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Game Complete Analysis of Symmetric Cournot Duopoly

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Abstract. In this paper we apply the *Complete Analysis of Differentiable Games* (introduced by D. Carfi in [3], [6], [8], [9], and already employed by himself and others in [4], [5], [7]) to the classic Cournot Duopoly (1838), classic oligopolistic market in which there are two enterprises producing the same commodity and selling it in the same market. In this classic model, in a competitive background, the two enterprises employ, as possible strategies, the quantities of the commodity produced. The main solutions proposed in literature for this kind of duopoly are the *Nash equilibrium* and the *Collusive Optimum*, without any subsequent critical exam about these two kinds of solutions. The absence of any critical quantitative analysis is due to the relevant lack of knowledge regarding the set of all possible outcomes of this strategic interaction. On the contrary, by considering the Cournot Duopoly as a *differentiable game* (a game with differentiable payoff functions) and studying it by the new topological methodologies introduced by D. Carfi, we obtain an exhaustive and complete vision of the entire payoff space of the Cournot game (this also in asymmetric cases with the help of computers) and this total view allows us to analyze critically the classic solutions and to find other ways of action to select Pareto strategies. In order to illustrate the application of this topological methodology to the considered infinite game, several compromise decisions are considered, and we show how the complete study gives a real extremely extended comprehension of the classic model.

Keywords: Duopoly; Normal-form Games; Microeconomic Policy; Complete Study of Differentiable games; Bargaining solutions.

JEL Classification: D7; C71; C72; C78

1. Introduction

This paper is organized into 7 sections:

- section 2 contains the **formal description and the definitions of payoff functions of our symmetric Cournot's duopoly**;
- in section 3, we start **the study of the symmetric game and find the critical zone**;
- in section 4, we show **the transformation of the strategic space**;
- in section 5, we study **the non-cooperative friendly phase** and find the extremes and Pareto boundaries of payoff space;
- in section 6, we study **the properly non-cooperative phase**, and obtain Nash equilibrium and the best reply correspondences;
- section 7 considers **the defensive and offensive phases**, obtained through the worst gain functions;
- in section 8, we study **the cooperative phase**, through the analysis of the best cooperative strategies.

2. Formal description of Cournot's normal-form game

Our model will be a non-linear two-player loss game $G = (f, <)$ (see also [6], [8] and [9]). The two players (enterprises) shall be called *Emil* and *Frances* (following Aubin books [1] and [2]).

Assumption 1 (strategy sets). The two players produce and offer the same commodity by the following quantities: $x \in \mathbb{R}_{\geq}$ for Emil and $y \in \mathbb{R}_{\geq}$ for Frances. In more precise terms: the payoff function f of the game is defined on a subset of the positive cone of the Cartesian plane \mathbb{R}^2 , interpreted as a space of bi-quantities. We assume (by simplicity) that the set of all strategies (of each player) is the interval $E = [0, +\infty[$.

Assumption 2 (symmetry of the game). The game will be assumed symmetric with respect to the players. In other terms, the payoff pair $f(x, y)$ is the symmetric of the pair $f(y, x)$.

Assumption 3 (form of price functions). We assume the price function as a function $p : \mathbb{R}^2 \rightarrow \mathbb{R}$, depending only upon the total production $x + y$. Precisely, we assume:

$$p(x, y) = a - b(x + y), \quad (2.1)$$

for every productive bi-strategy (x, y) , where $a \geq 0$ is the fixed price and $b > 0$ is the *absolute marginal price* with respect to the total production

$$P(x, y) = x + y. \quad (2.2)$$

Assumption 4 (form of cost functions). Let the cost function C_1 (defined on E^2) of the first player be given by

$$C_1(x, y) = cx + d, \quad (2.3)$$

for every positive strategy pair (x, y) and let, analogously, be the cost function of the second enterprise given by

$$C_2(x, y) = cy + d, \quad (2.4)$$

for every positive bi-strategy (x, y) , where $c > 0$ is the marginal cost and $d \geq 0$ the fixed cost.

Assumption 5 (payoff functions). First player's *net cost function* is defined, classically, by the opposite of the revenue

$$f_1(x, y) = C_1(x, y) - p(x, y)x = bx(x + y - (a - c)/b) + d, \quad (2.5)$$

for every positive bi-strategy (x, y) . Symmetrically, for Frances, the loss function is defined by

$$f_2(x, y) = C_2(x, y) - p(x, y)y = by(x + y - (a - c)/b) + d, \quad (2.6)$$

for every positive bistrategy (x, y) .

Assumption 6 (critical production of duopoly: u). Both the rival players will be able to obtain a positive net profit, only if the following quantity:

$$u = (a - c)/b, \quad (2.7)$$

is positive. If u were non-positive, for every $(x, y) \geq 0$, the value $x + y - u$ would be non-negative and it was the same for the loss-values $f_1(x, y)$ and $f_2(x, y)$.

Each players, in order to obtain non-negative profits, has to exclude all strategies strictly greater than u . If $x > u$, the value

$$x + y - u$$

is positive and strictly greater than y ; then the loss $f_1(x, y)$ is positive too (and it is the same for Frances when y is strictly greater than u). The strategic space is so reduced from the unbounded interval $[0, +\infty[$ to the compact interval $[0, u]$.

The players, to obtain losses below the fixed cost, have to exclude, collectively, all bi-strategies having total productivity strictly greater than u . The relation $f_1(x, y) < d$ is equivalent to

$$bx(x + y - u) < 0,$$

which is equivalent to $x + y < u$ because of $by > 0$ (it is the same for Frances).

The bi-loss function is so defined by

$$f(x, y) = b(x(x + y - u), y(x + y - u)) + (d, d), \quad (2.8)$$

for every bistrategy (x, y) of the game in the unbounded square E^2 .

3. Study of the Cournot's normal form game

Payoff function to examine. When the fixed cost d is zero and the absolute marginal price b is one (this assumption determines only a “reversible” translation of the loss space), the bi-loss function f from the compact square $[0, u]^2$ into the bi-loss plane \mathbb{R}^2 is defined by

$$f(x, y) = (x(x + y - u), y(x + y - u)), \quad (3.1)$$

for every bistrategy (x, y) in the square $S = [0, u]^2$ which is the convex envelope of its vertices, denoted by A, B, C, D starting from the origin and going anticlockwise.

