The existence of equilibrium without fixed-point arguments

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29. March 2010

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Abstract

This paper gives a proof of the existence of general equilibrium without the use of a fixed point theorem. Unlike other results of this type, the conditions we use do not imply that the set of equilibrium prices is convex. We use an assumption on the excess demand correspondence that is related to, but weaker than, the weak axiom of revealed preference (WARP). The proof is carried out for compact and convex valued upper hemicontinuous excess demand correspondences satisfying this WARP-related condition and some other standard conditions. We also provide an algorithm for finding equilibrium prices.

KEYWORDS: existence of economic equilibrium, the weak axiom of revealed preference, excess demand correspondence, distribution economies, law of demand.

JEL CLASSIFICATION NUMBERS: C62; D50

☆I would like to thank an anonymous referee for comments and hints. In particular, the examples below assumption 4 and the algorithm for the computation of equilibrium prices are strongly based on the referee's remarks. All remaining errors are mine.

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1. Introduction

It is well-known that under standard convexity, continuity, and monotonicity assumptions on preferences, one may prove the existence of equilibrium using Kakutani's fixed point theorem. In a series of papers (Barbolla and Corchon (1989), Fraysse (2009), Greenberg (1977), John (1999), Quah (2008)) efforts were undertaken to deliver proofs of existence of economic equilibrium without the use of Kakutani's fixed point theorem. But this can only be achieved at the price of generality: the above-mentioned papers assume that the economy's aggregate excess demand function (or correspondence) satisfies a version of gross substitutability or the weak axiom of revealed preference (WARP).

A central feature of the assumptions in these papers is that they guarantee convexity of the equilibrium price set (see Arrow and Hahn (1971) pages 222 and 232, John (1998), Mas-Colell et al. (1995), p. 607, and in the case of multivalued excess demand - see the corollary after Lemma 2 in this paper). We show that it is possible to avoid convexity of the equilibrium price set and still use 'elementary' tools for proving the existence of equilibrium. However, the tools should not be too elementary because the existence of economic equilibrium under standard assumptions is equivalent to Brouwer's fixed point theorem (Toda (2006)) and if one reaches too far, then either one is wrong or a new proof of Brouwer's/Kakutani's fixed point theorem is delivered.

The crucial assumption we make on the excess demand correspondence (Assumption 4) is a weakening of the WARP assumption made by Quah (2008). Let p and p' be two price vectors such that the price of the last good is the same; formally $p_n = p'_n$. For such price pairs, our assumption states that if a bundle of goods y is an excess demand at price p and is *just* affordable at price p' (i.e., $p'y = 0$), then every bundle y' from the excess demand set at price p' is either not affordable at price p or just affordable (i.e., $py' \geq 0$). Our condition weakens the WARP assumption used in Quah (2008) in two ways: the first is that we only require the WARP condition to hold for price vector pairs in which the last good has the same price; the second is that we only require $py' \geq 0$ when $p'y = 0$ but not necessarily when $p'y < 0$.

There are economies in which WARP may fail but where our weaker version of WARP is satisfied. For example, it is well-known that WARP holds for the excess demand function of an exchange economy in which endowments are collinear and all agents have demand functions obeying the law of demand; we show that our weaker version of WARP allows for a weakening of the collinearity assumption.

Our proof of equilibrium existence (Theorem 1) uses induction and relies heavily on the connectedness of the unit interval. Our approach is a kind of generalization of the proof of Lemma 4.1 in John (1999) which, according to John (in the same paper), was employed by Wald in his proof of the existence of a competitive equilibrium. We also need a version of the separating hyperplane theorem to prove an intermediate step (Lemma 1). While the set of equilibrium prices is not necessarily convex under our assumptions, this 'convexity feature' is partially preserved. Specifically, we show in Lemma 2 that Assumption 4 guarantees the convexity of the equilibrium price set of a lower-dimensional excess demand correspondence. Finally, we present an algorithm for computing equilibrium prices under a strengthened version of Assumption 4.

