Solving payoff sets of perfect public equilibria: an example

Chuang Du

Institute of Economics, Chinese Academy of Social Sciences

7 May 2012

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(2011-05-07)

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Abstract

We study an example of infinitely repeated games in which symmetric duopolistic firms produce experience goods. After consuming the products, short-run consumers only observe imperfect public information about product quality. We characterize perfect public equilibrium payoff set $\mathrm{E}(\delta)$ of firms for each fixed discount factor $\delta \in [0,1)$ when each firm has two action choices, signals follow binomial distributions and the game has a product structure. The set $\mathrm{E}(\delta)$ turns out a single point or symmetric pentagon for fixed δ. And $\delta \in [0,1)$ can be divided into countable infinite subintervals in which $\mathrm{E}(\delta)$ remains constant. The strategies to implement payoffs in boundaries of $\mathrm{E}(\delta)$ are constructed in a recursive way, in which infinite repetition of Nash Equilibrium of stage game could be viewed as an absorbing state in a Markov Process where state transitions are controlled through public signals and optimal punishments in each period.

Keywords: repeated games, imperfect public monitoring, equilibrium payoff sets, duopoly

JEL Classification: C72, C73, D21, D82, L13, L15.

Author Information:

Institute of Economics,

Chinese Academy of Social Sciences,

No.2 Yuetanbeixiaojie Street, Xicheng District, Beijing, China (100836)

E-mail: duchovry@163.com; duchuang@cass.org.cn

1 Introduction

How to calculate payoff sets of perfect public equilibria for repeated games with long-run and short-run players and with imperfect public monitoring? Existing literature gives only the upper bound, or limit equilibrium payoff set as discount factor $\delta \to 1$, while remains silent on the exact equilibrium payoff sets for fixed $\delta < 1$ when folk theorem fails, and without description on the optimal strategy to implement payoffs on the bound of equilibrium payoff set if signal space is finite (Fudenberg and Levine, 1994; Fudenberg, Levine and Takahashi, 2007). Characterization of the precise form of equilibrium payoff set is valuable not only in theory, but also in many applications. Because of inevitable existence of multiple equilibria in repeated games, the optimal equilibrium payoff is generally used for prediction in many practical contexts such as IO. And it is equivalent to solving equilibrium payoff sets in the case of finite signal space since characterize the Pareto boundary is usually equivalent to characterize equilibrium payoff sets[1]. Moreover, it is more meaningful in real applications to clarify the mechanism through a detailed description of strategies to achieve boundaries on equilibrium payoff sets.

By a concrete example, we provide an idea to solve payoff sets of perfect public equilibria for fixed δ in repeated games with long-run and short-run players and with imperfect public monitoring. The idea is not a purely mathematical technique, equilibrium strategies constructed being also of empirical relevance.

Consider the case of two symmetric long-term players each having two hidden action choices in each stage game, and signals of actions follow binomial distributions. For example, in some duopolistic experience goods[2] market, firms may exert low effort or high effort level in the production. It is more likely to produce high quality products when a firm exerts high effort. However, high effort incurs higher costs. The firm's choice of effort level is hidden action. After consuming the products, consumers observe only product quality, which can be passed to other consumers, e.g., through consumer reports, word of mouth, etc. However, product quality is unverifiable, i.e., it is difficult to verified by courts and other third-party, thus consumers who buy low-quality products may find themselves difficult to resort to legal claims. Moral hazard problem naturally occurs. Should consumers believe that a firm choose high effort, the optimal response of the firm would be to choose low effort with lower costs and the expected quality of products would be low then. Rational consumers expect this and would not believe any firm. Thus the market is full of low quality products and may vanish.

In a market economy, the reputation mechanism can overcome firms' moral hazard problem. The basic idea is as follows. When firms are long-run players in repeated games, consumers can observe product quality of each firm in last period and make choices in accordance with it. Should the trusted firm deviate to low effort for saving costs in current period, the probability of losing reputation in next period will increase. If firms are patient enough, such a deviation will be not profitable. Thus patient firms will cherish their reputations and exert high efforts even if consumers are short-run participants. The reputation mechanism is an example of cooperation in

[1] If signals are continuous, then optimal equilibrium payoffs could be implemented by Bang-Bang strategy (Abreu, Pearce and Stacchetti, 1990).

[2] The concept of "experience goods" follows Nelson (1970).

repeated games with imperfect public monitoring. But how to characterize firms' equilibrium payoff sets in this repeated games? Note that conventional folk theorem fails here because of the existence of short-term participants. No matter how patient firms are (even $\delta \to 1$), equilibrium payoff set will strictly less than firms' feasible payoff set (Fudenberg, Kreps and Maskin, 1990).

Consider the following class of equilibria characterized by "states". Even in repeated games, consumers may not trust any firm, and the result is repetition of static game equilibrium, in which both firms exert low effort, consumers do not buy and the market vanishes. We call this "state 0 equilibrium". Consumers may also believe that only one of two firms, firm *A*, will exert high effort, while the other firm, firm *B*, is never trusted. If trusted firm *A* produces low quality products, with a certain probability it will be punished from the next period by consumers. Punishment here means repetition of static game equilibrium. We call the above situation "state 1 equilibrium". Consumers might also first trust firm *B* and try buying. If firm *B* produce high quality products, consumers will continue to purchase from *B;* otherwise, with a certain probability all consumers switch from *B* to *A* in the next period after low quality of B. Thereafter, if firm *A* also produce low quality products, all consumers will not trust any firm again. We call this "state 2 equilibrium". In addition, consumers may also trust two firms respectively twice, S times and even infinite times. Such as in "state 3 equilibrium", firm *A* is trusted twice, while firm *B* is only trusted one time. In "state ∞ equilibrium", consumers just switch between the two firms, never out of the market. We call the trusted firm "incumbent firm", and the other "waiting firm". Note that due to the uncertainty in production, any firm could not be "incumbent firm" forever.

We find that consumer beliefs described above could be defined and classified in a recursive way, so do accompanying equilibrium mechanisms. For instance, continuation equilibria of state s equilibrium include state s-1, state s-2 ... state 0 equilibrium. It also can be shown that the lowest discount factor δ_s to ensure existence of state s equilibrium is strictly increasing in s for $s \geq 2$. We call all such state s equilibrium "Recursive Belief Equilibrium", which is a special case of perfect public equilibrium. Under public randomization assumption, the convex hull of payoff set of Recursive Belief Equilibrium constitutes a complete characterization of payoff set of public perfect equilibrium for any fixed δ. Specifically, for some parameter value of cost function, product function and consumers' utility function, there exist $0 \leq \delta_s \leq 1$, $0 \leq s < \infty$, $\delta_0 = 0$ such that for any $\delta \in [\,\delta_s,\ \delta_{s+1})$, state s equilibrium exists, so do state $s-1, s-2 \ldots 1,0$ equilibrium. Define $\bar{V}_s$ as the convex hull of the set of payoff pairs (average discounted expected payoffs) of all possible state $s, s-1, s-2 \ldots 1,0$ equilibrium. We demonstrate that $\bar{V}_s$ is equal to $\mathrm{E}(\delta)$, payoff set of perfect public equilibria. Depending on parameter values, it may be that $\delta_\infty = \lim_{s\to\infty} \delta_s = 1$, then $\delta \in [\,\delta_s,\ \delta_{s+1})$, $s < \infty$, $\delta_0 = 0$ constitute a complete description of $\delta \in [0,1)$; it may also be that $\delta_\infty < 1$, then state ∞ equilibrium exists. In the latter case, for any $\delta \in [\,\delta_\infty, 1)$, $\bar{V}_\infty$ is equal to $\mathrm{E}(\delta)$. In addition, there will still be other parameter values such that only "state 0 equilibrium" exists for any $\delta \in [0,1)$. Thus payoff set of public perfect equilibrium is trivially equal to a single point, "state 0 equilibrium" payoff. Note that due to symmetry of the game, any state s equilibrium payoffs include two symmetric points: Let (v_{is}, v_{js}) is state s equilibrium payoff pairs, so is (v_{js}, v_{is}), where the first component in brackets represents firm i's payoff and second component firm j (j ≠ i). It can also be shown that, for any $\delta < 1$, if v_s is the largest payoff of incumbent firm in all possible state s equilibria, $v_s \equiv \mathrm{v} > 0$ for all $s \geq 1$. Thus $\mathrm{E}(\delta)$ is a single point or symmetrical pentagon for fixed δ, and five vertices of the pentagon are $(0,0), (\mathrm{v}, 0), (\mathrm{v}, w_s), (w_s, \mathrm{v}), (0, \mathrm{v})$, where

$w_s > 0$ is the payoff of "waiting firm" in state s equilibrium when incumbent's payoff is v .

The intuition is as follows. The game analyzed in this paper has the product structure, namely, the hidden action of each long-run player has a public signal and the signals are independent of each other. Each signal is a sufficient statistic of the player's hidden action, thus for all equilibrium payoff pairs, strategies to implement them could be equivalently expressed as if each firm's continuation payoffs rely solely on its own signal (Holmstrom, 1982). The range to search for the optimal equilibrium strategy is then greatly reduced. For instance, performance comparison evaluation is unnecessary. In recursive belief equilibrium, continuation payoffs of the incumbent firm depend only on its own signal. Although continuation payoffs of the waiting firm depend on the signal of the incumbent, the waiting firm is just waiting, no incentive problem accompanied. In addition, although there are only two firms, two actions, binomial distribution of signals in the example, it could easily be extended to N firms, N actions and any type of signal distribution, as long as firms are symmetric and the game has product structure.

The idea in this article is similar to Upwind Gauss-Seidel method in numerical computation of dynamic programming problem. The state transition between the incumbent and waiting firm from state s to $s-1$ may be regarded as a Markov Process, where state transition probabilities are endogenous, depending on probabilities of consumers' punishments after the incumbent firm producing low-quality products. And state 0 is the only absorbing state, the equilibrium payoffs of which is easy to calculate, equaling to infinitely repetitions of static game Nash equilibrium. State 1 equilibrium payoffs could be calculated according to state 0 equilibrium, while state 2 equilibrium payoffs could be calculated according to state 1 equilibrium, and so on. The following words in Judd (1998) provide a good description of the idea. (See also Figure 1)

"We see here that if the state flows in the directions of the solid arrows, the proper direction for us to construct the value function is in the opposite," upwind ", direction. Another way to express this is that information naturally flows in the upwind direction. For example, information about state 1 tells us much about states 2 ..., but we can evaluate state 1 without knowing anything about states 2 ... Again, a single pass through the states will determine the value of the indicated policy ... In general, once we ascertain the value of the problem at the stable steady states, we can determine the value at the remaining states. "

---- Judd (1998, pp.420-421)

(Insert Figure 1 Here)

In addition, the dynamics of reputations on oligopolistic markets are revealed in this paper in a pure moral hazard context, which may be an interesting complement to reputation models introducing private information of firms, such as Kreps and Wilson (1982), Milgrom and Roberts (1982). While they concentrate on how reputation is built step by step, firms lose reputation one-off in their models. On the contrary, we concentrate on how reputation vanishes step by step. For instance, the optimal probability of punishment in equilibrium (the probability of losing consumers when the incumbent firm producing low-quality products) will becomes lower and lower as chances to be incumbent again reduces. Consumers are more and more "tolerant", or less and less sensitive to incumbent firm's quality signal as from state s to s-1…2, 1. It is not some kind of "inertia" of consumers' belief due to long-term success of firm (Crips, Mailath and

Samuelson, 2004), but because firms have less and less opportunities to be incumbent again, which itself constitutes a growing potential punishment. For the same reason, the critical discount factor to support state s equilibrium is increasing in s for $s \geq 2$. We quoted some empirical literature in part 6 of this article to verify the relevance of reputation dynamics.