When the critical production u is 1, we will obtain the payoff vector function defined by

$$f(x, y) = (x(x + y - 1), y(x + y - 1)), \quad (3.2)$$

on the strategy square $S = [0, 1]^2$.

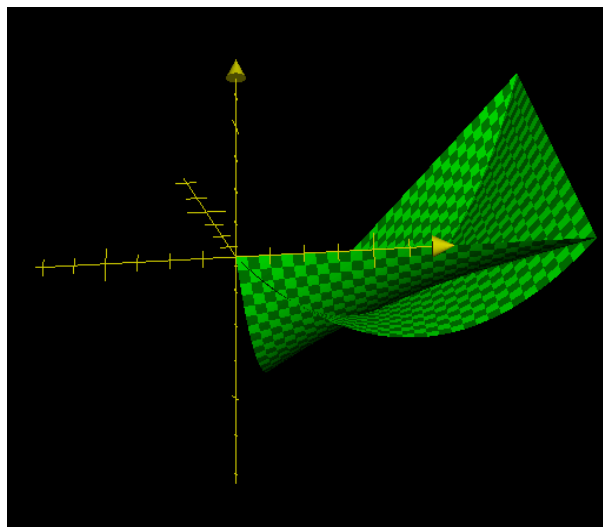


Figure 3.1. The Symmetric Cournot game.

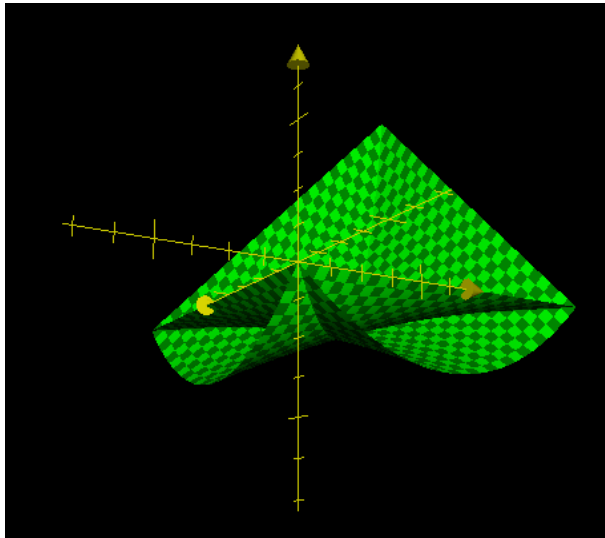


Figure 3.2. The Symmetric Cournot game.

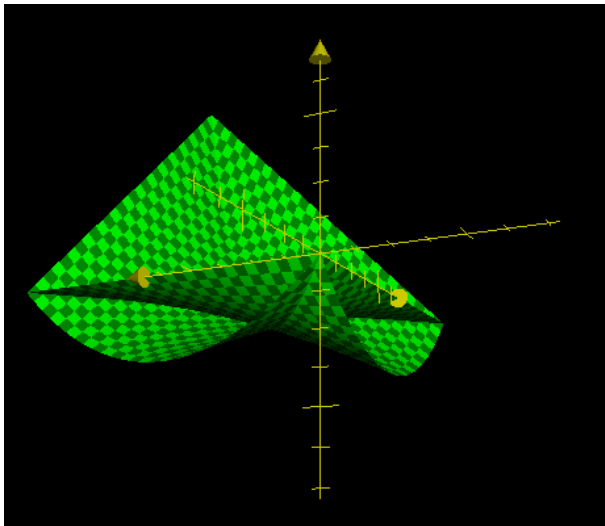


Figure 3.3 (the Cournot game)

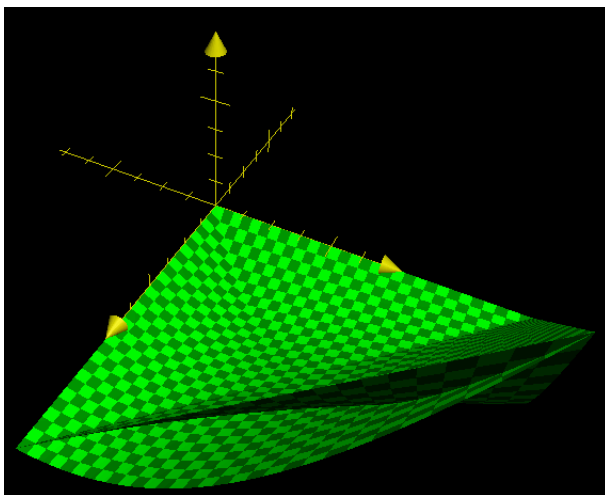


Figure 3.4. The Symmetric Cournot game.

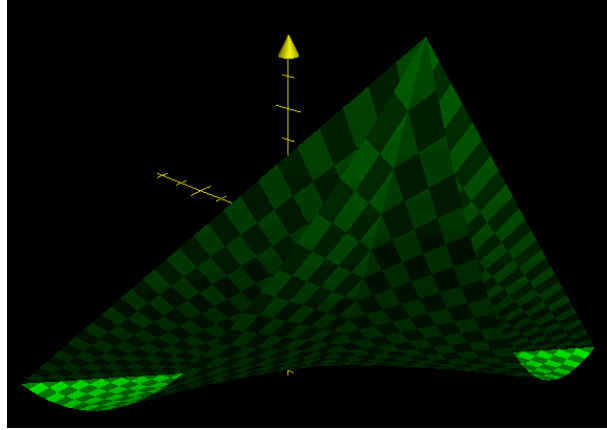


Figure 3.5. The Symmetric Cournot game.

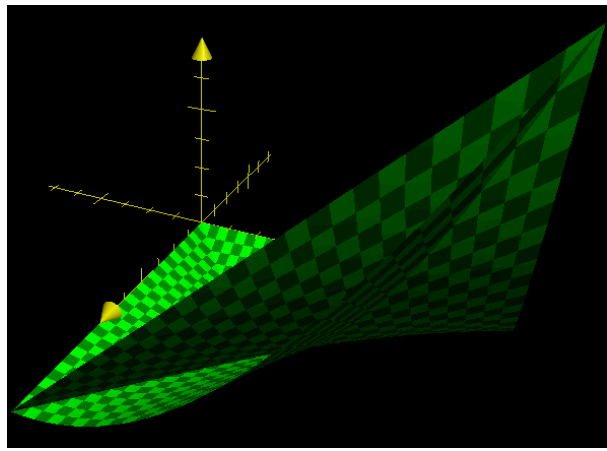


Figure 3.6. The Symmetric Cournot game.

