On the continuous equilibria of affiliated-value, all-pay auctions with private budget constraints

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20. March 2011

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January 4, 2012

Abstract

We consider all-pay auctions in the presence of interdependent, affiliated valuations and private budget constraints. For the sealed-bid, all-pay auction we characterize a symmetric equilibrium in continuous strategies for the case of N bidders and we investigate its properties. Budget constraints encourage more aggressive bidding among participants with large endowments and intermediate valuations. We extend our results to the war of attrition where we show that budget constraints lead to a uniform amplification of equilibrium bids. An example shows that with both interdependent valuations and private budget constraints, a revenue ranking between the two mechanisms is generally not possible.

JEL: D44

Keywords: All-Pay Auction, War of Attrition, Budget Constraints, Common Values, Private Values, Affiliation, Contests

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Suppose firms are lobbying for a lucrative government contract. Clearly, the contract's value to each firm has an idiosyncratic component since the firms likely have different operating costs. On the other hand, each firm also has a privately known limit on how much it is able or willing to spend on the lobbying game. Perhaps the management of one firm is approving of small restaurant meals with officials but expenditures or bribes beyond some threshold are morally too much to stomach. A competitor, in contrast, may be less hampered in its lobbying strategy. How does the lobbying game unfold when competitors differ in their valuation for the prize and in their ability or capacity to compete for it? Would some firms spend *more* on lobbying believing that their competitors have to navigate within some private constraints on actions?

In this essay we consider the class of situations like the above by analyzing the all-pay auction. In an all-pay auction, the highest bidder is the winner of the item for sale; however, all bidders incur a payment equal to their bid. As a stylized model of a lobbying contest, the all-pay auction has a long tradition in political economy (Baye *et al.*, 1993).

Despite the frequent application of the all-pay auction to contests, most analyses fail to capture the exogenous but private limits on actions that are relevant in many situations. We introduce these constraints into the all-pay auction and we identify sufficient (and from a practical perspective necessary) conditions for the existence of an equilibrium in continuous strategies. Our analysis isolates a general amplification of bids submitted by bidders with intermediate valuations. An extension of our model to the war of attrition shows the generality of the bid amplification phenomenon and allows for a comparison of the revenue potential of the two mechanisms. Generally, no revenue ranking exists between the two formats in the presence of both budget constraints and affiliated, interdependent values.

Although our model is phrased in the language of auctions (players are called "bidders," etc.), it applies to any situation where resources are irreversibly expended in pursuit of a goal or a prize. The goal or prize can have a value that has both private and common components. The private constraints on bids or effort that we introduce are often quite natural elements of the situation. For example, Hickman (2011) employs a version of the all-pay auction to model students competing for college admissions. It is clear that the value of a college degree varies across students due to personal preferences; it is also natural to assume that idiosyncratic shocks, such as health status, family background, or school location, place an exogenous, heterogenous, and private cap on the "effort" that a student can exert in the college admissions game. These shocks may be orthogonal to the value of the degree per se.

As another example consider a patent race between competing firms. Such competition is naturally modeled as either an all-pay auction or as a war of attrition (Leininger, 1991). The expected value of the invention and the budget available to the research division will determine the effort devoted to the race. However, information asymmetries or agency concerns can create a wedge between the available budget and the research division's assessment of the project's value. Moreover, a firm as a whole likely faces a hard, short-run resource constraint that will cap its feasible effort level. It is natural to suppose that this resource limit is also private information. The interaction between expected rewards and the heterogenous resource constraints will shape how firms engage in this competition.

While we are motivated by the range of applications of all-pay auctions in modeling social and economic situations, our study also fills a gap in the small but growing literature on auctions with private budget constraints. Our analysis begins with the work of Krishna & Morgan (1997) who study the all-pay auction and the war of attrition with interdependent and affiliated valuations. To this setting we introduce private budget constraints distributed continuously on an interval. Our environment parallels the setting of Fang & Parreiras (2002) and Kotowski (2010) who study the second-price and the first-price auction with private budget constraints respectively. Both of these studies build on Che & Gale (1998), which is the seminal paper in this strand of literature.

In light of this literature, our study contributes along several dimensions. First, by focusing on all-pay mechanisms we put under scrutiny an important allocation mechanism in resource-constrained environments. Many authors examining optimal auctions with budget-constrained participants have resorted to mechanisms that feature "all-pay" payment schemes (Maskin, 2000; Pai & Vohra, 2011). Our analysis therefore complements this literature but we do not attempt the mechanism design exercise here.

Second, by developing our model in a more general setting than traditionally employed we are able to identify additional features of the environment that affect the existence of a well-behaved and (relatively) tractable equilibrium. Previous studies lodged in the affiliated and interdependent-value paradigm, such as Fang & Parreiras (2002) and Kotowski (2010), have focused exclusively on the two-bidder case. While some of the intuition from the two-bidder case is relevant generally, the case of two bidders masks much of the nuance that we identify. For example, in the all-pay auction we document how changes in the number of bidders alone directly affects the existence of an equilibrium within the class of strategies traditionally considered by this literature.

Third, in our equilibrium characterization we relax many of the assumptions previously encountered in the analogous settings of Fang & Parreiras (2002) and Kotowski (2010).

In particular, we endeavor to employ more localized assumptions whenever possible. For example, we do not require differentiability or continuity conditions on the distribution of budgets over its entire range. Similarly, we do not need the existence of an unambiguously unconstrained bidder.[1] Accommodating weaker assumptions renders our argument more involved than previous analyses in this environment.

Although we relax many assumptions, we do not study the all-pay auction's equilibrium in its fullest generality. Indeed, from the onset we focus on the existence of an equilibrium that is continuous, symmetric, and monotone. We view this restricted scope to be a pragmatic but reasonable choice. From a technical perspective, this restriction allows us to define equilibrium behavior as a solution to a differential equation. This methodology offers a window on the mechanism's economic properties and gives precise predictions concerning bidder behavior. We believe that the set of cases covered is rich and it offers insights that would carry over to any discontinuous (but symmetric and monotone) equilibrium. Moreover, continuous equilibria would receive the bulk of attention in applications due to their relative tractability.

The remainder of the paper is organized as follows. Section 1 introduces the model and section 2 characterizes our symmetric equilibrium in the all-pay auction. We then consider the equilibrium's comparative static properties with focus on changes in the distribution of budgets, changes in the number of bidders, and changes in the public information surrounding the contest. The final section considers this model's second-price analogue, the war of attrition. We explore the symmetric equilibria of this model and we discuss the scope for a revenue ranking. Proofs and supporting lemmas are in the appendix.

1 The Environment

Let $\mathcal{N} = \{1, \ldots, N\}$ be the set of bidders. Each bidder $i \in \mathcal{N}$ observes a two-dimensional private signal $(s_i, w_i) \in \Theta \subset \mathbb{R}^2_+$. We allow for both $\Theta = [0, 1] \times [\underline{w}, \bar{w}]$ and $\Theta = [0, 1] \times [\underline{w}, \infty)$ but for brevity we will phrase our model in the former case. A bidder's realized *value-signal*, s_i, is her private information about the item for purchase. For example, in a patent race it would be an estimate of the invention's value. Let $\mathbf{s} = (s_1, \ldots, s_N)$ be a profile of realized value-signals.[2] We use capital letters—S_i, etc.—to refer to signals as random variables.

A bidder's realized *budget*, w_i, is a limit above which she cannot bid. We consider a

[1] There exists an unambiguously unconstrained bidder if with positive probability there exists a participant with a budget in excess of the maximum possible value of the item.

[2] We use standard notation: $\mathbf{s}_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_N)$, $\mathbf{s} = (s_i, \mathbf{s}_{-i})$, etc.

budget to be a hard constraint on expenditures. A budget may correspond to a bidder's cash holdings, her credit limit, or some other private limit on actions. Alternatively, budget constraints can be modeled as "soft" constraints acting through an increasing cost of bidding. For brevity, we do not explore this extension. Zheng (2001), among others, is an application of this specification of budget constraints.

We assume that bidder i's valuation for the item can be described by a random variable: $V_i = u(S_i, S_{-i})$. We assume that $u\colon [0,1] \times [0,1]^{N-1} \to [0,1]$ is strictly increasing in the first argument and nondecreasing and permutation-symmetric in the last $N-1$ arguments. As standard, we suppose u is continuously differentiable and normalized such that $u(0,\ldots,0) = 0$ and $u(1,\ldots,1) = 1$.

A bidding strategy is a (measurable) function $\beta_i\colon \Theta \to \mathbb{R}_+$. Let $\mathcal{S}$ be the set of strategies. A strategy is nondecreasing if $s_i' \geq s_i$ and $w_i' \geq w_i$ imply $\beta_i(s_i', w_i') \geq \beta_i(s_i, w_i)$. A strategy is strictly increasing if it is nondecreasing and $\beta_i(s_i', w_i') > \beta_i(s_i, w_i)$ when either $s_i' > s_i$ or $w_i' > w_i$. Throughout, we adopt Bayesian Nash equilibrium as our solution concept. An equilibrium is symmetric if all bidders follow the same bidding strategy. We focus on symmetric equilibria and we henceforth suppress player subscripts in our notation whenever possible.

We always assume that bidders are risk neutral. The introduction of risk aversion into this model introduces complications analogous to those seen in the first-price auction. As shown by Kotowski (2010), the interaction of a bidder's private budget with her risk preferences can introduces countervailing incentives rendering the existence of "monotone" equilibria a more involved question.

We begin with two assumptions on the ambient environment which we maintain throughout. Our initial set-up is standard. Subsequently, we will impose additional restrictions on the information structure that will be sufficient for equilibrium existence. The additional conditions will be specific to the auction format; therefore, we introduce them separately.

Assumption 1. *The distribution of value-signals satisfies the following conditions:*

(a) Value-signals have a joint density $h(s_1, \cdots, s_N)$ which is continuous and strictly positive: $0 < h(s_1, \cdots, s_N)$ for all $(s_1, \cdots, s_N) \in [0,1]^N$.

(b) $h(s_1, \cdots, s_N)$ is invariant to permutations of $(s_1, \ldots, s_N)$.

(c) $h(s_1, \cdots, s_N)$ is log-supermodular. Thus, value-signals are affiliated.

Affiliated signals are a standard assumption introduced to the auction literature by Milgrom & Weber (1982).[3] Except for budget constraints, our model corresponds to their classic environment. Affiliation allows some correlation in player's information. Although more general than independence, it is nonetheless a strong assumption (de Castro, 2010).

Our second assumption about the ambient environment concerns the distribution of a player's budget.

Assumption 2. *Players' budgets are independently and identically distributed according to the cumulative distribution function $G\colon [0,\infty) \to [0,1]$. If $\underline{w}$ ($\bar{w}$) is the infimum (supremum) of the support of G, then $G(\underline{w}) = 0$. Moreover, budgets are distributed independently of value-signals.*

While the independence assumption is strong, without it the model is not tractable. It is standard in studies of auctions with budget constraints. We emphasize that $\underline{w}$ need not be zero and $\bar{w}$ need not be infinite. Indeed, in many situations it would be unreasonable to suppose this to be the case. For example the parameters of the support of G may become common knowledge via posted bonds, (not modeled) participant selection, or obligatory financial disclosures as may occur in the context of political lobbying. In future work we intend to explore the effects of endogenous disclosure of $\underline{w}$ (or $\bar{w}$) but for now we take them as given and common knowledge.

2 A Symmetric, Continuous Equilibrium in the All-Pay Auction

The rules of the all-pay auction are well-known. Each bidder i will simultaneously submit a bid b_i. A bid must be feasible given the bidder's budget: $b_i \leq w_i$. If bidder i submits the highest bid she wins the game and her payoff under the realized signal profile $\mathbf{s} = (s_i, \mathbf{s}_{-i})$ is $u(s_i, \mathbf{s}_{-i}) - b_i$; otherwise, it is $-b_i$. Ties among high bidders are resolved by a uniform randomization to designate the winner.

We endeavor to identify a symmetric equilibrium where all bidders follow a strategy of the form

$$\beta(s, w) = \min\{b(s), w\} \tag{1}$$

where $b\colon [0,1] \to \mathbb{R}_+$ is strictly increasing, continuous, and piecewise differentiable. Let $\mathcal{M} \subset \mathcal{S}$ denote the set of strategies that meet these criteria. We will say that an auction has

[3] We refer the reader to this paper for a summary of the properties of affiliated random variables.

an equilibrium in $\mathcal{M}$ if there exists $\beta \in \mathcal{M}$ which is a symmetric Bayesian Nash equilibrium. Our focus on equilibria meeting these criteria is consistent with previous studies of auctions with private budget constraints. Che & Gale (1998), Fang & Parreiras (2002, 2003), and Kotowski (2010) examine equilibria that reside in this class of strategies.

To motivate the sufficient conditions for equilibrium existence that we propose below, we begin with an heuristic discussion. Suppose for the moment that there is a symmetric equilibrium $\beta(s, w) = \min\{b(s), w\} \in \mathcal{M}$ and consider a bidder of type (s, w). If this player bids $b(x) \leq w$ her bid will defeat two categories of opponents assuming all other bidders are following the strategy $\beta(s, w)$. First it defeats all opponents who have a value-signal $s < x$. Second, it defeats all opponents who have a budget $w < b(x)$ irrespective of value-signal. Since $\beta(s, w)$ is strictly increasing, ties are probability zero events.

To succinctly express a bidder's expected payoff from the bid $b(x)$ require new notation, which we use recurrently. First, for each k let

$$z_k(x|s) \equiv \underbrace{\int_0^1 \cdots \int_0^1}_{N-1-k} \underbrace{\int_0^x \cdots \int_0^x}_{k} u(s, y_1, \ldots, y_{N-1}) h(y_1, \ldots, y_{N-1}|s) dy_1 \cdots dy_{N-1}. \qquad (2)$$

In (2) we have omitted the subscript for player i and we have relabeled the $N-1$ signals of the other bidders as $(y_1, y_2, \ldots, y_{N-1})$.[4] Second, for each k define

$$F_k(x|s) = \underbrace{\int_0^1 \cdots \int_0^1}_{N-1-k} \underbrace{\int_0^x \cdots \int_0^x}_{k} h(y_1, \ldots, y_{N-1}|s) dy_1 \cdots dy_{N-1}. \qquad (3)$$

For $k \geq 1$, $F_k(x|s)$ is the cumulative distribution function of the random variable $\bar{Y}_k = \max(Y_1, \ldots, Y_k)$ given $S = s$. Let $f_k(x|s)$ be the associated density function. Third, if we let

$$v_k(s, y) \equiv \mathbb{E}[u(s, Y_1, \ldots, Y_{N-1})|S = s, \bar{Y}_k = y],$$

then by Lemma 17 we can write

$$z_k(x|s) = \int_0^x v_k(s, y) f_k(y|s) dy. \qquad (4)$$

Depending on the circumstances we may express $z_k(x|s)$ in form (2) or (4).[5]

[4]We do not reorder the signals of the other bidders from greatest to least.

[5]We adopt the following conventions: $z_0(x|s) = v_0(s, y) = \mathbb{E}[u(s, Y_1, \ldots, Y_{N-1})|S = s]$. Additionally,

Defining

$$\gamma_k(b) \equiv \binom{N-1}{k} G(b)^{N-1-k}(1-G(b))^k, \tag{5}$$

we can write the bidder's expected payoff from the bid $b(x)$ as

$$U(b(x)|s,w) = \sum_{k=0}^{N-1} \gamma_k(b(x)) z_k(x|s) - b(x). \tag{6}$$

The binomial terms account for the combinations of opponents who are defeated by $b(x)$ due to having a low value-signal or a low budget. We do not need to keep track of the precise identities of these bidders due to the symmetry assumptions on both valuations and information structure. Introducing asymmetries would necessitate a more detailed accounting of the different cases. The final term in (6) is the bidder's payment which she makes irrespective of the auction's outcome.

If $b(s) < w$ is indeed this player's equilibrium best response, a local first-order optimality condition must be satisfied. Specifically, if $U(b(x)|s,w)$ is sufficiently smooth then

$$\frac{d}{dx} U(b(x)|s,w)\bigg|_{x=s} = 0. \tag{7}$$

Adopting the notation

$$z_k'(x|s) \equiv \frac{\partial}{\partial x} z_k(x|s) = \begin{cases} 0 & \text{if } k=0 \\ v_k(s|x) f_k(x|s) & \text{if } k \neq 0 \end{cases} \tag{8}$$

we can evaluate (7) to derive the following differential equation:

$$b'(s) = \frac{\sum_{k=0}^{N-1} \gamma_k(b(s)) z_k'(s|s)}{1 - \sum_{k=0}^{N-1} \gamma_k'(b(s)) z_k(s|s)}. \tag{9}$$

Our subsequent discussion will identify conditions that ensure (9) has a sensible solution which we later confirm is characteristic of equilibrium bidding.

Two initial observations are worthwhile. First, when $b(s) < \underline{w}$ equation (9) reduces to

$$b'(s) = v_{N-1}(s,s) f_{N-1}(s|s)$$

when $x < 0$, then for $k \geq 1$, $z_k(x|s) = 0$ and $F_k(x|s) = 0$.

which is the differential equation identified by Krishna & Morgan (1997) as characterizing bidding behavior in the all-pay auction absent budget constraints. Therefore, our sufficient conditions must suitably generalize their assumptions. Second, when $b(s) > \underline{w}$, (9) accounts for the change in marginal incentives faced by unconstrained bidders. Slight bid increases not only defeat opponents with slightly higher valuations but they also defeat all opponents with sufficiently low budgets regardless of their valuation. This second effect ameliorates the well-known winner's curse phenomenon in interdependent-value settings.

Regrettably the derivation of (9) was heuristic and we made many implicit assumptions. First, to render the differential approach valid, we will need to impose some degree of smoothness on the distribution of budgets, $G(w)$, and we will need to ensure that the solution of (9) is contained in a region where $G(w)$ is smooth. Second, even if (9) can be solved, it is not clear that an *increasing* solution on all of $[0, 1]$ exists. For example, if the denominator of (9) is ever negative, then $b'(s) < 0$ contradicting our original hypothesis that $b(s)$ is increasing. Finally, we must also be mindful of boundary conditions, which we have not yet considered.

To address the above points and to identify sufficient conditions when (9) does characterize equilibrium bidding we introduce three additional assumptions. Our first assumption will make $G(w)$ continuously differentiable in a relevant range of values. The second assumption will limit the degree of affiliation among bidder's value-signals. The final assumption will place a restriction on the joint distribution of value-signals and budgets. While the first assumption is strictly technical, the latter two assumptions speak to the complicated interaction among incentives faced by bidders in the all-pay auction. We elaborate on these assumptions below.

We begin our analysis by introducing the function

$$\alpha(s) \equiv \int_0^s v_{N-1}(y, y) f_{N-1}(y|y) dy. \tag{10}$$

Krishna & Morgan (1997) show that under suitable conditions $\alpha(s)$ defines the equilibrium bidding strategy in the all-pay auction without private budget constraints. Let $\bar{\alpha} = \alpha(1)$. This value will emerge as an upper bound on equilibrium bids in our model and with reference to it, we can present our first restriction.

Assumption 3. *$G(\cdot)$ is continuously differentiable on $[\underline{w}, \bar{\alpha}]$ and its derivative $g(w)$ on this interval is strictly positive. Moreover, $\underline{w} < \bar{\alpha} < \bar{w}$.*

On absolute terms, the differentiability condition in Assumption 3 is weaker than seen elsewhere in the literature. Unlike Fang & Parreiras (2002), for example, we do not require

$G(w)$ to be differentiable for all $w \leq 1$. However, on terms relative the equilibrium and the auction format, we consider it to be of an equivalent character to their assumption.

The next assumption generalizes the sufficient condition proposed by Krishna & Morgan (1997) supporting $\alpha(s)$ as the equilibrium strategy in the all-pay auction without budget constraints.

Assumption 4. *For all $\mathbf{s}_{-i} \in [0,1]^{N-1}$, $u(\cdot, \mathbf{s}_{-i})h(\mathbf{s}_{-i}|\cdot) \to \mathbb{R}_+$ is nondecreasing and differentiable.*

Intuitively, Assumption 4 limits the degree of correlation between value-signals relative to the impact of a player's own signal on her valuation. The assumption always holds if signals are independent but it can hold in other cases as well. For example it is satisfied when there are two bidders, $u(s_i, s_j) = (s_i + s_j)/2$ and $h(s_i, s_j) \propto 1 + s_i s_j$.