2. Notation

In what follows $[0,1] \subset \mathbf{R}$ is unit interval of the real line and $\mathbf{R}_{++}$ denotes the set of positive real numbers. For vectors $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, $y = (y_1, \ldots, y_n) \in \mathbf{R}^n$ we write $x \geq y$, when $x_i \geq y_i$, $i = 1, \ldots, n$; $x < y$ is for strict component-wise inequalities: $x_i < y_i$, $i = 1, \ldots, n$. For all $x, y \in \mathbf{R}^n$, $xy := \sum_{i=1}^{n} x_i y_i$. For $x \in \mathbf{R}^n$ and $A \subset \mathbf{R}^n$ $xA = \{xy : y \in A\}$ and $xA \leq 0$ means $\forall y \in A : xy \leq 0$ and $xA = 0$ is equivalent to $\forall y \in A : xy = 0$. If $A, B \subset \mathbf{R}^n$, then $A + B := \{x + y : x \in A, y \in B\}$. $S \subset \mathbf{R}^n$ is open standard (price) simplex of dimension $n - 1$:

$$S^{n-1} := \{(p_1, \ldots, p_n) \in \mathbf{R}^n_{++} : p_1 + \ldots + p_n = 1\}.$$

For a fixed number $q \in (0,1)$ we define the section of S^{n-1} w.r.t. its last coordinate at q as $S^{n-1}_q := \{p \in S : p_n = q\}$.

3. Assumptions and results

We make the following assumptions.

Assumption 1. $Z : S^{n-1} \to \mathbf{R}^n$ *is an upper hemicontinuous multivalued mapping with convex, compact and non-empty values. Moreover,* Z *is bounded from below, i.e.* $\exists M \in \mathbf{R} \, \forall p \in S^{n-1} \forall y \in Z(p) : y_i \geq M$, $i = 1, \ldots, n$.

A mapping $Z : S^{n-1} \to \mathbf{R}^n$ is upper hemicontinuous if $\forall v = 1, 2, \ldots,\ p^v \in S^{n-1}$ and $p^v \to_v p \in S^{n-1}$ and $y^v \in Z(p^v)$, $v = 1, 2, \ldots$, imply that y^v has a limit point, which belongs to $Z(p)$.

Assumption 2. *The mapping Z satisfies Walras' Law, i.e.,*

$$\forall p \in S^{n-1} \qquad pZ(p) = 0.$$

Assumption 3. *The mapping Z obeys the boundary condition: if $\forall v = 1, 2, \ldots\ p^v \in S^{n-1}$ and $p^v \to_v p$ and $p_i = 0$ for some i and $\forall v = 1, 2, \ldots :$ $y^v \in Z(p^v)$, then $\max_{i=1,\ldots,n}\{y_i^v\} \to_v +\infty$.*

Assumptions (1)-(3) are sufficient for the existence of equilibrium though to prove it requires a fixed point theorem. To provide a proof without using a fixed point theorem, we impose a variant of the weak axiom of revealed preference on the excess demand correspondence. Recall that for an excess demand *function* Z, WARP says the following: given any pair of prices p, p', $pZ(p') \leq 0$ implies $p'Z(p) > 0$, whenever $Z(p) \neq Z(p')$. In Quah (2008) the following extension of WARP to correspondences is employed in his elementary proof of equilibrium existence: for any pair of prices p, p', if there exists $y' \in Z(p')$ s.t. $py \leq 0$, then $p'Z(p) \geq 0$. Our assumption, which is weaker than Quah's, is stated below.

Assumption 4. *If p, $p' \in S^{n-1}$ and $p_n = p'_n$, and $y \in Z(p)$ satisfies $p'y = 0$, then $pZ(p') \geq 0$.*

To see that this assumption is strictly weaker than WARP we need only note that it is trivially satisfied in *any* exchange economy with two commodities while it is well-known that WARP need not be (see Example 4.C.1 in

Mas-Colell et al. (1995)). We now provide two examples of economies that obey Assumption 4 but not necessarily WARP.