The rest of the paper is structured as follows. In Section 2 we discuss related literature. Section 3 spells out the model setup. In Section 4 we characterize recursive belief equilibrium (state s equilibrium) payoff set and in Section 5 we prove that the set constitutes a complete characterization of perfect public equilibrium payoff set in the model. Relevant studies are cited in Section 6 to verify the empirical relevance of recursive belief equilibrium. Section 7 contains the conclusions and further questions.

2 Related Literature

Our paper is closely related to the literature on computation of payoff set of perfect public equilibrium. Abreu, Pearce and Stacchetti (1990) first present a general algorithm of set iteration, similar to value function iteration in dynamic programming with a single decision-maker. However, set iteration is generally not tractable in specific operations. The direct application of set iteration algorithm for solving equilibrium payoff set is now mainly in the framework of repeated games with perfect monitoring, and can only obtain upper and lower bounds of equilibrium payoff set through numerical computation (Judd, et al, 2003) [3]. Abreu and Sannikov (2011) provide a new algorithm to calculate pure strategy SPE in 2-person repeated games with perfect monitoring and with public randomization. For repeated games with imperfect public monitoring, Fudenberg, Levine and Maskin (1994), Fudenberg and Levine (1994) give explicit methods to calculate equilibrium payoff set (upper bound) under limited signal space. Fudenberg and Levine (1994) presents a general algorithm for computing the upper bound of set of payoffs of perfect public equilibria of repeated games with long-run and short-run players; and shows that the upper bound is equal to limit equilibrium payoff set as $\delta \to 1$ if the game meets some condition of full rank. Fudenberg, Levine and Takahashi (2007) further solve the limit set of perfect public equilibrium payoff when full rank condition fails. Horner, Sugaya, Takahashi, and Vieille (2011) extends Fudenberg and Levine (1994)'s algorithm to general stochastic games. Horner, Takahashi and Vieille (2012) further provides a dual characterization of the limit set of perfect public equilibrium payoffs in stochastic games and shows that, in the context of repeated games, this limit set of payoffs is a polytope when attention is restricted to equilibria in pure strategies.

A second strand of relevant literature is that on reputation mechanisms of experience goods markets. There are a lot of studies on reputation mechanisms in different market structure, such as Klein and Leffler (1981), Shapiro (1983), Horner (2002), Rob and Fishman(2005), Rob and Sekiguchi (2006), Cai and Obara (2009). In this strand, our paper is perhaps most closely related to Rob and Sekiguchi (2006), which studies "turnover equilibrium" in a duopoly market, similar to state ∞ equilibrium in our paper. In addition, Rob and Fishman (2005) also shows the dynamics of reputation in a pure moral hazard framework, but it is in the context of monopolistic competition market with infinite number of firms. And the concern of Rob and Fishman (2005) is the gradual accumulation of reputation, while firms lose reputation one-off. The focus of

[3] For other numerical methods, see also Wang(1995), Athey and Bagwell (2001).

reputation dynamics in our article is how reputation vanishes gradually.

3 The Model

There are two long-lived firms in the market. Time is discrete and the horizon is infinite. In each period, each firm and consumers play the following stage game. The consumers decide whether to purchase one unit of the firm's products. If they do not buy from the firm, both the consumers and the firm get a payoff of zero. If they decide to buy from the firm, their payoffs decide on the firm's product quality. The firm decides whether to exert high effort a_h (or, provide high quality) or exert low effort a_l (or, provide low quality). The firm incurs an effort cost /quality cost of c for providing high effort/quality, 0 for providing low effort/quality, where $c > 0$. The consumers' expected benefit is u if the firm choose a_h, and is 0 if the firm choose a_l , where $u > 0$. Market price is fixed at $p = u$ [4]. The stage game is depicted below in the normal form.

		Firm: Low	Firm: High
Consumers	Not Buy	0， 0	0， 0
	Buy	$-u$, u	0, $u - c$

We assume $u - c > 0$, hence high effort/quality is more efficient than no trade. However, Should consumers choose "*Buy*", the optimal response of the firm would be "*Low*" (effort/quality). Thus the unique equilibrium outcome is (*Not Buy, Low*), resulting in payoffs (0, 0).

We suppose that in each period there are identical consumers with measure 1, who purchase the products only once. So consumers will maximize their current period payoffs. If consumers are indifferent between "Buy" and "Not Buy", they will choose "Buy".

If the firm's effort in each period were publicly observable, it would be straightforward to show that the efficient outcome (*Buy, High*) can be supported when firms are patient enough. Let δ be the firms' common discount factor. It can be easily checked that the efficient outcome is attainable in every period if and only if $\delta \geq c/u$.

We consider an environment in which a firm's effort is not public information, but rather its noisy public signal $y_i \in \{y_h, y_l\}, i = 1,2$ becomes available at the end of each period. For each firm i, the probability of $y = y_h$ is μ_h and $y = y_l$ is $1 - \mu_h$ if firm i exerts high effort; the

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[4] To get a simple expression, we abstract away the price factor in this illustrated example. The price is a continuous variable. The formulation of the game will become complicated if the price factor is introduced into the game as an endogenous variable, because this means that each firm's action will be two-dimensional: price and effort level. However, it can be proven that ignoring price competition between firms has not a substantial impact on characterization of equilibrium payoff sets. If in each period there is only one firm being trusted, such as recursive beliefs equilibrium described in Section 4, price selection and the choice of effort level will be actually separate; the only trusted firm will optimally set price p_i to the highest: $p_i = u$. If in some period there are two firms being trusted to exert high effort, the occurrence of price wars and $p_i < u$ will lead firms' payoffs not in the boundary of equilibrium payoff set; if the two firms collude to $p_i = u$, maintaining of collusions will require additional conditions on the discount factor, where joint payoffs of two firms will not be strictly higher than best recursive belief equilibrium payoffs.

probability of $y = y_h$ is μ_l and $y = y_l$ is $1 - \mu_l$ if firm i exerts low effort, where $\mu_h > \mu_l > 0$. Signals of different firm, y_1 and y_2, are independent. We can interpret public signals y_i as quality level of firm *i*'s products and there are uncertainties in production processes. Or we can interpret y_i as some subjective signals on quality because communications between current period consumers and future consumers are imperfect.

Following Fudenberg and Levine (1994), a rigorous description of the repeated games is as follows. Let number 1, 2 represent two long-run players, firms and SR represents short-run players, consumers. In each stage game, each player $i = 1,2, SR$ simultaneously chooses a (pure) action a_i from a finite set A_i. $A_i = \{a_h, a_l\}$ for $i = 1,2$, where a_h represents High Effort, a_l represents Low Effort. $A_i = \{\text{Buy}, \text{Not Buy}\}$ for $i = SR$. Note that $a_{SR} = \{a_{SR,1}, a_{SR,2}\}$ is a vector where $a_{SR,i}$ represents consumers action on firm i ($i = 1,2$). Each action profile $a = (a_1, a_2, a_{SR})$ induces a probability distribution over public signals $y = (y_1, y_2)$, $y_i \in \{y_h, y_l, y_{null}\}$, $i = 1,2$, where y_h represents high quality, y_l represents low quality and y_{null} represents no trade. For a given action profile a, let $\pi_y(a)$ denote the probability of y and $\pi_i(a)$, $i = 1,2$ denote the marginal probability for the public signals. Obviously in our model we have $\pi_y(a_1, a_2, a_{SR}) = \pi_1(a_1, \alpha_{SR,1})\pi_2(a_2, a_{SR,2})$, $\pi_1(a_1, a_{SR,1}) = \pi_2(a_1, a_{SR,1})$. And for $i = 1,2$

$$\pi_i(a_H, \text{Not Buy}) = \pi_i(a_L, \text{Not Buy}) = (0,0,1)$$
$$\pi_i(a_H, \text{Buy}) = (\mu_h, 1 - \mu_h, 0)$$
$$\pi_i(a_L, \text{Buy}) = (\mu_l, 1 - \mu_l, 0)$$

To each action profile a, player *i*'s expected payoff ($i = 1,2$) is $g_i(a) = \sum_{y \in Y} \pi_y(a) r_i(y)$, where $r_i(y)$ is player *i*'s payoff ($i = 1,2$) for signal y.

Let α_i be mixed actions for each player i. For each profile $\alpha = (\alpha_i, \alpha_j, \alpha_{SR})$ of mixed actions, we can compute the induced distribution over public signals, $\pi_y(\alpha) = \sum_{a \in A} \pi_y(a)\alpha(a)$; and the expected payoffs, $g_i(\alpha) = \sum_{y \in Y} \sum_{a \in A} \pi_y(a)\alpha(a) r_i(y)$. Denote the profile where player *i* plays α_i and all other players follow profile α by (α_i, α_{-i}). $\pi_y(\alpha_i, \alpha_{-i})$ and $g_i(\alpha_i, \alpha_{-i})$ are defined in the similar way.

The objective of each long-run player $i = 1,2$ is to maximize the average discounted value of per-period payoffs using the common discount factor δ. If $\{g_i(t)\}$ is a sequence of payoffs for long-run player i, player i's average discounted expected payoff will be: $(1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} g_i(t)$. In the repeated game, in each period $t = 1,2, \ldots$, the stage game is played, and the corresponding public outcome is then revealed. The public history at the end of period t is $h(t) = \{y_1, y_2, \ldots, y_t\}$. We also let $h(0)$ denote the null public history in which nothing has happened. A public strategy for long-run player i ($i = 1,2$) is a sequence of maps mapping public history to mixed actions. A public strategy for the period-t consumers is a map from the public information $h(t-1)$ to mixed actions. We define $E(\delta) \subset \mathbb{R}^2$ to be the set of average present values for the long-run player that can arise in perfect public equilibria when the discount factor is δ.

4 Recursive Belief Equilibrium

Define "recursive beliefs" as follows.

Definition 1: A recursive belief s $(0 \leq s \leq \infty)$ includes the following state s to state 0.

(i) state s $(s \geq 2)$: all consumers believe that only one firm, say firm 1, exert high effort. If the public signal of firm 1 is y_h at the end of current period, consumers' beliefs will remain unchanged in next period; if the public signal of firm 1 is y_l at the end of current period, next period consumers' beliefs will transit into state s-1 with probability ρ_s $(0 < \rho_s \leq 1)$ and remain in state s with probability $1 - \rho_s$. Call the trusted firm "incumbent firm" in state s , the other firm "waiting firm" in state s. Let v_s, w_s be (per period) average discounted expected payoffs of "incumbent firm" and "waiting firm" from state s to state 0, respectively.

(ii) state s-1: all consumers believe that only the "waiting firm" in state s, firm2, will high effort. If the public signal of firm 2 is y_h at the end of current period, consumers' beliefs will remain unchanged in next period; if the public signal of firm 2 is y_l at the end of current period, next period consumers' beliefs will transit into state s-2 with probability ρ_{s-1} $(0 < \rho_{s-1} \leq 1)$ and remain in state s with probability $1 - \rho_{s-1}$. Again, call the trusted firm, firm 2, "incumbent firm" in state $s - 1$; the other firm, firm 1, "waiting firm" in state $s - 1$. Let v_{s-1}, w_{s-1} be average discounted expected payoffs of "incumbent firm" and "waiting firm" from state $s - 1$ to state 0, respectively.