Whereas Assumption 4 places a restriction on the correlation among only value-signals, we additionally require an assumption structuring the joint distribution of value-signals and budgets. Assumption 5 presents this restriction. We defer interpreting this assumption until after presenting our main result and an example illustrating the identified equilibrium.

Definition 1. The value $\tilde{s} \in [0,1)$ is the unique solution to $\alpha(\tilde{s}) = \underline{w}$.

Assumption 5. *For all $(x,s) \in [0,1]^2$ and for all $w \in [\underline{w}, \bar{\alpha}]$, define*

$$\xi(x,w|s) = \gamma_0(w) + \sum_{k=1}^{N-1} \gamma_k(w) \left(1 - \frac{kg(w)}{1-G(w)}(z_{k-1}(x|s) - z_k(x|s))\right). \tag{11}$$

Then the following conditions hold:

(a) For all $s \geq \tilde{s}$, $\exists w_s \in [\underline{w}, \bar{\alpha})$ such that $w \in [\underline{w}, w_s) \implies \xi(s,w|s) < 0$ and $w \in (w_s, \bar{\alpha}] \implies \xi(s,w|s) > 0$.

(b) If $\tilde{s} > 0$, there exists $\epsilon > 0$ such that for all $s \in (\tilde{s} - \epsilon, \tilde{s} + \epsilon)$, $\xi(s, \underline{w}|s) > 0$.

Proposition 1. *Suppose Assumptions 1–5 are satisfied. Then there exists a continuous, symmetric equilibrium of the all-pay auction with budget constraints of the form $\beta(s,w) = \min\{b(s), w\}$ where $b(s)$ is a continuous, strictly increasing, and piecewise differentiable function such that*

$$b(\tilde{s}) = \inf\{w > \underline{w} \colon \xi(\tilde{s}, w|\tilde{s}) > 0\}, \tag{12}$$

and for almost every s, $b'(s) = \frac{\sum_{k=0}^{N-1} \gamma_k(b(s)) z_k'(s|s)}{1 - \sum_{k=0}^{N-1} \gamma_k'(b(s)) z_k(s|s)}$.

Remark 1. When $\underline{w} > 0$, condition (12) reduces to $b(\tilde{s}) = \underline{w}$.

The following example highlights several features of the equilibrium of the all-pay auction.

Example 1. Suppose $N = 2$ and that value-signals are given by $S_i \overset{i.i.d.}{\sim} U[0, 1]$ while budgets $W_i \overset{i.i.d.}{\sim} U[\frac{2}{25}, \frac{3}{4}]$. Let $u(s_i, s_j) = (s_i + s_j)/2$.

It is readily verified that $b(s) = s^2/2$ for $s < \tilde{s} = \frac{2}{5}$. Of course, $\alpha(s) = s^2/2$ is also the equilibrium strategy in this model absent budget constraints. For $s > \frac{2}{5}$, $b(s)$ is the solution to the differential equation

$$b'(s) = \frac{25(3 - 4b(s))s}{25s(3s - 2) + 42}$$

satisfying the boundary condition $b\left(\frac{2}{5}\right) = \frac{2}{25}$. The resulting equilibrium strategy is

$$\beta(s, w) = \begin{cases} \frac{s^2}{2} & s \leq \frac{2}{5} \\ \min\{b(s), w\} & s > \frac{2}{5} \end{cases}$$

[Figure 1 about here.]

Figure 1 plots the functions $b(s)$ and $\alpha(s) = \frac{s^2}{2}$.[6] The introduction of budget constraints rendered $b(s)$ concave for $s > \tilde{s}$ while $\alpha(s)$ is convex. This pronounced change is not a general phenomenon; in other examples $b(s)$ remains convex throughout. What is a general phenomenon, however, is the more aggressive bidding of some types of bidders. To the right of $\tilde{s} = \frac{2}{5}$, $b(s) > \alpha(s)$. Additionally, bidders with very high valuations $s \to 1$ bid less aggressively than in the corresponding equilibrium in this environment without budget constraints. These equilibrium characteristics apply generally as shown by the following corollaries.

Corollary 1. *Suppose the conditions of Proposition 1 hold and* $\underline{w} > 0$.

(a) $\lim_{s \to \tilde{s}^+} b'(s) > \lim_{s \to \tilde{s}^-} b'(s) = \lim_{s \to \tilde{s}^-} \alpha(s)$.

(b) For all (s, w), $\beta(s, w) \leq \alpha(1)$.

The encouragement of more aggressive bidding by bidders with relatively large budgets and intermediate valuations is due to the change in marginal incentives that bidders experience in the presence of budget constraints. The prospect of defeating additional opponents who are budget constrained increases the marginal return of a higher bid; therefore, some types of bidders respond to this incentive with more aggressive bidding.

[6] In all examples in this paper, graphs of numerical solutions are obtained using the Runge-Kutta method.

Discussion and Interpretation

To interpret the sufficient conditions behind Proposition 1 it is useful to examine in detail the role of Assumption 5. Assumption 5(a) asserts that the function $\xi(s, \cdot|s)$ satisfies a single crossing condition and is strictly positive for w sufficiently close to $\bar{\alpha}$.[7] As shown by Lemma 14, $\xi(x, w|s) = 1 - \sum_{k=0}^{N-1} \gamma_k'(w) z_k(x|s)$. Thus, the assumption ensures that the righthand side of the differential equation (9) is eventually strictly positive. While this is Assumption 5's technical role, it also has an economic interpretation that we outline below.

With Lemma 14, we can write the condition $\xi(s, w|s) > 0$ as

$$g(w)(N-1)\left[\sum_{k=0}^{N-2} \binom{N-2}{k} G(w)^{N-2-k}(1-G(w))^k (z_k(s|s) - z_{k+1}(s|s))\right] < 1.$$

To simplify further suppose values are private and value-signals are independent draws from a common distribution with c.d.f. $H(s)$. In this case:

$$z_k(s|s) - z_{k+1}(s|s) = u(s)H(s)^k(1 - H(s)).$$

$\xi(s, w|s) > 0$ now becomes

$$\begin{aligned} g(w)(N-1)u(s)(1-H(s))\left(G(w) + H(s) - G(w)H(s)\right)^{N-2} &< 1 \\ \iff u(s)\frac{d}{dw}\left[G(w) + H(s) - G(w)H(s)\right]^{N-1} &< 1. \end{aligned}$$

The term $[G(w) + H(s) - G(w)H(s)]^{N-1}$ is the probability of all other bidders having a signal less than s and/or a budget less than w. If $b = \beta(s, w)$ as equilibrium bidding assumes, this would be the probability with which bidder i wins the auction with a bid of b. We can therefore regard Assumption 5 as imposing a limit on the rate of change in the probability of winning owing only to defeating opponents who have a smaller budget. If this probability increases too rapidly at some point $\hat{w}$—for instance, due to an "atom"[8] in the distribution of budgets—then as $b(s)$ crosses $\hat{w}$, $b'(s)$ becomes undefined or negative and the continuous strategy we are considering can no longer be an equilibrium.

In an interdependent-value setting, the preceding intuition continues to apply. However,

[7] $\xi(s, \cdot|s)$ may become negative for $w > \bar{\alpha}$.

[8] We are assuming atomless distributions of budgets, but the intuition in the case of an atom offered here is illuminating. In the atomless framework, suppose that $g(\hat{w}) > N - 1$ while $g(w) < \frac{1}{N-1}$ when $w \notin (\hat{w} - \epsilon, \hat{w} + \epsilon)$ for $\epsilon > 0$ but sufficiently small.

it must be extended to incorporate the winner's curse. Defeating low-budget opponents is generally "good news" concerning the expected value of the item. Therefore in its fullest form, (11) additionally incorporates a weighted average controlling for these marginal effects on an opponent-by-opponent basis.

Since the sufficient conditions in Assumption 5 may be difficult to verify in practice, a simple (but exceptionally conservative) alternative is given by the following lemma.

Lemma 1. *Assumption 5 is satisfied if* $g(w)(N-1)\mathbb{E}[u(1, Y_1, \ldots, Y_{N-1})|S = 1] < 1$ *for all* $w \in [\underline{w}, \bar{\alpha}]$.

Effectively, this lemma places a uniform limit on the preponderance of budget constraints in the relevant range of bids. Of course, this strict limit is not necessary for equilibrium existence. Example 1 presented above does not satisfy the condition of Lemma 1.

The proof of Proposition 1 is presented in the Appendix and it follows several steps. First, we consider the situation where $g(w)$ exists and is continuous for all $[\underline{w}, \bar{w}]$. In addition, it satisfies a specific bound when $w > \bar{\alpha}$. This bound is defined in a self-referential manner; therefore, we verify that there exist distributions that indeed meet it and fulfill our other assumptions. The bound ensures that the single-crossing assumption imposed on ξ is sufficient to ensure that relevant solutions to (9) are increasing. We then argue that an appropriate solution to (9) is defined for all $[\tilde{s}, 1]$ when the distribution of budgets satisfies our bound. Finally we show that the solution, $b(s)$, is bounded above by $\bar{\alpha}$. To confirm this fact requires a definition for $b(s)$ for all $[\tilde{s}, 1]$ and for $b(s)$ to be strictly increasing; therefore, our sequencing of the argument is important.

We conclude by considering the wider class of distribution functions allowed by Proposition 1. We show how any G meeting the maintained assumptions can be transformed to a distribution meeting the aforementioned bound while preserving its values for $w \leq \bar{\alpha}$. The solution for $b(s)$ in the transformed situation will be a solution in the original environment. Having established that $b(s)$ exists and that it satisfies (9), confirming that $\beta(s, w) = \min\{b(s), w\}$ is an equilibrium is a relatively straightforward case-by-case argument.

Necessity

A natural question to pose is to what extent our assumptions are necessary to support a continuous symmetric equilibrium. First, any assumptions concerning the differentiability of $G(w)$ on $[\underline{w}, \bar{\alpha}]$ are needed to ensure that the differential approach we adopt is possible.

It is therefore more apt to examine the extent to which Assumption 5 is necessary since it is the most unusual of the three.

First suppose $\underline{w} > 0$ but that $\xi(s, \underline{w}|s) < 0$ in a neighborhood of $\tilde{s}$. In this situation, the solution $b(s)$ cannot be extended continuously to bids in the range above $\underline{w}$. All solutions to the differential equation (9) will be decreasing in a neighborhood immediately above $\underline{w}$. In this regard, Assumption 5(b) cannot be relaxed while ensuring an equilibrium in continuous strategies.

From a formal point of view Assumption 5(a) is not necessary for the existence of the equilibrium that we identify. From a practical perspective we view it as necessary. It is the weakest assumption that *guarantees* increasing solutions to (9) on the domain $[\tilde{s}, 1]$ without referring to the solution of (9) itself. In this vein, the weakest alternative assumption in lieu of Assumption 5(a) would be:

> The differential equation (9) has a strictly increasing solution defined for all $[\tilde{s}, 1]$ satisfying the boundary condition $b(\tilde{s}) = \inf\{w > \underline{w} \colon \xi(\tilde{s}, w|\tilde{s}) > 0\}$.

Such an alternative statement would allow $\xi(s, \cdot|s)$ to fail its single crossing condition provided the failure did not substantively affect the desired solution to (9). Although this alternative statement is (technically) an assumption on model primitives, we find an assumption imposed directly on a solution to a differential equation to be too specific and devoid of an economic interpretation about the general environment. As we have explained above, Assumption 5(a) at least fulfills this latter criterion.

2.1 Comparative Statics in the All-Pay Auction

To place the equilibrium in context and to foster intuition for its properties we investigate several comparative statics. Throughout we focus on the effect of changes of the environment on changes in individual bidder behavior. Given the indirect characterization of the equilibrium bidding strategy, we cannot offer detailed conclusions concerning aggregate auction performance, such as expected revenue. In this regard we do not differ from previous studies of auctions with budget constraints. None has yet arrived at a concise description of aggregate statistics allowing for affiliated and interdependent values. Only Fang & Parreiras (2003) are able to document the failure of the linkage principle through an extended example.

Changes in the Distribution of Budgets

Consider a change in the environment that makes budget constraints more lax. For example, the distribution of budgets may vary exogenously with broader economic or social conditions. In principle, this relaxation can lead to two competing effects. On one hand, when budget constraints are relaxed, bidders may be encouraged to bid more—constraints on competition have been removed and on the margin a bidder can bid more to influence the auction outcome. The countervailing force, however, draws on the amelioration of the winner's curse associated with budget constraints. Conditional on winning, the item is of relatively higher value when budget constraints bind since there is a good chance of having defeated a budget-constrained opponent. Relaxing budget constraints would dampen this effect which would tend to pull bids down. In the context of the second-price auction, Fang & Parreiras (2002) conclude that the latter effect can dominate.

In the all-pay auction, however, there does not exist a standard and general ordering of bidder's strategies as we change G. This is true even under very restrictive stochastic orders. To appreciate this conclusion, suppose $N = 2$, let $\underline{w} > 0$, and fix a distribution of budgets G on $[\underline{w}, \bar{w}]$. Consider the family of distribution functions $G_a(w) \equiv G(w)^a$ for $a \geq 1$. If $a' > a$, then $G_{a'}$ likelihood-ratio dominates G_a. Intuitively, higher values of a imply more relaxed budget constraints. Denote by $\beta_a(s, w)$ an equilibrium strategy parameterized by a. Suppose $\beta_1 \in \mathcal{M}$ is well defined and is an equilibrium. Then for all a sufficiently large $1 - g(w)aG(\bar{\alpha})^{a-1} > 0$ for all $w \in [\underline{w}, \bar{\alpha}]$ since $g(w)$ is bounded. Therefore, for a sufficiently large, $\exists \beta_a \in \mathcal{M}$ which is a symmetric equilibrium when budgets are distributed according to G_a. By examining the main differential equations defining $b_a(s)$ as $a \to \infty$, we see that

$$\frac{(1 - G(b)^a)v_1(s,s)f_1(s|s)}{1 - ag(b)G(b)^{a-1}\int_s^1 v_1(s,y)f_1(y|s)dy} \to v_1(s,s)f_1(s|s)$$

uniformly for all s and $b \leq \bar{\alpha}$. Therefore $b_a(s) \to \int_0^s v_1(y,y)f_1(y|y)dy$.

Recall however that for each a, $b_a(\tilde{s}+\epsilon) > \alpha(\tilde{s}+\epsilon)$ while $b_a(1) < \bar{\alpha}$. Therefore a bidder's strategy adjustment is not monotonic across types and in general $b_a(\cdot)$ is neither greater nor less than $b_{a'}(\cdot)$ for $a' \neq a$. Thus, the same qualitative ordering that exists for the second-price auction does not carry over to the case of the all-pay auction.

Changes in the Bidder Population

How will changes in the bidder population affect the auction's equilibrium? While original studies of auctions with budget constraints, such as Che & Gale (1998), allowed for variation in the number of bidders, comparative statics exploring the sensitivity of equilibrium to changes in N were not pursued systematically. Studies by Fang & Parreiras (2002) and Kotowski (2010) of the second-price and first-price auction did not extend the model beyond two bidders. The main conclusion from our study is that the existence of an equilibrium in $\mathcal{M}$ is very sensitive to the number of bidders in the auction. This holds for even independent, private-value environments.

Fix an auction environment and suppose there is an equilibrium in the class $\mathcal{M}$ for some $N \geq 2$. Changing N can lead to two possible violations of Assumption 5. First, due to a change in N at the (new) critical value $\tilde{s}$, the (new) expression (11) is such that $\xi(\tilde{s}, \underline{w}|\tilde{s}) < 0$, which violates Assumption 5(b). Second, even if 5(b) is satisfied or not applicable, following a change in the number of bidders $\xi(s, w|s)$ may instead violate the single-crossing crossing condition of Assumption 5(a) in a *relevant range of values*. The violation can preclude the existence of a strictly increasing solution to (9) for all $s \geq \tilde{s}$. We illustrate both failures with an example.

Example 2. Suppose there are N bidders with private values, $u(s_i, \mathbf{s}_{-i}) = s_i$. Value-signals are distributed uniformly and independently on the unit interval. Budgets are distributed independently according to the distribution $G(w) = 1 - \exp(-4(w - \underline{w}))$ with support $[\underline{w}, \infty)$. Choose $\underline{w} = 0.1$.

As a function of N we can express $b(s)$ for bids below $\underline{w}$ as

$$b_N(s) = \frac{N-1}{N} s^N$$

The associated critical value is $\tilde{s}_N = \sqrt[N]{\frac{N\underline{w}}{N-1}}$. Similarly, for each $N \geq 2$, we can compute $\xi(s, w|s)$ to be

$$\xi_N(s, w|s) = 1 + 4(N-1)(s-1)se^{\frac{2}{5}-4w}\left((s-1)e^{\frac{2}{5}-4w} + 1\right)^{N-2}.$$

Suppose $N = 2$, then $\xi_2(s, w|s) = 4(s-1)se^{\frac{2}{5}-4w} + 1$ which is strictly positive for all $(s, w) \in [0, 1] \times [\underline{w}, \infty)$ except at the point $(s, w) = \left(\frac{1}{2}, \frac{1}{10}\right)$ where it is zero. Since $\tilde{s}_2 = \frac{1}{\sqrt{5}} \approx 0.447$, Assumption 5 is satisfied and an equilibrium in $\mathcal{M}$ exists.

Keeping the environment otherwise the same, suppose $N = 3$. Now $\tilde{s}_3 = \frac{\sqrt[3]{\frac{3}{5}}}{2^{2/3}} \approx 0.531$. At this value, $\xi_3(\tilde{s}_3, \underline{w}|\tilde{s}_3) = \frac{11}{5} - 2\left(\frac{6}{5}\right)^{2/3} < 0$. This is a volition of Assumption 5(b) and a continuous extension of $b(s)$ at $\tilde{s}$ into the range above $\underline{w}$ is not possible.

Finally, suppose $N = 10$. In practical terms this would be a setting with a large number of bidders. Now, $\tilde{s}_{10} = \frac{1}{\sqrt[5]{3}} \approx 0.803$ and $\xi_{10}(\tilde{s}_{10}, \underline{w}|\tilde{s}_{10}) = 1 + 4\sqrt[5]{3}\left(\frac{1}{\sqrt[5]{3}} - 1\right) > 0$. Thus, Assumption 5(b) is met. However, Assumption 5(a) fails. We illustrate this failure with Figure 2. The figure shows the function $b(s)$ in this case along with its solution satisfying the boundary condition $b(\tilde{s}_{10}) = \underline{w}$.[9] This extension of $b(s)$ above $\underline{w}$ necessarily needs to traverse a region, illustrated in gray, where $\xi_{10}(s, w|s) < 0$. Therefore, there does not exist a strictly increasing solution $b(s)$ for all $s > \tilde{s}_{10}$.

[Figure 2 about here.]

The main implication stemming from Example 2 concerns the possibilities and opportunities for inference in auction environments where bidders may be budget constrained. While there does not exist a good theory of inference and identification in auctions with budget constraints (and it is far beyond the scope of this study to develop one), changes in N are a common source of variation exploited in empirical auction studies.[10] Fully exploiting this variation in auctions with budget constraints may be problematic (or at best challenging) due to the qualitative differences of equilibrium bidding as the environment changes with N. For example, for some values of N (depending on the distribution of budgets and valuations), one would not be able to employ first-order conditions to fully characterize a bidder's optimal bid. Much more research is required to develop precise conclusions and restrictions accounting for such concerns.

Public Signals

Suppose prior to bidding players observe the realization of some public signal S_0. For example, this signal may be some piece of information released non-strategically by the auctioneer. We begin by distinguishing two types of public signals that the bidders may observe.

Definition 2. A signal S_0 is said to be (strictly) *value-relevant* if for all $(s_i, \mathbf{s}_{-i})$, $u(s_0, s_i, \mathbf{s}_{-i})$ is (strictly) increasing in s_0. S_0 is said to be *value-irrelevant* if for all $(s_i, \mathbf{s}_{-i})$, $u(s_0, s_i, \mathbf{s}_{-i})$ is constant in s_0.

[9] Precisely, we plot the solution of the inverse function—$q(b)$—as explained in the proof of Proposition 1.

[10] See Athey & Haile (2007) for a recent survey of identification in auction models. See Bajari & Hortaçsu (2005) for an implementation.

Definition 3. A signal S_0 is said to be *information-relevant* if conditional on $S_i = s_i$, S_0 and S_{-i} are not independent. S_0 is said to be *information-irrelevant* if conditional on $S_i = s_i$, S_0 and S_{-i} are independent.