It is well known that WARP holds in exchange economies where endowments are collinear and all agents have demand functions obeying the law of demand. Let the individual demand function of consumer j, denoted by $f^j(p, w^j)$, where $w^j > 0$ is consumer's j income, $j = 1, \ldots, m$, satisfy the *law of demand*,[1] i.e, given any two non-equal price vectors p and p',

$$(p' - p)\left(f^j(p', w^j) - f^j(p, w^j)\right) < 0.$$

Aggregating demands across the economy we obtain under the given incomes $w^1, \ldots, w^m$, that

$$(p' - p)\left(\sum_{i=1}^{m}(f^j(p', w^j) - f^j(p, w^j)\right) < 0. \tag{1}$$

In an exchange economy, $w^j = pa^j$, where $a^j \in \mathbf{R}^n_{++}$ is the endowment of consumer j, $j = 1, \ldots, m$. Without loss of generality we can assume that $\sum_{j=1}^m a^j = (\underbrace{1, \ldots, 1}_{\times n}) =: \mathbf{1}_n$, which implies that the aggregate income is independent of prices $p \in S^{n-1}$ though individual incomes may change as prices change. If endowments a^j are collinear, i.e. $a^j = \alpha^j \mathbf{1}_n$ and $\sum_{j=1}^m \alpha^j = 1$, then $\forall p \in S^{n-1}\, \forall j : pa^j = \alpha^j = w^j$, so income does not depend on prices[2] and we obtain $\forall p, p' \in S^{n-1}$, $p \neq p' : (p' - p)(Z(p') - Z(p)) < 0$. This in turn implies that Z obeys WARP and consequently Assumption 4 is valid too.

[1] Sufficient conditions for the Law of Demand are some curvature properties of the consumer's utility function (Mas-Colell et al. (1995), proposition 4.C.3, p. 112) or homothetic preferences (Mas-Colell et al. (1995), p. 112, and Moore (2007), p. 287).

[2] This defines a distribution economy, see Hildenbrand (1983), p.1002-1009.

Now suppose that the total endowment of the economy is $\mathbf{1}_n$, but $a^j = (\alpha^j \mathbf{1}_{n-1}, b_j), j = 1, \ldots, m$ – so that the endowment vectors are collinear in the first $n-1$ commodities but not necessarily in all n commodities. We shall call such an exchange economy an $n-1$-*distribution economy.* If $p, p' \in S_q^{n-1}$ for some $q \in (0,1)$, then $w^j(p) = pa^j = \alpha^j(1-q) + qb_j = p'a^j = w^j(p')$ and it follows that if the price of the n-th good is fixed, then the income $w^j(p)$ is constant. Therefore, it follows from (1) that

$$\forall q \in (0,1)\, \forall p, p' \in S_q^{n-1}\, p \neq p' : (p'-p)(Z(p') - Z(p)) < 0. \qquad (2)$$

This guarantees that Assumption 4 is satisfied, even though WARP need not hold.

Another justification for Assumption 4 is motivated by Quah (1997). Quah shows that Z obeys WARP in an exchange economy where preferences and endowments are independently distributed and all preferences are homothetic. We weaken his assumptions along the following lines. Consider an exchange economy in which all agents have homothetic preferences and the distribution of endowments of goods $1, \ldots, n-1$ is independent of the distribution of preferences. With no loss of generality assume that that the aggregate endowment is $\mathbf{1}_n$. Let A denote the set of different preference types in the economy and denote the demand function for preference type $\alpha \in A$ by $f(p, w, \alpha)$. Given the distribution of preferences among agents it follows from the independence assumption that the aggregate/mean income of agents with type α equals $p(\mathbf{1}_{n-1}, e^n(\alpha))$, where $e^n(\alpha)$ is the aggregate endowment of n-th good owned by consumers of type α, $\alpha \in A$. Since preferences are homothetic, the demand functions are linear in income and the aggregate demand of consumers with type α is $f(p, p(\mathbf{1}_{n-1}, e^n(\alpha)), \alpha)$. The aggregate demand in this economy is obtained by summing the aggregate demands across the types $\alpha \in A$. Clearly we are back in a situation of an $n-1$-distribution economy discussed in the previous paragraph. Furthermore, since preferences are homothetic, they generate demand functions that obey the law of demand, so the reasoning we used in the previous example may be applied again to obtain (2) (and hence Assumption 4).

From now on we take Assumptions 1-4 as granted. To prove equilibrium existence we require the following two lemmas.

Lemma 1. *Let $C \subset \mathbf{R}^n$ be a non-empty compact and convex set s.t.* $\forall p \in S^{n-1} \, \exists y \in C : py \leq 0$ *and for some* $\overline{p} \in S^{n-1} : \overline{p}C = 0$. *Then* $0 \in C$.