Critical space of the game. Now, we must find *the critical space of the game* and its image by the function f , before representing $f(S)$. For, we determine (as explained in [3], [6], [8] and [9]) firstly the *Jacobian matrix* of the function f at any point (x, y) - denoted by $J_f(x, y)$. We have, in vector form, the pair of gradients

$$J_f(x, y) = ((y + 2x - 1, x), (y, 2y + x - 1)), \quad (3.3)$$

and concerning the determinant of the above pair of vectors

$$\begin{aligned} \det J_f(x, y) &= (y + 2x - 1)(2y + x - 1) - xy = \\ &= 2y^2 + 4xy - 3y + 2x^2 - 3x + 1, \end{aligned} \quad (3.4)$$

for every bi-strategy (x, y) in $[0, 1]^2$.

The *Jacobian determinant* is zero at those points (x_1, y_1) and (x_2, y_2) of the strategy square such that

$$y_1 = 1/2 - x_1, \quad (3.5)$$

and

$$y_2 = 1 - x_2. \quad (3.6)$$

From a geometrical point of view, we will obtain two straight lines (Figure 3.1); their union represents *the critical zone of Cournot Game*.

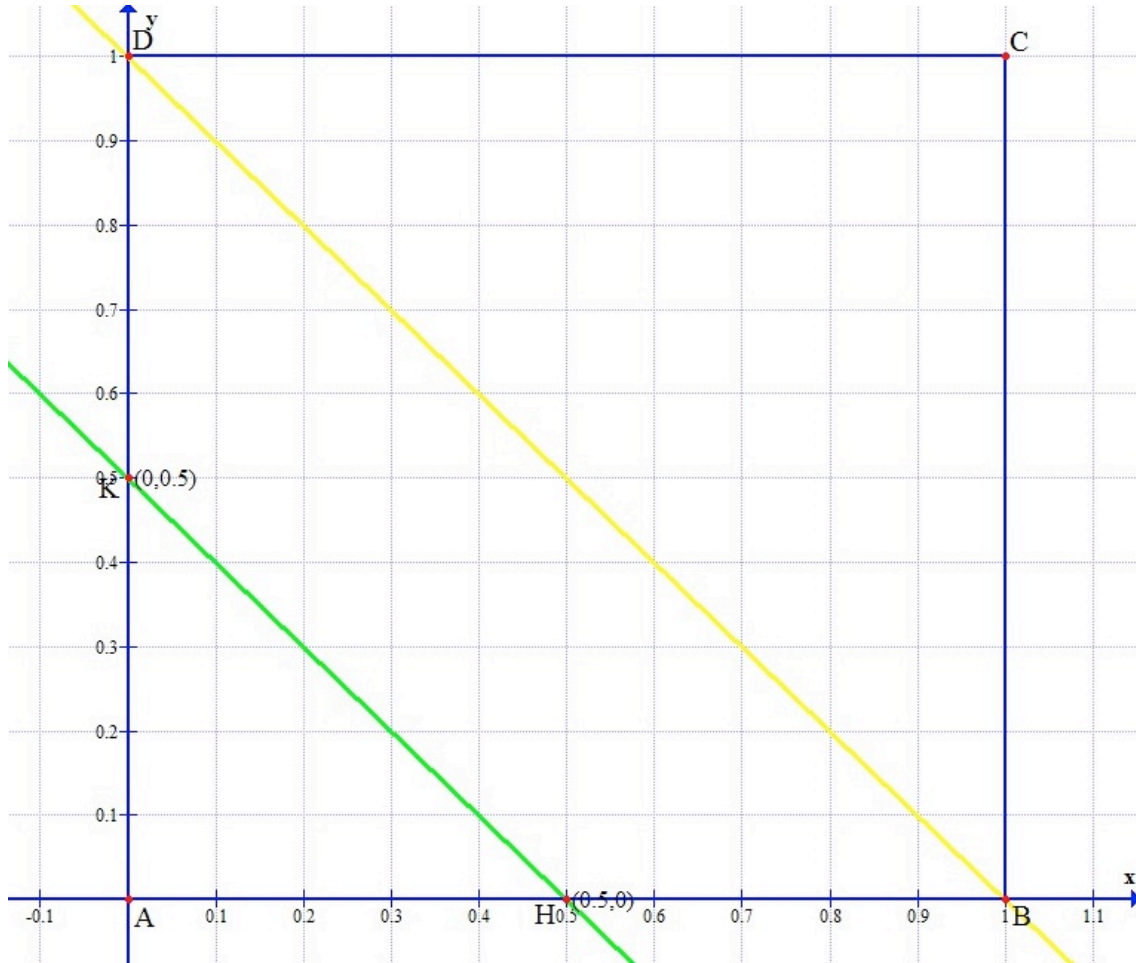


Figure 3.7. Critical zone of Cournot game

4. Transformation of the strategy space and the Cournot “fish”

We start from the strategy space in Figure 3.1.

The transformations of the bi-strategy square vertices and of the points H, K are the following:

- $A' = f(A) = f(0, 0) = (0, 0)$;
- $B' = f(B) = f(1, 0) = (0, 0)$;
- $C' = f(C) = f(1, 1) = (1, 1)$;
- $D' = f(D) = f(0, 1) = (0, 0)$;
- $H' = f(H) = f(1/2, 0) = (-1/4, 0)$;
- $K' = f(K) = f(0, 1/2) = (0, -1/4)$.

Starting from Figure 3.1, with $S = [0, 1]^2$, we can transform its sides.

Side [A, H]. Its parameterization is the function sending any point $x \in [0, 1/2]$ into the point $(x, 0)$; the transformation of this side can be obtained by transformation of its generic point $(x, 0)$, we have:

$$f(x, 0) = (x(x-1), 0). \quad (4.1)$$

We obtain the line segment with end points A' and H' , with parametric equations

$$X = x^2 - x \text{ and } Y = 0, \quad (4.2)$$

with x in the compact interval $[0, 1/2]$.