A signal that is value-relevant conveys information about the value of the item directly; its realized value is effectively a parameter of the bidder's utility function. An information-relevant signal is correlated with other bidders' private information. Therefore, it conveys additional information about others' signals beyond the information contained already in S_i. While nothing precludes a signal from being both value- and information-relevant—indeed, we consider this to be the norm—we will focus only on extreme cases where public signals are either value- or information-relevant, but not both. This dichotomy allows us to characterize the competing effects of information in the all-pay auction. Signals that are purely value-relevant encourage bidders to respond in the intuitive manner—"good news" will encourage uniformly more aggressive bidding. In contrast, high realizations of signals that are purely information-relevant are a discouragement leading some bidders to bid less.

Proposition 2. *Suppose the conditions of Proposition 1 are satisfied and let* $\underline{w} > 0$*. Let* $s_0' > s_0$ *be realizations of a public signal* S_0 *observable to all bidders. Let* $\beta(s, w|s_0')$ *[*$\beta(s, w|s_0)$*] be the equilibrium strategy in the all-pay auction when the public signal is high [low].*

(a) If the public signal is value-relevant and information-irrelevant, then $\beta(s, w|s_0') \geq \beta(s, w|s_0)$*.*

(b) If the public signal is information-relevant and affiliated with players' value-signals but is value-irrelevant, then there exists an $\hat{s} > 0$ *such that for all* $s < \hat{s}$*,* $\beta(s, w|s_0') \leq \beta(s, w|s_0)$*.*

Consider first the case of purely value-relevant information. Noting the preceding discussion, and viewing s_0 as a parameter entering u it is clear that our equilibrium characterization remains the same with statements conditional on s_0 replacing the unconditional statements. An implicit assumption, of course, is that changes in s_0 are sufficiently small to ensure (via an appeal to continuity) that we maintain an equilibrium in $\mathcal{M}$. The associated comparative static is intuitive.

In turning to information-relevant signals, we observe a different reaction. This conclusion is independent of the presence of budget constraints per se but is instead a general feature of the all-pay auction. The intuition is straightforward. Conditional on observing a high public signal s_0' bidder i can infer that her opponent likely has a high signal and will in consequence bid high. A high bid by the opponent decreases the probability with which bidder i wins the auction, discouraging her from bidding aggressively (recall, in an all-pay auction she must pay her bid irrespective of the outcome). In contrast, if the public signal also has a direct effect on a bidder's value for the item, the resulting boost in expected payoff may be enough to counteract the discouragement effect.

3 The War of Attrition

Given that the first-price, second-price, and all-pay auctions have symmetric equilibria of the form $\beta(s, w) = \min\{b(s), w\}$, a natural conjecture is that the war of attrition—the second-price, all-pay auction—also has an equilibrium in this class. In this section we extend our baseline model to accommodate this auction format as well. Many of the qualitative features of the all-pay auction's equilibrium find natural analogues in the war of attrition. The major distinction is that under a very mild technical condition the war of attrition features a uniform amplification of bids following the introduction of budget constraints. In the all-pay auction, such an amplification was present generally only for a subset of types with intermediate valuations. The section concludes by noting the prospects for a revenue ranking between the all-pay auction and the war of attrition. Generally, such a ranking is not possible if both budget constraints and affiliated interdependent values are present.

We maintain our assumptions concerning the environment from Section 1. Again, bidders will simultaneously submit bids and the highest bidder will be deemed the winner. The winning bidder will make a payment equal to the second-highest bid. All losing bidders continue to incur a cost equal to their bid. Our static treatment of the war of attrition mirrors the treatment in Krishna & Morgan (1997). Therefore, we do not model the war of attrition as an extensive game where bidders sequentially submit additional (incremental) bids. Leininger (1991) and Dekel *et al.* (2006) consider such models with budget limits and perfect information. Extending our analysis in this direction would introduce many interesting complications such as the role of jump bidding in signaling valuations *and* budget levels.

As in the case of the all-pay auction we first derive a differential equation that we will later prove characterizes equilibrium behavior. Suppose that all bidders $j \neq i$ are following the bidding strategy $\beta(s, w) = \min\{b(s), w\}$ where $b(s)\colon [0, 1] \to \mathbb{R}_+$ is again a strictly increasing and piecewise differentiable function. Since the range of $b(s)$ may not equal the range of $\beta(s,w)$, we extend its domain and range as follows. Let

$$\hat{b}(x) = \begin{cases} b(0) + x & \text{if } x \in [-b(0), 0) \\ b(x) & \text{if } x \in [0, 1] \end{cases} \tag{13}$$

$\hat{b}(x)$ is also piecewise differentiable and strictly increasing. Then for all $x \in [-b(0), 1]$ the probability that $\max_{j \neq i} \beta(s,w)$ is less than $\hat{b}(x)$ is given by

$$\hat{H}(x, \hat{b}(x)|s) = \sum_{k=0}^{N-1} \binom{N-1}{k} G(\hat{b}(x))^{N-1-k} (1 - G(\hat{b}(x)))^k F_k(x|s). \tag{14}$$

Since $\hat{b}(x)$ is fixed, we can let $\hat{H}_{\hat{b}}(x|s) \equiv \hat{H}(x, \hat{b}(x)|s)$. $\hat{H}_{\hat{b}}(x|s)$ is a cumulative distribution function. Suppose its density is $\hat{h}_{\hat{b}}(x|s)$. Using $\hat{H}_{\hat{b}}(x|s)$ we can write the expected payoff of a bidder of type (s,w) when she chooses to bid $b(x) \leq w$ and $x > 0$, as

$$\begin{aligned} U(b(x)|s,w) = & \sum_{k=0}^{N-1} \gamma_k(b(x)) z_k(x|s) - (1 - \hat{H}_{\hat{b}}(x|s)) b(x) \\ & - \int_{-b(0)}^{0} (b(0) + y) \hat{h}_{\hat{b}}(y|s) dy - \int_0^x b(y) \hat{h}_{\hat{b}}(y|s) dy. \end{aligned}$$

The first term is the same as in the all-pay auction. It is the expected benefit of winning the auction. The second term is the payment the bidder must make if she loses the auction. This is her own bid. The third and fourth terms define her payment when she wins the auction.

Proceeding analogously to the case of the all-pay auction, computing $\frac{d}{dx} U(b(x)|s,w)\big|_{x=s} = 0$ leads to the following differential equation:

$$b'(s) = \frac{\sum_{k=0}^{N-1} \gamma_k(b(s)) z_k'(s|s)}{1 - \hat{H}_{\hat{b}}(s|s) - \sum_{k=0}^{N-1} \gamma_k'(b(s)) z_k(s|s)}.$$

Since $\hat{H}_{\hat{b}}(s|s) = \hat{H}(s, b(s)|s)$ for $s > 0$, we arrive at

$$b'(s) = \frac{\sum_{k=0}^{N-1} \gamma_k(b(s)) z_k'(s|s)}{1 - \hat{H}(s, b(s)|s) - \sum_{k=0}^{N-1} \gamma_k'(b(s)) z_k(s|s)}. \tag{15}$$

In the following discussion we will show that under suitable assumptions (15) will characterize equilibrium behavior in the war of attrition, much like (9) did in the all-pay auction. The assumptions we impose will again relate to both the nature of correlation among bidder's private information and on the interaction between budgets and value-signals.

Suppose for the moment that $b(s) < \underline{w}$, then (15) reduces to $b'(s) = \frac{v_{N-1}(s,s)f_{N-1}(s|s)}{1-F_{N-1}(s|s)}$. This again is the differential equation identified by Krishna & Morgan (1997) as describing a symmetric equilibrium of the war of attrition without budget constraints. In our notation, this strategy is

$$\omega(s) \equiv \int_0^s \frac{v_{N-1}(y,y)f_{N-1}(y|y)}{1-F_{N-1}(y|y)}dy. \tag{16}$$

Recalling that $\lim_{s\to 1}\omega(s) = \infty$, leads us to anticipate an analogous development in our model. Therefore, we strengthen Assumption 3 relating to the distribution of budgets as follows:

Assumption 6. *For all $w \geq \underline{w}$ in the support of G, $G(w)$ admits a continuous, strictly positive density $g(w)$.*

Like in the all-pay auction, we will need a condition that limits the degree of affiliation among value-signals. The sufficient condition rendering $\omega(s)$ an equilibrium strategy in the absence of budget constraints is that

$$\frac{v_{N-1}(\cdot,y)f_{N-1}(y|\cdot)}{1-F_{N-1}(y|\cdot)} \colon [0,1] \to \mathbb{R} \tag{17}$$

is nondecreasing. Assumption 7 generalizes this condition to our setting.

Definition 4. The value $\tilde{\sigma} \in [0,1)$ is the unique solution to $\omega(\tilde{\sigma}) = \underline{w}$.

Assumption 7. *Let $q_k(x,w|s) = \frac{\gamma_k(w)(1-F_k(x|s))}{\sum_{n=1}^{N-1}\gamma_n(w)(1-F_n(x|s))}$. Define the function*

$$\Phi(x,w|s) = \sum_{k=1}^{N-1} q_k(x,w|s)\left[\frac{v_k(s,x)f_k(x|s)}{1-F_k(x|s)}\right]. \tag{18}$$

For all $(x,w) \in [\tilde{\sigma},1) \times [\underline{w},\bar{w})$, $\Phi(x,w|\cdot)\colon [0,1] \to \mathbb{R}$ is nondecreasing.

Like Assumption 4, Assumption 7 is a restriction on the degree of affiliation among value-signals. It always holds if value-signals are independent. For $w < \underline{w}$, (18) reduces to (17). Also, when $N = 2$ it again reduces to (17) and is the same restriction imposed by Fang & Parreiras (2002) in their study of the two-bidder, second-price auction with budget constraints. As with Assumption 8 below, we present (18) as a weighted average emphasizing the interaction between the number of bidders and the limit on affiliation that needs to hold for subsets of signals.

Assumption 8. *Let* $q_k(x,w|s) = \frac{\gamma_k(w)(1-F_k(x|s))}{\sum_{n=1}^{N-1}\gamma_n(w)(1-F_n(x|s))}$. *Define the function*

$$\Xi(x,w|s) = \sum_{k=1}^{N-1} q_k(x,w|s)\left[1 - \frac{kg(w)}{1-G(w)}\left(\frac{z_{k-1}(x|s) - z_k(x|s)}{1-F_k(x|s)}\right)\right]. \tag{19}$$

$\Xi(x,w|s)$ *satisfies the following properties:*

(a) For all $\tilde{\sigma} \leq s < 1$, $\exists w_s \in [\underline{w},\bar{w})$ *such that* $w < w_s \implies \Xi(s,w|s) < 0$ *and* $w \in (w_s,\bar{w}) \implies \Xi(s,w|s) > 0$.

(b) If $\tilde{\sigma} > 0$, *there exists* $\epsilon > 0$ *such that for all* $s \in (\tilde{\sigma}-\epsilon,\tilde{\sigma}+\epsilon)$, $\Xi(s,\underline{w}|s) > 0$.

(c) For all $(x,w) \in [\tilde{\sigma},1) \times [\underline{w},\bar{w})$, $\Xi(x,w|\cdot)\colon [0,1] \to \mathbb{R}$ *is non-increasing.*

Assumptions 8(a) and 8(b) fulfill the same role as Assumptions 5(a) and 5(b) in the all-pay auction. They ensure that there exists an increasing solution to the differential equation (15) in the appropriate range of values. The underlying intuition is also the same. Assumption 8(c) is particular to the war of attrition. The analogous requirement in the all-pay auction was implied by Assumption 4.[11] In the war of attrition, we must impose this condition explicitly since conditional on a given bid $\beta(s,w)$, the expected payment of a bidder is a nontrivial function of (s,w). This contrasts with the all-pay auction where a bid of $\beta(s,w)$ is associated with a certain payment of exactly that amount irrespective of (s,w).

Some simplifications of Assumption 8 are possible in special cases of interest. For example, Assumption 8(c) is satisfied automatically if value-signals are independent. Likewise, if $N = 2$, the following lemma is simple to confirm.

Lemma 2. *Suppose* $N = 2$. *Then Assumption 1 implies Assumption 8*(c).

Turning to the equilibrium characterization in the war of attrition, Proposition 3 identifies sufficient conditions for the existence of an equilibrium in the class $\mathcal{M}$. The equilibrium strategy resembles the equilibrium of the all-pay auction. Low value-signal bidders will follow the usual no-budget-constraints equilibrium strategy. Only for bids above $\underline{w}$ will the change in marginal incentives introduced by budget constraints modify equilibrium behavior. Unlike the all-pay auction, bidders with sufficiently large value-signals will desire to expend an arbitrarily large (but feasible) amount in equilibrium.

[11] The analogous assumption in the all-pay auction would be that for all $(x,w) \in [0,1) \times [0,\bar{\alpha})$, $\xi(x,w|\cdot)\colon [0,1] \to \mathbb{R}$ is non-increasing.

Proposition 3. *Suppose Assumptions 1, 2, and 6–8 hold. Then there exists a symmetric equilibrium in the war of attrition of the form* $\beta(s,w) = \min\{b(s), w\}$. *The function* $b(s)\colon [0,1) \to [0,\bar{w}]$ *is defined as:*

$$b(s) = \begin{cases} \omega(s) & s < \tilde{\sigma} \\ b_1(s) & s \in [\tilde{\sigma}, \hat{\sigma}) \\ \bar{w} & s > \hat{\sigma} \end{cases} \tag{20}$$

where $b_1(s)\colon [\tilde{\sigma}, \hat{\sigma}) \to [\underline{w}, \bar{w})$ *is a strictly increasing, absolutely continuous function, such that*

(a) $b_1(\tilde{\sigma}) = \inf\{w > \underline{w}\colon \Xi(\tilde{\sigma}, w|\tilde{\sigma}) > 0\}$;

(b) $\lim_{s\to\hat{\sigma}^-} b_1(s) = \bar{w}$; *and,*

(c) For almost every s, $b_1'(s) = \frac{\sum_{k=0}^{N-1} \gamma_k(b_1(s)) z_k'(s|s)}{1-\hat{H}(s,b_1(s)|s) - \sum_{k=0}^{N-1} \gamma_k'(b_1(s)) z_k(s|s)}$.

Remark 2. We leave $b(s)$ not defined at $s = \hat{\sigma}$. It can assume any value without affecting the equilibrium properties of this strategy profile. If $\bar{w} < \infty$ we could set $b(\hat{\sigma}) = \bar{w}$ and make $b(s)$ continuous.

The following example illustrates an equilibrium strategy profile in the war of attrition. Occasionally, closed form expressions for the equilibrium strategy are available.

Example 3. Suppose $N = 2$ and that value-signals $S_i \overset{i.i.d.}{\sim} U[0,1]$. Let $u(s_i, s_j) = (s_i + s_j)/2$ and suppose budgets follow the cumulative distribution $G(w) = 1 - e^{-(w-\underline{w})}$ on $[\underline{w}, \infty)$. Choose $\underline{w} = -\frac{7}{20} + \log\left(\frac{20}{13}\right) \approx 0.081$. With these parameters, our equilibrium strategy in the war of attrition is $\beta(s,w) = \min\{b(s), w\}$ where

$$b(s) = \begin{cases} -s - \log(1-s) & \text{if } s \le \frac{7}{20} \\ \int_{\frac{7}{20}}^{s} \frac{4y}{3(y-1)^2} dy + \underline{w} & \text{if } s > \frac{7}{20} \end{cases}$$

For $s > 7/20$, we can integrate the above expression to see that

$$b(s) = \frac{1040(s-1)\log(1-s) + 1820(s-1)\log\left(\frac{20}{13}\right) - 1873s + 833}{780(s-1)}.$$

For comparison, Figure 3 presents the functions $b(s)$ and $\omega(s) = -s - \log(1-s)$.

[Figure 3 about here.]

As seen in Example 3, bidders with a value-signal of only 0.65 desire to commit to a bid greater than 1, which is the maximum possible value of the available prize. Such "overbidding" is a particular feature of the war of attrition (Albano, 2001). The effect of budget constraints is to amplify this phenomenon further. The following corollary formalizes this observation.

Corollary 2. *Suppose the conditions of Proposition 3 hold and let $\underline{w} > 0$.*

(a) $\lim_{s\to\tilde{\sigma}^+} b'(s) > \lim_{s\to\tilde{\sigma}^-} b'(s) = \lim_{s\to\tilde{\sigma}^-} \omega'(s)$.

(b) If $\frac{f_k(s|s)}{1-F_k(s|s)} \geq \frac{f_{N-1}(s|s)}{1-F_{N-1}(s|s)}$ *for all* k.[12] *Then for all* s, $b(s) \geq \omega(s)$.

Although the sufficient conditions supporting the equilibrium in the war of attrition are more restrictive, the equilibrium enjoys similar comparative statics. Again, the equilibrium strategy identified here will converge to the equilibrium in an environment without budget constraints if the constraints are relaxed. However, noting Corollary 2(b), a sufficient relaxation of budget constraints will lead to an ordering of equilibrium strategies, much like in the second-price auction. Additionally, the same bidder-level comparative statics apply concerning information revelation. The distinction between value-relevant and information-relevant public signals continues to be important in appreciating a bidder's equilibrium reaction to public information.

Proposition 4. *Suppose the conditions of Proposition 3 are satisfied and let* $\underline{w} > 0$. *Let* $s_0' > s_0$ *be realizations of a public signal* S_0 *observable to all bidders. Let* $\beta(s, w|s_0')$ *[*$\beta(s, w|s_0)$*] be the equilibrium strategy in the all-pay auction when the public signal is high [low].*

(a) If the public signal is value-relevant and information-irrelevant, then $\beta(s, w|s_0') \geq \beta(s, w|s_0)$.

(b) If the public signal is information-relevant and affiliated with players' value-signals but is value-irrelevant, then there exists an $\hat{s} > 0$ *such that for all* $s < \hat{s}$, $\beta(s, w|s_0') \leq \beta(s, w|s_0)$.

3.1 Comparing of the All-Pay Auction and the War of Attrition

We conclude our investigation with a brief comparison of the two auction formats. Naturally, we restrict attention to environments where the all-pay auction has an equilibrium of the form $\beta_A(s, w) = \min\{b_A(s), w\} \in \mathcal{M}$ and the war of attrition has an equilibrium of the form

[12] The condition is satisfied, for example, when $S_i \overset{i.i.d.}{\sim} U[0, 1]$.

$\beta_W(s,w) = \min\{b_W(s), w\} \in \mathcal{M}$. Our first comparison considers an ordering of the bidding strategies.

Proposition 5. *If $\underline{w} > 0$ then $\beta_W(s,w) \geq \beta_A(s,w)$ for all (s,w). Moreover, $b_W(s) > b_A(s)$ for all $s > 0$.*

Noting Proposition 5, we can employ the arguments in Che & Gale (1998) and also outlined in Krishna (2002) to conclude that the all-pay auction will be more efficient on average than the war of attrition when preferences are reflective of the ordering of bidder's value-signals.[13]

We close with a discussion of revenue comparisons between the two formats. There does not exist a general revenue ranking between the war of attrition and the all pay auction in the presence of budget constraints and affiliated valuations. One can draw this conclusion by documenting the results in extreme cases. First, suppose that budget constraints are very lax. For example, suppose budgets are distributed according to the exponential distribution with a mean that is arbitrarily large. Since valuations are bounded the equilibrium bids submitted in both formats are essentially those submitted in the case of a no-budget constraints situation. In this case, it is known that in the presence of value interdependence the war of attrition will revenue dominate the all-pay auction (Krishna & Morgan, 1997).

When budget constraints are more meaningful, and they constrain bidders with non-vanishing probability, the all-pay auction can generate more revenue. Consider the following case. Suppose there are two bidders and value signals are distributed independently according to the uniform distribution. Suppose budget constraints follow the exponential distribution $G(w) = 1 - e^{-(w-\underline{w})}$ on $[\underline{w}, \infty)$. Choose $\underline{w} = \log(\frac{10}{3}) - \frac{7}{10} \approx 0.5039$. Finally, assume bidders have private values: $u_i(s_i, s_j) = s_i$.

In this situation, budget constraints are irrelevant in the case of the all-pay auction. The equilibrium strategy is $\beta_A(s,w) = \min\{b_A(s), w\}$ where $b_A(s) = \frac{s^2}{2}$. The expected revenue in the all-pay auction is $R_A = \frac{1}{3}$.