Proof: Let C and $\overline{p}$ satisfy the hypothesis. Suppose that $0 \notin C$. This implies $0 \notin (\mathbf{R}^n_+ + C)$. Since C is compact and convex, $\mathbf{R}^n_+$ is closed and convex, then $\mathbf{R}^n_+ + C$ is closed and convex. By the separating hyperplane theorem there exists $\overline{p}' \in \mathbf{R}^n \backslash \{0\}$ s.t. for all $y \in C$ and $x \in \mathbf{R}^n_+$ we have $\overline{p}'y + \overline{p}'x > 0$ (Florenzano and LeVan (2001), p. 24, Proposition 2.1.6). Therefore $\overline{p}' \geq 0$ and $\overline{p}'C > 0$. W.l.o.g. we may assume that $\overline{p} \in clS^{n-1}$, the closure of S^{n-1}. Take any $t \in (0,1)$ and let $p^t := t\overline{p}' + (1-t)\overline{p} \in S^{n-1}$. We have $\forall y \in C \; p^t y > 0$, which implies $p^t C > 0$ for $p^t \in S^{n-1}$ - contradiction. $\square$

Lemma 2. *Fix* $q \in (0,1)$. *Suppose that* $p', p'' \in S^{n-1}_q$ *and* $y' \in Z(p')$, $y'' \in Z(p'')$, *with* $y'_i + \frac{q}{1-q}y'_n = 0$, $y''_i + \frac{q}{1-q}y''_n = 0$, $i = 1, \ldots, n-1$. *Then* $\forall t \in [0,1] \, \exists y \in Z(tp' + (1-t)p'') :$

$$y_i + \frac{q}{1-q}y_n = 0, \; i = 1, \ldots, n-1. \tag{3}$$

Proof: Let $p', p'' \in S_q^{n-1}$ and $y' \in Z(p')$, $y'' \in Z(p'')$ satisfy the hypothesis for some $q \in (0,1)$. For every $t \in (0,1)$ put $p^t := tp' + (1-t)p''$. Since for all $p \in S_q^{n-1}$ it holds

$$py' = py'' = p_1\left(y_1'' + \frac{q}{1-q}y_n''\right) + \ldots + p_{n-1}\left(y_{n-1}'' + \frac{q}{1-q}y_n''\right) = 0 \quad (4)$$

then by Assumption 4 $\forall p \in S_q^{n-1}$ we get $p'Z(p) \geq 0$ and $p''Z(p) \geq 0$, from which it follows $\forall p \in S_q^{n-1} : p^t Z(p) \geq 0$. For any arbitrarily fixed $p \in S_q^{n-1}$ put $p^\lambda := \lambda p^t + (1-\lambda)p$, where $\lambda \in (0,1)$. We obtain

$$0 = p^\lambda Z(p^\lambda) \Leftrightarrow \forall y \in Z(p^\lambda) : \quad \lambda p^t y + (1-\lambda)py = 0.$$

Since $p^\lambda \in S_q^{n-1}$ and $y \in Z(p^\lambda)$ imply $p^t y \geq 0$, then $\forall \lambda \in (0,1)\, \forall y \in Z(p^\lambda) : py \leq 0$.[3] By upper hemicontinuity of Z, we obtain in the limit $\lambda \to 1$ that for each $p \in S_q^{n-1}$ there exists $y \in Z(p^t)$ s.t. $py \leq 0$. Since $p^t Z(p^t) = 0$, $p^t \in S_q^{n-1}$ and $\forall p \in S_q^{n-1}\, \exists y \in Z(p_t) : py \leq 0$, then we may apply lemma 1 to compact convex set

$$C = (1-q)\left\{\left(y_i + \frac{q}{1-q}y_n\right)_{i=1}^{n-1} : y \in Z(p^t)\right\}$$

with $\overline{p} = (p_1^t, \ldots, p_{n-1}^t)$. Therefore $0 \in C$ and this implies the existence of $y \in Z(p^t)$, which satisfies (3). $\square$

Corollary 1. *The WARP axiom (defined in (Quah (2008))) implies that our Assumption 4 holds. We immediately conclude from the proof of Lemma 2 that under WARP the equilibrium price set is convex.*

[3]This part of proof is motivated by the proof of proposition 2.1 in John (1998).

The main result of the paper follows.