Side [A, K]. The side $[A, K]$ is parametrized by the function

from $[0, 1/2]$ into the Cartesian plane, mapping the strategy y into the point $(0, y)$;

the image of the generic point is the bi-loss

$$f(0, y) = (0, y(y - 1)). \quad (4.3)$$

We obtain the segment with end points A' and K' , with parametric equations

$$X = 0 \text{ and } Y = y^2 - y. \quad (4.4)$$

It is readily seen that the transformation of the trapezium denoted by H, B, D, K is again the triangle A', H', K' .

Side [B, C]. Its parameterization is

$$(x = 1, y \in [0, 1]);$$

the transformation of its generic point is

$$f(1, y) = (y, y^2). \quad (4.5)$$

We obtain the parabola passing through the points B', C' with

$$X = y \text{ and } Y = y^2, \quad (4.6)$$

with y in the unit interval.

The side [C, D] is parametrized by

$$(x \in [0, 1], y = 1);$$

the transformation of its generic point is

$$f(x, 1) = (x^2, x). \quad (4.7)$$

We obtain the parabola passing through the points C', D' with parametric equations

$$X = x^2 \text{ and } Y = x. \quad (4.8)$$

Now, we find *the transformation of the critical zone*. The parameterization of the critical zone is defined by the equations

$$y_1 = (1/2) - x_1 \text{ with } x_1 \text{ in } [0, 1/2], \quad (\text{see (3.5)})$$

and

$$y_2 = 1 - x_2 \text{ with } x_2 \text{ in } [0, 1]. \quad (\text{see (3.6)})$$

A natural Monge parametrization of the **green segment** $[H, K]$ is defined by

$$(x \in [0, 1/2], y = 1/2 - x);$$

the transformation of its generic point is

$$f(x, 1/2 - x) = (x(x + 1/2 - x - 1), (1/2 - x)(x + 1/2 - x - 1)) = (-1/2 x, -1/2 (1/2 - x)), \quad (4.9)$$

a natural Monge parametrization of the **yellow segment** [B, D] is defined by

$$(x \in [0, 1], y = 1 - x);$$

the transformation of its generic point is the bi-loss

$$f(x, 1 - x) = (x(x + 1 - x - 1), (1 - x)(x + 1 - x - 1)) = (0, 0). \quad (4.10)$$

The transformation of the Green Zone has parametric equations

$$X = -1/2 x, \quad Y = -1/2 (1/2 - x). \quad (4.11)$$

We obtain the segment [H', K'] colored in *green* and the origin colored in *red*, that coincides with the point A' (Figure 4.1).

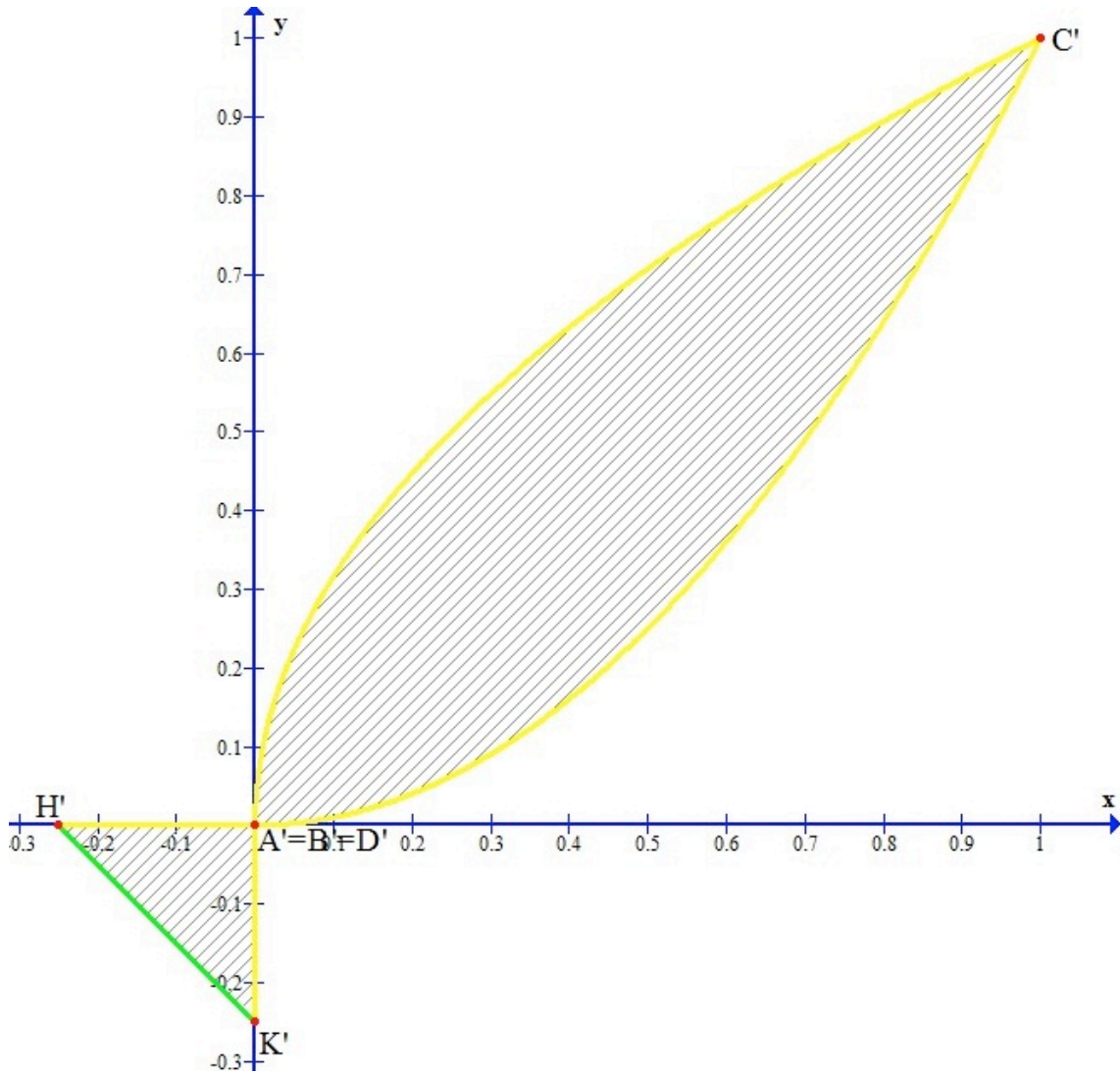


Figure 4.1. Payoff space of Cournot game (the Cournot “fish”).