Since the bidding strategy in the war of attrition is not bounded, the introduced budget constraints will directly affect the equilibrium strategy. It is straightforward to show that the equilibrium bidding strategy is $\beta_W(s,w) = \min\{b_W(s), w\}$ where

$$b_W(s) = \begin{cases} s & \text{if } s < \frac{7}{10} \\ \int_{\frac{7}{10}}^{s} \frac{y}{(y-1)^2} dy + \log(\frac{10}{3}) - \frac{7}{10} & \text{if } s \geq \frac{7}{10} \end{cases}$$

[13]For all $\mathbf{s} = (s_1, \ldots, s_N)$, $s_i \geq s_j \implies u(s_i, \mathbf{s}_{-i}) \geq u(s_j, \mathbf{s}_{-j})$.

When $s > \frac{7}{10}$, we can write $b_W(s)$ in closed form as

$$b_W(s) = \frac{s\left(\log\left(\frac{1000}{27}\right) - 10\right) + 7 + \log(27) - 3\log(10)}{3(s-1)} + \log\left(\frac{10}{3} - \frac{10}{3}s\right) - \frac{7}{10}.$$

A direct calculation for the revenue in this case now gives

$$R_W = \frac{9}{250}\left(5\log\left(\frac{10}{3}\right) - 1 - 5e^{20/3}\int_{\frac{20}{3}}^{\infty}\frac{e^{-t}}{t}dt\right) + \frac{581 + 270\log(3) - 270\log(10)}{1500}.$$

The terms in R_W are straightforward to approximate accurately to conclude that $R_W < 0.327$. In this example the impact on revenue following the introduction of budget constraints is very slight for two reasons. First, only about 30% of bidders in the war of attrition are somehow directly impacted by the budget constraint. Additionally, those who are impacted adjust their bidding upward (Corollary 2) which partially ameliorates the revenue decline. This adjustment is not sufficient to preclude a strict revenue decline.

While tractability has guided our discussion of revenues towards comparisons of extreme scenarios, its conclusions apply more generally. It is clear that we can modify our final example by perturbing the distribution of budgets slightly such that it has full support on $[0, \infty)$ without changing the conclusion. Similarly, one can perturb the distribution of value-signals such that they are strictly but "slightly" affiliated while maintaining the strict difference in expected revenues. For example, consider the distribution $h(s_1, s_2) \propto K + s_i s_j$ on $[0, 1]^2$ and let K be very large. Finally, one can introduce strict value interdependence by endowing bidders with the preferences $u(s_i, s_j) = (1 - \epsilon)s_i + \epsilon s_j$. As we have shown, the boundary of the revenue dominance of one auction format over the other will lie somewhere in between these extreme situations. We hope to explore this boundary further, along with its implications for auction and contest design, in future research.

A Proofs from Section 2 (All-Pay Auction)

Proof of Proposition 1

The proof of Proposition 1 has several steps which we present as a sequence of lemmas. Proofs of additional minor lemmas, referenced throughout, are in Appendix C.

We first establish the existence of the function $b(s)$. We begin by identifying a useful bound in Lemma 3. If the distribution of budgets satisfies this bound our main differential equation, equation (9), will have increasing solutions for all $b(s)$ sufficiently large. With a fixed point argument, Lemma 4 shows that the introduction of this bound (which has a self-referential definition) is consistent with our maintained assumptions.

Considering only distributions of budgets that satisfy the derived bound and whose cumulative distribution function is continuously differentiable, we proceed to the define the function $b(s)$. By restricting the distribution of budgets in the manner that we have, we can guarantee that $b(s)$ can be defined for all $\in [0, 1]$ and that it is nondecreasing throughout this entire domain. Throughout this step we frequently work with the inverse of $b(s)$. This is a common technique in analysis of equilibria in auctions.[14]

Having defined $b(s)$ in a satisfactory manner for all $[0, 1]$, we establish that it is indeed bounded from above by $\bar{\alpha}$. Arriving at this conclusion beforehand is generally not possible. Our argument at this step requires $b(s)$ to be defined for all $s \in [0, 1]$ and satisfying all possible contingencies with regards to its definition necessitated the restrictions we imposed on $G(\cdot)$ in the preceding discussion.

As a final step we observe that since $b(s)$ is bounded by $\bar{\alpha}$ the modification of the distribution of budgets for $w > \bar{\alpha}$ was unnecessary. G does not need to satisfy the introduced bound nor does it have to be differentiable in this upper range of values.

Remark 4 and Lemma 11 concludes the proof by confirming that $\beta(s, w) = \min\{b(s), w\}$ is indeed a symmetric equilibrium strategy.

Throughout the discussion, we suppose Assumptions 1–5 are satisfied.

Defining a Special Class of Distributions

Lemmas 3 and 4 serve to define a special class of distribution functions for budgets. The key conclusion from these lemmas is that when player's budgets are distributed according to a distribution that meets the conditions of Lemma 3, then the differential equation (9) will have nondecreasing solutions in the range of values relevant for our problem.

Lemma 3. *Suppose $G(w)$ meets Assumptions 2, 3, and 5. Additionally, suppose*

1. *$G(w)$ is continuously differentiable on $[\underline{w}, \bar{w})$ with a strictly positive derivative $g(w)$.*

2. *For all $w \in [\bar{\alpha}, \bar{w})$, $G(w)$ satisfies the bound*

$$\frac{g(w)}{g(\bar{\alpha})} \leq \frac{\sup_{s\in[\tilde{s},1]} \sum_{k=0}^{N-2} \binom{N-2}{k} G(\bar{\alpha})^{N-2-k}(1-G(\bar{\alpha}))^k \left[z_k(s|s) - z_{k+1}(s|s)\right]}{\sup_{s\in[\tilde{s},1]} \sum_{k=0}^{N-2} \binom{N-2}{k} G(w)^{N-2-k}(1-G(w))^k \left[z_k(s|s) - z_{k+1}(s|s)\right]}. \tag{21}$$

[14] See, for example, Lebrun (1999) or Maskin & Riley (2003).

Then the function $\xi(x,w|s)$ defined in (11) exists for all $w \in [\underline{w},\bar{w})$ and is continuous. Moreover, for all $s \geq \tilde{s}$, $\exists w_s \in [\underline{w},\bar{\alpha})$ such that $w < w_s \implies \xi(s,w|s) < 0$ and $w \in (w_s,\bar{w}] \implies \xi(s,w|s) > 0$.

Proof of Lemma 3. That ξ is well defined for all $w \in [\underline{w},\bar{w})$ is a consequence of the additional differentiability assumption on $G(w)$. We therefore only need to verify the single crossing condition.

Suppose $s \geq \tilde{s}$. From Assumption 5(a), there exists $w_s < \bar{\alpha}$ such that for all $w \in (w_s,\bar{\alpha}]$, $\xi(s,w|s) > 0$. Therefore, it is sufficient to establish that if (21) is satisfied then $\xi(s,w|s) > 0$ for $w > \bar{\alpha}$.

Using Lemma 14 and since $\xi(s,\bar{\alpha}|s) > 0$ for all $s \in [\tilde{s},1]$, we can write $\Delta z_k(s) = z_k(s|s) - z_{k+1}(s|s)$ to derive the following implications:

$$
\begin{aligned}
&\forall s \in [\tilde{s},1], \xi(s,\bar{\alpha}|s) > 0 \\
&\implies \forall s \in [\tilde{s},1],\ g(\bar{\alpha}) \sum_{k=0}^{N-2} \binom{N-2}{k} G(\bar{\alpha})^{N-2-k}(1-G(\bar{\alpha}))^k \Delta z_k(s) < \frac{1}{N-1} \\
&\implies g(\bar{\alpha}) \sup_{s\in[\tilde{s},1]} \sum_{k=0}^{N-2} \binom{N-2}{k} G(\bar{\alpha})^{N-2-k}(1-G(\bar{\alpha}))^k \Delta z_k(s) < \frac{1}{N-1} \\
&\implies g(w) \sup_{s\in[\tilde{s},1]} \sum_{k=0}^{N-2} \binom{N-2}{k} G(w)^{N-2-k}(1-G(w))^k \Delta z_k(s) < \frac{1}{N-1}, \qquad \forall w > \bar{\alpha} \\
&\implies \forall s \in [\tilde{s},1],\ g(w) \sum_{k=0}^{N-2} \binom{N-2}{k} G(w)^{N-2-k}(1-G(w))^k \Delta z_k(s) < \frac{1}{N-1}, \qquad \forall w > \bar{\alpha}
\end{aligned}
$$

Thus, for all $w > w_s$ and $\tilde{s} \leq s$, $\xi(s,w|s) > 0$. ■

Lemma 4. *Suppose $G(w)$ is any distribution satisfying Assumptions 2, 3, and 5. Then there exists a cumulative distribution function $\tilde{G}(w)$ with support $[\underline{w},\tilde{w})$, $\tilde{w} > \bar{\alpha}$, such that $\tilde{G}$ meets the conditions of Lemma 3 and for all $w \leq \bar{\alpha}$, $\tilde{G}(w) = G(w)$.*

Proof of Lemma 4. Let $G(w)$ be any cumulative distribution function meeting Assumptions 2, 3, and 5 with density $g(w)$ defined for $[\underline{w},\bar{\alpha}]$. For any cumulative distribution function $F\colon [0,\infty) \to [0,1]$ define the functional

$$\delta(F,w) = \sup_{s\in[\tilde{s},1]} \sum_{k=0}^{N-2} \binom{N-2}{k} F(w)^{N-2-k}(1-F(w))^{N-2-k} \left[z_k(s|s) - z_{k+1}(s|s)\right] \tag{22}$$

Note that for any F, $\delta(F,w) \leq \sup_{s\in[\tilde{s},1]} z_0(s|s) \leq 1$ and $\delta(F,w) \geq \min_k \sup_{s\in[\tilde{s},1]} z_k(s|s) - z_{k+1}(s|s) \equiv D > 0$.

Let $\mathcal{K}$ be the set of all continuous, nondecreasing functions $F\colon [\bar{\alpha},\infty) \to [0,1]$ such that $F(\bar{\alpha}) = G(\bar{\alpha})$, $F(w) = 1$ for all $w > \bar{\alpha} + \frac{1-G(\bar{\alpha})}{g(\bar{\alpha})\delta(G(\bar{\alpha}))}$, and $|F(w) - F(w')| \leq \frac{g(\bar{\alpha})\delta(G(\bar{\alpha}))}{D}|w - w'|$. The set $\mathcal{K}$ is equicontinuous and uniformly bounded; hence, compact. It is also convex.

Consider the mapping $F_0 \xrightarrow{\Lambda} F_1$ defined by

$$F_1(w) = \Lambda(F_0) \equiv \min\left(1, G(\bar{\alpha}) + \int_{\bar{\alpha}}^{w} g(\bar{\alpha}) \frac{\delta(G,\bar{\alpha})}{\delta(F_0,t)} dt\right)$$

(To preclude confusion, F_0 and F_1 do not necessarily correspond to the distributions defined in (3).) We record several facts about Λ.

1. $\Lambda(\mathcal{K}) \subset \mathcal{K}$. To see this conclusion note that $\Lambda(F_0) = F_1(w)$ is nondecreasing, continuous and when $w > \bar{\alpha}$ yet $F(w) < 1$,

$$\begin{aligned} F_1(w) &= G(\bar{\alpha}) + \int_{\bar{\alpha}}^{w} g(\bar{\alpha}) \frac{\delta(G, \bar{\alpha})}{\delta(F_0, t)} dt \\ &\geq G(\bar{\alpha}) + \int_{\bar{\alpha}}^{w} g(\bar{\alpha}) \delta(G, \bar{\alpha}) dt \\ &= G(\bar{\alpha}) + (w - \bar{\alpha}) g(\bar{\alpha}) \delta(G, \bar{\alpha}) \end{aligned}$$

Since F_1 is bounded above by 1, $F_1(w) = 1$ when $w > \bar{\alpha} + \frac{1-G(\bar{\alpha})}{g(\bar{\alpha})\delta(G(\bar{\alpha}))}$. To verify the Lipschitz condition, note that $g(\bar{\alpha}) \frac{\delta(G(\bar{\alpha}))}{\delta(F_0, t)} \leq \frac{g(\bar{\alpha})\delta(G(\bar{\alpha}))}{D}$.

2. Λ is continuous. Define $M = \max_k \sup_{w \in [G(\bar{\alpha}), 1]} |\frac{d}{dw}(w^{N-2-k}(1-w)^k)|$. Then, $M < \infty$. Taking $F_0, F_1 \in \mathcal{K}$, we see that

$$\begin{aligned} &\left| \frac{g(\bar{\alpha})\delta(G, \bar{\alpha})}{\delta(F_0, w)} - \frac{g(\bar{\alpha})\delta(G, \bar{\alpha})}{\delta(F_1, w)} \right| \\ &\leq \frac{g(\bar{\alpha})\delta(G(\bar{\alpha}))}{D^2} |\delta(F_0, w) - \delta(F_1, w)| \\ &\leq \sup_{s \in [0,1]} z_0(s|s) \sum_{k=0}^{N-2} \binom{N-2}{k} \left[F_0(w)^{N-2-k}(1 - F_0(w))^k - F_1(w)^{N-2-k}(1 - F_1(w))^k \right] \\ &\leq \sup_{s \in [0,1]} z_0(s|s)(N-1)M|F_0(w) - F_1(w)|. \end{aligned}$$

The continuity of Λ in the sup-norm follows immediately.

Since Λ is a continuous self-map acting on a compact, convex set, it has a fixed point, say $\tilde{F} = \Lambda(\tilde{F})$. With this function define the distribution function $\tilde{G}(w)$ as follows:

$$\tilde{G}(w) = \begin{cases} G(w) & \text{if } w \leq \bar{\alpha} \\ \tilde{F}(w) & \text{if } w > \bar{\alpha} \end{cases} \tag{23}$$

It is simple to verify that $\tilde{G}$ has the properties claimed by the Lemma. In particular, its density function $\tilde{g}(w)$ is continuous and it satisfies the required bound with equality for $w \geq \bar{\alpha}$. ■

Defining the Function $b(s)$ (Case of $\underline{w} > 0$)

We henceforth assume $\underline{w} > 0$ and we develop a working definition $b(s)$. (The required amendments for $\underline{w} = 0$ are outlined in Lemma 10.) In our working definition $b(s)$ will be a continuous function composed of three components:

$$b(s) = \begin{cases} \alpha(s) & s \in [0, \tilde{s}) \\ b_1(s) & s \in [\tilde{s}, \bar{s}_1] \\ b_2(s) & s \in (\bar{s}_1, 1] \end{cases} \tag{24}$$

The relationship between components is illustrated in Figure 4. $\alpha(s)$ is defined as in (10). We define b_1 as the inverse of the solution of the differential equation $q'(b) = \psi(b, q(b))$,

$$\psi(b, s) = \frac{1 - \sum_{k=0}^{N-1} \gamma_k'(b) z_k(s|s)}{\sum_{k=1}^{N-1} \gamma_k(b) z_k'(s|s)}, \tag{25}$$

subject to the boundary condition $q(\underline{w}) = \tilde{s}$. Moreover, we restrict $q(b)$ to the region $[\underline{w}, \bar{\alpha}] \times [0, 1]$. As we show, this solution will be a strictly increasing function $q\colon [\underline{w}, \bar{b}_1] \to [0, 1]$. There are now two possible cases:

1. If, as illustrated in Figure 4, $\bar{b}_1 = \bar{\alpha}$ (and thus $q(\bar{b}_1) = \bar{s}_1 < 1$), we define $b_2(s)$ as the strictly increasing solution to $b_2'(s) = \rho(s, b_2(s))$,

$$\rho(s, b) = \frac{\sum_{k=1}^{N-1} \gamma_k(b) z_k'(s|s)}{1 - \sum_{k=0}^{N-1} \gamma_k'(b) z_k(s|s)}, \tag{26}$$

   meeting the boundary condition $b_2(\bar{s}_1) = \bar{\alpha}$.

2. If instead $\bar{b}_1 < \bar{\alpha}$, we argue that $q(\bar{b}_1) = \bar{s}_1 = 1$. In this case the definition of $b_2(s)$ above not needed.

Whatever the case we are able to define a function $b\colon [0, 1] \to [0, \bar{w}]$ by (24). We have separated the definition of $b(s)$ into the components b_1 and b_2 to be able to identify solutions to (25) and (26). Both ψ and ρ may be undefined in certain regions of $[\tilde{s}, 1] \times [\underline{w}, \bar{w}]$; however, in a region where one fails the other does not.

[Figure 4 about here.]

Lemma 5 (Case 1: $\underline{w} > 0$)**.** *Suppose $\underline{w} > 0$ and that $G(w)$ satisfies the hypotheses of Lemma 3. Consider the function*

$$\psi(b, q) = \frac{1 - \sum_{k=0}^{N-1} \gamma_k'(b) z_k(s|s)}{\sum_{k=1}^{N-1} \gamma(b) z_k'(s|s)}$$

defined for $(b, s) \in [\underline{w}, \bar{\alpha}] \times [0, 1]$. Then there exists a strictly increasing function $q\colon [\underline{w}, \bar{b}_1] \to [\tilde{s}, 1]$ which is a solution to the differential equation $q'(b) = \psi(b, q(b))$ satisfying the boundary condition $q(\underline{w}) = \tilde{s}$.

Proof of Lemma 5. It is simple to verify that $\psi(b, s)$ is continuous on $[\underline{w}, \bar{\alpha}] \times [\epsilon, 1]$ for all $0 < \epsilon < \tilde{s}$. Thus, the boundary value problem has a solution in this restricted domain. Let $q(b)\colon [\underline{w}, \bar{b}_1] \to [0, 1]$ be the maximal solution in the sense of largest domain.[15]

[15] If there are multiple solutions with the same maximal domain choose any of them.

We now argue that $q(b)$ is strictly increasing. By Assumption 5 and Lemma 14, there exists $\epsilon > 0$ sufficiently small such that $q'(b) > 0$ for all $b \in [\underline{w}, \underline{w}+\epsilon)$. Therefore $q(b)$ is initially increasing and for all $b \in [\underline{w}, \underline{w}+\epsilon)$, $q(b) > q(\underline{w}) = \tilde{s}$.

Suppose there exists $\underline{w} < b' < b'' < \eta$ such that $q(b') > q(b'')$. Without loss of generality, we may suppose that $q(b') > \tilde{s}$. Then, there exists $\hat{b} \in (b', b'')$ such that $q'(\hat{b}) < 0$. Moreover, since $q'(b)$ is continuous, there exists an interval $(\hat{b}', \hat{b}'') \subset (b', b'')$ such that $b \in (\hat{b}', \hat{b}'') \implies q'(b) < 0$.

Also without loss of generality, we may suppose that $q(\hat{b}') > \tilde{s}$. Choose any $b \in (\hat{b}', \hat{b}'')$. Then, $q'(b) < 0 \implies \xi(q(b), b|q(b)) < 0$. By Assumption 5 $\xi(q(b), w|q(b)) < 0$ for all $w \in [\underline{w}, b)$. Hence, $q(b)$ must be decreasing whenever it is in the range $(q(\hat{b}'), q(\hat{b}''))$. This is a contradiction since otherwise it would never be able to assume a value in excess of $q(\hat{b}')$ at any $b < \hat{b}'$ but it does so at b'.

Next suppose that $q(b)$ is constant on some interval (b', b''). Since $q'(b)$ is continuous, $q'(b) = 0$ for $b \in (b', b'')$. However, this implies $\xi(q(b''), w|q(b'')) = 0$ for all $w \in (b', b'')$ which violates Assumption 5. Therefore, $q(b)$ is strictly increasing.