(iii) state 0: all firms are not to be trusted. Payoff pair of two firms is $(v_0, w_0) = (0, 0)$.

The above definition means that consumers trust the same one of two firms in state s, s-2, ... and trust the other firm in state s-1 s-3, ... And v_1, v_2, v_3, ... denote the average payoff of incumbent firm from state 1, 2, 3, ... to state 0 respectively, while w_1, w_2, w_3, ... denote the average payoff of waiting firm from state 1, 2, 3, ...to state 0 respectively. To facilitate the presentation, we call a perfect public equilibrium with recursive beliefs as "recursive belief equilibrium". The formal definition is as follows.

Definition 2: A recursive belief equilibrium is a perfect public equilibrium where consumers and firms take the following strategies: (i) the consumers' strategies are in line with recursive beliefs; (ii) each firm exerts high effort if it is a "incumbent firm", and exerts low effort if it is a "waiting firm" or if it is in state 0.

We also call a recursive belief equilibrium "state s equilibrium" if the consumers' initial belief is at state s. The critical discount factor of stats s equilibrium, the minimum discount factor to support state s equilibrium, is denoted by δ_s. According to the definition of recursive belief, it is obviously that if state s equilibrium exists, state s' equilibrium exists for any $s' < s$. Thus the critical discount factor δ_s is non-decreasing in s.

Obviously, "state 0 equilibrium" always exists for any discount factor. It is just infinite repetition of static game Nash equilibrium.

In any state s $(s \geq 1)$, it is clearly rational for consumers taking recursive beliefs given firms' strategies. The question is: is it optimal for the incumbent firm to choose high effort given the consumers' and waiting firm's strategy? Here we look at the trade-off faced by the incumbent. According to Abreu, Pearce and Stacchetti (1990), the average discounted payoff of the incumbent, v_s, can be decomposed into a weighted sum of current period payoff and continuation payoffs from next period, weighted by discount factor:

$$v_s = (1-\delta)(u-c) + \delta\{[1-(1-\mu_h)\rho_s]v_s + (1-\mu_h)\rho_s w_{s-1}\} \quad (1)$$

The first term on the right of (1) is the current period payoff of the incumbent firm when exerts high-effort, the second item is expected continuation payoffs from next period when exerts high effort, discounted to current period. $(1-\mu_h)\rho_s$ is the probability to be punished in next period when the incumbent firm exerts high effort, i.e., the transition probability from state s to state s-1. $[1-(1-\mu_h)\rho_s]$ is the probability of remaining in state s when the incumbent exerts high effort. Note that the incumbent firm in state s is just the waiting firm in state s-1.

Accordingly, to make optimal the incumbent choosing high effort, the following incentive compatibility conditions is necessary:

$$v_s \geq (1-\delta)u + \delta\{[1-(1-\mu_l)\rho_s]v_s + (1-\mu_l)\rho_s w_{s-1}\} \quad (2)$$

Should the incumbent deviate to low effort, the cost savings in current period is c, but in the next period the transition probability from state s to state s-1 would increase from $(1-\mu_h)\rho_s$ to $(1-\mu_l)\rho_s$. Inequality (2) means that such a deviation is not profitable. It is easy to prove that the incumbent's payoff is maximized when (IC) condition binds.

In addition, in state s the waiting firm has no incentive problem and is just waiting passively: waiting for the failure of the incumbent. The waiting firm's average payoff is:

$$w_s = \delta\{[1-(1-\mu_h)\rho_s]w_s + (1-\mu_h)\rho_s v_{s-1}\} \quad (3)$$

According to (1), (2) and (3), there are multiple equilibria for state s since incentive compatibility condition need not bind. The following Proposition 3 gives closed solutions to v_s, w_s and optimal probability of punishment ρ_s when the incumbent firm's payoff is maximized, (IC) binds, in all possible state s equilibria.

Proposition 3: If state s $(s \geq 1)$ equilibrium exists, the following results hold when the incumbent firm's equilibrium payoff is maximized,

$$\mathbf{v_s(\delta) = v \equiv u - c - \frac{(1-\mu_h)c}{(\mu_h-\mu_l)}}$$

$$\rho_s(\delta) = \begin{cases} \eta(\delta)\left[1-\left(\frac{1-\mu_h}{r}\right)^{s-1}\right], & \text{if } s \geq 2 \text{ and } r \neq 1-\mu_h \\ \frac{(s-1)(1-\delta)}{\delta r}, & \text{if } s \geq 2 \text{ and } r = 1-\mu_h \\ \frac{(1-\delta)}{\delta r}, & \text{if } s = 1 \end{cases}$$

$$w_s(\delta) = w_s^* \equiv \begin{cases} \frac{(1-\mu_h)\left[1-\left(\frac{1-\mu_h}{r}\right)^{s-1}\right]v}{[r-(1-\mu_h)]+(1-\mu_h)\left[1-\left(\frac{1-\mu_h}{r}\right)^{s-1}\right]}, & \text{if } s \geq 2, r \neq 1-\mu_h \\ \frac{(1-\mu_h)(s-1)v}{r+(1-\mu_h)(s-1)}, & \text{if } s \geq 2, r = 1-\mu_h \\ \frac{(1-\mu_h)v}{r+(1-\mu_h)}, & \text{if } s = 1 \end{cases}$$

where $\mathbf{r} \equiv \frac{\mathbf{v}(\mu_h - \mu_l)}{\mathbf{c}}$, $\boldsymbol{\eta}(\boldsymbol{\delta}) \equiv \left\{\frac{(1-\delta)}{\delta} \cdot \frac{1}{[\mathbf{r}-(1-\mu_h)]}\right\}$.

Proof:

Combine (1) and (2), we have

$$\frac{(1-\delta)c}{\delta} \leq (\mu_h - \mu_l)\rho_s(v_s - w_{s-1}) \tag{4}$$

It is equivalent to

$$w_{s-1} \leq v_s - \frac{(1-\delta)c}{\delta(\mu_h - \mu_l)\rho_s}$$

Substitute above inequality for w_{s-1} in equation (1), we can obtain

$$v_s \leq (1-\delta)(u-c) + \delta\left\{[1-(1-\mu_h)\rho_s]v_s + (1-\mu_h)\rho_s\left[v_s - \frac{(1-\delta)c}{\delta(\mu_h - \mu_l)\rho_s}\right]\right\}$$

Rearrange it,

$$v_s \leq u - c - \frac{(1-\mu_h)c}{(\mu_h - \mu_l)}$$

Obviously v_s is maximized when IC condition binds,

$$v_s = u - c - \frac{(1-\mu_h)c}{(\mu_h - \mu_l)} \equiv v \qquad (s \geq 1)$$

$$w_{s-1} = \left[u - c - \frac{(1-\mu_h)c}{(\mu_h - \mu_l)}\right] - \frac{(1-\delta)c}{\delta(\mu_h - \mu_l)\rho_s} \tag{5}$$

And by (3), we have

$$w_s = \frac{\delta(1-\mu_h)\rho_s \cdot v}{[1-\delta+\delta(1-\mu_h)\rho_s]}$$

Or equivalently, for $s \geq 2$

$$w_{s-1} = \frac{\delta(1-\mu_h)\rho_{s-1} \cdot v}{1-\delta+\delta(1-\mu_h)\rho_{s-1}} \tag{6}$$

Combine (5) and (6),

$$\frac{\delta(1-\mu_h)\rho_{s-1} \cdot v}{1-\delta+\delta(1-\mu_h)\rho_{s-1}} = v - \frac{(1-\delta)c}{\delta(\mu_h - \mu_l)\rho_s}$$

Thus,

$$\rho_s = \frac{1}{v(\mu_h - \mu_l)/c} \cdot \left[(1-\mu_h)\rho_{s-1} + \frac{1-\delta}{\delta}\right] \equiv \frac{1}{r} \cdot \left[(1-\mu_h)\rho_{s-1} + \frac{1-\delta}{\delta}\right] \tag{7}$$

Moreover, when (IC) condition binds, (4) becomes

$$\frac{(1-\delta)c}{\delta} = (\mu_h - \mu_l)\,\rho_s(v_s - w_{s-1}) \tag{8}$$

By initial conditions $v_1 = v,\ w_0 = 0$ and (8), we have

$$\rho_1^* = \frac{(1-\delta)c}{\delta(\mu_h - \mu_l)v} \equiv \frac{(1-\delta)}{\delta r}$$

When $s = 2$, $v_2 = v,\ w_1 = 0$, by (8) we obtain

$$\rho_2^* = \rho_1^*$$

And from (3) we have

$$w_2^* = \frac{\delta(1-\mu_h)\rho_2^* \cdot v}{[1-\delta+\delta(1-\mu_h)\rho_2^*]} = \frac{(1-\mu_h)v}{r+(1-\mu_h)}$$

When $s = 3$, $v_3 = v$, $w_2 = w_2^*$, by (8) we obtain

$$\rho_3^* = \frac{(1-\delta)c}{\delta(\mu_h-\mu_l)(v-w_2^*)} = \left\{\frac{(1-\delta)}{\delta}\cdot\frac{1}{[r-(1-\mu_h)]}\right\}\left\{1-\left[\frac{(1-\mu_h)}{r}\right]^2\right\}$$

$$w_3^* = \frac{\delta(1-\mu_h)\rho_3^* \cdot v}{[1-\delta+\delta(1-\mu_h)\rho_3^*]} = \frac{(1-\mu_h)[r+(1-\mu_h)]v}{r^2+(1-\mu_h)[r+(1-\mu_h)]}$$

When $s = 4$, $v_4 = v$, $w_3 = w_3^*$, by (8) we obtain

$$\rho_4^* = \frac{(1-\delta)c}{\delta(\mu_h-\mu_l)(v-w_3^*)} = \frac{(1-\delta)}{\delta}\frac{[r^2+(1-\mu_h)[r+(1-\mu_h)]]}{r^3}$$

$$= \left\{\frac{(1-\delta)}{\delta}\cdot\frac{1}{[r-(1-\mu_h)]}\right\}\left\{1-\left[\frac{(1-\mu_h)}{r}\right]^3\right\}$$

Solving difference equation on ρ_s, (7), with initial condition $\rho_2^* = \frac{(1-\delta)}{\delta r}$, we obtain, for $s \geq 2$

$$\rho_s = \rho_s^* \equiv \begin{cases} \eta(\delta)\left[1-\left(\frac{1-\mu_h}{r}\right)^{s-1}\right], & \text{if } r \neq 1-\mu_h \\ \frac{(s-1)(1-\delta)}{\delta r}, & \text{if } r = 1-\mu_h \end{cases}$$

Where $r \equiv \frac{v(\mu_h-\mu_l)}{c}$, $\eta(\delta) \equiv \left\{\frac{(1-\delta)}{\delta}\cdot\frac{1}{[r-(1-\mu_h)]}\right\}$.