5. Non-cooperative friendly phase

Examining the Figure 4.1 we will note that the game has both extremes, that is the *shadow minimum* $\alpha = (-1/4, -1/4)$ and the *maximum* $\beta = C' = (1, 1)$.

The *Pareto minimal boundary of the payoff space* $f(S)$ is the *green segment*

$$\partial (f(S)) = [H', K'],$$

it is showed in the Figure 5.1.

The *Pareto maximal boundary of the payoff space* $f(S)$ coincides with the *maximum* $\beta = C' = (1, 1)$.

Both Emil and Frances do not control the Pareto minimal boundary; they could reach the point H' and K' of the boundary, but the solution is not many satisfactory for them. In fact, a player will suffer the maximum loss.

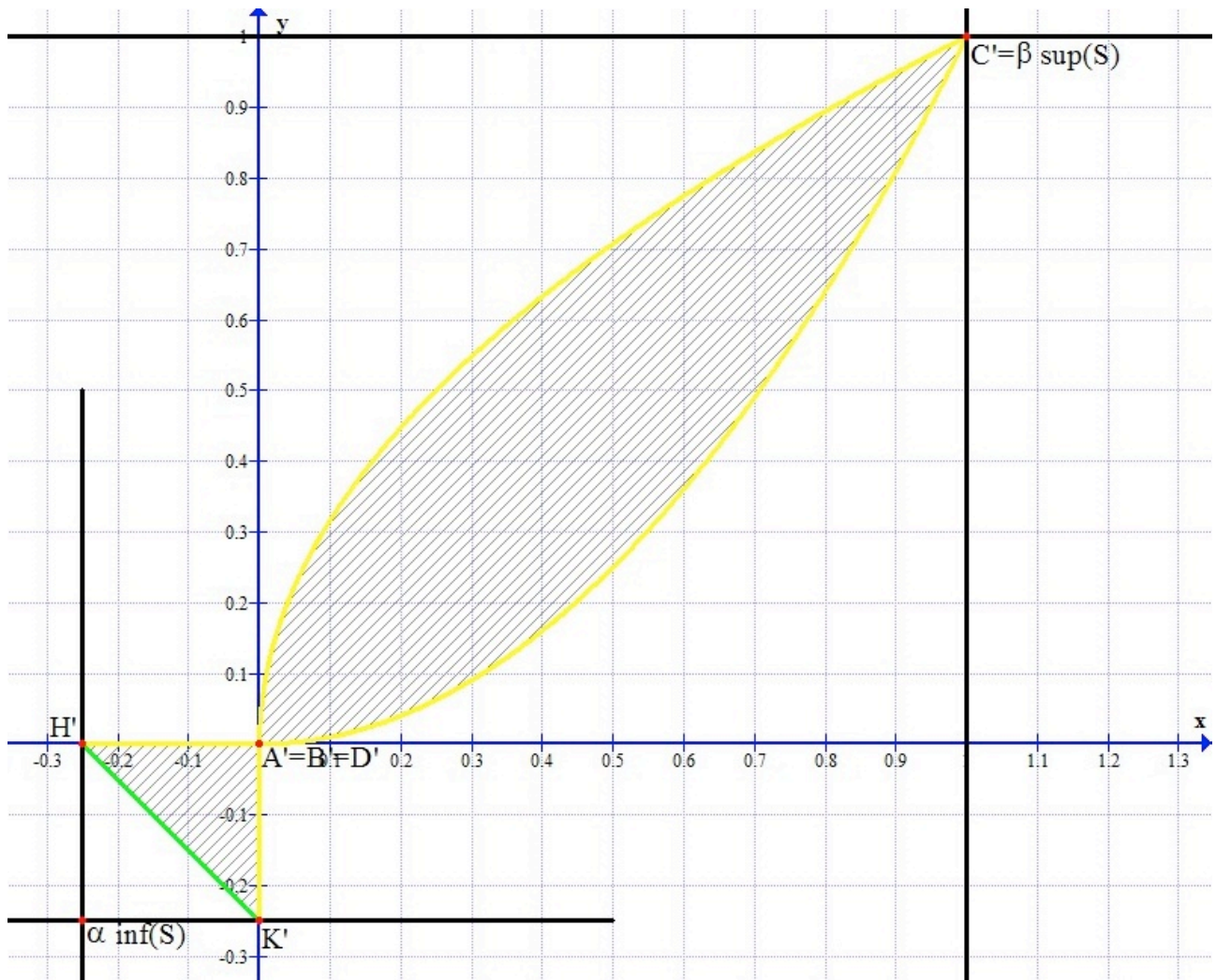


Figure 5.1. Extrema and payoff Pareto boundary of the Cournot game.

6. Properly non-cooperative (or selfish) phase

Now, we will consider *the best reply correspondences* of the two players Emil and Frances.

If Frances produces (and sells) the quantity y of the commodity, Emil - in order to reply rationally - should minimize his *partial cost function*

$$f_1(\cdot, y) : x \mapsto x(x + y - 1), \quad (6.1)$$

on the compact interval $[0, 1]$.

According to the Weierstrass theorem, there exists at least one Emil's strategy minimizing that partial cost function and, by Fermat theorem, the *Emil's best reply strategy to Frances' strategy y* is the unique quantity

$$B_1(y) := (1/2)(1 - y). \quad (6.2)$$

Indeed, the partial derivative

$$f_1(\cdot, y)'(x) = 2x + y - 1, \quad (6.3)$$

is negative for every $x < (1/2)(1 - y)$ and positive for each $x > (1/2)(1 - y)$.

