaling procedure, an arbitrarily selected factor that is sufficiently large (input orientation), or a sufficiently small factor (output orientation) is utilized. The scaling procedure removes the unit from the reference set to avoid any infeasibility problem. The super-efficiency score of the modified unit is calculated after rescaling the assigned score.

The method introduced by Lovell and Rouse (2003) copes with the infeasibility problem. However, concerns are raised about the desirability of the results of this method and the role of exogenous intervention to the procedure (Chen et al., 2011; Cook et al., 2009; Ray, 2008). To be more precise, the super-efficiency scores of the efficient DMUs with infeasible solutions are identical to the scaling factor. Therefore, these particular results should not be interpreted while the target levels for inputs and outputs are fictitious. The results obtained solely reflect an arbitrary choice of the scaling factor. In addition, Lovell and Rouse's method fails to classify the efficient units in that the infeasible DMUs are assigned equal super-efficiency scores.

Chen (2005)'s model relies on the substitution of the inefficient units with their efficient projections, under the assumption of variable returns to scale (VRS). Chen argues that infeasibility is eliminated either in the input- or the output-oriented expression of super-efficiency models, though, not in both simultaneously. As a result, both orientations should be applied to tackle the infeasibility problem and calculate the overall super-efficiency score of a unit. The overall super-efficiency score derives from the combination of the two SE-DEA orientations with suitable weights. Chen's method provides partial solution to the infeasibility problem of SE-DEA models because in some cases it fails to define a feasible solution in both orientations (Chen et al., 2011; Ray, 2008).

Cook et al. (2009) introduced an approach which proposes one-directional input-output movements (i.e. decreases when input-orientation is applied, and increases in case of output-orientation) so that the unit under evaluation that experiences infeasibility in super-efficiency models reaches the frontier formed by the rest of DMUs. Lee et al. (2011) extended Cook et al.'s method by introducing a two-stage method to achieve Cook et al.'s solution.

In addition to the above oriented solutions for infeasibility, Ray (2008) put forth a non-oriented super-efficiency model drawing on the directional distance function introduced by Chambers et al. (1996). Ray's approach allows synchronous proportional output reductions and input expansions by an unrestricted factor which is determined by the optimization procedure. Despite this particular method resolving the infeasibility problem, it is not an oriented analysis.

Chen et al. (2011) proposed a combinatorial input- and output-oriented method that provides targets for the evaluated DMU with radial movements of both inputs and outputs. The aggregated super-efficiency score is defined as a ratio of optimal input- and output-oriented super-efficiency components. Hence, it is the result of an optimization procedure without requiring arbitrary selections on a factor. Chen et al., as Ray, introduce a non-oriented analysis for tackling the infeasibility problem at VRS SE-DEA models.

4 A proxy approach to dealing with infeasibility of SE-VRS model

4.1 A proxy approach to input-oriented SE-VRS model

As discussed in the previous section, the essential reason for the infeasibility in the input-oriented SE-VRS model is that the efficient DMU_k does not belong to the output set S^y formed by the remaining DMUs.

$$y_k \notin S^y = \{ y : y \le \sum_{\substack{j=1 \\ j \ne k}}^{n} \lambda_j y_{rj}, \sum_{\substack{j=1 \\ j \ne k}}^{n} \lambda_j = 1 \} \quad (5)$$

The concept of the proxy approach is to find a virtual proxy unit for the efficient $DMU_{k.}$ The proxy of the DMU_k (x_k,y_k) is indicated by $DMU_{k'}$ $(x_{k'},y_{k'})$.The $DMU_{k'}$ is the nearest point to DMU_k at the frontier, and its outputs $y_{k'}$ belong to the output set S^y.