Therefore, for $s \geq 2$

$$w_s = \frac{\delta(1-\mu_h)\rho_s \cdot v}{[1-\delta+\delta(1-\mu_h)\rho_s]}$$

$$= \begin{cases} \dfrac{(1-\mu_h)\left[1-\left(\frac{1-\mu_h}{r}\right)^{s-1}\right]v}{[r-(1-\mu_h)]+(1-\mu_h)\left[1-\left(\frac{1-\mu_h}{r}\right)^{s-1}\right]}, & \text{if } r \neq 1-\mu_h \\ \dfrac{(1-\mu_h)(s-1)v}{r+(1-\mu_h)(s-1)}, & \text{if } r = 1-\mu_h \end{cases}$$

Q.E.D

By Proposition 3, the maximal payoff that the incumbent firm can obtain in any recursive belief equilibrium is less than the maximal feasible payoff $(v < u - c)$ even $\delta \to 1$. Folk Theorem fails here because there are short-run players and uncertainty in production process, punishment is eventually unavoidable to maintain firm's incentive to exert high effort although consumers know that on the equilibrium path bad signal comes only from bad luck. And w_s^* is a weighted sum of v and 0, where the weight depends on belief state s.

Taking mixed strategy does not improve the incumbent firm's payoff, because consumers' expected payoff will be negative when the incumbent firm exerts low effort with positive probability in the equilibrium path, "Buy" will not be an optimal response then.

Note that $r \equiv \frac{v(\mu_h-\mu_l)}{c}$, so "$r = 1-\mu_h$ " is equivalent to "$u - c = \frac{2(1-\mu_h)c}{(\mu_h-\mu_l)}$ ". For $s \geq 2$, let

$$\delta_s \equiv \begin{cases} \dfrac{1}{1+r}, & \text{if } s = 1 \\ \dfrac{1}{1+\dfrac{[r-(1-\mu_h)]}{\left[1-\left(\frac{1-\mu_h}{r}\right)^{s-1}\right]}}, & \text{if } s \geq 2 \text{ and } u - c \neq \dfrac{2(1-\mu_h)c}{(\mu_h-\mu_l)} \\ \dfrac{s-1}{r+s-1}, & \text{if } s \geq 2 \text{ and } u - c = \dfrac{2(1-\mu_h)c}{(\mu_h-\mu_l)} \end{cases}$$

Proposition 4: Necessary and sufficient conditions for the existence of state s equilibria.

(i) State $s\ (1 \leq s < \infty)$ equilibrium exists if and only if $u - c > \frac{(1-\mu_h)c}{(\mu_h-\mu_l)}$ and $\delta \geq \delta_s$.

(ii) State ∞ equilibrium exists if and only if $u - c > \frac{2(1-\mu_h)c}{(\mu_h-\mu_l)}$ and $\delta \geq \frac{1}{r+\mu_h}$.

(iii) Only state 0 equilibrium exists if $u - c < \frac{(1-\mu_h)c}{(\mu_h-\mu_l)}$ or $\delta < \delta_1$.

Proof:

By the proof of Proposition 3, a necessary condition for the existence of state s equilibrium is:

$$v \equiv u - c - \frac{(1-\mu_h)c}{(\mu_h-\mu_l)} > 0$$

It is equivalent to $u > \frac{(1-\mu_l)c}{(\mu_h-\mu_l)}$.

Moreover, when $s = 1$, the minimum δ satisfying inequality (4) could be obtained by letting $\rho_1 = 1$ and (4) binds. Thus the critical discount factor for state 1 equilibrium is:

$$\delta_1 = \frac{1}{1+\frac{v(\mu_h-\mu_l)}{c}} \equiv \frac{1}{1+r}$$

When $s \geq 2$, the existence of state s equilibrium requires $\rho_s \leq 1$. Hence for $r \neq 1-\mu_h$, or equivalently for $u - c \neq \frac{2(1-\mu_h)c}{(\mu_h-\mu_l)}$,

$$\left\{\frac{(1-\delta)}{\delta}\cdot\frac{1}{[r-(1-\mu_h)]}\right\}\left[1-\left(\frac{1-\mu_h}{r}\right)^{s-1}\right] \leq 1$$

$$\Leftrightarrow \qquad \delta \geq \frac{1}{1+\dfrac{[r-(1-\mu_h)]}{\left[1-\left(\frac{1-\mu_h}{r}\right)^{s-1}\right]}} = \delta_s$$

Hence, $\delta_\infty = \frac{1}{r+\mu_h} < 1$ and state ∞ equilibrium exists if $r > 1-\mu_h$ or equivalently $u - c >$

$\frac{2(1-\mu_h)c}{(\mu_h-\mu_l)}$. $\delta_\infty = \lim_{s\to\infty}\delta_s = 1$ and state ∞ equilibrium does not exist for $\delta < 1$ if $r < 1-\mu_h$.

For $r = 1-\mu_h$ or equivalently $u - c = \frac{2(1-\mu_h)c}{(\mu_h-\mu_l)}$,

$$\rho_s^* = \frac{(s-1)(1-\delta)}{\delta r} \leq 1,$$

$$\Leftrightarrow \qquad \delta \geq \frac{s-1}{r+s-1} = \delta_s$$

Moreover, $\delta_\infty = \lim_{s\to\infty}\delta_s = 1$ and state ∞ equilibrium does not exist for $\delta < 1$ if $r = 1-\mu_h$.

The conclusions hold by the definition of δ_s.

Q.E.D

By Proposition 3 and 4, the critical discount factor $\delta_1 = \delta_2$, optimal probability of punishment $\rho_2^* = \rho_1^*$. Because in "state 2 equilibrium", the incumbent firm will become a waiting firm in "state 1 equilibrium" and its continuation payoff will be 0 after punished in state 2; while in state 1 equilibrium, the incumbent firm will never be trusted and its continuation payoff will also be 0 after punished in state 1. The incentives faced by them are the same, so are optimal punishment probabilities and the critical discount factors. However, $\delta_4 \neq \delta_3$, $\rho_4^* \neq \rho_3^*$. The incumbent firm in state 4 equilibrium has another chance, to be incumbent in state 2, and the incumbent firm in state 3 equilibrium has also another chance, to be incumbent in state 1; but continuation payoffs of the two after being punished are different, hence different for optimal probabilities of punishment and critical discount factors. In state 4 equilibrium the incumbent firm's continuation payoff after being punished, w_3^*, depends on the waiting time in state 3, which in turn depends on the optimal probability of punishment on the incumbent firm in state 3; while in state 3 equilibrium the incumbent firm's continuation payoff after being punished, w_2^*, depends on the waiting time in state 2, which in turn depends on the optimal probability of punishment on the incumbent firm in state 2. Nevertheless, the optimal probability of punishment on the incumbent firm in state 3 is different from that of state 2, for the incumbent firm in state 3 will have another chance, to be incumbent in state 1, while the incumbent in state 2 will be no chance after being punished.

More generally, by Proposition 3 and 4, we have the following corollary.

Corollary 5: For all $s \geq 2$,
(i) The critical discount factor δ_s is strictly increasing in s.
(ii) The optimal probability of punishment ρ_s^* is strictly increasing in s.
(iii) The equilibrium payoff of the waiting firm w_s^* is strictly increasing in s..

Proof:

(i) If $s \geq 2$ and $r > 1-\mu_h$, $\frac{r-(1-\mu_h)}{1-\left(\frac{1-\mu_h}{r}\right)^{s-1}}$ will be strictly decreasing in s, hence by definition of δ_s, δ_s will be strictly increasing in s. If $s \geq 2$ and $r < 1-\mu_h$, $\frac{r-(1-\mu_h)}{1-\left(\frac{1-\mu_h}{r}\right)^{s-1}}$ will also be strictly decreasing in s，hence δ_s strictly increasing in s. If $s \geq 2$ and $r =$

$1-\mu_h$, $\delta_s = \frac{s-1}{r+s-1}$ will obviously be strictly increasing in s.

(ii) If $s \geq 2$ and $r = 1-\mu_h$, $\rho_s^* = \frac{(s-1)(1-\delta)}{\delta r}$ will obviously be strictly increasing in s. If $s \geq 2$ and $r \neq 1-\mu_h$,

$$\frac{d\rho_s^*}{ds} = -\eta(\delta)\left(\frac{1-\mu_h}{r}\right)^{s-1}\ln\left(\frac{1-\mu_h}{r}\right)$$

Thus when $1-\mu_h > r$, $\eta(\delta) \equiv \left\{\frac{(1-\delta)}{\delta}\cdot\frac{1}{[r-(1-\mu_h)]}\right\} < 0$, $\ln\left(\frac{1-\mu_h}{r}\right) > 0$, $\frac{d\rho_s^*}{ds} > 0$; when $1-\mu_h < r$, $\eta > 0$, $\ln\left(\frac{1-\mu_h}{r}\right) < 0$, $\frac{d\rho_s^*}{ds} > 0$.

(iii) If $s \geq 2$ and $r = 1-\mu_h$, $w_s^* = \frac{(1-\mu_h)(s-1)v}{r+(1-\mu_h)(s-1)}$ will obviously be strictly increasing in s. If $s \geq 2$ and $r \neq 1-\mu_h$,

$$w_s^* = \frac{(1-\mu_h)\left[1-\left(\frac{1-\mu_h}{r}\right)^{s-1}\right]v}{[r-(1-\mu_h)]+(1-\mu_h)\left[1-\left(\frac{1-\mu_h}{r}\right)^{s-1}\right]}$$

$$= \frac{v}{\frac{[r-(1-\mu_h)]}{(1-\mu_h)\left[1-\left(\frac{1-\mu_h}{r}\right)^{s-1}\right]}+1}$$

As in (i), $\frac{[r-(1-\mu_h)]}{\left[1-\left(\frac{1-\mu_h}{r}\right)^{s-1}\right]}$ is strictly decreasing in s, hence w_s^* is strictly increasing in s.

Q.E.D

Corollary 5 reveals interesting dynamics on firm's reputation. When we consider the timing of the game in reality, then for any state s equilibrium, it must be first to experience the state s, then the state s-1, s-2... 1, 0 in turn. So the optimal probability of punishment will decrease in the process of equilibrium state. Consumers are more and more "tolerant", or less and less sensitive to incumbent firm's quality signal as from state s to s-1…2, 1. It is not any kind of "inertia" of consumers' belief due to long-term success of firm (such as in Crips, Mailath and Samuelson, 2004), but because firms have less and less opportunities to be incumbent again, which itself constitutes a growing potential punishment. For the same reason, the critical discount factor to support state s equilibrium is strictly decreasing in the process of equilibrium state.

Moreover, from state s to state 1, the payoff of waiting firm is decreasing in the process of equilibrium state. Because the payoff of waiting firm is just from waiting for the failure of the incumbent firm and substitute it. With the state proceeds from s to 2, this opportunity is getting smaller.

5 Perfect Public Equilibrium Payoff Set

Consider a Euclidean Plane, where horizontal coordinate represents firm 1's payoff, vertical coordinate represents firm 2's payoff. Since consumers are short-run players and maximize current period payoff, the joint payoff of two firms in each period cannot be more than $\mathrm{u}-\mathrm{c}$. The feasible payoff set of two firms is therefore a triangle in this plane: $\{(\mathrm{v}_1, \mathrm{v}_2): 0 \le \mathrm{v}_1 \le \mathrm{u}-\mathrm{c},$ $0 \le \mathrm{v}_2 \le \mathrm{u}-\mathrm{c},\ 0 \le \mathrm{v}_1 + \mathrm{v}_2 \le \mathrm{u}-\mathrm{c}\}$.