So, the Emil's best reply correspondence is the function B_1 , from the interval $[0, 1]$ into the interval $[0,1]$, defined by

$$y \mapsto (1/2)(1 - y). \quad (6.4)$$

As we already have observed, our Cournot game is a symmetric game, therefore Frances' best reply correspondence is the function B_2 , from $[0, 1]$ into $[0,1]$, defined by the association

$$x \mapsto (1/2)(1 - x). \quad (6.5)$$

The *Nash equilibrium* is the fixed point of the function B , determined by the pair of reaction functions (B_1, B_2) and defined, from the Cartesian product $E \times F$ of the of the domains of the pair (B_1, B_2) (in inverse order) into the Cartesian product $E \times F$ of the codomains (in direct order), by

$$B : [0, 1]^2 \rightarrow [0, 1]^2 : (x, y) \mapsto (B_1(y), B_2(x)); \quad (6.6)$$

Nash equilibrium is so the is the unique bi-strategy (x, y) satisfying the below system of linear equations

$$x = (1/2)(1 - y) \quad \text{and} \quad y = (1/2)(1 - x), \quad (6.7)$$

that is the point $N = (1/3, 1/3)$ - as we can see also from the Figures 6.1 - which determines a bi-loss

$$N' = (-1/9, -1/9),$$

(hence a bi-gain $(1/9, 1/9) = -N'$) as Figures 6.2 is showing.

The Nash equilibrium is not collectively satisfactory, because it is not a Pareto optimal bi-strategy, but it represents the unique *properly non-cooperative* game solution.

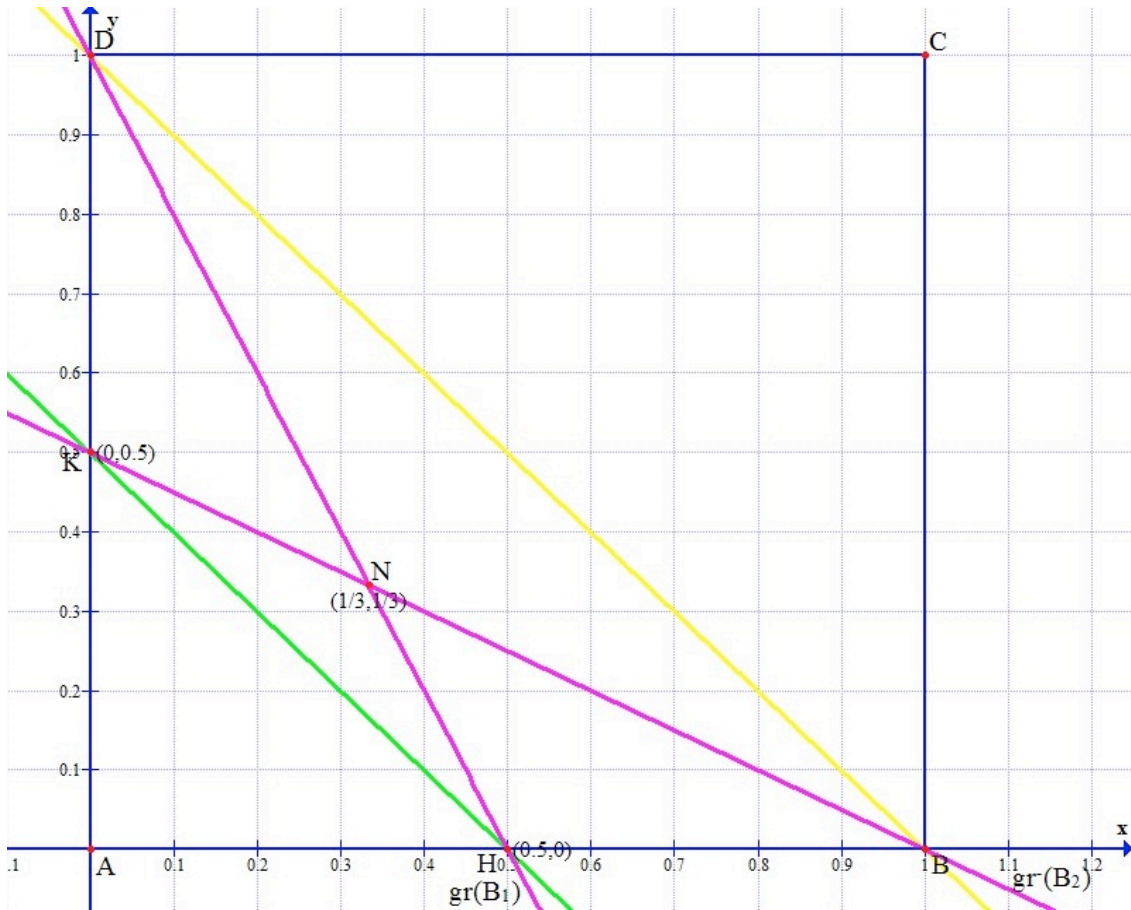


Figure 6. 1. Nash Equilibrium of Cournot game.

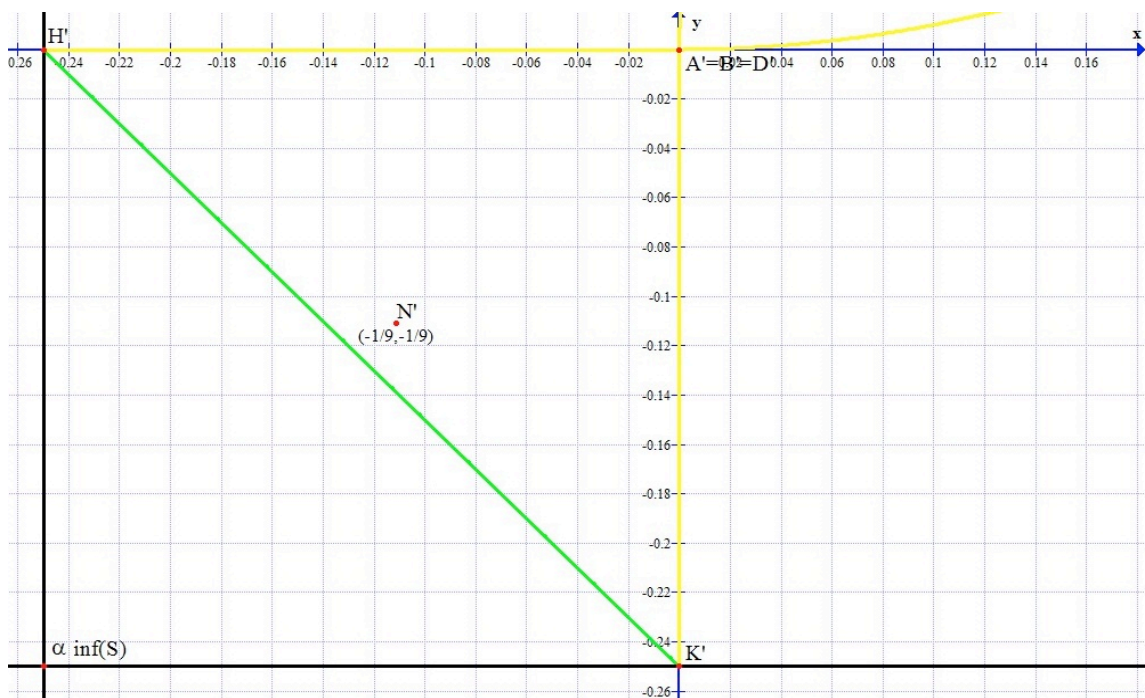


Figure 6.2. Payoff at the Nash equilibrium of Cournot game.