The process applied for determining the proxy of the efficient DMU_k has two steps. In the first step, an intermediate $DMU_{k''}$ $(x_{k''},y_{k''})$ is defined. The intermediation process is expressed by a vertical movement from point K to K'' in Fig. 1, or, a scaling down of the output levels of DMU_k holding the inputs fixed. In this context, the first step of the proxy approach can be written as follows

$$\min \beta$$

$$s.t. \ (1-\beta) y_{rk} \le \sum_{\substack{j=1 \\ j \ne k}}^{n} \lambda_j y_{rj}$$

$$\sum_{\substack{j=1 \\ j \ne k}}^{n} \lambda_j = 1$$

$$\lambda_j, \beta \ge 0 \quad (6)$$

The inputs and outputs of the intermediate $DMU_{k''}$ are defined by

$$x_{ik''} = x_{ik}, \quad y_{rk''} = (1-\beta) y_{rk}.$$

Having already identified the intermediate $DMU_{k''}$, we solve the following linear programming in order to determine the inputs and outputs of the proxy $DMU_{k'}$ in a second step

$$\max \alpha$$

$$s.t. \sum_{j=1}^{n} \lambda_j x_{ij} \le (1-\alpha) x_{ik"}, \quad i = 1, 2, \ldots, m$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{rk"}, \quad r = 1, 2, \ldots, s$$

$$\sum_{j=1}^{n} \lambda_j = 1$$

$$\lambda_j \ge 0, \quad j = 1, 2, \ldots, n \quad (9)$$

The inputs and outputs of the proxy $DMU_{k'}$ are defined as follows

$$x_{ik'} = (1-\alpha) x_{ik"}, \quad y_{rk'} = y_{rk"}.$$

The above discussion is expressed graphically by the horizontal movement from point K" to K' in Fig. 1.

If the efficient DMU_k is feasible in the traditional SE-DEA model, its proxy $DMU_{k'}$ will be the same point as DMU_k, i.e., there are neither vertical nor horizontal movements in the above two steps.

At last, by replacing the original DMU_k with its proxy unit $DMU_{k'}$ in the sample and by solving the following super-efficiency model we define a feasible super-efficiency score for every efficient DMU

$$\min \theta$$

$$s.t. \sum_{\substack{j=1 \\ j \ne k'}}^{n} \lambda_j x_{ij} \le \theta x_{ik'}, \quad i = 1, 2, \ldots, m$$

$$\sum_{\substack{j=1 \\ j \ne k'}}^{n} \lambda_j y_{rj} \ge y_{rk'}, \quad r = 1, 2, \ldots, s \qquad (10)$$

$$\sum_{\substack{j=1 \\ j \ne k'}}^{n} \lambda_j = 1$$

$$\lambda_j \ge 0, \quad j = 1, 2, \ldots, n \quad (j \ne k')$$

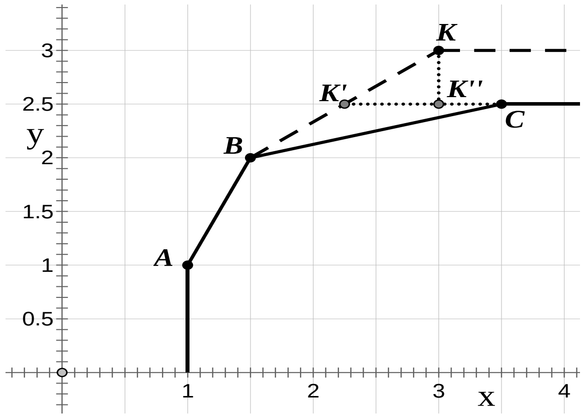

Figure 1. Proxy approach to infeasibility in input-oriented SE-DEA under VRS

4.2 A proxy approach to output-oriented SE-VRS model

In the output-oriented VRS SE-DEA model, infeasibility is present in case an efficient DMU_k does not belong to the input set S^x determined by the rest of DMUs.

$$x_k \notin S^x = \{x : x \geq \sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j x_{rj}, \sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j = 1\} \qquad (11)$$

Similarly, a proxy of DMU_k is defined, as indicated by $DMU_{k'}$, whose inputs belong to the input set S^x.