Define $\overline{\mathrm{V}}_{\mathrm{s}}$ as the pentagon connected by following vertices: $(0,0), (\mathrm{v}, 0), (0, \mathrm{v}), (\mathrm{v}, w_s^*), (w_s^*, \mathrm{v})$, where $(\mathrm{v}, w_s^*), (w_s^*, \mathrm{v})$ are payoff profiles for state s equilibrium when the incumbent firm's payoff is maximized and firm 1 or 2 is the incumbent firm respectively; $(\mathrm{v}, 0), (0, \mathrm{v})$ are payoff profiles for state 1 equilibrium when firm 1 or 2 is the incumbent firm respectively and $(0,0)$ is payoff profile for state 0 equilibrium. Note that points at the connected lines and inner regions could be supported as equilibrium payoffs through a public randomization on above five points.

$\overline{V}_s$ degenerates to a single point $(0,0)$ when $\mathrm{s}=0$. And when $\mathrm{s}=1$, $\overline{\mathrm{V}}_{\mathrm{s}}$ degenerates to a triangle since $w_1^* = 0$. However, by proposition 3 and 4, state 2 equilibrium exists whenever state 1 equilibrium exists. Thus we omit $\overline{\mathrm{V}}_1$

By proposition 3 and 4, $\overline{V}_s$ are as following Figure 2 (shaded region) when state $\mathrm{s} \to \infty$. $\mathrm{v}_\infty + \mathrm{w}_\infty = \mu_\mathrm{h} - \mathrm{c}$ if $\mathrm{v} > \frac{(1-\mu_\mathrm{h})c}{(\mu_\mathrm{h}-\mu_\mathrm{l})}$, so the boundary of feasible payoff set is reached. Otherwise, $\mathrm{v} \le \frac{(1-\mu_\mathrm{h})c}{(\mu_\mathrm{h}-\mu_\mathrm{l})}$, $\lim_{s\to\infty} w_s^* = \mathrm{v}$, $\mathrm{v} + \lim_{s\to\infty} w_s^* = 2\mathrm{v} < \mu_\mathrm{h} - \mathrm{c}$, thus the boundary of feasible payoff set may not be supported as recursive belief equilibrium.

(Insert Figure 2 here)

More generally, for $2 \le \mathrm{s} < \infty$, $\overline{\mathrm{V}}_{\mathrm{s}}$ are as following Figure 3.

(Insert Figure 3 here)

Our purpose is to prove that, $\overline{\mathrm{V}}_{\mathrm{s}} = E(\delta)$ for any δ and $0 \le \mathrm{s} \le \infty$, where $E(\delta)$ is the payoff set of perfect public equilibria of the game given δ . Thus any point similar to A in Figure 3 could not be an PPE payoff profile, and any point similar to B, C in Figure 3 would not be PPE payoff profiles if they could not be supported by recursive equilibrium payoff profiles. We first prove an important lemma before providing the critical proposition and its proof.

Lemma 6: for any $\{v_1, v_2\} \in E(\delta)$, there exist $\{w_1(y_1), w_2(y_2)\} \in E(\delta)$ and $a \in \mathbf{A}$ such that, $(\mathbf{i}, \mathbf{j} = 1, 2)$

$$\boldsymbol{v_i = (1-\delta) g_i(a_i, \alpha_{-i}) + \delta \sum_{y_i \in \{y_h, y_l, y_{null}\}} \pi_i(a_i, \alpha_{SR,i}) w_i(y_i) \quad for\ any\ a_i\ s.t.\ \alpha_i(a_i) > 0}$$

And if $a_i = a_h$, $a_j = a_l$, there will be $v_j = \delta \sum_{y_i} \pi_i(a_i, \alpha_{SR,i}) w_j(y_i)$.

Proof:

According to APS (1990), for any $\{v_1, v_2\} \in E(\delta)$, there exist $\{w_i(y_i, y_j), w_j(y_i, y_j)\} \in E(\delta)$ and a ∈ A such that

$$v_i = (1-\delta) g_i(a_i, \alpha_{-i}) + \delta \sum_{y} \pi_y(a_i, a_j, \alpha_{SR}) w_i(y_i, y_j)$$
$$for\ any\ a_i\ s.t.\ \alpha_i(a_i) > 0$$

By assumption signals of two firms are independent, $\pi_y(a_i, a_j, \alpha_{SR}) = \pi_i(a_i, \alpha_{SR,i}) \pi_j(a_j, \alpha_{SR,j})$, it follows that,

$$\begin{aligned} v_i &= (1-\delta) g_i(a_i, \alpha_{-i}) + \delta \sum_{y} \pi_i(a_i, \alpha_{SR,i}) \pi_j(a_j, \alpha_{SR,j}) w_i(y_i, y_j) \\ &= (1-\delta) g_i(a_i, \alpha_{-i}) + \delta \sum_{y_i} \pi_i(a_i, \alpha_{SR,i}) \sum_{y_j} \pi_j(a_j, \alpha_{SR,j}) w_i(y_i, y_j) \\ &\equiv (1-\delta) g_i(a_i, \alpha_{-i}) + \delta \sum_{y_i} \pi_i(a_i, \alpha_{SR,i}) w_i(y_i) \end{aligned}$$

where $w_i(y_i) \equiv \sum_{y_j} \pi_j(a_j, \alpha_{SR,j}) w_i(y_i, y_j)$. For $j \neq i$, it can also be demonstrated that there exists $w_j(y_j)$ such that

$$v_j = (1-\delta) g_j(a_j, \alpha_{-j}) + \delta \sum_{y_j} \pi_j(a_j, \alpha_{SR,j}) w_j(y_j)$$

By Proposition 3 we know that "$a_i = a_h$ and $a_j = a_l$" could be supported as perfect public equilibrium. If $a_i = a_h$ and $a_j = a_l$, $\pi_j(a_j, \alpha_{SR,j}) = 1$ for $y_j = y_{null}$, $g_j(a_j, \alpha_{-j}) = 0$, for it could not be on the equilibrium path that "Consumers choose Buy when Firm j exert Low Effort", i.e., "$a_j = a_l$, $\alpha_{SR,j} > 0$ for $a_{SR,j} = \text{Buy}$" could not be on the equilibrium path. It follows that

$$v_j = \delta w_j(y_{null})$$

$w_j(y_{null})$ should also be an equilibrium payoff by APS(1990), and firm j is just a waiting firm in current period, thus for each public signal of i, $y_i \in \{y_h, y_l\}$, there exists $w_j(y_i)$ such that $\{w_i(y_i), w_j(y_i)\} \in E(\delta)$ and

$$w_j(y_{null}) = \sum_{y_i} \pi_i(a_i, \alpha_{SR,i}) w_j(y_i)$$

Thus when $a_i = a_h$, $a_j = a_l$,

$$v_j = \delta \sum_{y_i} \pi_i(a_i, \alpha_{SR,i}) w_j(y_i)$$

Q.E.D

The intuition behind Lemma 6 is actually very simple. The continuation payoffs of firms after realization of public signals could be interpreted as implicit incentive contracts between firms and consumers. Public signals of two firms are independent of each other, only depending on their own actions (effort level), thus the signal of each firm is a sufficient statistics for its own action. It is unnecessary to make a firm's continuation payoff depend on the other firm's public signals, such as tournaments and other forms of comparative performance evaluation.

Now we present the critical proposition. The following Proposition 7 constitutes a complete description of payoff sets of public perfect equilibria of the game. And it shows that each point on the boundary of the PPE payoff set can be generated by a class of equilibria with recursive nature, recursive belief equilibrium, or their convex combinations.

Proposition 7:

(i) if $\mathbf{u-c \leq \frac{(1-\mu_h)c}{(\mu_h-\mu_l)}}$, $\boldsymbol{E(\delta)} = \overline{\mathbf{V}}_\mathbf{0} = \{\mathbf{0,0}\}$ for any $\boldsymbol{\delta} \in [\mathbf{0,1})$.

(ii) if $\mathbf{u-c > \frac{(1-\mu_h)c}{(\mu_h-\mu_l)}}$, $\boldsymbol{E(\delta)} = \overline{\mathbf{V}}_\mathbf{s}$ for $\boldsymbol{\delta} \in [\boldsymbol{\delta_s, \delta_{s+1}})$, $\mathbf{0 \leq s < \infty}$, where $\delta_0 = 0$.

Moreover, if $\mathbf{u-c > \frac{2(1-\mu_h)c}{(\mu_h-\mu_l)}}$, $\boldsymbol{\delta_\infty} = \frac{\mathbf{1}}{\mathbf{r+\mu_h}} < \mathbf{1}$ and $\mathbf{E(\boldsymbol{\delta}) = \overline{V}_\infty}$ for $\boldsymbol{\delta} \in [\boldsymbol{\delta_\infty}, \mathbf{1})$; if $\mathbf{\frac{(1-\mu_h)c}{(\mu_h-\mu_l)} < u-c \leq \frac{2(1-\mu_h)c}{(\mu_h-\mu_l)}}$, $\boldsymbol{\delta_\infty} \to \mathbf{1}$.

The whole proof will be divided into four steps. Step 1: in all perfect public equilibria, the maximal payoff of firm i $(i = 1,2)$ given δ is equal to 0 or the incumbent firm's maximal payoff in recursive belief equlibria, $\max_{v_i \in E(\delta)} v_i = \mathrm{v}$ or $\max_{v_i \in E(\delta)} v_i = 0$. This means that any point similar to point A in Figure 3 cannot be a PPE payoff. Meanwhile, if state ∞ equilibrium exists, then the recursive belief equilibria constitute a complete description of perfect public equilibrium. Step 2: in all perfect public equilibria the maximal payoff of firm j $(j = 1,2)$, given $\delta \in [\delta_s, \delta_{s+1})$, $2 \leq \mathrm{s} < \infty$ and the other firm *i*'s payoff equal to v, is equal to the waiting firm's payoff in recursive belief equlibirum, i.e, $w_s^* = \max_{v_j \in E(\delta)} v_j \ s.t. v_i = v, i \neq j$. Remember that $\delta_1 = \delta_2$. This means that any point similar to point B in Figure 3 will not be a PPE payoff if it could not be supported as a recursive belief equilibrium payoff. Step 3: in all recursive belief equilibria, maximal joint payoff is equal to $\mathrm{v} + w_s^*$ or 0 for $\delta \in [\delta_s, \delta_{s+1})$, $2 \leq \mathrm{s} < \infty$, i.e., $\max_{(v_1,v_2) \in \overline{V}_s} v_1 + v_2 = \mathrm{v} + w_s^*$ or $\max_{(v_1,v_2) \in \overline{V}_s} v_1 + v_2 = 0$. Step 4: in all perfect public equilibria, maximal joint payoff is equal to $\mathrm{v} + w_s^*$ or 0 for $\delta \in [\delta_s, \delta_{s+1})$, $2 \leq \mathrm{s} < \infty$, i.e., $\max_{(v_1,v_2) \in E(\delta)} v_1 + v_2 = \mathrm{v} + w_s^*$ or 0. This means that any point similar to point C in Figure 3 will not be a PPE payoff if it could not be supported as a recursive belief equilibrium payoff.