7. Defensive and offensive phase

Players' conservative values are obtained through their *worst loss functions*.

Worst loss functions. On the strategic square $S = [0, 1]^2$, *Emil's worst loss function* is defined by

$$f^{\#_1}(x) = \sup_{y \in F} x(x + y - 1) = x^2, \quad (7.1)$$

its minimum is the loss

$$v^{\#_1} = \inf_{x \in E} (f^{\#_1}(x)) = \inf_{x \in E} (x^2) = 0, \quad (7.2)$$

attained at the unique *conservative strategy* $x^{\#} = 0$.

Symmetrically, *Frances's worst loss function* is defined by

$$f^{\#_2}(y) = y^2, \quad (7.3)$$

its minimum is the loss $v^{\#_2} = 0$, attained at the unique *conservative strategy* $y^{\#} = 0$.

Conservative bivalue. The *conservative bivalue* is then the bi-loss

$$v^{\#} = (v^{\#_1}, v^{\#_2}) = (0, 0).$$

The worst offensive multi-functions are determined by the study of the *worst loss functions*.

The Frances's worst offensive reaction multifunction $O_2 : E \rightarrow F$ is defined by $O_2(x) = 1$, for every Emil's strategy x ; indeed, fixed an Emil's strategy x , the Frances's strategy 1 strictly maximizes the partial cost function $f_1(x, \cdot)$. Symmetrically, the Emil's worst offensive correspondence versus Frances $O_1 : F \rightarrow E$ is defined by $O_1(y) = 1$, for every Frances's strategy y .

The dominant offensive strategy is 1 for both players, indeed the offensive correspondences are constant.

The offensive (dominant) equilibrium $C = (1,1)$ brings to the point $C' = (1,1)$, in which a severe loss is registered for both players.

The core of the payoff space (in the sense introduced by J. P. Aubin) is the part of Pareto minimal boundary, contained in the cone of lower bounds of the conservative bi-value $v^{\#}$. The core of our game is exactly the whole of the Pareto minimal boundary, so the conservative bi-value doesn't give us any bound for the choice of cooperative bistrategies.

Remark. Let us note that, in this case, the Nash equilibrium gives a very small area of the Pareto minimal boundary in which an agreement can be reasonable attained: we have a very small *Nash payoff core* of our game (the part of Pareto minimal boundary of the payoff space, contained in the cone of lower bounds of the Nash equilibrium payoff $f(N)$).

The **conservative knot zone** of the game is the set of all strategy profiles that lead to the conservative bi-value $v^{\#}$: it is the union

$$S^{\#} = [B, D] \cup \{A\}.$$

8. Cooperative phase

A possible agreement between the two players is *the best compromise solution* (in the sense introduced by J.P. Aubin); it is the pair of strategies leading to the unique payoff at the intersection between the segment joining the conservative bi-value with the infimum of the payoff space and the Pareto minimal boundary of the payoff space itself: in our specific case, it is $(1/4, 1/4)$, showed graphically in the Figures 8.2. This compromise bi-strategy determines just the bi-loss $(-1/8, -1/8)$ showed in Figures 8.1.

Remark. In the gain games, the best compromise solution in the sense of J. P. Aubin is the intersection of the segment joining the conservative bi-value and the supremum of the game with the Pareto boundary, it is essentially the Kalai-Smorodinsky solution of the bargaining problem $(\partial^*(f(S)), \sup(f(S)), v^\#)$, with choice space $f(S)$, utopia point $\sup(f(S))$ and threat point the point $v^\#$.

Other compromise solutions? Besides, the best compromise solution coincides with:

- the *core best compromise*,
- the *Nash bargaining solution*,
- the *bistrategy with closest bi-loss to the shadow minimum*;
- the *Kalai-Smorodinsky bargaining solution*.

It is also one of the *transferable utility solutions* (strategies minimizing the aggregate utility function $f_1 + f_2$) which in this case constitute the entire Pareto minimal boundary, this can be easily viewed by geometric evidences considering on the payoff universe the levels of that aggregate function, which are affine lines parallel to the vector $(1, -1)$.

Selection of Pareto solutions. The Nash equilibrium can help in the selection of Pareto solutions (Figures 8.1). Indeed, if Emil and Frances decide to cooperate their possible choices will lead to those points of Pareto boundary which are also lower bounds of the Nash Equilibrium, in order to obtain a compromise solution strictly better than the non-cooperative one.