In the first step, an intermediate $DMU_{k''}$ $(x_{k''}, y_{k''})$ is identified after scaling up the inputs of DMU_k holding the outputs fixed. As illustrated in Fig. 2, the intermediate $DMU_{k''}$ is determined after a rightward shift from K to K'' so that the inputs of the intermediate $DMU_{k''}$ to be identical to the lowest input level of the reference set. To achieve this, we solve the following linear programming model

$$\min \alpha$$

$$s.t. \quad (1+\alpha) x_{ik} \geq \sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j x_{ij}$$

$$\sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_j = 1$$

$$\lambda_j, \alpha \geq 0 \qquad (12)$$

The inputs and outputs of the intermediate DMU$_{k''}$ are defined by

$$x_{ik''} = (1+\alpha)x_{ik}, \quad y_{rk''} = y_{rk} .$$

In the second step, the proxy unit DMU$_{k'}$ is identified by projecting the intermediate DMU$_{k''}$ to the original frontier with the following programming

$$\max \beta$$

$$s.t. \quad \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{ik''}, \quad i = 1, 2, \ldots, m$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} \geq (1+\beta) y_{rk''}, \; r = 1, 2, \ldots, s$$

$$\sum_{j=1}^{n} \lambda_j = 1$$

$$\lambda_j \geq 0, \; j = 1, 2, \ldots, n \qquad (13)$$

The above procedure is depicted by the upward movement from K'' to K' in Fig. 2.

The inputs and outputs of the proxy DMU$_{k'}$ are defined by

$$x_{ik'} = x_{ik''}, \quad y_{rk'} = (1+\beta) y_{rk''}.$$

Like the input-oriented proxy approach, DMU$_{k'}$ replaces DMU$_k$ in the sample and is evaluated against the super-efficiency reference set

$$\max \varphi$$

$$s.t. \quad \sum_{\substack{j=1\\ j\neq k'}}^{n} \lambda_j x_{ij} \leq x_{ik'}, \quad i = 1, 2, \ldots, m$$

$$\sum_{\substack{j=1 \\ j \neq k'}}^{n} \lambda_j y_{rj} \geq \varphi y_{rk'}, \ \ r = 1, 2, \ldots, s$$

$$\sum_{\substack{j=1 \\ j \neq k'}}^{n} \lambda_j = 1$$

$$\lambda_j \geq 0, \ \ j = 1, 2, \ldots, n \ \ (j \neq k') \qquad (14)$$

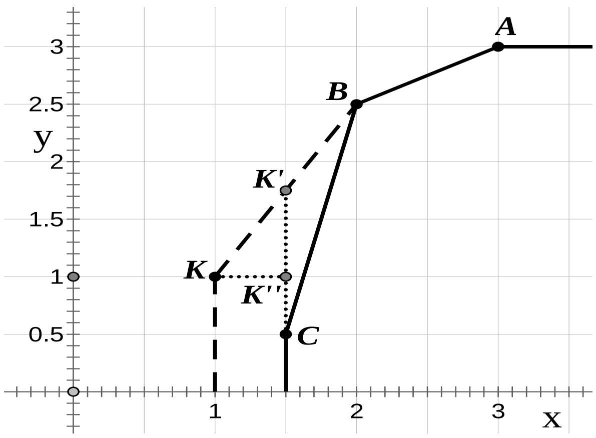

Figure 2. Proxy approach to infeasibility in output-oriented SE-DEA under VRS

5. Illustrative example

In Section 4, we apply our approach to a dataset used in Bal et al. (2010) (see Appendix 1). This dataset consists of 30 OECD countries that utilize three inputs (Input 1: unemployment ratio (2006), Input 2: rate of inflation (2005), and Input 3: infant mortality (2005)) in order to generate five outputs (Output 1: national income per capita (US dollars, 2006), Output 2: human development index: life expectancy from birth (2006), Output 3: human development index: education index (2006), Output 4: contribution rate to labor force of female population (2006), and Output 5: health expenditure per capita (US dollars, 2005)).