The intuition for Step 4 is as follows. Due to Lemma 6 and step 1-3, to obtain a joint payoff over $\mathrm{v} + w_s^*$, the only case needed to be considered is: at least in one period, two firms are both trusted and exert high efforts. Without loss of generality, we can regard that period as the first period of the repeated game. Again by Lemma 6, comparative performance evaluation would not generate higher payoff, therefore we need only consider the following situation. Proportion *m* $(0 < m < 1)$ consumers only trust firm 1 and buy from 1, and will punish firm 1 with a certain probability after a low-quality signal, namely exit the market. Proportion *1-m* consumers only trust firm 2 and buy form 2, and will punish firm 2 in a similar way. It is not difficult to prove that the highest payoff of firm 1 in a PPE is mv, the highest payoff of firm 2 in a PPE is (1-m) v, and the highest joint payoff is only v.

By Proposition 7 and the definition of $\overline{\mathrm{V}}_\mathrm{s}$, $\delta \in [0, 1)$ can be divided into countable infinite number of sub-intervals $\delta \in [\delta_s, \delta_{s+1})$, such that in each subinterval the maximal possible PPE payoff $v_s = \mathrm{v}, w_s = w_s^*$ is independent of δ, so $\overline{\mathrm{V}}_\mathrm{s}$ is unchanged in each subinterval, and thus $E(\delta)$. Discontinuity occurs only on the boundaries among subintervals, $\delta = \delta_s$, $\mathrm{s} = 2,3,4$ ... Note that $\delta_1 = \delta_2$. Therefore the following corollary is obvious.

Corollary 8：if $\mathbf{u-c>\frac{(1-\mu_h)c}{(\mu_h-\mu_l)}}$, $E(\delta)$ is upper hemicontinuous for all $\delta\in[0,1)$, and $E(\delta)$ is lower hemicontinuous on $\delta\in[0,1)$ except on countable infinite number of points, $\{\delta: \delta=\delta_s,\ s=2,3,\dots\}$.

In the remaining part of this section, we will give the complete proof of Proposition7.

Proof of Proposition 7.

Step 1: $\max\ v_i=\mathrm{v}$, or $\max\ v_i=0$, $\mathrm{i}=1,2$ for any $\{v_1,v_2\}\in E(\delta),\delta\in[0,1)$. $\bar{\mathbf{V}}_\infty=\mathbf{E}(\delta)$ for (i) $\mathbf{u-c>\frac{2(1-\mu_h)c}{(\mu_h-\mu_l)}}$, $\delta\geq\frac{1}{r+\mu_h}$, or (ii) $\frac{(1-\mu_h)c}{(\mu_h-\mu_l)}<\mathbf{u-c}<\frac{2(1-\mu_h)c}{(\mu_h-\mu_l)}$, $\delta\to 1$.

Assume $v_i^*=\max_{\{v_1,v_2\}\in E(\delta)} v_i>0, i=1,2$, it follows that firm *i* will exert High effort in current period and consumers will buy from firm *i* with some positive probability $0<m_i\leq 1$.

Let $\alpha_{SR,i}=\rho_{i,h}\ for\ a_{SR,i}=Not\ Buy\ and\ y_i=y_h$, $\alpha_{SR,i}=1-\rho_{i,h}\ for\ a_{SR,i}=Buy\ and\ y_i=y_h$, $\alpha_{SR,i}=\rho_{i,l}\ for\ a_{SR,i}=Not\ Buy\ and\ y_i=y_l$, $\alpha_{SR,i}=1-\rho_{i,l}\ for\ a_{SR,i}=Buy\ and\ y_i=y_l$ （$0\leq\rho_{ih},\ \rho_{il}\leq 1$）.

In our model, *i*'s current period payoff when exert High Effort, g_{ih} , could be represented as $g_{iH}=m_i(u-c)$, and *i*'s current period payoff when is trusted but exert Low Effort, is $g_{il}=m_iu$. By Lemma 6, it follows that, for any $v_i\in E(\delta)$, there exists $w_{ih},w_{il}\in E(\delta)$ such that

$$\begin{aligned} v_i=&(1-\delta)m_i(u-c)\\ &+\delta\{[\mu_h(1-\rho_{i,h})w_{ih}+\mu_h\rho_{i,h}w_{il}]\\ &+[(1-\mu_h)(1-\rho_{i,l})w_{ih}+(1-\mu_h)\rho_{il}w_{il}]\} \qquad (9)\\ v_i\geq&(1-\delta)m_iu\\ &+\delta\{[\mu_l(1-\rho_{i,h})w_{ih}+\mu_l\rho_{i,h}w_{il}]\\ &+[(1-\mu_l)(1-\rho_{i,l})w_{ih}+(1-\mu_l)\rho_{i,l}w_{il}]\} \qquad (10)\end{aligned}$$

Combine (9), (10) we have

$$\frac{(1-\delta)m_ic}{\delta}\leq(\mu_h-\mu_l)(\rho_{i,l}-\rho_{i,h})(w_{ih}-w_{il})$$

It is equivalent to

$$\mathrm{w_{il}}\leq\mathrm{w_{ih}}-\frac{(1-\delta)m_ic}{\delta(\mu_h-\mu_l)(\rho_{i,l}-\rho_{i,h})} \qquad (11)$$

Similar to proof of Proposition 3 we obtain,

$$\begin{aligned} v_i\leq&(1-\delta)m_i(u-\mathrm{c})\\ &+\delta\left\{[\mu_h(1-\rho_{i,h})+(1-\mu_h)(1-\rho_{i,l})]w_{ih}\right.\\ &\left.+[\mu_h\rho_{i,h}+(1-\mu_h)\rho_{il}]\left[\mathrm{w_{ih}}-\frac{(1-\delta)m_ic}{\delta(\mu_h-\mu_l)(\rho_{i,l}-\rho_{i,h})}\right]\right\}\\ =&(1-\delta)m_i(u-\mathrm{c})+\delta w_{ih}-\left\{\frac{m_i[\mu_h\rho_{i,h}+(1-\mu_h)\rho_{il}]}{(\rho_{i,l}-\rho_{i,h})}\frac{(1-\delta)c}{(\mu_h-\mu_l)}\right\}\\ \leq&(1-\delta)m_i(u-\mathrm{c})+\delta\max v_i-\left\{\frac{m_i[\mu_h\rho_{i,h}+(1-\mu_h)\rho_{il}]}{(\rho_{i,l}-\rho_{i,h})}\frac{(1-\delta)c}{(\mu_h-\mu_l)}\right\} \qquad (12)\end{aligned}$$

(12) hold for all v_i , Hence

$$\max_{m_i,\rho_{i,l},\rho_{i,h}} v_i \leq m_i(u-\mathrm{c}) - \left\{\frac{m_i[\mu_h\rho_{i,h} + (1-\mu_h)\rho_{il}]}{(\rho_{i,l}-\rho_{i,h})}\frac{c}{(\mu_\mathrm{h}-\mu_\mathrm{l})}\right\}$$

If $u - c > \frac{(1-\mu_H)c}{(\mu_H-\mu_L)}$, it follows that $m_i = 1$ and $\rho_{i,h} = 0$ when v_i is maximized.

$$\max_{m_i,\rho_{i,l},\rho_{i,h}} v_i = (u-c) - (1-\mu_H)\left(\frac{c}{(\mu_H-\mu_L)}\right) \equiv \mathrm{v}$$

If $u - c < \frac{(1-\mu_H)c}{(\mu_H-\mu_L)}$, $m_i = 0$, $\max_{m_i,\rho_{i,l},\rho_{i,h}} v_i = 0$。

Moreover, by Proposition 4, when $\mathrm{v} > \frac{(1-\mu_\mathrm{h})c}{(\mu_\mathrm{h}-\mu_\mathrm{l})}$, $\delta > \frac{1}{\mathrm{r}+\mu_\mathrm{h}}$, state ∞ equilibrium exists and $w_\infty = \frac{(1-\mu_h)c}{(\mu_h-\mu_l)}$, $v_\infty + w_\infty = u - c$. When $0 < \mathrm{v} < \frac{(1-\mu_\mathrm{h})c}{(\mu_\mathrm{h}-\mu_\mathrm{l})}$ and $\delta \to 1$, $\lim_{s\to\infty} w_s \to \mathrm{v}$. In both cases the only differences between payoff set of recursive belief equilibrium, $\bar{V}_\infty$, and feasible payoff set, $\{(v_1, v_2): 0 \leq v_1 + v_2 \leq \mu_\mathrm{h} - \mathrm{c}\}$, lie in regions of $v_1 > v$ and $v_2 > v$. (See Figure 2.) However, $v_1 > v$ or $v_2 > v$ could not be supported as PPE payoffs by above proof. Thus $\bar{V}_\infty = E(\delta)$ under these conditions.

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Step 2: for any $\boldsymbol{\delta} \in [\boldsymbol{\delta_s}, \boldsymbol{\delta_{s+1}})$, $\mathbf{2} \leq \mathbf{s} < \infty$, $\max_{j\neq i} \boldsymbol{v_j} = \boldsymbol{w_s^*}$ if $\boldsymbol{v_i} = \mathbf{v}$, $\mathbf{i} = \mathbf{1}, \mathbf{2}$.

By Step 1, $m_i = 1$, $\rho_{i,h} = 0$ if $v_i = \mathrm{v}$. And there exists $w_{iL} \in E(\delta)$ s.t. (let $\rho \equiv \rho_{il}$)

$$\mathrm{v} = (1-\delta)(u-c) + \delta\{\mu_h\mathrm{v} + (1-\mu_h)[(1-\rho)\mathrm{v} + \rho w_{il}]\}$$

$$\mathrm{v} = (1-\delta)u + \delta\{\mu_l\mathrm{v} + (1-\mu_l)[(1-\rho)\mathrm{v} + \rho w_{il}]\}$$

There are three unknown variables, v, ρ, w_{il} , for two equations[5]. Let w_{il} be undetermined, we have

$$\mathrm{v} = u - c - \frac{c(1-\mu_H)}{(\mu_H-\mu_L)}$$

$$\rho = \frac{(1-\delta)c}{\delta(1-\mu_h)(\mathrm{v}-w_{il})}$$

The latter is equivalent to

$$w_{il} = \mathrm{v} - \frac{(1-\delta)c}{\rho\delta(1-\mu_h)} \qquad (13)$$

Moreover, $m_j = 0$ if $v_i = \mathrm{v}$. Hence $a_j = a_l$. By Lemma 6 there exists $w_{jL} \geq 0$ s.t. $\{w_{iL},\ w_{jL}\} \in E(\delta)$ and,

$$v_j = \delta\{\mu_H v_j + (1-\mu_H)[(1-\rho)v_j + \rho w_{jl}]\}$$

Therefore

$$\max_{\rho,w_{jl}} v_j = \max_{\rho,w_{jl}} \frac{\delta(1-\mu_H)\rho w_{jl}}{1-\delta+\delta(1-\mu_H)\rho} \qquad (14)$$

Obviously at the optimum $w_{jl} = \mathrm{v} > v_j$ and ρ should be maximized under the following constraints

$$w_{il} = \mathrm{v} - \frac{(1-\delta)c}{\rho\delta(\mu_H-\mu_L)} \leq \max v_j$$

It follows that, the larger w_{il} , the larger ρ given $w_{il} \leq \max\ v_j$. Thus the premise to

[5] Here we cannot say that there are two unknown variables, ρ, w_{il} , for two equations and given $v_i = v = \max\ v_i$. Because the exact value of $v = \max\ v_i$ is just determined by the following two equations which mean that (IC) binds.

solve $\max\ v_j\ \ s.t. v_i = \mathrm{v}$ is to solve $\max\ w_{il}\ \ s.t.\ w_{jl} = v$. It means that we can repeat the previous step by swapping letters *i* and *j*.