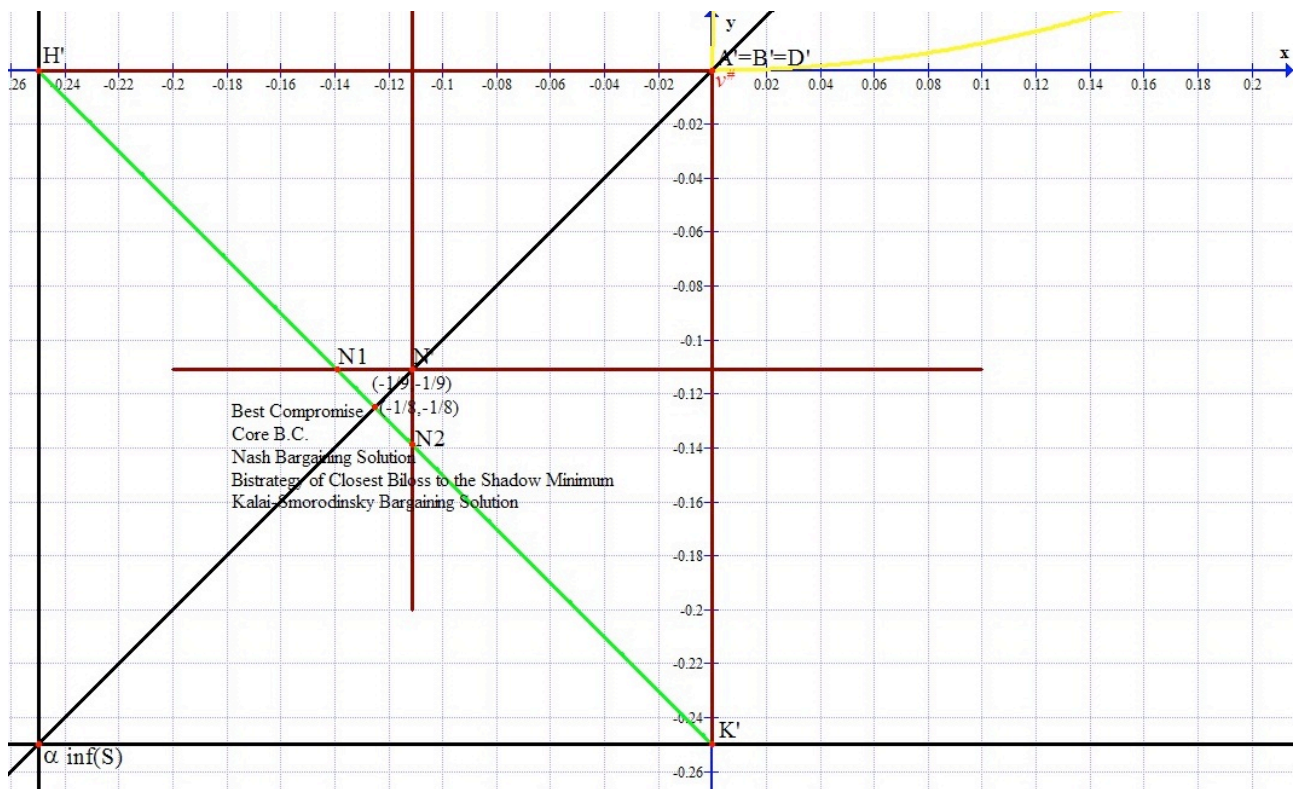


Figure 8. 1. Conservative study: the core and the Kalai-Smorodinsky payoff of Cournot game

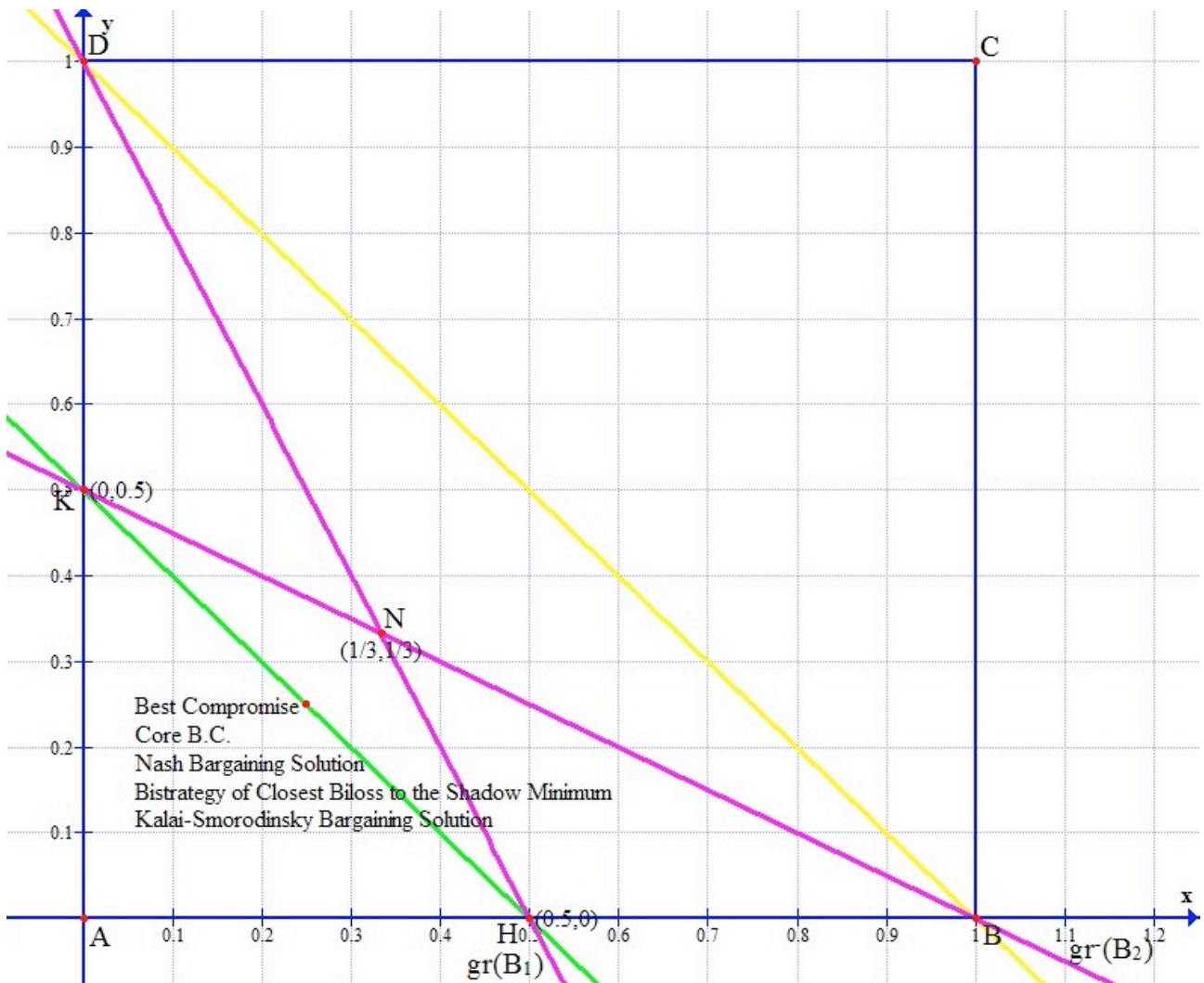


Figure 8.2. Compromise solutions of Cournot game.

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