Table 1. Super-efficiency measures

DMU	Traditional SE-DEA		Lovell & Rouse		Ray	Chen et al.			Proxy Approach	
	Input-oriented	Output-oriented	Input-oriented	Output-oriented	Non-oriented			Non-oriented	Input-oriented	Output-oriented
	θ	φ	θ	φ	ψ	θ	φ	ρ	θ	φ
1	Infeasible	0.9952	16.2222	0.9952	1.0048	1.0000	0.9952	1.0049	**1.4385**	0.9952
2	0.6707	1.0165	0.6707	1.0165	0.9837	0.6707	1.0000	0.6707	0.6707	1.0165
3	0.7347	1.0147	0.7347	1.0147	0.9853	0.7347	1.0000	0.7347	0.7347	1.0147
4	Infeasible	0.9991	16.2222	0.9991	1.0009	1.0000	0.9991	1.0009	**1.1114**	0.9991
5	0.6000	1.0572	0.6000	1.0572	0.9433	0.6000	1.0154	0.5909	0.6000	1.0572
6	Infeasible	0.9985	16.2222	0.9985	1.0015	1.0000	0.9985	1.0015	**1.1451**	0.9985
7	1.1550	Infeasible	1.1550	0.0575	1.1550	1.1550	1.0000	1.1550	1.1550	**0.9984**
8	Infeasible	0.9987	16.2222	0.9987	1.0013	1.0000	0.9987	1.0013	**1.4826**	0.9987
9	0.8732	1.0023	0.8732	1.0023	0.9978	0.8732	1.0000	0.8732	0.8732	1.0023
10	0.6000	1.0220	0.6000	1.0220	0.9781	0.6000	1.0042	0.5975	0.6000	1.0220
11	0.6000	1.0210	0.6000	1.0210	0.9793	0.6000	1.0082	0.5951	0.6000	1.0210
12	0.4281	1.0365	0.4281	1.0365	0.9635	0.4288	1.0031	0.4275	0.4281	1.0365
13	Infeasible	Infeasible	16.2222	0.0575	1.5556	1.7290	0.8790	1.9670	**1.9444**	**0.6947**
14	Infeasible	0.9992	16.2222	0.9992	1.0008	1.0000	0.9992	1.0008	**1.2780**	0.9992
15	0.5253	1.0153	0.5253	1.0153	0.9847	0.5253	1.0000	0.5253	0.5253	1.0153
16	Infeasible	0.9879	16.2222	0.9879	1.0121	1.0000	0.9879	1.0122	**5.2174**	0.9879
17	Infeasible	0.7145	16.2222	0.7145	1.2005	1.0000	0.7145	1.3997	**1.2642**	0.7145
18	0.6392	1.0854	0.6392	1.0854	0.9159	0.6392	1.0615	0.6022	0.6392	1.0854
19	Infeasible	0.9984	16.2222	0.9984	1.0016	1.0000	0.9984	1.0016	**1.7876**	0.9984
20	0.7863	1.0042	0.7863	1.0042	0.9958	0.7863	1.0000	0.7863	0.7863	1.0042
21	Infeasible	0.7924	16.2222	0.7924	1.0831	1.0554	0.9162	1.1519	**2.6922**	0.7924
22	0.4971	1.0442	0.4971	1.0442	0.9558	0.4971	1.0000	0.4971	0.4971	1.0442
23	0.5000	1.0505	0.5000	1.0505	0.9496	0.5000	1.0227	0.4889	0.5000	1.0505
24	0.8006	1.0390	0.8006	1.0390	0.9615	0.8006	1.0231	0.7825	0.8006	1.0390
25	0.3750	1.0782	0.3750	1.0782	0.9218	0.3750	1.0362	0.3619	0.3750	1.0782

26	0.8540	1.0024	0.8540	1.0024	0.9976	0.8540	1.0000	0.8540	0.8540	1.0024
27	1.2573	0.9673	1.2573	0.9673	1.0286	1.0000	0.9673	1.0338	1.2573	0.9673
28	Infeasible	Infeasible	16.2222	0.0575	1.3053	1.5485	0.9765	1.5856	**4.1306**	**0.6485**
29	0.2263	1.1527	0.2263	1.1527	0.8473	0.2263	1.1229	0.2016	0.2263	1.1527
30	Infeasible	0.6687	16.2222	0.6687	1.3313	1.0000	0.6687	1.4953	**1.0227**	0.6687