It may be an infinite loop, then $w_{il} = \max\ \mathrm{v_j}$. Let $\mathrm{v_j^*} = \max\ \mathrm{v_j}$, then

$$\mathrm{v} - \frac{(1-\delta)c}{\rho\delta(\mu_H - \mu_L)} = v_j^* = \frac{\delta(1-\mu_H)\rho \mathrm{v}}{1-\delta+\delta(1-\mu_H)\rho}$$

We know from the proof of Proposition 3 and 4, the above equation requires $\mathrm{u - c} > \frac{2(1-\mu_h)c}{(\mu_h-\mu_l)}$ and $\delta \geq \delta_\infty$, namely, state ∞ equilibrium exists. And it is easily to prove that $v_j^* = w_\infty$.

Otherwise, the solution will be obtained by finite iteration. Suppose it needs *s* times of iteration to solve $\max\ v_j$, then we can define:

$$\mathrm{q_s} = \max_{\rho} \mathrm{v_j}$$

$$\mathrm{s.t.}\ \ \mathrm{v_i} = \mathrm{v}$$

$$\mathrm{q_{s-1}} = \max_{\rho} w_{il}$$

$$\mathrm{s.t.}\ w_{jl} = \mathrm{v}$$

$$\dots\dots$$

And so on. Note that constraint conditions of above maximization problem have nested forms, namely constraint conditions of solving q_s is solving q_{s-1}, constraint conditions of solving q_{s-1} is solving q_{s-2},… Let the solution to $q_s = \max_{\rho} v_j\ \ s.t. \dots$ is $\rho = \rho_s'$, then by (14)

$$\mathrm{q_s} = \mathrm{v_j^*} = \frac{\delta(1-\mu_H)\rho_s'\mathrm{v}}{1-\delta+\delta(1-\mu_H)\rho_s'}$$

And by (13)

$$\mathrm{q_{s-1}} = \mathrm{w_{il}^*} = \mathrm{v} - \frac{(1-\delta)\mathrm{c}}{\rho_s'\delta(\mu_H-\mu_L)}$$

Thus we get differential equations about ρ_s'. Remaining steps are similar to the proof of Proposition 3, easy to prove that $\rho_s' = \rho_s$, where ρ_s is the optimal probability of punishment in recursive belief equilibria (Proposition 3). It also shows that the solution for above nested maximization problems is just the solution for optimal recursive belief equilibrium.

In a word, given a firm obtaining the maximal possible equilibrium payoff, recursive belief equilibrium is the best equilibrium in all PPE, that is, payoff of the other firm has also reached the maximal point of PPE payoff set.

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In a recursive belief equilibrium, if the incumbent's payoff reduces slightly, can joint payoffs of two firms be increased? For instance, when punishment probability is greater than optimal, so expected waiting time of the waiting firm is reduced and payoff improved, thus the overall effect is not clear. Step 3 provides an answer for this question.

Step 3: for any $\delta \in [\delta_s, \delta_{s+1})$, $0 \leq s < \infty$, $v_s + w_s \leq \mathrm{v} + w_s^*$ for any $\{v_s, w_s\} \in \overline{V}_s$.

In any recursive belief equilibrium,

$$v_s + w_s = \frac{(1-\delta)(\mu_h - c) + \delta(1-\mu_h)\rho_s(v_{s-1}+w_{s-1})}{1-\delta[1-(1-\mu_h)\rho_s]}$$

Given incentive compatibility constraints, the only variable is ρ_s. Consider a change of ρ_s in the margin when $\{v_s, w_s\} = \{v,\ w_s^*\}$, then incentive compatibility condition will not bind, $v_s <$ v and

$$\frac{\partial(v_s + w_s)}{\partial \rho_s} = \frac{\delta(1-\delta)(1-\mu_h)}{\{1-\delta[1-(1-\mu_h)\rho_s]\}^2}[(v_{s-1} + w_{s-1}) - (\mu_h - c)] < 0$$

Thus total payoff of two firms reduces.

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Step 4: for any $\boldsymbol{\delta} \in [\boldsymbol{\delta_s}, \boldsymbol{\delta_{s+1}})$, $\mathbf{0} \le \mathbf{s} < \infty$, $\boldsymbol{v_i + v_j \le \mathrm{v} + w_s^*}$ for any $\{\boldsymbol{v_i}, \boldsymbol{v_j}\} \in \boldsymbol{E(\delta)}$.

Proof of Step 4 is equivalent to solve (15)

$$\max_{i \neq j}\ v_i + v_j$$

$$\text{s.t. } v_i = (1-\delta)g_i(a_i, \alpha_{-i}) + \delta \sum\nolimits_{y_i} \pi_i(a_i, \alpha_{SR,i})w_i(y_i)$$

$$v_i \ge (1-\delta)g_i(a_i{'}, \alpha_{-i}) + \delta \sum\nolimits_{y_i} \pi_i(a_i', \alpha_{SR,i})w_i(y_i)$$

$$v_j = (1-\delta)g_j(a_j, \alpha_{-j}) + \delta \sum\nolimits_{y_j} \pi_j(a_j, \alpha_{SR,j})w_j(y_j)$$

$$v_j \ge (1-\delta)g_j(a_j{'}, \alpha_{-j}) + \delta \sum\nolimits_{y_j} \pi_j(a_j{'}, \alpha_{SR,j})w_i(y_j)$$

$$\{w_i, w_j\} \in E(\delta)$$

Therefore,

$$v_i +\ v_j$$
$$= (1-\delta)[g_i(a_i, \alpha_{-i}) + g_j(a_j, \alpha_{-j})] + \delta\left[\sum\nolimits_{y_i} \pi_i(a_i, \alpha_{SR,i})w_i(y_i) + \sum\nolimits_{y_j} \pi_j(a_j, \alpha_{SR,j})w_j(y_j)\right]$$

Note constraints imposed by consumers seeking short-term optimization, "$a_i = a_L$ and consumers buy from i with positive probability" cannot be on the equilibrium path. All possibilities are following three cases.

(i) $\boldsymbol{a_i = a_h}$, $\boldsymbol{a_j = a_h}$

Let $\alpha_{SR,i} = \rho_{i,h}$ $for\ a_{SR,i} = Not\ Buy\ and\ y_i = y_h$, $\alpha_{SR,i} = 1 - \rho_{i,h}$ for $a_{SR,i} = Buy\ and\ y_i = y_h$, $\alpha_{SR,i} = \rho_{i,l}$ for $a_{SR,i} = Not\ Buy\ and\ y_i = y_l$, $\alpha_{SR,i} = 1 - \rho_{i,l}\ for\ a_{SR,i} = Buy\ and\ y_i = y_l$ ($0 < \rho_{ih}, \rho_{il} \le 1$). Then by (12) in Step 1,

$$v_i \le (1-\delta)m_i(u - \mathrm{c}) + \delta w_{ih} - \left\{\frac{m_i[\mu_h\rho_{i,h} + (1-\mu_h)\rho_{il}]}{(\rho_{i,l} - \rho_{i,h})}\frac{(1-\delta)c}{(\mu_h - \mu_l)}\right\}$$

Similarly we have

$$v_j \le (1-\delta)m_j(u - \mathrm{c}) + \delta w_{jh} - \left\{\frac{m_j[\mu_h\rho_{j,h} + (1-\mu_h)\rho_{jl}]}{(\rho_{j,l} - \rho_{j,h})}\frac{(1-\delta)c}{(\mu_h - \mu_l)}\right\}$$

Therefore

$$v_i + v_j \le (1-\delta)(m_i + m_j)(u - \mathrm{c}) + \delta(w_{ih} + w_{jh})$$
$$- \frac{(1-\delta)c}{(\mu_h - \mu_l)}\left\{\frac{m_i[\mu_h\rho_{i,h} + (1-\mu_h)\rho_{il}]}{(\rho_{i,l} - \rho_{i,h})} + \frac{m_j[\mu_h\rho_{j,h} + (1-\mu_h)\rho_{jl}]}{(\rho_{j,l} - \rho_{j,h})}\right\}$$

$$\le (1-\delta)(u - \mathrm{c}) + \delta \max(v_i + v_j)$$

$$- \frac{(1-\delta)c}{(\mu_h - \mu_l)}\left\{\frac{m_i[\mu_h\rho_{i,h} + (1-\mu_h)\rho_{il}]}{(\rho_{i,l} - \rho_{i,h})} + \frac{m_j[\mu_h\rho_{j,h} + (1-\mu_h)\rho_{jl}]}{(\rho_{j,l} - \rho_{j,h})}\right\}$$

The above inequality holds for all $v_i + v_j$, hence

$$\max(v_i + v_j) \le (1-\delta)(u-\mathrm{c}) + \delta\max(v_i + v_j)$$
$$-\frac{(1-\delta)c}{(\mu_h-\mu_l)}\left\{\frac{m_i[\mu_h\rho_{i,h} + (1-\mu_h)\rho_{il}]}{(\rho_{i,l}-\rho_{i,h})} + \frac{m_j[\mu_h\rho_{j,h} + (1-\mu_h)\rho_{jl}]}{(\rho_{j,l}-\rho_{j,h})}\right\}$$

Rearranging terms and noting that the inequality binds at max $(v_i + v_j)$, we have,

$$\max_{m_i,m_j,\rho_{i,h},\rho_{i,l},\rho_{j,h},\rho_{j,l}}(v_i + v_j)$$
$$= \max_{m_i,m_j,\rho_{i,h},\rho_{i,l},\rho_{j,h},\rho_{j,l}}\left\{(u-\mathrm{c}) - \frac{c}{(\mu_h-\mu_l)}\left\{\frac{m_i[\mu_h\rho_{i,h} + (1-\mu_h)\rho_{il}]}{(\rho_{i,l}-\rho_{i,h})} + \frac{m_j[\mu_h\rho_{j,h} + (1-\mu_h)\rho_{jl}]}{(\rho_{j,l}-\rho_{j,h})}\right\}\right\}$$

Obviously at the optimum $\rho_{i,h} = \rho_{j,h} = 0$, the above problem reduces to

$$\max(v_i + v_j) = \max_{m_i,m_j}\left\{(u-\mathrm{c}) - \frac{c(1-\mu_h)}{(\mu_h-\mu_l)}(m_i + m_j)\right\}$$
$$s.t. \qquad m_i + m_j \le 1$$

Hence

$$\max(v_i + v_j) = (u-\mathrm{c}) - \frac{c(1-\mu_h)}{(\mu_h-\mu_l)} \equiv \mathrm{v} < \mathrm{v} + w_2^*$$

Similar to the proof of Proposition 3, max $(v_i + v_j) > 0$ requires $(u-\mathrm{c}) > \frac{c(1-\mu_h)}{(\mu_h-\mu_l)}$ and $\delta > \delta_2$. It means that, given that both firms have positive sales (both exert high effort), the sum of equilibrium payoffs of two firms is strictly less than that in state 2 equilibrium.