Since $q'(b) \geq 0$ for all $b \geq \underline{w}$, there are two cases. Either $\bar{b}_1 = \bar{\alpha}$ and thus $q(\bar{b}_1) = \bar{s}_1$ for some $\bar{s}_1 \in (\tilde{s}, 1]$. Alternatively, $\bar{b}_1 < \bar{\alpha}$ and therefore $\bar{s}_1 = 1$. These two cases are easily seen in Figure 4 where we have illustrated the case of $\bar{b}_1 = \bar{\alpha}$. ■

Lemma 6. *Let $q\colon [\underline{w}, \bar{b}_1] \to [0,1]$ be defined as in Lemma 5. Consider the function $b_1 = q^{-1}$. Then $b_1\colon [\tilde{s}, \bar{s}_1] \to [\underline{w}, \bar{\alpha}]$ is strictly increasing and absolutely continuous.*

Proof of Lemma 6. b_1 is strictly increasing since q has this property. To establish absolute continuity, it is sufficient to show that $\{b\colon q'(b) = 0\}$ has measure zero. Noting Lemma 24 it is sufficient to show that $q'(b)$ is not equal to zero for any interval (x, y). Suppose $q'(b) = 0$ for all $b \in (x, y)$. Then for all $b \in (x, y)$, $q(b) = \bar{q}$ a constant. This implies $\xi(\bar{q}, b|\bar{q}) = 0$ for all $b \in (x, y)$. But, this contradicts Assumption 5. ■

Remark 3. The function b_1 in Lemma 6 is differentiable almost everywhere since it is nondecreasing. Since $q'(b) > 0$ except on a set of measure zero, for almost every s,

$$b_1'(s) = \frac{\sum_{k=1}^{N-1} \gamma_k(b_1(s)) z_k'(s|s)}{1 - \sum_{k=0}^{N-1} \gamma_k'(b_1(s)) z_k(s|s)}.$$

Lemma 7. *Let $b_1\colon [\tilde{s}, \bar{s}_1] \to [\underline{w}, \bar{b}_1]$ be defined as in Lemma 6. Suppose that $G(w)$ satisfies the conditions of Lemma 3 and that $\bar{s}_1 < 1$. Let*

$$\rho(s, b) = \frac{\sum_{k=1}^{N-1} \gamma_k(b) z_k'(s|s)}{1 - \sum_{k=0}^{N-1} \gamma_k'(b) z_k(s|s)}$$

be defined for $(s, b) \in [\bar{s}_1, 1] \times [\bar{b}_1, \bar{w}]$.

There exists a strictly increasing solution $b_2\colon [\bar{s}_1, 1] \to [\bar{b}_1, \bar{w}]$ to the differential equation $b_2'(s) = \rho(s, b_2(s))$ satisfying the boundary condition $b_2(\bar{s}_1) = \bar{b}_1$.

Proof of Lemma 7. First observe the $\rho(s, b)$ is continuous in both b and s. By Lemmas 3 and 14, $\rho(s, b) > 0$ for all $b < \underline{w}$. If $\underline{w} < \infty$, $\rho(s, \bar{w}) = 0$. So, $\lim_{s \to 1} b(s) \leq \underline{w}$. The existence of a strictly increasing solution $b_2\colon [\bar{s}_1, 1] \to [\bar{b}_1, \bar{w})$ is an application of standard results in the theory of ordinary differential equations. ■

Lemma 8. *Suppose $G(w)$ satisfies the conditions of Lemma 3 and define $b(s)$ as in equation (24). Then $b(s)$ is strictly increasing, differentiable almost everywhere, absolutely continuous, and bounded above: $b(s) \leq \bar{\alpha}$.*

Proof of Lemma 8. That $b(s)$ is increasing, differentiable almost everywhere, and absolutely continuous follows from Lemmas 5, 6, and 7 and Remark 3.

By Lemma 24, $\gamma_k(b(s))$ is absolutely continuous.[16] Therefore, $U_\beta(s) = \sum_{k=0}^{N-1} \gamma_k(b(s)) z_k(s|s) - b(s)$ is absolutely continuous. Using Lemma 19, some computations show that for almost every $s > \tilde{s}$,

$$\begin{aligned} U'_\beta(s) &= \sum_{k=0}^{N-1} \gamma_k(b(s)) \int_0^s \frac{\partial}{\partial s} \left[v_k(s,y) f_k(y|s) \right] dy \\ &= \int_0^{\tilde{s}} \frac{\partial}{\partial s} \left[v_{N-1}(s,y) f_{N-1}(y|s) \right] dy \\ &\quad + \sum_{k=0}^{N-1} \gamma_k(b(s)) \int_{\tilde{s}}^s \frac{\partial}{\partial s} \left[v_k(s,y) f_k(y|s) \right] dy \end{aligned}$$

In the first line we have cancelled terms using the definition of $b'(s)$. In the second line we have used the fact that $s < \tilde{s} \implies b(s) < \underline{w} \implies \gamma_k(b(s)) = 0 \; \forall k \neq N-1$.

Similarly, let $\alpha(s) = \int_0^s v_{N-1}(y,y) f_{N-1}(y|y) dy$ and $U_\alpha(s) = \int_0^s v(s,y) f_{N-1}(y|s) dy - \alpha(s)$. By an analogous argument, $U'_\alpha(s) = \int_0^s \frac{\partial}{\partial s} \left[v_{N-1}(s,y) f_{N-1}(y|s) \right] dy$. Then,

$$U'_\beta(s) - U'_\alpha(s) = \sum_{k=0}^{N-1} \gamma_k(b(s)) \int_{\tilde{s}}^s \left(\frac{\partial}{\partial s} \left[v_k(s,y) f_k(y|s) \right] - \frac{\partial}{\partial s} \left[v_{N-1}(s,y) f_{N-1}(y|s) \right] \right) dy.$$

By Lemma 20, $\int_0^x \frac{\partial}{\partial s} \left[v_k(s,y) f_k(y|s) \right] dy \geq \int_0^x \frac{\partial}{\partial s} \left[v_{N-1}(s,y) f_{N-1}(y|s) \right] dy$. Thus for almost every s, $U'_\beta(s) \geq U'_\alpha(s)$. But then,

$$U_\beta(s) = U_\beta(\tilde{s}) + \int_{\tilde{s}}^s U'_\beta(x) dx \geq U_\alpha(\tilde{s}) + \int_{\tilde{s}}^s U'_\alpha(x) dx = U_\alpha(s).$$

Taking $s \to 1$ and noting that $z_k(1|1) = z_{N-1}(1|1)$ for all k, this implies

$$U_\beta(1) = \sum_{k=0}^{N-1} \gamma_k(b(1)) z_k(1|1) - b(1) = z_{N-1}(1|1) - b(1) \geq U_\alpha(1) = z_{N-1}(1|1) - \alpha(1).$$

Thus, $\bar{\alpha} = \alpha(1) \geq b(1)$. Therefore, $b(s) \leq \bar{\alpha}$. ■

[16] The composition of absolutely continuous functions need not be absolutely continuous. $G(w)$ is additionally continuously differentiable for $w > \underline{w}$ which is sufficient.

Lemma 9. *Suppose $G(w)$ meets only Assumptions 2, 3, and 5. Let $b_1\colon [\tilde{s}, \bar{s}_1] \to [\underline{w}, \bar{\alpha}]$ be defined as in Lemma 6. Then, $\bar{s}_1 = 1$ and the function*

$$b(s) = \begin{cases} \alpha(s) & s \in [0, \tilde{s}) \\ b_1(s) & s \in [\tilde{s}, 1] \end{cases}.$$

is strictly increasing, absolutely continuous, and for almost every s has derivative equal to

$$b'(s) = \frac{\sum_{k=1}^{N-1} \gamma(b(s)) z_k'(s|s)}{1 - \sum_{k=0}^{N-1} \gamma_k'(b(s)) z_k(s|s)}.$$

Proof of Lemma 9. Given G, we use the procedure in Lemma 4 to construct the distribution $\tilde{G}$. For all $w \le \bar{\alpha}$, $\tilde{G}(w) = G(w)$ and $\tilde{g}(w) = g(w)$. The distribution $\tilde{G}$ meets the additional conditions allowing us to apply Lemmas 6, 7, and 8. Let $\tilde{b}(s)$ be the resulting expression for equation (24) in this auxiliary environment with components $\tilde{\alpha}$, $\tilde{b}_1$ and $\tilde{b}_2$.

Clearly, $\tilde{\alpha} = \alpha$. Since $\tilde{b} \le \bar{\alpha}$, the definition of $\tilde{b}_2$ is superfluous and $\tilde{b}_1\colon [\tilde{s}, 1] \to [\underline{w}, \bar{\alpha}]$. However since $\tilde{G}(w) = G(w)$ for all $w \le \bar{\alpha}$, $b_1(s) = \tilde{b}_1(s)$ as they are solutions to the same differential equation. The lemma's conclusion then follows. ■

Defining the Function $b(s)$ (Case of $\underline{w} = 0$)

Defining $b(s)$ when $\underline{w} = 0$ is analogousto the case of $\underline{w} > 0$. The only modification is the definition of $b_1(s)$ and identifying the appropriate boundary conditions. Therefore it suffices to replace Lemma 5 with Lemma 10.

Lemma 10 (Case 2: $\underline{w} = 0$)**.** *Suppose $\underline{w} = 0$ and that $G(w)$ satisfies the hypotheses of Lemma 3. Let*

$$\psi(b, s) = \frac{1 - \sum_{k=0}^{N-1} \gamma_k'(b) z_k(s|s)}{\sum_{k=1}^{N-1} \gamma_k(b) z_k'(s|s)}.$$

For each s, let $\mu_s = \inf_{w \ge 0}\{w\colon \xi(s, w|s) > 0\}$. There exists a strictly increasing function $q\colon [\mu_0, \bar{b}_1] \to [0, 1]$ such that for all $\epsilon > 0$, $q(b)$ is a solution to the differential equation $q'(b) = \psi(b, q(b))$ with domain $[\mu_0 + \epsilon, \bar{b}_1]$ and $\lim_{b \to \mu_0^+} q(b) = 0$.

Proof of Lemma 10. Our argument follows Fang & Parreiras (2002) who propose a similar argument for the case of the second-price auction.

For any $(b^*, s^*) \in [0, \bar{\alpha}] \times [\epsilon, 1]$ consider the solution of the boundary value problem $q'(b) = \psi(b, q(b))$, $q(b^*) = s^*$, which is confined to the region $[\underline{w}, \bar{\alpha}] \times [\epsilon, 1]$. Taking $\epsilon \to 0$, we can extend such a solution to the boundary continuously. We will follow this procedure in defining functions of interest in the arguments that follow.

For $s > 0$, let μ_s be defined as in the lemma's statement and consider the solution to $q'(b) = \psi(b, q(b))$ satisfying $q(\mu_s) = s$. Call this solution $q_s(b)$. By arguments analogous to those in Lemma 5, this solution is strictly increasing when $b > \mu_s$ and if $s' < \hat{s}$, then $q_{s'}(b) \le q_{\hat{s}}(b)$.

Next consider the function

$$\rho(s, b) = \frac{\sum_{k=1}^{N-1} \gamma_k(b) z_k'(s|s)}{1 - \sum_{k=0}^{N-1} \gamma_k'(b) z_k(s|s)} \tag{27}$$

defined on the plane $[0,1] \times [\mu_0, \bar{\alpha}]$. For $\hat{x} > \mu_0$, let $b_{\hat{x}}(s)$ be a solution to the differential equation $b'(s) = \rho(s, b(s))$ for the initial condition $(0, \hat{x})$.

Since $\hat{x} > \mu_0$, $b_{\hat{x}}(s)$ is strictly increasing. This follows from a similar argument to that employed in Lemma 5.

[Figure 5 about here.]

For any $\hat{s} > 0$ and any $\hat{x} > \mu_0$ loci of points in the plane corresponding to the solutions $q_{\hat{s}}$ and $b_{\hat{x}}$ are disjoint. The relationship between these two functions on the plane is sketched in Figure 5. Let $\hat{x} \to \mu_0$ and $s \to 0$ and consider the limiting function $b_{\mu_0} = \lim_{\hat{x} \to \mu_0} b_{\hat{x}}$ and $q_0 = \lim_{s \to 0} q_s$. (s^*, b^*) be a point contained in the region bounded by q_0, b_{μ_0}, and the space's boundary. It is simple to verify that the solution to the differential equation $q'(s) = \psi(b, q(s))$ satisfying the condition $q(b^*) = s^*$ will have the properties claimed by the lemma. Since there are (possibly) multiple solutions meeting these properties, as a convention we will choose the greatest solution (although our results apply to any such solution). ■

Confirming $\beta(s, w) = \min\{b(s), w\}$ is an equilibrium.

Remark 4. Together, Lemmas 3–10 imply that under our maintained assumptions, there exists a function $b(s)\colon [0,1] \to \mathbb{R}$ such that

1. $b(s)$ is absolutely continuous, strictly increasing, and bounded above by $\bar{\alpha}$;
2. For almost every s, the $b'(s) = \frac{\sum_{k=0}^{N-1} \gamma_k(b(s)) z_k'(s|s)}{1 - \sum_{k=0}^{N-1} \gamma_k'(b(s)) z_k(s|s)}$. If $b(s) < \underline{w}$, this reduces to $b'(s) = v_{N-1}(s,s) f_{N-1}(s|s)$.
3. If $\underline{w} > 0$, then $\tilde{s} > 0$ and $b(\tilde{s}) = \underline{w} = \inf_{w > \underline{w}}\{w\colon \xi(\tilde{s}, w|\tilde{s}) > 0\}$;
4. If $\underline{w} = 0$, then $\tilde{s} = 0$ and $\tilde{s} = 0$, then $b(0) = \mu_0 = \inf_{w>0}\{w\colon \xi(0, w|0) > 0\}$.

Lemma 11. *Suppose $b(s)\colon [0,1] \to \mathbb{R}$ satisfies the properties outlined in Remark 4. Then $\beta(s,w) = \min\{b(s), w\}$ is a symmetric equilibrium of the all-pay auction with budget constraints.*

Proof of Lemma 11. We view the all-pay auction as a revelation mechanism and we show that no type can benefit by mimicking the bid of any other type. Two claims underly this conclusion.

Claim 1. A bidder of type (s, w) has no profitable deviations to any bid in the range of $b(s)$. First suppose $s < \tilde{s}$. Then $\beta(s, w) = \int_0^s v_{N-1}(y, y) f_{N-1}(y|y) dy$. From Krishna & Morgan (1997) we know that this bidder has no profitable deviation to any bid $b(x)$ where $x \in [0, \tilde{s}]$. ($\diamond$)

Suppose that w is sufficiently large to allow this bidder to bid $b(x)$ and $x > \tilde{s}$. THe utility from this bid is $U(b(x)|s, w)$, or equivalently,

$$U(b(x)|s,w) = U(b(\tilde{s})|s,w) + \int_{\tilde{s}}^{x} \frac{d}{dt} U(b(t)|s,w)\bigg|_{t=y} dy.$$

Therefore, it is sufficient to show that $\frac{d}{dt}U(b(t)|s,w) \leq 0$ when $t > \tilde{s}$. A direct calculation then gives

$$\begin{aligned}
&\frac{d}{dt}U(b(t)|s,w) \\
&= \sum_{k=0}^{N-1} \gamma_k'(b(t))z_k(t|s)b'(t) + \sum_{k=0}^{N-1} \gamma_k(b(t))z_k'(t|s) - b'(t) \\
&= \sum_{k=0}^{N-1} \gamma_k(b(t))z_k'(t|s) - \sum_{k=0}^{N-1} \gamma_k(b(t))z_k'(t|t) \left[\frac{1-\sum_k \gamma_k'(b(t))z_k(t|s)}{1-\sum_k \gamma_k'(b(t))z_k(t|t)}\right] \\
&\leq \sum_{k=0}^{N-1} \gamma_k(b(t))z_k'(t|t) - \sum_{k=0}^{N-1} \gamma_k(b(t))z_k'(t|t) \left[\frac{1-\sum_k \gamma_k'(b(t))z_k(t|t)}{1-\sum_k \gamma_k'(b(t))z_k(t|t)}\right] = 0
\end{aligned}$$

The inequality follows from noting that $s < t$ and then applying Lemmas 18, 21, and 22. Hence, deviating to a bid $b(x)$ for $x > s$ is not profitable.

Suppose instead that $s > \tilde{s}$. The same argument as above shows that this bidder will not deviate to any feasible bid $b(x)$ for $x > s$. Consider instead a deviation to a lower bid $b(x) < s$ where $x \in [\tilde{s}, s)$. The expected payoff from doing so is

$$U(b(x)|s,w) = U(b(\hat{s})|s,w) - \int_x^{\hat{s}} \frac{d}{dt}U(b(t)|s,w)\bigg|_{t=y} dy$$

where we let $\hat{s} \leq s$ be such that $b(\hat{s}) = \beta(s,w)$. Therefore, it is sufficient to verify that $\frac{d}{dt}U(b(t)|s,w) \geq 0$ when $t < s$.

$$\begin{aligned}
&\frac{d}{dt}U(b(t)|s,w) \\
&= \sum_{k=0}^{N-1} \gamma_k'(b(t))z_k(t|s)b'(t) + \sum_{k=0}^{N-1} \gamma_k(b(t))z_k'(t|s) - b'(t) \\
&= \sum_{k=0}^{N-1} \gamma_k(b(t))z_k'(t|s) - \sum_{k=0}^{N-1} \gamma_k(b(t))z_k'(t|t) \left[\frac{1-\sum_k \gamma_k'(b(t))z_k(t|s)}{1-\sum_k \gamma_k'(b(t))z_k(t|t)}\right]
\end{aligned}$$

By Lemma 18, $\sum_{k=0}^{N-1} \gamma_k(b(t))z_k'(t|s) \geq \sum_{k=0}^{N-1} \gamma_k(b(t))z_k'(t|t)$. There are two cases:

1. Suppose $1 - \sum_k \gamma_k'(b(t))z_k(t|s) < 0$. Then we conclude immediately that $\frac{d}{dt}U(b(t)|s,w) \geq 0$.
2. Suppose $1 - \sum_k \gamma_k'(b(t))z_k(t|s) \geq 0$. An application of applying Lemma 22 gives $1 - \sum_k \gamma_k'(b(t))z_k(t|s) \leq 1 - \sum_k \gamma_k'(b(t))z_k(t|t)$ and so $\frac{d}{dt}U(b(t)|s,w) \geq 0$.

Therefore, there is no profitable deviation to a bid $b(x)$ when $x \in [\tilde{s}, s)$.

Finally, consider a deviation to a bid of $b(x)$ for $x < \tilde{s}$. It is sufficient to show that $\frac{d}{dt}U(b(t)|s,w)\big|_{t=y} \geq$

0 for all $y < \tilde{s}$. Again, a direct calculation and an application of Lemma 18 gives

$$\begin{aligned}\frac{d}{dt}U(b(t)|s,w) &= z'_{N-1}(t|s) - z'_{N-1}(t|t)\\ &= v_{N-1}(s,t)f_{N-1}(t|s) - v_{N-1}(t,t)f_{N-1}(t|t)\\ &\geq v_{N-1}(t,t)f_{N-1}(t|t) - v_{N-1}(t,t)f_{N-1}(t|t) = 0.\end{aligned}$$

Claim 2. A bidder of type (s,w) has no profitable deviations to any bid outside the range of $b(s)$. Since $b(1)$ is equal to the maximum submitted bid and wins with probability 1, no bidder can benefit by submitting a strictly greater bid. Consider alternative bids $\hat{b} < b(0)$. First, note that if $b(0) = 0$, this situation is vacuous. Therefore suppose $b(0) > 0$. This is only possible, however, if $\tilde{s} = 0$ and $\underline{w} = 0$. As a consequence, $b(0) = \mu_0 = \inf\{w > 0 \colon \xi(0,w|0) > 0\}$.

Suppose $\hat{b} \in [0,\mu_0)$. A bidder who submits this bid receives an expected payoff of $U(\hat{b}|s,w) = G(\hat{b})^{N-1}z_0(s|s) - \hat{b}$. Since $G(\cdot)$ is differentiable in this range of values,

$$\begin{aligned}\frac{d}{d\hat{b}}U(\hat{b}|s,w) &= (N-1)g(\hat{b})G(\hat{b})^{N-2}z_0(s|s) - 1\\ &= (N-1)g(\hat{b})G(\hat{b})^{N-2}z_0(0|s) - 1.\end{aligned}$$

This final expression is nondecreasing in s. To establish that $\frac{d}{d\hat{b}}U(\hat{b}|s,w) \geq 0$, it is therefore sufficient to establish that $\frac{d}{d\hat{b}}U(\hat{b}|0,w) \geq 0$. Working from its definition,

$$\xi(0,\hat{b}|0) = 1 - \sum_{k=0}^{N-1}\gamma'_k(\hat{b})z_k(0|0) = 1 - (N-1)g(\hat{b})G(\hat{b})^{N-2}z_0(0|0).$$

But we know from Assumption 5 that $\xi(0,w|0)$ crosses zero once from below at some w_0. But this occurs at μ_0. Thus $\xi(0,\hat{b}|0) < 0$ and therefore $\frac{d}{d\hat{b}}U(\hat{b}|s,w) \geq 0$. As $\frac{d}{d\hat{b}}U(\hat{b}|s,w) \geq 0$, the (constrained) optimal bid in the range $(0,\mu_0]$ is $\min\{b(0),w\}$. Any bidders with a budget greater than $b(0)$ prefer to bid $\beta(s,w) \geq b(0)$ since by the above arguments $U(\beta(s,w)|s,w) \geq U(b(0)|s,w)$. Bidders with a budget $w < b(0)$ are bidding as best they can by choosing $\beta(s,w) = w$.