Theorem 1. *Fix some integer $n \geq 1$. If $Z : S^{n-1} \to \mathbf{R}^n$ satisfies Assumptions 1-4, then*

$$\exists p \in S^{n-1} : \ 0 \in Z(p).$$

Proof: It is clear that the theorem is true for $n = 1$. Suppose that the thesis is valid for $n-1$, $n \geq 1$. We shall prove that it holds true for n. The proof goes by contradiction. So suppose that the thesis is false for n. Fix any $q \in (0,1)$ - q plays the role of n-th good price. If Z satisfies Assumptions 1-4, then the mapping $\widetilde{Z}^q : S^{n-2} \to \mathbf{R}^{n-1}$ defined as

$$\widetilde{Z}^q(\underbrace{p_1, \ldots, p_{n-1}}_{\widetilde{p}}) := \{\underbrace{(1-q)(y_i + q(1-q)^{-1} y_n)_{i=1}^{n-1}}_{\widetilde{y}} : \ y \in Z((1-q)\widetilde{p}, q)\}$$

satisfies them too: Assumptions 1 and 3 are satisfied since Z^q is compact, convex and non-empty valued since it may be viewed as the composition $(g \circ Z \circ h)(p_1, \ldots, p_{n-1})$ of linear function $g(y_1, \ldots, y_n) = (1-q)\left(y_i + \frac{q}{1-q} y_n\right)_{i=1}^{n-1}$, mapping Z and affine function $h(p_1, \ldots, p_{n-1}) = ((1-q)p_1, \ldots, (1-q)p_{n-1}, q)$ restricted to S^{n-2}; Walras' Law $\widetilde{p}\widetilde{y} = 0$, $\widetilde{y} \in \widetilde{Z}^q(\widetilde{p})$ comes easily from construction of points in $\widetilde{Z}^q(\widetilde{p})$ and expansion (4) of the scalar product of vectors $((1-q)\widetilde{p}, q)$ and $y \in Z((1-q)\widetilde{p}, q)$ corresponding to $\widetilde{y}$. Assumption 4 is also met: suppose $\widetilde{p}\widetilde{y}' = 0$ some $\widetilde{y}' \in \widetilde{Z}^q(\widetilde{p}')$, where $p'_{n-1} = p_{n-1}$. It holds that $p = ((1-q)\widetilde{p}, q)$, $p' = ((1-q)\widetilde{p}', q) \in S_q^{n-1}$ and we have $\widetilde{p}\widetilde{y}' = py'$ for some $y' \in Z(p')$ corresponding to $\widetilde{y}'$ (again by expansion (4) and the definition of $\widetilde{Z}^q(\widetilde{p}')$). Using Assumption 4 (applied to Z), we obtain that $p'Z(p) \geq 0$. But $y \in Z(p)$, $p' \in S_q^{n-1}$ imply $0 \leq p'y = \sum_{i=1}^{n-1} p_i[(1-q)(y_i + q(1-q)^{-1} y_n)] = \widetilde{p}'\widetilde{y}$, which proves the claim, since to each $\widetilde{y} \in \widetilde{Z}^q(\widetilde{p})$ corresponds some $y \in Z(p)$.

So, by the inductive assumption we conclude that $\exists p \in S_q^{n-1}\, \exists y \in Z(p)$ which satisfies (3). We have that $\forall q \in (0,1)$ set $L(q)$ defined as

$$L(q) := \{p \in S_q^{n-1} : \exists y \in Z(p) \text{ which satisfies (3)}\}$$

is non-empty. By the contradictory assumption, for all points $p \in S^{n-1}$ s.t. $\exists y \in Z(p)$ satisfying (3), it holds: $y_n \neq 0$. Define

$$A := \{q \in (0,1) : \forall p \in L(q) \forall y \in Z(p) \text{ satisfying (3) it holds } y_n > 0\},$$

$$B := \{q \in (0,1) : \forall p \in L(q) \forall y \in Z(p) \text{ satisfying (3) it holds } y_n < 0\}.$$