(ii) $\boldsymbol{a_i = a_h,\ a_j = a_l\ , \mathrm{i}, \mathrm{j} = 1, 2, \mathrm{i} \ne \mathrm{j}}$

Since $a_j = a_l$, firm j has no sale in current period. Let m_i be firm i’s sale in current period. Let $\alpha_{SR,i} = \rho_{i,h}$ *for* $a_{SR,i} = Not\ Buy\ and\ y_i = y_h$, $\alpha_{SR,i} = 1-\rho_{i,h}$ for $a_{SR,i} = Buy\ and\ y_i = y_h$, $\alpha_{SR,i} = \rho_{i,l}$ for $a_{SR,i} = Not\ Buy\ and\ y_i = y_l$, $\alpha_{SR,i} = 1-\rho_{i,l}$ *for* $a_{SR,i} = Buy\ and\ y_i = y_l$ $(0 < \rho_{ih}, \rho_{il} \le 1)$. By Lemma 6 we have

$$v_i = (1-\delta)m_i(u-c) + \delta\{[\mu_h(1-\rho_{i,h})w_{ih} + \mu_h\rho_{i,h}w_{il}] + [(1-\mu_h)(1-\rho_{i,l})w_{ih} + (1-\mu_h)\rho_{il}w_{il}]\}$$
$$v_j = \delta\{[\mu_h(1-\rho_{i,h})w_{jh} + \mu_h\rho_{i,h}w_{jl}] + [(1-\mu_h)(1-\rho_{i,l})w_{jh} + (1-\mu_h)\rho_{il}w_{jl}]\}$$

Thus the maximization problem (15) reduces to

$$\max_{m_i,\rho_{i,h},\rho_{i,l},w}(v_i + v_j)$$
$$= (1-\delta)m_i(u-\mathrm{c}) + \delta\{[\mu_h(1-\rho_{i,h}) + (1-\mu_h)(1-\rho_{i,l})][w_{ih} + w_{jh}] + [\mu_h\rho_{i,h} + (1-\mu_h)\rho_{il}][w_{il} + w_{jl}]\}$$

$$\mathrm{s.t.} \qquad \mathrm{w_{il}} \le \mathrm{w_{ih}} - \frac{(1-\delta)m_i c}{\delta(\mu_h-\mu_l)(\rho_{i,l}-\rho_{i,h})} \qquad (11)$$
$$w_{ih} + w_{jh} \le v_i + v_j$$

$$w_{il} + w_{jl} \leq v_i + v_j$$
$$w_{il} + w_{jl} \leq w_{ih} + w_{jh}$$

The objective function is linear in $\rho_{i,h}$, and the coefficient is $\delta\mu_h\left[\left(w_{il} + w_{jl}\right) - \left(w_{ih} + w_{jh}\right)\right] \leq 0$. Thus the smaller $\rho_{i,h}$, the larger the objective function. And according to the constraint condition (11), the smaller $\rho_{i,h}$, the larger RHS of (11). So $\rho_{i,h} = 0$ when $\left(v_i + v_j\right)$ is maximized. Moreover, the objective function is linear in $\rho_{i,l}$, and the coefficient is $\delta(1 - \mu_h)\left[\left(w_{il} + w_{jl}\right) - \left(w_{ih} + w_{jh}\right)\right] \leq 0$, thus the smaller $\rho_{i,l}$, the larger the objective function. But according to the constraint condition (11), the smaller $\rho_{i,l}$, the smaller RHS of (11), and $\mathrm{RHS} = -\infty$ if $\rho_{i,l} = 0$ (since $\rho_{i,h} = 0$). So inequality (11) must be binding to get an optimal $\rho_{i,l}$. Binding of (11) means that v_i is maximized. Therefore, $\max_{m_i,,\rho_{i,h},\rho_{i,l},w}\left(v_i + v_j\right)$ is equivalent to $\max_{m_i,,\rho_{i,h},\rho_{i,l}} v_i$ first, and then $\max\ \ v_j\ \ s.t.\ v_i\ is\ maximized$. Combine Step 1, 2 and 3, we know that $\max\ \left(v_i + v_j\right) = \mathrm{v} + w_s^*$.

(iii) $a_i = a_l,\ a_j = a_l$ .trivial. $\max\ \left(v_i + v_j\right) = 0.$

Q.E.D

6 Empirical Relevance

A typical example of experience goods is civil air transport services. Air transport service is clearly experience goods, because consumers could not know whether flight problems occur when buying the ticket. After travel, flight problems such as delays are signals easily known to the public: there are many internet sites of aviation and transport in the United States to provide such signals as average "on time" performance, namely ratio of delays less than 15 minutes over a period of time. It also provides detailed information on price, voyage arrangements. Each month, the Department of Transportation of U.S. government published the "Air Travel Consumer Report", including comprehensive statistics of flight delays, luggage misplaced, consumers complaints, and so on. All such information can be openly and freely available from its official website.

Empirical studies have shown that reputation mechanism works in air transport services markets, and the revealed mechanism is similar to recursive belief equilibrium descripted in our paper. For example, (i) sometimes it is effective to implement the punishment by consumers' switch between airlines. Suzuki (2000), using data on Atlanta- O'Hare route in the United States, found that market share fluctuations of major airlines in that route from 1990 to 1997 can be explained by the "on time" performance; On average, passengers are more inclined to switching to another air company after experienced flight delays. (ii) Sometimes it is necessary to make air companies lose reputation completely and be driven out of the market. Foreman and Shea (1999) analyzed 14 major U.S. airlines in 1988 – 1995, found that the market achieved better performance and improved qualities after publication of performance information; Four of 14 air companies exited U.S. market in 1988-1995.

7 Conclusions

In this paper, we build a simple model of duopolistic experience goods markets, and then use the model as an example to solve payoff sets of perfect public equilibria for repeated games with imperfect public monitoring and short-run players. Although there are only two firms, two actions, binomial distribution of signals in the example, intuitively it could easily be extended to

N firms, N actions and any type of signal distribution, as long as firms are symmetric and the game has product structure. It will be possible in the future to obtain a general algorithm to compute payoff sets of perfect public equilibria for such kind of repeated games based on the idea of this paper.

Acknowledgements

This paper is a revised version of the author's Ph.D. dissertation paper submitted to Peking University, China. I am grateful to Professor Hongbin Cai, Homou Wu, Hengpeng Zhu and others for useful comments.

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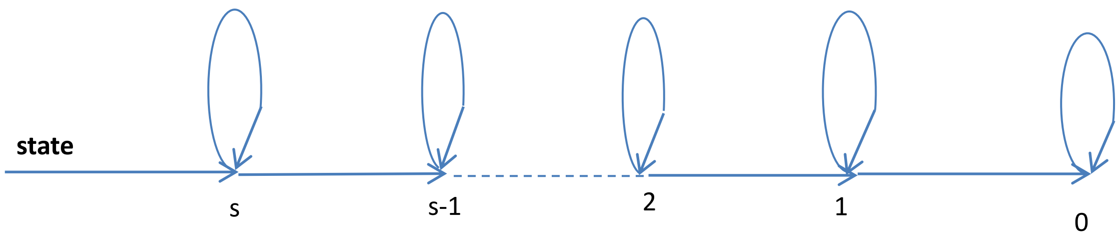
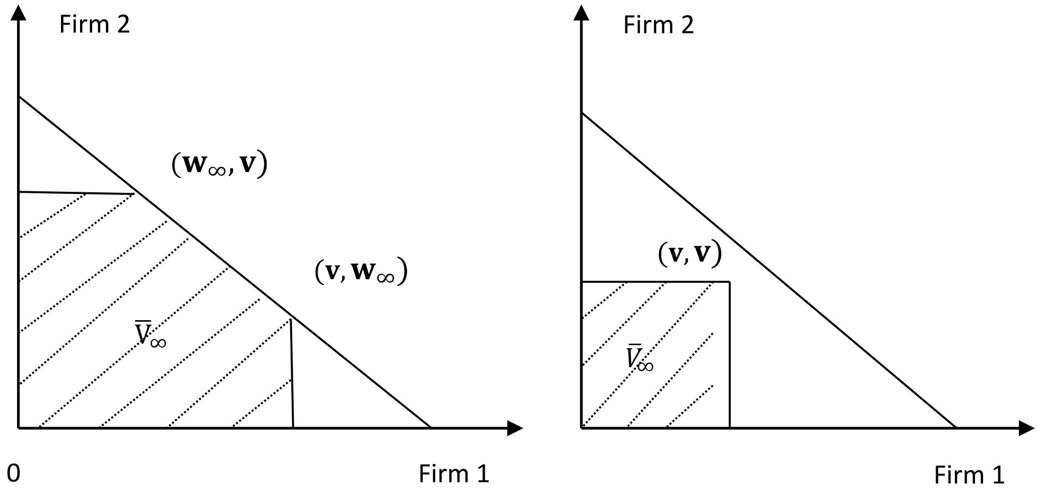


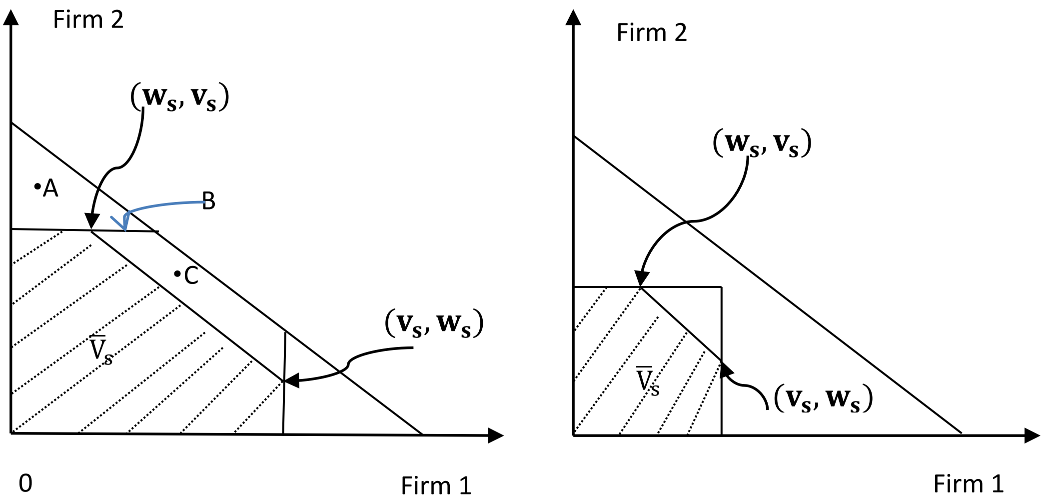
Figure 1: State s as a Markov Process



(i) $\mathbf{u} - \mathbf{c} > \frac{2(1-\mu_h)c}{(\mu_h-\mu_l)}$

(ii) $\frac{(1-\mu_h)c}{(\mu_h-\mu_l)} < \mathbf{u} - \mathbf{c} \leq \frac{2(1-\mu_h)c}{(\mu_h-\mu_l)}$

Figure 2: Payoff Sets of State ∞ equilibra (shaded regions)



(i) $\mathbf{u} - \mathbf{c} > \frac{2(1-\mu_h)c}{(\mu_h - \mu_l)}$

(ii) $\frac{(1-\mu_h)c}{(\mu_h-\mu_l)} < \mathbf{u} - \mathbf{c} \leq \frac{2(1-\mu_h)c}{(\mu_h-\mu_l)}$

Figure 3: Payoff Sets of State s $(2 \leq s < \infty)$ equilibra (shaded regions)