The two cases exhaust all possible deviations to alternative bids. Therefore, $\beta(s,w)$ is indeed an equilibrium. ■

Other Proofs from Section 2

Proof of Lemma 1. From its definition in (2), it follows that $z_k(x|s) \geq z_{k+1}(x|s)$ for all k. Recalling that $z_0(s|s) = \mathbb{E}[u(s, Y_1, \ldots, Y_{N-1})|S = s]$, we conclude that

$$\begin{aligned}
&g(w)(N-1)\left[\sum_{k=0}^{N-2}\binom{N-2}{k}G(w)^{N-2-k}(1-G(w))^k(z_k(s|s)-z_{k+1}(s|s))\right] \\
&\leq g(w)(N-1)\left[\sum_{k=0}^{N-2}\binom{N-2}{k}G(w)^{N-2-k}(1-G(w))^k\right]z_0(s|s) \\
&\leq g(w)(N-1)\mathbb{E}[u(1, Y_1, \ldots, Y_{N-1})|S = 1].
\end{aligned}$$

Thus, if the final expression is bounded above by 1, a rearrangement of terms gives the desired bound via Lemma 14 ■

Proof of Corollary 1. When $\underline{w} > 0$, then $\tilde{s} > 0$. Using Assumption 5(b) and noting that $\sum_{k=0}^{N-1}\gamma_k'(b(\tilde{s}))z_k(\tilde{s}|\tilde{s}) \geq 0$ gives $0 < 1 - \sum_{k=0}^{N-1}\gamma_k'(b(\tilde{s}))z_k(\tilde{s}|\tilde{s}) < 1$. Also, for all $k \leq N-2$, $\gamma_k(\underline{w}) = 0$, while $\gamma_{N-1}(\underline{w}) = 1$. Since $\lim_{s\to\tilde{s}} b(s) = \underline{w}$, and $b'(s)$ is continuous from the left and the right, we see that

$$\begin{aligned}
\lim_{s\to\tilde{s}^+} b'(s) &\geq \frac{\sum_{k=0}^{N-2}\gamma_k(\underline{w})v_k(\tilde{s},\tilde{s})f_k(\tilde{s}|\tilde{s}) + \gamma_{N-1}(\underline{w})v_{N-1}(\tilde{s},\tilde{s})f_{N-1}(\tilde{s}|\tilde{s})}{1 - \sum_{k=0}^{N-1}\gamma_k'(b(\tilde{s}))z_k(\tilde{s}|\tilde{s})} \\
&> v_{N-1}(\tilde{s},\tilde{s})f_{N-1}(\tilde{s}|\tilde{s}) \\
&= \lim_{s\to\tilde{s}^-} b'(s).
\end{aligned}$$

This proves part (a). Part (b) is subsumed in the proof of Proposition 1. ■

Proof of Proposition 2. To prove part (a) it is sufficient to verify that $b(s|s_0') \geq b(s|s_0)$ for all s. Since the value-signal is only value-relevant we can let $\bar{u}$ ($\underline{u}$) be a bidder's utility function when the public signals is high (low). The values $\bar{z}_k$, $\bar{v}_k$, $\underline{z}_k$, and $\underline{v}_k$ are defined in the obvious way. Note that for $\bar{u}(\mathbf{s}) > \underline{u}(\mathbf{s})$ for all $\mathbf{s} > 0$.

Let $\tilde{s}'$ solve $\int_0^{\tilde{s}'}\bar{v}_{N-1}(y,y)f_{N-1}(y|y)dy = \underline{w}$. Define $\tilde{s}$ analogously when the public signal is s_0. Then for any $s < \min\{\tilde{s}', \tilde{s}\}$,

$$b(s|s_0') - b(s|s_0) = \int_0^s \left[\bar{v}_{N-1}(y,y) - \underline{v}_{N-1}(y,y)\right] f_{N-1}(y|y)dy > 0.$$

Therefore, for $0 < s < \min\{\tilde{s}', \tilde{s}\}$ we have $b(s|s_0') \geq b(s|s_0)$. Hence, $\tilde{s}' < \tilde{s}$.

Consider next the range of bids above $\underline{w}$. Since $b(s|s_0')$ is strictly increasing, $b(\tilde{s}|s_0') > b(\tilde{s}|s_0)$.

Suppose there exists a (s^*, b^*) such that $b(s^*|s_0') = b(s^*|s_0) = b^*$. At this point,

$$\begin{aligned} b'(s^*|s_0) &= \frac{\sum_{k=1}^{N-1} \gamma_k(b^*)\underline{v}_k(s^*, s^*) f_k(s^*|s^*)}{1 - \sum_{k=0}^{N-1} \gamma_k'(b^*)\underline{z}_k(s^*|s^*)} \\ &< \frac{\sum_{k=1}^{N-1} \gamma_k(b^*)\bar{v}_k(s^*, s^*) f_k(s^*|s^*)}{1 - \sum_{k=0}^{N-1} \gamma_k'(b^*)\bar{z}_k(s^*|s^*)} = b'(s^*|s_0'). \end{aligned}$$

The inequality follows from Lemmas 21 and 22.[17] Since $b'(\cdot|s_0)$ and $b'(\cdot|s_0')$ are continuous functions, this implies that for $\epsilon > 0$ sufficiently small when $s \in (s^* - \epsilon, s^*)$ then $b(s|s_0) > b(s|s_0')$. But this would be a contradiction since it would have to apply to every such (s^*, b^*). Thus $b(s|s_0) \leq b(s|s_0')$ as needed.

Turning to part (b), we know that given the public signal s_0 in equilibrium bidders with signals $s < \tilde{s}(s_0)$ will bid $b(s|s_0) = \int_0^s v_{N-1}(y, y) f_{N-1}(y|y, s_0) dy$. Since $\bar{Y}_{N-1}$, S_1 and S_0 are affiliated, an application of Lemma 15 implies that if $s_0' > s_0$, then $f_{N-1}(y|y, s_0') \leq f_{N-1}(y|y, s_0)$ for all $0 \leq y < \hat{y}$. Therefore, $b(s|s_0) \geq b(s|s_0')$ for all $s < \hat{y}$. ■

B Proofs from Section 3 (War of Attrition)

Proof of Lemma 2. In the case of $N = 2$, then $f_1(y|s) = h(y_1|s) = \frac{h(s,y_1)}{\int_0^1 h(s,y_1)dy_1}$. Let $H(x|s) = \int_0^x h(y_1|s)dy_1$. With this notation, we can write

$$\begin{aligned} \Xi(x, w|s) &= \frac{(1 - G(w))(1 - F_1(x|s))}{(1 - G(w))(1 - F_1(x|s))} \left[1 - \frac{g(w)}{1 - G(w)} \frac{\int_x^1 v_1(s, y) f_1(y|s) dy}{1 - F_1(x|s)} \right] \\ &= \frac{(1 - G(w))(1 - F_1(x|s))}{(1 - G(w))(1 - F_1(x|s))} \left[1 - \frac{g(w)}{1 - G(w)} \frac{\int_x^1 u(s, y_1) h(y_1|s) dy_1}{1 - H(x|s)} \right] \\ &= 1 - \frac{g(w)}{1 - G(w)} \mathbb{E}[u(s, Y_1)|S = s, Y_1 \geq x] \end{aligned}$$

Since (S, Y_1) are affiliated, $\mathbb{E}[u(s, Y_1)|Y_1 \geq x, S = s]$ is nondecreasing in s. Therefore, $\Xi(x, w|\cdot)$ is non-increasing. ■

Lemmas 12 and 13 are used in the proof of Proposition 3.

Lemma 12. *There exists a continuous, nondecreasing function $b_1(s)\colon [\tilde{\sigma}, \hat{\sigma}) \to [\underline{w}, \bar{w})$ such that*

(a) $b_1(\tilde{\sigma}) = \inf\{w > \underline{w} \colon \Xi(\tilde{\sigma}, w|\tilde{\sigma}) > 0\}$.

(b) For almost every s, $b_1'(s) = \frac{\sum_{k=0}^{N-1} \gamma_k(b_1(s)) z_k'(s|s)}{1 - \hat{H}(s, b_1(s)|s) - \sum_{k=0}^{N-1} \gamma_k'(b_1(s)) z_k(s|s)}$.

(c) $\lim_{s \to \hat{\sigma}} b_1(s) = \bar{w}$.

[17] If $b'(s^*|s_0)$ is not defined at (s^*, b^*) then we can arrive at the same conclusions working with the inverse functions $\phi(b|s_0)$ and $\phi(b|s_0')$ as in the proof of Proposition 1.

Proof of Lemma 12. The existence of the function $b_1(s)$ satisfying condition (a) and (b) follow from an argument that is full analogous to that presented for the all-pay auction. The argument here is considerably simplified owing to the stronger assumptions on the distribution $G(w)$. Again it is convenient to phrase the argument in terms of the inverse of $b_1(s)$ and define $b_1(s)$ appropriately thereafter. As before if $\underline{w} > 0$, then $b_1(\tilde{\sigma}_1) = \underline{w}$.

Point (c) is specific to the war of attrition and it ensures that the strategy defined in Proposition 3 is a.e. continuous. We first apply Lemma 16 multiple times we see that $\frac{f_k(x|s)}{1-F_k(x|s)} \geq \left(\frac{k}{N-1}\right) \frac{f_{N-1}(x|s)}{1-F_{N-1}(x|s)} \geq \left(\frac{1}{N-1}\right) \frac{f_{N-1}(x|s)}{1-F_{N-1}(x|s)}$. Then, we note that

$$\begin{aligned}
b_1'(s) &= \frac{\sum_{k=1}^{N-1} \gamma_k(b_1(s)) v_k(s,s) f_k(s|s)}{\sum_{k=1}^{N-1} \gamma_k(b_1(s))(1-F_k(s|s)) \left[1 - \frac{kg(b_1(s))}{1-G(b_1(s))} \left(\frac{z_{k-1}(s|s)-z_k(x|s)}{1-F_k(s|s)}\right)\right]} \\
&\geq v_{N-1}(s,s) \frac{\sum_{k=0}^{N-1} \gamma_k(b_1(s))(1-F_k(s|s)) \frac{f_k(s|s)}{1-F_k(s|s)}}{\sum_{k=1}^{N-1} \gamma_k(b_1(s))(1-F_k(s|s))} \\
&= \frac{v_{N-1}(s,s)}{N-1} \frac{f_{N-1}(s|s)}{1-F_{N-1}(s|s)} = \frac{1}{N-1} \omega'(s)
\end{aligned}$$

Since $b_1(\tilde{\sigma}) = \omega(\tilde{\sigma})$, we have that for all $s \in [\tilde{\sigma}, \hat{\sigma})$, $b_1(s) = b_1(\tilde{\sigma}) \int_{\tilde{\sigma}}^{s} b_1'(x)dx \geq \omega(\tilde{\sigma}) + \frac{1}{N-1} \int_{\tilde{\sigma}}^{s} \omega'(x)dx = \omega(\tilde{\sigma}) + \frac{1}{N-1}(\omega(s) - \omega(\tilde{\sigma}))$. From Krishna & Morgan (1997), $\lim_{s\to 1} \omega(s) = \infty$. Thus, since $b_1(s)$ is continuous and strictly increasing, $\lim_{s\to\hat{\sigma}} b_1(s) = \bar{w}$. ■

Lemma 13. *Let $b(s)\colon [\tilde{\sigma}, \hat{\sigma}) \to [\underline{w}, \bar{w})$ be a strictly increasing solution to* (15)*. Then for almost every s, $b'(s) = \frac{\Phi(s,b(s)|s)}{\Xi(s,b(s)|s)}$.*

Proof of Lemma 13. Suppose $b'(s) > 0$. Consider first the numerator in (15). Since $z_0'(s|s) = 0$ and $F_k(s|s) < 1$,

$$\begin{aligned}
\sum_{k=0}^{N-1} \gamma_k(b(s)) z_k'(s|s) &= \sum_{k=1}^{N-1} \gamma_k(b(s)) z_k'(s|s) \\
&= \sum_{k=1}^{N-1} \gamma_k(b(s)) v_k(s,s) f_k(s|s) \\
&= \sum_{k=1}^{N-1} \gamma_k(b(s))(1-F_k(s|s)) \left[\frac{v_k(s,s) f_k(s|s)}{1-F_k(s|s)}\right] \\
&= \Phi(s,b(s)|s) \sum_{n=1}^{N-1} \gamma_n(b(s))(1-F_n(s|s)).
\end{aligned}$$

Consider next the denominator. Since the numerator is always strictly positive, the denominator must also be strictly positive. Using Lemma 14,

$$
\begin{aligned}
&1-\hat{H}(s,b(s)|s)-\sum_{k=0}^{N-1}\gamma_k'(w)z_k(s|s)\\
&=\sum_{k=1}^{N-1}\gamma_k(b(s))(1-F_k(s|s))\left[1-\frac{kg(b(s))}{1-G(b(s))}\left(\frac{z_{k-1}(s|s)-z_k(s|s)}{1-F_k(s|s)}\right)\right]\\
&=\Xi(s,b(s)|s)\sum_{n=1}^{N-1}\gamma_n(b(s))(1-F_n(s|s)).
\end{aligned}
$$

Thus, $b'(s)=\frac{\Phi(s,b(s)|s)\sum_{n=1}^{N-1}\gamma_n(b(s))(1-F_n(s|s))}{\Xi(s,b(s)|s)\sum_{n=1}^{N-1}\gamma_n(b(s))(1-F_n(s|s))}=\frac{\Phi(s,b(s)|s)}{\Xi(s,b(s)|s)}$. ■

Proof of Proposition 3. Noting Lemma 12 here we only verify that the proposed strategy is indeed an equilibrium. The argument proceeds similarly to the case of the all-pay auction. Again, we verify two claims that confirm that a player of type (s,w) has no profitable deviation to any other bid in the range of $\beta(s,w)$.

Claim 1. A bidder of type (s,w) has no profitable deviation to any (feasible) bid $b(x)$ for $x\in[0,\hat{\sigma})$. First suppose $s<\tilde{\sigma}$. Then $\beta(s,w)=\int_0^s\frac{v_{N-1}(y,y)f_{N-1}(y|y)}{1-F_{N-1}(y|y)}dy$. Since $b(s)<\underline{w}$ and using Assumption 7, by results in Krishna & Morgan (1997) this bidder has no profitable deviation to any bid $b(x)$ where $x\in[0,\tilde{\sigma}]$.

Suppose instead that w is sufficiently large to allow this bidder to bid $b(x)$ and $x\in(\tilde{\sigma},\hat{\sigma})$. Since the payoff from submitting this bid is

$$U(b(x)|s,w)=U(b(\tilde{\sigma})|s,w)+\int_{\tilde{\sigma}}^{x}\frac{d}{dt}U(b(t)|s,w)\bigg|_{t=y}dy,$$

it is sufficient to verify that $\frac{d}{dt}U(b(t)|s,w)\leq 0$ for almost every $t\geq\tilde{\sigma}$. A direct calculation and final a simplification with Lemma 14 yields

$$
\begin{aligned}
&\frac{d}{dt}U(b(t)|s,w) \\
&= \sum_{k=1}^{N-1} \gamma_k(b(t))v_k(s,t)f_k(t|s) - b'(t)\left(1 - \hat{H}_{\hat{b}}(t|s) - \sum_{k=0}^{N-1}\gamma'_k(b(t))z_k(t|s)\right) \\
&= \sum_{k=1}^{N-1} \gamma_k(b(t))v_k(s,t)f_k(t|s) \\
&\qquad - b'(t)\sum_{k=1}^{N-1}\gamma_k(w)(1-F_k(t|s))\left[1 - \frac{kg(w)}{1-G(w)}\left(\frac{z_{k-1}(t|s)-z_k(t|s)}{1-F_k(t|s)}\right)\right] \\
&= \left(\sum_{k=1}^{N-1}\frac{\gamma_k(b(t))(1-F_k(t|s))\left[\frac{v_k(s,t)f_k(t|s)}{1-F_k(t|s)}\right]}{\sum_{n=1}^{N-1}\gamma_n(b(t))(1-F_n(t|s))} - b'(t)\Xi(t,b(t)|s)\right) \\
&\qquad \times \left(\sum_{n=1}^{N-1}\gamma_n(b(t))(1-F_n(t|s))\right) \\
&= \left(\Phi(t,b(t)|s) - b'(t)\Xi(t,b(t)|s)\right)\left(\sum_{n=1}^{N-1}\gamma_n(b(t))(1-F_n(t|s))\right)
\end{aligned}
$$

Since $b'(t) > 0$ for almost every t with Lemma 13 we know that for almost every t, $\Xi(t,b(t)|t) > 0$. Then using Assumption 8(c), $s < t \implies \Xi(t,b(t)|s) \geq \Xi(t,b(t)|t) > 0$. Hence, we can write

$$
\begin{aligned}
&\frac{d}{dt}U(b(t)|s,w) \\
&= \left(\frac{\Phi(t,b(t)|s)}{\Xi(t,b(t)|s)} - b'(t)\right)\Xi(t,b(t)|s)\left(\sum_{n=1}^{N-1}\gamma_n(b(t))(1-F_n(t|s))\right) \\
&\leq \left(\frac{\Phi(t,b(t)|x)}{\Xi(t,b(t)|t)} - b'(t)\right)\Xi(t,b(t)|s)\left(\sum_{n=1}^{N-1}\gamma_n(b(t))(1-F_n(t|s))\right) = 0
\end{aligned}
$$

Therefore if $s < t$, $\frac{d}{dt}U(b(t)|s,w) \leq 0$. This is true for almost every $t \in [\tilde{\sigma},\hat{\sigma})$. Thus, $U(b(t)|s,w) \leq U(b(\tilde{\sigma})|s,w) \leq U(b(s)|s,w)$.

Suppose instead that $s > \tilde{\sigma}$. Suppose this bidder contemplates placing an alternative bid $b(x)$ such that $x \in (s,\hat{\sigma})$. Then, the exact same reasoning as above concludes that this too will not be a profitable deviation. Suppose instead this bidder contemplates bidding $b(x)$ for $x \in (\tilde{\sigma},\hat{s})$ where $\hat{s} \leq s$ is defined as the value such that $b(\hat{s}) = \beta(s,w)$. The expected payoff from this alternative bid is

$$U(b(x)|s,w) = U(b(\hat{s})|s,w) - \int_{\hat{s}}^{x} \frac{d}{dt}U(b(t)|s,w)\bigg|_{t=y} dy,$$

Thus it is sufficient to verify that $\frac{d}{dt}U(b(t)|s,w) \geq 0$ for almost every $t \in (x,\hat{s})$. Proceeding as in the case above we can conclude that again,

$$\begin{aligned}
&\frac{d}{dt}U(b(t)|s,w)\\
&= \big(\Phi(t,b(t)|s) - b'(t)\Xi(t,b(t)|s)\big)\left(\sum_{n=1}^{N-1}\gamma_n(b(t))(1-F_n(t|s))\right)
\end{aligned}$$

As before $b'(t) > 0$ for almost every t. Considering such a t, there are two cases:

1. Suppose $\Xi(t,b(t)|s) \le 0$. Then as $\Phi(t,b(t)|s) \ge 0$, $\frac{d}{dt}U(b(t)|s,w) \ge 0$.

2. Suppose $0 < \Xi(t,b(t)|s)$. Then

$$\begin{aligned}
&\frac{d}{dt}U(b(t)|s,w)\\
&= \left(\frac{\Phi(t,b(t)|s)}{\Xi(t,b(t)|s)} - b'(t)\right)\left(\sum_{n=1}^{N-1}\gamma_n(b(t))(1-F_n(t|s))\right)\Xi(t,b(t)|s)\\
&\ge \left(\frac{\Phi(t,b(t)|t)}{\Xi(t,b(t)|t)} - b'(t)\right)\left(\sum_{n=1}^{N-1}\gamma_n(b(t))(1-F_n(t|s))\right)\Xi(t,b(t)|s) = 0
\end{aligned}$$

Therefore $U(b(t)|s,w) \le U(b(\hat{s})|s,w)$.

Finally, consider a bid $b(x)$ when $x \in [0,\tilde{\sigma})$. The same argument as presented by Krishna & Morgan (1997) for the war of attrition without budget constraints show that for such a bid $U(b(x)|s,w) \le U(b(\tilde{\sigma})|s,w) \le U(b(\bar{s})|s,w)$. Therefore this too will not be a profitable deviation. We have thus shown that no type of bidder has a profitable deviation to any other bid in the range of $b(s)$.