In Table 1, the second and third columns report super-efficiency scores measured by the traditional input- and output-oriented SE-DEA models (2) and (4), respectively. Columns four and five represent input- and output-oriented SE-DEA scores obtained by Lovell and Rouse (2003)'s measure. The next column presents super-efficiency scores according to Ray (2008)'s approach. Columns seven to nine report the movements of inputs and outputs and the super-efficiency scores, as defined by Chen et al. (2011)'s measure. The final two columns illustrate the input- and output-oriented super-efficiency scores yielded by the proxy approach.

The new approach successfully overcomes the infeasibility problem of the traditional SE-DEA method in both orientations and its results are fully consistent with those of the traditional method for feasible DMUs. The super-efficiency scores assigned to the infeasible DMUs by the proxy SE-DEA model are displayed in bold numbers in the last two columns of Table 1. The new proxy approach provides differentiated scores for every DMU enabling their ranking. To be more precise, when input orientation is selected, the most efficient DMU among the thirty counties of the sample is Japan (DMU 16) which obtains 5.2174, followed by Switzerland (DMU 28) with 4.1306, and Norway (DMU 21) with 2.6922. When output orientation is applied, the most efficient country is Switzerland (DMU 28), receiving a score of 0.6485, followed by the United States (DMU 30) and Iceland (DMU 13), obtaining scores of 0.6687 and 0.6947, respectively.

The United States (DMU 30), which is a feasible DMU in the output-oriented traditional SE-DEA method, is ranked No. 2, above the infeasible DMU 13 (i.e., Iceland). In addition, there are 5 feasible DMUs which are ranked higher than the infeasible DMU 7 (i.e., England). Such cases can also be found in the results of the input-oriented proxy model. This reveals that infeasibility under the traditional SE-DEA models does not always mean extreme super-efficiency. A similar conclusion is deduced by Ray's and Chen et al.'s measures.

The input- and output-oriented proxy approach yields completely consistent super-efficiency scores with the respective traditional SE-DEA models for every feasible unit. As a result, the new approach provides rankings identical to that obtained by the traditional measures for the feasible DMUs.

Lovell and Rouse (2003) method eliminates infeasibility but fails to provide an ordering procedure for the DMUs deemed infeasible by the traditional SE-DEA models. For instance, under input-oriented Lovell and Rouse's measure, units that are deemed infeasible, according to conventional SE-DEA measures, obtain a unique score of 16.2222; and under the respective output-oriented measure, the three infeasible units are assigned a score of 0.0575. Both scores reflect the scaling factor that is arbitrarily decided rather than the results of the super-efficiency assessment process. Therefore, the obtained scores for the traditionally infeasible efficient DMUs are unlikely to be interpreted. Additionally, this method yields consistent results for the DMUs that are regarded as feasible by the traditional SE-DEA method. This is just because those feasible DMUs are actually not involved in the scaling procedure. Essentially, Lovell and Rouse's method is applied exclusively to the infeasible DMUs.

Ray (2008) measure has a twofold interpretation referring both to inputs and outputs. For instance, England obtains a super-efficiency score of 1.1550, which denotes that the inputs of this country can be increased by 15.5% and its outputs reduced by 15.5% without affecting its efficiency status. Acknowledging that this method is non-oriented, it is not desirable to compare its results with the traditional SE-DEA and the Lovell & Rouse's measures, which are oriented.