Obviously, $A \cap B = \emptyset$. Moreover - by Assumptions 1 and 3 - for q sufficiently close to 0 we have $q \in A$: to see this suppose that $q \to 0$ and $y^q \in Z(p^q)$ satisfies (3) for some $p^q \in S_q^{n-1}$. Since $q(1-q)^{-1} \to_q 0$, then it must be that $y_n^q \to_q +\infty$ - if not, then by Assumption 3 $y_i^q \to_q +\infty$, $i = 1, \ldots, n-1$, which entails $y_n^q < 0$ and by boundedness from below $q(1-q)^{-1} y_n^q \to_q 0$, so that for small values of q equation (3) could not hold - therefore $A \neq \emptyset$. Moreover, from Assumption 3 it follows that $q \to 0 \Rightarrow y_n^q \to +\infty$, for y^q satisfying (3). If $q \to 1$ and $y^q \in Z(p^q)$ satisfies (3) for some $p^q \in S_q^{n-1}$ and $q(1-q)^{-1} y_n^q > 0$ (for q's close to 1) then $y_n^q \to_q +\infty$, so that $q(1-q)^{-1} y_n^q \to_q +\infty$ and assumption 1 implies contradiction. Therefore $A \neq \emptyset$, $B \neq \emptyset$. Suppose that there exists $q \in (0,1) \backslash (A \cup B)$. So, it holds that for some $p, p' \in L(q)$ there exist vectors $y \in Z(p)$ and $y' \in Z(p')$ meeting conditions (3) with $y_n > 0$ and $y'_n < 0$. It follows from Lemma 2 that $\forall t \in (0,1) \exists y^t \in Z(tp + (1-t)p')$ which fulfills (3). From convexity of $Z^t := Z(tp + (1-t)p')$ and the contradictory assumption it follows that if $y, y' \in Z^t$ satisfy (3), then y_n and y'_n are of the same sign. If

$$A' := \{t \in (0,1) : \forall y \in Z^t \text{ s.t. (3) is satisfied} \Rightarrow y_n > 0\},$$

$$B' := \{t \in (0,1) : \forall y \in Z^t \text{ s.t. (3) is satisfied} \Rightarrow y_n < 0\},$$

then we get $(0,1) = A' \cup B'$. Since both sets A' and B' are open (by the contradictory assumption and upper hemicontinuity of Z (Assumption 1)) and disjoint - this leads to contradiction with connectedness of $(0,1)$, if $A' \neq \emptyset \neq B'$. Let's suppose for a while that $B' = (0,1)$. For vectors $y^t \in Z^t$ satisfying equation (3), where $t \to 0$, we get by upper hemicontinuity of Z that there exists $\overline{y} \in Z(p)$, which meets (3) with $\overline{y}_n < 0$. Since $y \in Z(p)$ satisfies (3) with $y_n > 0$, then by convexity of $Z(p)$ we again are led to contradiction. If $A' = (0,1)$, then the similar reasoning results in contradiction. From this follows that $A \cup B = (0,1)$. But for the same reasons as in the case of sets A' and B' both sets A and B are open. We know that $A \neq \emptyset \neq B$ and $(0,1) = A \cup B$, which is contradiction and the proof is finished. $\square$

To develop an algorithm for finding an equilibrium price vector, we now assume that Z is a function (rather than a correspondence) from S^{n-1} to $\mathbf{R}^n$ that satisfies Assumptions 1-3. We replace Assumption 4 with the following assumption.

Assumption 5. *If $q \in (0,1)$ and $p, p' \in S^{n-1}_q$, then $p'Z(p) \leq 0$ implies $pZ(p') > 0$.*

Assumption 5 is stronger than 4 but it is implied by (2) and hence by the two examples given earlier in which (2) holds. Assumption 5 implies the uniqueness of equilibrium of the dimension-reduced excess demand given price of the last good $p_n = p'_n = q$ (see function $\widehat{Z}^q$ below).

Let us fix $q \in (0,1)$ and define a function $\widehat{Z}^q : \mathbf{R}^{n-1}_{++} \to \mathbf{R}^{n-1}$ as

$$\forall p \in \mathbf{R}^{n-1}_{++} \quad \widehat{Z}^q(p) := (\widehat{Z}^q_1(p), \ldots, \widehat{Z}^q_{n-1}(p)),$$

where

$$\widehat{Z}^q_i(p) := (1-q)\left(Z_i(\widehat{p}(p,q)) + \frac{q}{1-q} Z_n(\widehat{p}(p,q))\right),$$