Claim 2. A bidder of type (s,w) has no profitable deviations to any bid outside the range of $b(s)$. If $\lim_{s\to\hat{\sigma}} b(s) < \infty$, the bid $\lim_{s\to\hat{\sigma}^-} b(s)$ is equal to the maximum submitted bid and thus wins with probability 1; no bidder can benefit by submitting a strictly greater bid. Therefore we need only consider deviations to bids $\tilde{b} < b(0)$. If $\underline{w} > 0$, this situation is not consistent with our proposed strategy. Therefore, $\underline{w} = 0$ and $\tilde{\sigma} = 0$.

Working from the definition, $b(0) = \mu_0 \equiv \inf\{w > 0\colon \Xi(0,w|0) > 0\}$. Using Assumption 8 we note that for all $0 < w < \mu_0$

$$\begin{aligned}
\Xi(0,w|0) &= \frac{1-G(w)^{N-1}-(N-1)G(w)^{N-2}g(w)z_0(0|0)}{1-G(w)^{N-1}} < 0\\
&\Longrightarrow 1-G(w)^{N-1}-(N-1)G(w)^{N-2}g(w)z_0(0|s) < 0, \forall s \ge 0. \qquad (28)
\end{aligned}$$

Consider a bidder who bids $\hat{b} \in (0,\mu_0)$. This bidder's expected payoff is

$$U(\hat{b}|s,w) = G(\hat{b})^{N-1}z_k(0|s) - (1-G(\hat{b})^{N-1})\hat{b} - \int_0^{\hat{b}} y(N-1)g(y)G(y)^{N-2}dy$$

Since $G(\cdot)$ is differentiable in this range of values, we can calculate

$$\frac{d}{d\hat{b}}U(\hat{b}|s,w) = (N-1)g(\hat{b})G(\hat{b})^{N-2}z_0(0|s) - (1-G(\hat{b})^{N-1}) \geq 0.$$

The inequality is due to (28). As $\frac{d}{d\hat{b}}U(\hat{b}|s,w) \geq 0$, the (constrained) optimal bid in the range $(0,\mu_0]$ is $\min\{b(0),w\}$. Bidders with a budget greater than $b(0)$ would therefore prefer to place their original bids of $\beta(s,w) \geq b(0)$ while bidders with a budget less than $b(0)$ are playing a best response by bidding $\beta(s,w) = w$.

Together, the two claims exhaust all possible deviations to alternative bids. Therefore, $\beta(s,w)$ is indeed an equilibrium. ■

Proof of Corollary 2. For part (a) note that the function

$$b'(s) = \frac{\sum_{k=0}^{N-1}\gamma_k(b(s))z_k'(s|s)}{1-\hat{H}(s,b(s)|s) - \sum_{k=0}^{N-1}\gamma_k'(b(s))z_k(s|s)}$$

has both left and right limits at $s = \tilde{\sigma}$. Thus, taking the limit from either side of $\tilde{\sigma}$ gives:

$$\begin{aligned}
\lim_{s\to\tilde{\sigma}^-} b'(s) &= \lim_{s\to\tilde{\sigma}^-}\frac{v_{N-1}(s|s)f_{N-1}(s|s)}{1-F_{N-1}(\tilde{\sigma}|\tilde{\sigma})} \\
&= \frac{v_{N-1}(\tilde{\sigma}|\tilde{\sigma})f_{N-1}(\tilde{\sigma}|\tilde{\sigma})}{1-F_{N-1}(\tilde{\sigma}|\tilde{\sigma})} \\
&< \frac{v_{N-1}(\tilde{\sigma}|\tilde{\sigma})f_{N-1}(\tilde{\sigma}|\tilde{\sigma})}{(1-F_{N-1}(\tilde{\sigma}|\tilde{\sigma}))\left[1-(N-1)g(\underline{w})\frac{z_{N-2}(\tilde{\sigma}|\tilde{\sigma})-z_{N-1}(\tilde{\sigma}|\tilde{\sigma})}{1-F_{N-1}(\tilde{\sigma}|\tilde{\sigma})}\right]} \\
&= \lim_{s\to\tilde{\sigma}^+} b'(s)
\end{aligned}$$

The strict inequality is due to the assumption that $g(w)$ is strictly positive for all w.

For part (b) it is sufficient to verify that $b'(s) \geq \omega'(s)$ for almost every $s \geq \tilde{\sigma}$. Suppose $b'(s) < \infty$, then we can write

$$\begin{aligned}
b'(s) &= \frac{\sum_{k=0}^{N-1}\gamma_k(b(s))z_k'(s|s)}{1-\hat{H}_{\hat{b}}(s|s) - \sum_{k=0}^{N-1}\gamma_k'(b(s))z_k(s|s)} \\
&= \frac{\sum_{k=0}^{N-1}\gamma_k(b(s))v_k(s,s)f_k(s|s)}{\sum_{k=1}^{N-1}\gamma_k(b(s))(1-F_k(s|s))\left[1-k\frac{g(b(s))}{1-G(b(s))}\left(\frac{z_{k-1}(s|s)-z_k(x|s)}{1-F_k(s|s)}\right)\right]} \\
&\geq \frac{\sum_{k=0}^{N-1}\gamma_k(b(s))(1-F_k(s|s))v_k(s,s)\frac{f_k(s|s)}{1-F_k(s|s)}}{\sum_{k=1}^{N-1}\gamma_k(b(s))(1-F_k(s|s))\left[1-k\frac{g(b(s))}{1-G(b(s))}\left(\frac{z_{k-1}(s|s)-z_k(x|s)}{1-F_k(s|s)}\right)\right]} \\
&\geq v_{N-1}(s,s)\frac{\sum_{k=0}^{N-1}\gamma_k(b(s))(1-F_k(s|s))\frac{f_k(s|s)}{1-F_k(s|s)}}{\sum_{k=1}^{N-1}\gamma_k(b(s))(1-F_k(s|s))} \\
&\geq v_{N-1}(s,s)\frac{f_{N-1}(s|s)}{1-F_{N-1}(s|s)} = \omega'(s).
\end{aligned}$$

■

Proof of Proposition 4. We follow the notation introduced in the proof of Proposition 2.

(a) As in the case of the all-pay auction it is sufficient to verify that $b(s|s_0) \leq b(s|s_0')$.

In an argument parallel to that in the proof of Proposition 2 and adapting there notation therein as needed, we can easily conclude that $b(s|s_0) < b(s|s_0')$ for all $s < \tilde{\sigma}' < \tilde{\sigma}$. Clearly, $b(\tilde{\sigma}|s_0') > b(\tilde{\sigma}|s_0)$.

Suppose that at some (s^*, b^*), $s^* > \tilde{\sigma}$, $b^* = b(s^*|s_0') = b(s^*|s_0)$. Noting the argument in the proof of Proposition 2 it suffices to show $b'(s^*|s_0') > b(s^*|s_0)$. By Lemma 13 we can write $b'(s) = \frac{\Phi(s,b(s)|s)}{\Xi(s,b(s)|s)}$. It therefore suffices to show that $\bar{\Phi}(s^*, b^*|s^*) > \underline{\Phi}(s^*, b^*|s^*)$ and $\bar{\Xi}(s^*, b^*|s^*) \leq \underline{\Xi}(s^*, b^*|s^*)$.

Considering Φ first,

$$
\begin{aligned}
\underline{\rho}(s^*, b^*|s^*) &= \sum_{k=1}^{N-1} \left(\frac{\gamma_k(b^*)(1 - F_k(s^*|s^*))}{\sum_{n=1}^{N-1} \gamma_n(b^*)(1 - F_n(s^*|s^*))} \right) \left[\frac{\underline{v}_k(s^*, s^*) f_k(s^*|s^*)}{1 - F_k(s^*|s^*)} \right] \\
&< \sum_{k=1}^{N-1} \left(\frac{\gamma_k(b^*)(1 - F_k(s^*|s^*))}{\sum_{n=1}^{N-1} \gamma_n(b^*)(1 - F_n(s^*|s^*))} \right) \left[\frac{\bar{v}_k(s^*, s^*) f_k(s^*|s^*)}{1 - F_k(s^*|s^*)} \right] \\
&= \bar{\Phi}(s^*, b^*|s^*).
\end{aligned}
$$

And applying Lemma 21 we can consider Ξ,

$$
\begin{aligned}
&\underline{\Xi}(s^*, b^*|s^*) \\
&= \sum_{k=1}^{N-1} \left(\frac{\gamma_k(b^*)(1 - F_k(s^*|s^*))}{\sum_{n=1}^{N-1} \gamma_n(b^*)(1 - F_n(s^*|s^*))} \right) \left[1 - \frac{kg(b^*)}{1 - G(b^*)} \left(\frac{\underline{z}_{k-1}(s^*|s^*) - \underline{z}_k(s^*|s^*)}{1 - F_k(s^*|s^*)} \right) \right] \\
&\geq \sum_{k=1}^{N-1} \left(\frac{\gamma_k(b^*)(1 - F_k(s^*|s^*))}{\sum_{n=1}^{N-1} \gamma_n(b^*)(1 - F_n(s^*|s^*))} \right) \left[1 - \frac{kg(b^*)}{1 - G(b^*)} \left(\frac{\bar{z}_{k-1}(s^*|s^*) - \bar{z}_k(s^*|s^*)}{1 - F_k(s^*|s^*)} \right) \right] \\
&= \bar{\Xi}(s^*, b^*|s^*).
\end{aligned}
$$

(b) An immediate consequence of Lemma 15(b) like in the proof of Proposition 2.

■

Proof of Proposition 5. Since $v_{N-1}(s,s) f_{N-1}(s|s) < v_{N-1}(s,s) \frac{f_{N-1}(s|s)}{1 - F_{N-1}(s|s)}$, $b_W(s) > b_A(s)$ for all $s \in (0, \tilde{s}]$. Mimicking the argument in the proof of Propositions 2 and Propositions 4 it is sufficient to verify that if there exists a (s^*, b^*) such that $b^* = b_W(s^*) = b_A(s^*)$, then $b_W'(s^*) > b_A'(s^*)$ which will imply a contradiction. This however is straightforward since

$$
\begin{aligned}
b_W'(s^*) &= \frac{\sum_{k=0}^{N-1} \gamma_k(b^*) z_k'(s^*|s^*)}{1 - \hat{H}(s^*, b^*|s^*) - \sum_{k=0}^{N-1} \gamma_k'(b^*) z_k(s^*|s^*)} \\
&> \frac{\sum_{k=0}^{N-1} \gamma_k(b^*) z_k'(s^*|s^*)}{1 - \sum_{k=0}^{N-1} \gamma_k'(b^*) z_k(s^*|s^*)} = b_A'(s^*)
\end{aligned}
$$

$\blacksquare$

C Miscellaneous Lemmas

Lemma 14. *Let $G \equiv G(w)$, $g \equiv g(w)$, $z_k \equiv z_k(x|s)$, $F_k \equiv F_k(x|s)$, $\hat{H} \equiv \hat{H}(x,w|s)$, and $\gamma_k \equiv \gamma_k(w)$. The following are equivalent versions of frequently used expressions:*

a) $1 - \sum_{k=0}^{N-1} \gamma_k' z_k \equiv \gamma_0 + \sum_{k=1}^{N-1} \gamma_k \left[1 - \frac{kg}{1-G}(z_{k-1} - z_k)\right]$.

b) $\sum_{k=0}^{N-1} \gamma_k' z_k \equiv \sum_{k=0}^{N-2} g(N-1)\binom{N-2}{k} G^{N-2-k}(1-G)^k (z_k - z_{k+1})$.

c) $1 - \hat{H} - \sum_{k=0}^{N-1} \gamma_k' z_k \equiv \sum_{k=1}^{N-1} \gamma_k (1 - F_k)\left[1 - \frac{kg}{1-G}\left(\frac{z_{k-1} - z_k}{1-F_k}\right)\right]$.

Proof of Lemma 14. Differentiating $\gamma_k(w)$ gives

$$\begin{aligned}\gamma_k' = &\binom{N-1}{k}(N-1-k)G^{N-2-k}g(1-G)^k \\ &- \binom{N-1}{k}G^{N-1-k}gk(1-G)^{k-1}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{k=0}^{N-1} \gamma_k' z_k = &\sum_{k=0}^{N-2}\binom{N-1}{k}(N-1-k)G^{N-2-k}g(1-G)^k z_k \\ &- \sum_{k=1}^{N-1}\binom{N-1}{k}kG^{N-1-k}g(1-G)^{k-1}z_k\end{aligned}$$

For $k \le N-2$, $\binom{N-1}{k}(N-1-k) = \frac{(N-1)!}{(N-1-k)!k!}(N-1-k) = (N-1)\frac{(N-2)!}{(N-2-k)!k!} = (N-1)\binom{N-2}{k}$ and for $k \ge 1$, $\binom{N-1}{k}k = \frac{(N-1)!}{(N-1-k)!(k-1)!} = (N-1)\frac{(N-2)!}{(N-2-(k-1))!(k-1)!} = (N-1)\binom{N-2}{k-1}$. Shifting the index of summation we see that

$$\begin{aligned}&(N-1)g\sum_{k=1}^{N-1}\binom{N-2}{k-1}G^{N-1-k}(1-G)^{k-1}z_k \\ &\qquad = (N-1)g\sum_{k=0}^{N-2}\binom{N-2}{k}G^{N-2-k}(1-G)^k z_{k+1}.\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{k=0}^{N-1} \gamma'_k z_k &= g(N-1) \sum_{k=0}^{N-2} \binom{N-2}{k} G^{N-2-k} g(1-G)^k z_k \\
&\quad - g(N-1) \sum_{k=0}^{N-2} \binom{N-2}{k} G^{N-2-k} (1-G)^k z_{k+1} \\
&= \sum_{k=1}^{N-1} g(N-1) \binom{N-2}{k-1} G^{N-1-k} (1-G)^{k-1} (z_{k-1} - z_k) \qquad (29) \\
&= \sum_{k=1}^{N-1} \gamma_k \frac{kg}{1-G} (z_{k-1} - z_k)
\end{aligned}$$

Since $1 = \gamma_0 + \sum_{k=1}^{N-1} \gamma_k$,

$$1 - \sum_{k=0}^{N-1} \gamma'_k z_k = \gamma_0 + \sum_{k=1}^{N-1} \gamma_k \left[1 - \frac{kg}{1-G} (z_{k-1} - z_k) \right]$$

which shows part (a). Shifting the index of summation at (29) gives part (b)

For part (c), note that $\hat{H} = \gamma_0 + \sum_{k=1}^{N-1} \gamma_k F_k$ since $F_0 = 1$. Therefore, $1 - \hat{H} = \sum_{k=1}^{N-1} \gamma_k (1 - F_k)$. So,

$$\begin{aligned}
1 - \hat{H} - \sum_{k=0}^{N-1} \gamma'_k z_k &= \sum_{k=1}^{N-1} \gamma_k (1 - F_k) - \sum_{k=1}^{N-1} \gamma_k \frac{kg}{1-G} (z_{k-1} - z_k) \\
&= \sum_{k=1}^{N-1} \gamma_k (1 - F_k) \left[1 - \frac{kg}{1-G} \left(\frac{z_{k-1} - z_k}{1 - F_k} \right) \right].
\end{aligned}$$

■

Lemma 15. *Suppose (X, Y, Z) are affiliated random variables with a strictly positive, bounded, continuous density $f(x, y, z)$ defined on $[0,1]^3$. Define*

$$f(x|y,z) = \frac{f(x,y,z)}{\int_0^1 f(x,y,z)dx}.$$

Let $z' > z$ and suppose that $f(\cdot|y,z) \neq f(\cdot|y,z')$ for all y. Then there exists $0 < \hat{y} < 1$ such that:

(a) $f(x|x,z') < f(x|x,z)$ for all $x < \hat{y}$.

(b) $\frac{f(x|x,z')}{1-F(x|x,z')} < \frac{f(x|x,z)}{1-F(x|x,z)}$ for all $x < \hat{y}$.

Proof of Lemma 15. Let $z' > z$ and fix y. Since $(y, z') \geq (y, z)$, but the properties of affiliated random variables (see Milgrom & Weber (1982) or Krishna (2002)) the function

$$\frac{f(\cdot|y,z')}{f(\cdot|y,z)}\colon [0,1] \to \mathbb{R} \tag{30}$$

is nondecreasing. (It is clearly continuous since $f(\cdot|y,z)$ is continuous and non-zero.)

Suppose $f(0|y,z') = f(0|y,z)$. Then $f(x|y,z') \geq f(x|y,z)$ for all $x \in [0,1]$. Since $f(\cdot|y,z) \neq f(\cdot|y,z')$ there must exist an open set $\mathcal{X} \subset [0,1]$ such that $f(\cdot|y,z) > f(\cdot|y,z')$. But this implies $1 = \int_0^1 f(x|y,z)dx < \int_0^1 f(x|y,z')dx$ which is a contradiction. Therefore $f(0|y,z') < f(0|y,z)$. By continuity, there exists a $\hat{x}(y)$ such that if $x < \hat{x}(y)$, then $f(x|y,z') < f(x|y,z)$. Let $\hat{y} = \inf_y \hat{x}(y)$. Clearly, $\hat{y} > 0$. Therefore, for all $x < \hat{y}$ and for all y, $f(x|y,z') < f(x|y,z)$ $(\star)$. Since, the inequality holds for all y letting $y = x$ we see that for all $x < \hat{y}$, $f(x|x,z') < f(x|x,z)$.

To derive the second conclusion, begin with $(\star)$ above and choose $y < \hat{y}$. Then $f(x|y,z') < f(x|y,z)$ for all $x < \hat{y}$ implies $F(y|y,z') = \int_0^y f(x|y,z') \leq \int_0^y f(x|y,z) = F(y|y,z)$. Thus,

$$\frac{1}{1 - F(y|y,z')} \leq \frac{1}{1 - F(y|y,z)}.$$

Combining this observation with the first conclusion gives the second result. ∎

Lemma 16. *If* $F_k(x|s) = \underbrace{\int_0^1 \cdots \int_0^1}_{N-1-k} \underbrace{\int_0^x \cdots \int_0^x}_{k} h(y_1,\ldots,y_{N-1}|s)dy_1 \cdots dy_{N-1}$ *and* $f_k(x|s) = \frac{d}{dx}F_k(x|s)$. *Then for all* k, $\frac{f_k(s|s)}{1-F_k(s|s)} \geq \left(\frac{k}{k+1}\right)\frac{f_{k+1}(s|s)}{1-F_{k+1}(s|s)}$.

Proof of Lemma 16. Clearly, $F_k(x|s) \geq F_{k+1}(x|s)$. Then, using the symmetry of the density $h(\cdot)$, a direct calculation gives

$$\begin{aligned}
\frac{f_k(x|s)}{1 - F_k(x|s)} &= \frac{k \overbrace{\int_0^1 \cdots \int_0^1}^{N-1-k} \overbrace{\int_0^x \cdots \int_0^x}^{k-1} h(y_1,\ldots,y_{N-2},x|s)dy_1 \ldots dy_{N-2}}{1 - F_k(x|s)} \\
&\geq \frac{\frac{k}{k+1}(k+1) \overbrace{\int_0^1 \cdots \int_0^1}^{N-2-k} \overbrace{\int_0^x \cdots \int_0^x}^{k} h(y_1,\ldots,y_{N-2},x|s)dy_1 \ldots dy_{N-2}}{1 - F_{k+1}(x|s)} \\
&= \left(\frac{k}{k+1}\right) \frac{f_{k+1}(x|s)}{1 - F_{k+1}(x|s)}
\end{aligned}$$

∎

Lemma 17. *Let* $k \in \{1,\ldots,N-1\}$. *Then,* $z_k(x|s) = \int_0^x v_k(s,y)f_k(y|s)dy$.