Similar to Ray's measure, Chen et al. (2011) developed a non-oriented method which defines the super-efficiency score (ρ) as a ratio of the input change (θ) to the output change (φ). Drawing on the results obtained by Chen et al.'s method, Switzerland (DMU 28) is ranked second, receiving an overall super-efficiency score (ρ) of 1.5856. By decomposing the super-efficiency score, we find that Switzerland will remain efficient by scaling up its inputs by 54.85% and simultaneously scaling down its outputs by 2.35%. The results yielded by Chen et al.'s method are not comparable with those of the traditional SE-DEA models due to the incompatible orientation concepts that underlie the two approaches.

The super-efficiency scores displayed in Table 1 are evidence of the incompatibility of Ray's and Chen et al.'s measures with the traditional SE-DEA models.

Table 2. Step-by-step results of the proxy approach

DMU	Input Oriented			Output Oriented		
	Step 1	Step 2	Proxy Approach	Step 1	Step 2	Proxy Approach
	Beta	Alpha	θ	Alpha	Beta	φ
1	0.004547	0.204935	**1.4385**	0	0	0.9952
2	N/A	N/A	0.6707	N/A	N/A	1.0165
3	N/A	N/A	0.7347	N/A	N/A	1.0147
4	0.000912	0.100621	**1.1114**	0	0	0.9991
5	N/A	N/A	0.6000	N/A	N/A	1.0572
6	0.001199	0.016073	**1.1451**	0	0	0.9985
7	0	0	1.1550	0.154952	0.01513	**0.9984**
8	0.000055	0.000702	**1.4826**	0	0	0.9987
9	N/A	N/A	0.8732	N/A	N/A	1.0023
10	N/A	N/A	0.6000	N/A	N/A	1.022
11	N/A	N/A	0.6000	N/A	N/A	1.021
12	N/A	N/A	0.4281	N/A	N/A	1.0365
13	0.067551	0	**1.9444**	0.555556	0	**0.6947**
14	0.000224	0.031724	**1.2780**	0	0	0.9992
15	N/A	N/A	0.5253	N/A	0.015303	1.0153
16	0.009721	0.08	**5.2174**	0	0	0.9879
17	0.200466	0.065156	**1.2642**	0	0	0.7145
18	N/A	N/A	0.6392	N/A	N/A	1.0854
19	0.000223	0.007139	**1.7876**	0	0	0.9984
20	N/A	N/A	0.7863	N/A	N/A	1.0042
21	0.047005	0.003766	**2.6922**	0	0	0.7924
22	N/A	N/A	0.4971	N/A	N/A	1.0442
23	N/A	N/A	0.5000	N/A	N/A	1.0505

24	N/A	N/A	0.8006	N/A	N/A	1.039
25	N/A	N/A	0.3750	N/A	N/A	1.0782
26	N/A	N/A	0.8540	N/A	N/A	1.0024
27	0	0	1.2573	0	0	0.9673
28	0.012261	0	**4.1306**	0.288889	0	**0.6485**
29	N/A	N/A	0.2263	N/A	N/A	1.1527
30	0.331259	0.271462	1.0227	0	0	0.6687

Detailed results of the proxy approach are presented in Table 2. In particular, columns two and three demonstrate the movements of outputs (betas) and inputs (alphas) of both the infeasible and the feasible efficient DMUs to their proxies when the input-oriented proxy SE-DEA model is applied. The columns five and six display the movements of inputs (alphas) and outputs (betas) to their proxies when the output-oriented proxy model is utilized. Note that there are no movements for the feasible efficient DMUs, which means that the proxies of the feasible efficient DMUs are themselves. The infeasible DMUs, as defined by the traditional SE-DEA models, are displayed in bold numbers in Table 2.

Taking an example in the input-oriented proxy approach, Japan (DMU 16) is originally deemed infeasible by the traditional input-oriented SE-DEA model. If it decreases its outputs by 0.97% and scales down its inputs by 8%, its proxy can obtain a feasible super-efficiency score of 5.2174. For feasible efficient DMUs, i.e., England (DMU 7) and Sweden (DMU 27), there is no need for input and output adjustments. In such a case, the proxy unit is the same as the original one, and the proxy approach will yield consistent super-efficiency scores as the traditional input-oriented SE-DEA model.