$$\widehat{p}(p,q) := \left(\frac{(1-q)}{\sum_{i=1}^{n-1} p_i} p, q\right).$$

The above construction is correct since $\forall p, p' \in \mathbf{R}^{n-1}_{++} : \widehat{p}(p,q) \in S^{n-1}$. The construction of $\widehat{Z}^q$ - which is analogous to the construction of $\widetilde{Z}^q$ in the above proof - implies that it exhibits at least the same properties as $\widetilde{Z}^q$. Moreover, the function $\widehat{Z}^q$ is homogeneous of degree 0 and since $\forall p \in \mathbf{R}^{n-1}_{++}$: $p\widehat{Z}^q(p) = \widehat{p}(p,q)Z(\widehat{p}(p,q))$, then - by Assumption 5 - it satisfies a version of the WARP axiom for excess demand functions: $\forall p, p' \in \mathbf{R}^{n-1}_{++}$ which are not collinear it holds $p\widehat{Z}^q(p') \leq 0$ implies $p'\widehat{Z}^q(p) > 0$ - this property and Theorem 1 guarantee that given $q \in (0,1)$ there exists exactly one structure of equilibrium prices, say $p \in \mathbf{R}^{n-1}_{++}$ s.t. $\widehat{Z}^q(p) = 0$. The tâtonnement dynamics of prices

$$\frac{dp(t)}{dt} = \widehat{Z}^q(p),\ p(0) = p_0, \tag{5}$$

where $p_0 \in \mathbf{R}^{n-1}_{++}$ is a fixed initial prices vector, implies convergence of prices $p(t)$ to the equilibrium price vector $p \in \mathbf{R}^{n-1}_{++}$ (see proposition 17.H.1 in Mas-Colell et al. (1995), p. 623), whose (Euclidean) length equals the length of p_0. Assumption 5 entails that for any pair p, $p' \in \mathbf{R}^{n-1}_{++}$ of equilibrium vectors (given the same q) it holds: $\widehat{p}(p,q) = \widehat{p}(p',q)$ - this implies that given q there exists unique $p \in (1-q)S^{n-2}$ s.t. $\widehat{Z}^q(p) = 0$. Let $p(q) \in (1-q)S^{n-2}$ denote the unique equilibrium price and put $L(q) := Z_n(p(q), q)$. The proof guarantees that if $q \to 0$ then $L(q) \to +\infty$, and if $q \to 1$ then $L(q) < 0$ [4]. Moreover $L(\cdot)$ is a continuous function of q - this comes from continuity of $\widehat{Z}^q$ and from the fact that $L(\cdot)$ is a function.

Now we can state an algorithm for finding an equilibrium price vector in economies with an excess demand function that satisfies Assumptions 1-3 and 5.

`Step 0:` `Fix` $v = 1$, $q_0 \in (0,1)$. `Compute` $p^0 := p(q_0)$ `and` $L_0 := L(q_0)$. `Go to step 1.`

`Step 1:` `Put`

$$q_v := \begin{cases} q_{v-1} + (1 - q_{v-1})/2, & \text{if } L_0 < 0 \\ q_{v-1}/2, & \text{if } L_0 > 0 \end{cases}$$

`Put` $p^v := p(q_v)$, $L_v := L(q_v)$, $v := v + 1$. `If` $L_{v-1}L_{v-2} > 0$, `then repeat this step. In other case go to step 2.`

`Step 2:` `Put`

$$q_v := \begin{cases} (q_{v-1} + q_{v-2})/2, & \text{if } L_{v-1}L_{v-2} < 0 \\ (q_{v-1} + q_{v-3})/2, & \text{otherwise} \end{cases}$$

`Put` $v := v+1$, $p^v := p(q_{v-1}), L_v := L(q_{v-1})$ `and repeat this step.`

At each step of the algorithm, Walras' tâtonnement (5) may be employed for finding equilibrium prices - the last found equilibrium price vector is then used as initial price vector in the next iteration. At Step 1 the first pair of consecutive values of L with opposite signs is found - such pair exists by properties of $L(\cdot)$. Step 2 bisects intervals with ends at which values of L_{q-1}, L_q are opposite in sign and determines next interval with the same

[4] This comes from non-emptiness of the sets A and B and the boundary condition.

property. It is assumed that $L_q \neq 0$ when the algorithm is executed (if $L_q = 0$ - equilibrium has been found). A stopping rule could be e.g. $|L(q_v)| < \epsilon$ for some $\epsilon > 0$.

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