Proof of Lemma 17. Let $\mathbf{y} = (y_1, \ldots, y_{N-1})$. Then, working from the definition:

$$\begin{aligned}
z_k(x|s) &\equiv \underbrace{\int_0^1 \cdots \int_0^1}_{N-1-k} \underbrace{\int_0^x \cdots \int_0^x}_{k} u(s, \mathbf{y}) h(\mathbf{y}|s) dy_1 \cdots dy_{N-1} \\
&= \Pr[\bar{Y}_k \leq x | S = s] \mathbb{E}[u(S, Y_1, \ldots Y_N) | S = s, \bar{Y}_k \leq x] \\
&= \int_0^x \mathbb{E}[u(S, Y_1, \ldots Y_N) | S = s, \bar{Y}_k = y] f_k(y|s) dy \\
&= \int_0^x v_k(s, y) f_k(y|s) dy
\end{aligned}$$

■

Lemma 18. *Suppose for all $\mathbf{s}_{-i}$, $u(\cdot, \mathbf{s}_{-i})h(\mathbf{s}_{-i}|\cdot)$ is nondecreasing. Then for all k, $v_k(\cdot, y) f_k(y|\cdot)$ is nondecreasing for every y.*

Proof of Lemma 18. Since $u(\cdot, \mathbf{s}_{-i})h(\mathbf{s}_{-i}|\cdot)$ is nondecreasing, $z_k(x|s) - z_k(x|s') \geq 0$ when $s > s'$. Moreover since $u(s, \mathbf{s}_{-i})h(\mathbf{s}_{-i}|s) - u(s', \mathbf{s}_{-i})h(\mathbf{s}_{-i}|s') \geq 0$, it is easy to conclude that $z_k(x|s) - z_k(x|s') \geq z_k(x'|s) - z_k(x'|s')$ for $x > x'$. Therefore,

$$\begin{aligned}
\lim_{x' \to x} \frac{1}{x - x'} \int_{x'}^x v_k(s, y) f_k(y|s) dy &\geq \lim_{x' \to x} \frac{1}{x - x'} \int_{x'}^x v_k(s', y) f_k(y|s') dy \\
\implies v_k(s, x) f_k(x|s) &\geq v_k(s', x) f_k(x|s').
\end{aligned}$$

■

Lemma 19. *Let $\mathbf{y} = (y_1, \ldots, y_{N-1})$. For all k,*

$$\begin{aligned}
&\int_0^s \frac{\partial}{\partial s} \big(v_k(s, y) f_k(y|s) \big) dy \\
&= \underbrace{\int_0^1 \cdots \int_0^1}_{N-1-k} \underbrace{\int_0^s \cdots \int_0^s}_{k} \frac{\partial}{\partial s} \big(u(s, \mathbf{y}) h(\mathbf{y}|s) \big) dy_1 \cdots dy_{N-1}.
\end{aligned}$$

Proof of Lemma 19. Applying Leibniz's Rule to (4), we see that

$$\frac{d}{ds} z_k(s|s) = v_k(s, s) f_k(s|s) + \int_0^s \frac{\partial}{\partial s} \big(v_k(s, y) f_k(y|s) \big) dy.$$

Working from the definition of $z_k(x|s)$ in (2) and using (8), we can compute

$$
\begin{aligned}
&\frac{d}{ds} z_k(s|s) \\
&= \frac{\partial}{\partial x} z_k(x|s)\bigg|_{x=s} + \frac{\partial}{\partial s} z_k(x|s)\bigg|_{x=s} \\
&= v_k(s|s) f_k(s|s) \\
&\qquad + \frac{\partial}{\partial s} \underbrace{\int_0^1 \cdots \int_0^1}_{N-1-k} \underbrace{\int_0^x \cdots \int_0^x}_{k} u(s,\mathbf{y}) h(\mathbf{y}|s) dy_1 \cdots dy_{N-1} \Bigg|_{x=s} \\
&= v_k(s|s) f_k(s|s) \\
&\qquad + \underbrace{\int_0^1 \cdots \int_0^1}_{N-1-k} \underbrace{\int_0^s \cdots \int_0^s}_{k} \frac{\partial}{\partial s} \left[u(s,\mathbf{y}) h(\mathbf{y}|s) \right] dy_1 \cdots dy_{N-1}
\end{aligned}
$$

Equating the two derivations, gives the desired result. ■

Lemma 20. *Suppose for all $\mathbf{s}_{-i}$, $u(\cdot,\mathbf{s}_{-i})h(\mathbf{s}_{-i}|\cdot)$ is nondecreasing. For all $k \leq N-1$,*

$$\int_0^s \frac{\partial}{\partial s} \left[v_k(s,y) f_k(y|s) \right] dy \geq \int_0^s \frac{\partial}{\partial s} \left[v_{k+1}(s,y) f_{k+1}(y|s) \right] dy.$$

Proof of Lemma 20. Let $\mathbf{y} = (y_1, \ldots, y_{N-1})$. Then,

$$
\begin{aligned}
&\int_0^s \frac{\partial}{\partial s} \left[v_k(s,y) f_k(y|s) \right] dy \\
&= \underbrace{\int_0^1 \cdots \int_0^1 \int_0^1}_{N-1-k} \underbrace{\int_0^s \cdots \int_0^s}_{k} \frac{\partial}{\partial s} \left[u(s,\mathbf{y}) h(\mathbf{y}|s) \right] dy_1 \cdots dy_{N-1} \\
&\geq \underbrace{\int_0^1 \cdots \int_0^1}_{N-2-k} \underbrace{\int_0^s \int_0^s \cdots \int_0^s}_{k+1} \frac{\partial}{\partial s} \left[u(s,\mathbf{y}) h(\mathbf{y}|s) \right] dy_1 \cdots dy_{N-1} \\
&= \int_0^s \frac{\partial}{\partial s} \left[v_{k+1}(s,y) f_{k+1}(y|s) \right] dy
\end{aligned}
$$

■

Lemma 21. *Let $s' \geq s$ and suppose for all $(s_i, \mathbf{s}_{-i})$, $\bar{u}(s_i,\mathbf{s}_{-i}) \geq \underline{u}(s_i,\mathbf{s}_{-i})$. Define $\bar{z}_k(x|s)$ as in (2) with respect to the utility function $\bar{u}$. $\bar{v}_k$, $\underline{z}_k$, $\underline{v}_k$ are defined analogously.*

(a) $\bar{v}_k(x|s) \geq \underline{v}_k(x|s)$.

(b) $\bar{v}_k(x|s') \geq \bar{v}_k(x|s)$.

(c) *If* $\bar{u}(s', \mathbf{s}_{-i})h(\mathbf{s}_{-i}|s') \geq \underline{u}(s, \mathbf{s}_{-i})h(\mathbf{s}_{-i}|s)$*, then for all* $k \leq N-2$*,* $\bar{z}_k(x|s') - \bar{z}_{k+1}(x|s') \geq \underline{z}_k(x|s) - \underline{z}_{k+1}(x|s)$*.*

(d) *If either* $s' > s$ *or* $\bar{u}(\mathbf{s}) > \underline{u}(\mathbf{s})$ *for all* $\mathbf{s} > 0$*, then if* $x > 0$*, the above weak inequalities hold strictly.*

Proof of Lemma 21. Claim (a) is immediate from the definition of $v_k(x|s)$. (b) is a consequence of affiliation. To prove part (c) we adopt the notation $\mathbf{y} = (y_1, \ldots, y_{N-1})$. Then,

$$\begin{aligned}
&\underline{z}_k(x|s) - \underline{z}_{k+1}(x|s) \\
&= \underbrace{\int_0^1 \cdots \int_0^1}_{N-1-k} \underbrace{\int_0^x \cdots \int_0^x}_{k} \underline{u}(s, \mathbf{y})h(\mathbf{y}|s)dy_1 \cdots dy_{N-1} \\
&\quad - \underbrace{\int_0^1 \cdots \int_0^1}_{N-1-(k+1)} \underbrace{\int_0^x \cdots \int_0^x}_{k+1} \underline{u}(s, \mathbf{y})h(\mathbf{y}|s)dy_1 \cdots dy_{N-1} \\
&= \int_x^1 \left(\underbrace{\int_0^1 \cdots \int_0^1}_{N-1-(k+1)} \underbrace{\int_0^x \cdots \int_0^x}_{k} \underline{u}(s, \mathbf{y})h(\mathbf{y}|s)dy_1 \cdots dy_{N-2} \right) dy_{N-1} \\
&\leq \int_x^1 \left(\underbrace{\int_0^1 \cdots \int_0^1}_{N-1-(k+1)} \underbrace{\int_0^x \cdots \int_0^x}_{k} \bar{u}(s', \mathbf{y})h(\mathbf{y}|s')dy_1 \cdots dy_{N-2} \right) dy_{N-1} \\
&= \bar{z}_k(x|s') - \bar{z}_{k+1}(x|s')
\end{aligned}$$

Part (d) is trivial since $\bar{u}$ and $\underline{u}$ are strictly increasing in their first argument. ■

Lemma 22. *Suppose* $\bar{z}_k(x|s') - \bar{z}_{k+1}(x|s') \geq \underline{z}_k(x|s) - \underline{z}_{k+1}(x|s)$*. Then* $1 - \sum_{k=0}^{N-1} \gamma_k'(w)\underline{z}_k(x|s) \geq 1 - \sum_{k=0}^{N-1} \gamma_k'(w)\bar{z}_k(x|s')$*.*

Proof of Lemma 22. Noting Lemma 14, we can write

$$\begin{aligned}
&\sum_{k=0}^{N-1} \gamma_k'(w)\underline{z}_k(x|s) \\
&= g(w)(N-1) \sum_{k=0}^{N-2} \binom{N-2}{k} G(w)^{N-2-k}(1-G(w))^k [\underline{z}_k(x|s) - \underline{z}_{k+1}(x|s)] \\
&\leq g(w)(N-1) \sum_{k=0}^{N-2} \binom{N-2}{k} G(w)^{N-2-k}(1-G(w))^k [\bar{z}_k(x|s') - \bar{z}_{k+1}(x|s')] \\
&= \sum_{k=0}^{N-1} \gamma_k'(w)\bar{z}_k(x|s').
\end{aligned}$$

Rearranging terms gives the conclusion. ■

Lemma 23. *Let $g(x)\colon [a,b] \to \mathbb{R}$ be continuously differentiable and $f\colon [a',b'] \to [a,b]$ be absolutely continuous. Then $(g \circ f)(x)\colon [a',b'] \to \mathbb{R}$ is absolutely continuous.*

Proof of Lemma 23. Let $\epsilon > 0$. Since f is absolutely continuous, there exists a $\delta > 0$ such that $\sum_{k=1}^{K} |y_k - x_k| < \delta \implies \sum_{k=1}^{K} |f(y_k) - f(x_k)| < \frac{\epsilon}{\sup_{x\in[a,b]}|g'(x)|}$ where $\{(y_k, x_k)\}$ is a finite collection of disjoint intervals of $[a',b']$. Then, $\sum_{k=1}^{K} |y_k - x_k| < \delta \implies \sum_{k=1}^{K} |g(f(y_k)) - g(f(x_k))| \leq \sup_{x\in[a,b]}|g'(x)| \sum_{k=1}^{K} |f(y_k) - f(x_k)| < \epsilon$. ■

Lemma 24. *Let $F(x)$ be absolutely continuous and nondecreasing on $[a,b]$ with continuous derivative $f(x)$. Suppose that for all $a \leq x < x' \leq b$, $\infty > \int_x^{x'} f(z)dz > 0$. Then $f(z) > 0$ for a.e. $z \in [a,b]$.*

Proof of Lemma 24. Let $M = \{z \in [a,b]\colon f(z) \leq 0\}$. Since f is continuous, M is closed. Thus M^c is open (in $[a,b]$) and it can be written as the countable union of disjoint open intervals $M^c = \cup_{k=1}^{\infty}(x_k, y_k)$. Without loss of generality suppose $x_k < y_k \leq x_{k+1} < \cdots$. We notice three facts:

1. $x_0 = a$. Suppose otherwise. Then $f(z) \leq 0$ for $z \in (a, x_0)$. But then $\int_a^{x_0} f(z)dz \leq 0$, which is a contradiction.

2. $y_k = x_{k+1}$. Suppose otherwise. Then $f(z) \leq 0$ for $z \in (y_k, x_{k+1})$. But then $\int_{y_k}^{x_{k+1}} f(z)dz \leq 0$, which is a contradiction.

3. $\lim_{k\to\infty} y_k = b$. $\{y_k\}$ must have a limit since it is a bounded nondecreasing sequence. Suppose $\lim_{k\to\infty} y_k < b$. Then there exists $\epsilon > 0$ such that for $z \in (b-\epsilon, b)$, $f(z) \leq 0$. But this again is a contradiction.

Thus $M \subset \{a,b\} \cup \{y_k\}_{k=1}^{\infty}$. Hence M is countable and therefore has measure zero. ■

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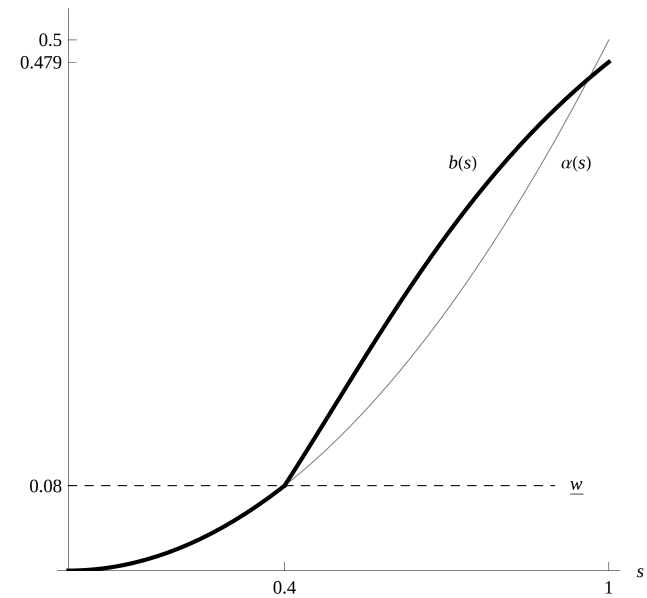

Figure 1: The functions $b(s)$, solid black, and $\alpha(s)$, gray, in the characterization of equilibrium bidding in Example 1.

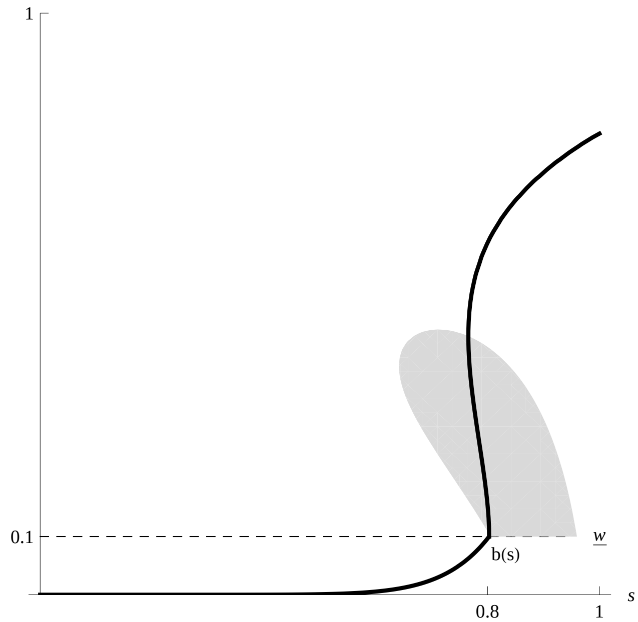

Figure 2: Failure of Assumption 5(a). The gray region denotes the set $\{(s, w)\colon \xi(s, w|s) < 0\}$. Elsewhere, $\xi(s, w|s) \geq 0$. $\xi(s, \cdot|s)$ crosses zero multiple times at values of s slightly less than 0.8.

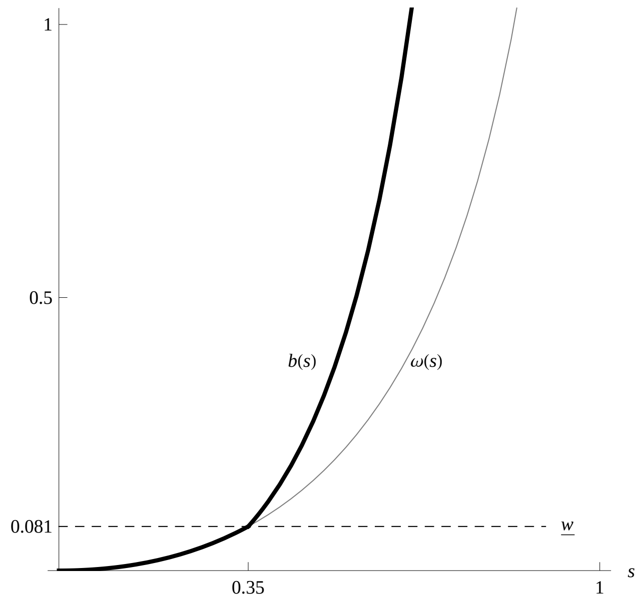

Figure 3: The functions $b(s)$, solid black, and $-s - \log(1 - s)$, gray, in the characterization of equilibrium bidding in Example 3. Both functions are not bounded.

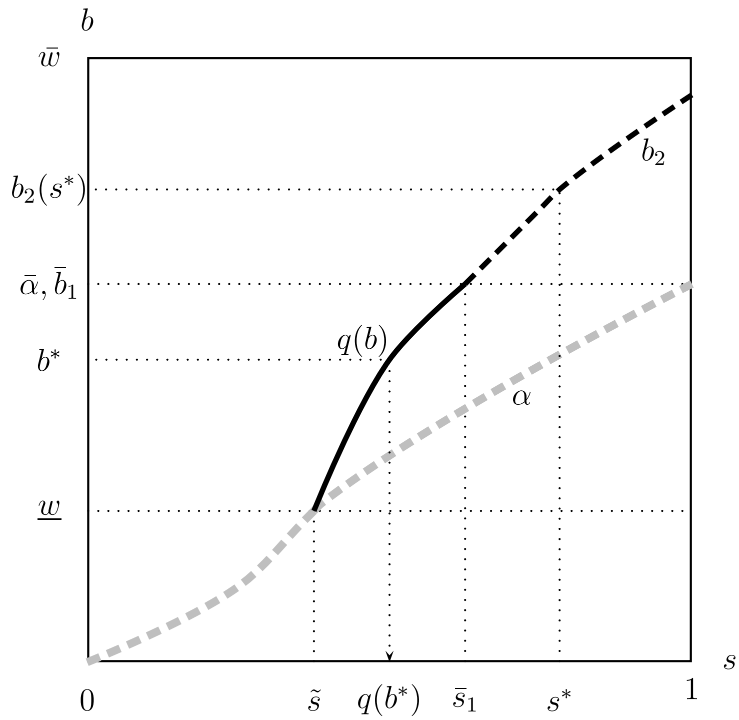

Figure 4: Components that define the function $b(s)$ when $\underline{w} > 0$.

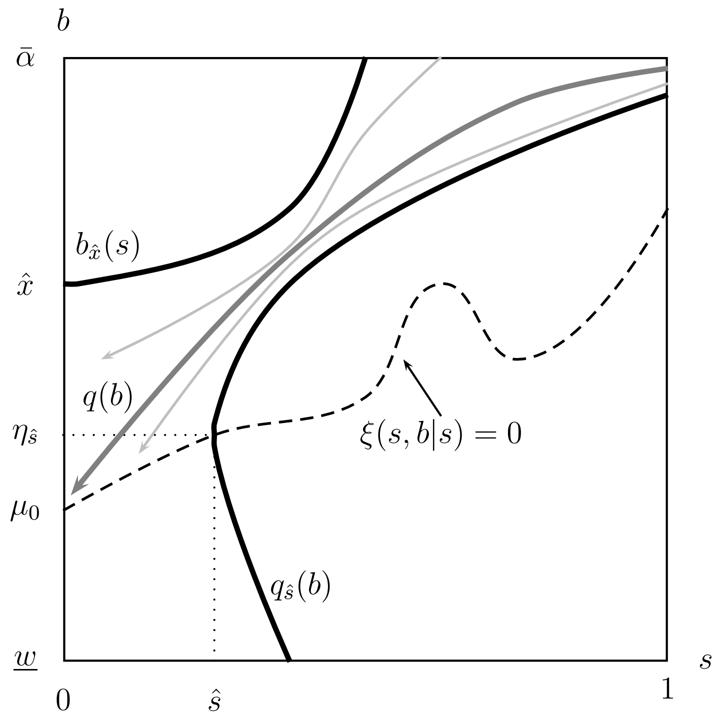

Figure 5: Regions and solutions $q_{\hat{s}}(b)$ and $b_{\hat{x}}(s)$ identified in the proof of Lemma 10. The solution $q(b)$ is bounded between $b_{\hat{x}}(s)$ and $q_{\hat{s}}(b)$ for all $\hat{x} > \mu_0$ and $\hat{s} > 0$.