Turning to an example in the output-oriented proxy approach, Switzerland (DMU 28) should scale up its inputs by 28.89% without adjusting its output levels to find its proxy, and the proxy can get a feasible super-efficiency score of 0.6485.

6. Conclusions

The current paper deals with the infeasibility problem that is present in traditional VRS SE-DEA models. Our approach holds the original orientation of the SE-DEA model and identifies an optimal virtual proxy unit that replaces the original infeasible DMU in the evaluation process. The proxy unit is defined by applying a two-stage procedure which secures that the proxy unit is an optimal derivative of the original unit. By applying the proposed method, 1) The proxy approach can yield a super-efficiency score in cases where the traditional super-efficiency model is infeasible; and 2) The proxy approach yields the same results as the traditional super-efficiency model when it is applied to cases where the traditional super-efficiency model is feasible. The properties of the proposed approach are presented in a numerical example. Utilizing a dataset found in Bal et al. (2010), we demonstrate the advantages of the proxy approach over some existing methods developed for tackling the infeasibility problem.

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Appendix 1

Table A1. Input and output data of 30 OECD countries

DMU No	Countries	Input1	Input2	Input3	Output1	Output2	Output3	Output4	Output5
DMU01	Australia	5.1	3	6	34740	80.9	0.993	67.4	2036
DMU02	Austria	7.2	1.8	5	37117	79.4	0.966	63.8	1968
DMU03	Belgium	12.1	1.6	6	35712	78.8	0.977	57.3	2081
DMU04	Canada	6.8	2.2	6	35133	80.3	0.991	72.8	2312
DMU05	Czech Republic	8.9	1.8	5	12152	75.9	0.936	64	930
DMU06	Denmark	5.6	2.4	4	47984	77.9	0.993	74.2	2133
DMU07	England	2.8	1.6	6	37023	79	0.97	69.3	1461
DMU08	Finland	8.4	1.7	4	37504	78.9	0.993	72.8	1502

DMU09	France	9.1	1.9	4	33918	80.2	0.982	62.4	2055
DMU10	Germany	9.2	2.3	5	33854	79.1	0.953	67.4	2424
DMU11	Greece	9.9	4.6	5	20327	78.9	0.97	56	1167
DMU12	Hungary	7.2	5.3	8	10814	72.9	0.958	53.5	705
DMU13	Iceland	1.8	4.8	4	52764	81.5	0.978	82.9	2103
DMU14	Ireland	4.3	4.7	6	48604	78.4	0.993	62.2	1436
DMU15	Italy	7.7	2.5	6	30200	80.3	0.958	50.1	1783
DMU16	Japan	4.4	1	4	35757	82.3	0.946	60.5	1822
DMU17	Luxembourg	4.2	1.1	5	80288	78.4	0.942	55.7	2215
DMU18	Mexico	3.6	5	25	7298	75.6	0.863	42.6	356
DMU19	New Zealand	3.7	2.7	6	26464	79.8	0.993	71.2	1424
DMU20	Netherlands	4.3	3.5	5	38618	79.2	0.988	69.5	2070
DMU21	Norway	3.5	1.3	4	64193	79.8	0.991	77.3	2330
DMU22	Poland	18.2	1.9	9	7946	75.2	0.951	57.6	496
DMU23	Portugal	7.6	3.5	6	17456	77.7	0.925	67.8	1237
DMU24	South Korea	3.7	2.8	5	16308	79	0.904	49.9	730
DMU25	Slovak Republic	11.7	3.3	8	8775	74.2	0.921	62.4	930
DMU26	Spain	9.2	3.1	5	27226	80.5	0.987	57.2	1218
DMU27	Sweden	5.8	2.2	3	39694	80.5	0.978	74.9	1746
DMU28	Switzerland	3.8	0.9	3	50532	81.3	0.946	75.3	2794
DMU29	Turkey	10.3	13.7	38	5816	71.4	0.812	26.5	255
DMU30	USA	5.1	1.6	7	42000	77.9	0.971	70.1	4178