# Balancing pairs of interfering elements

David Carfì and Gianfranco Gambarelli and Angelo Uristani

University of California Riverside

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**David Carfì**

**(Corresponding Author)**

*Dept. of Mathematics, University of California at Riverside, UCRiverside,* ***USA***

*Dept. DESMAS "V. Pareto", University of Messina,* ***Italy***

email: davidcarfi71@yahoo.it

**Gianfranco Gambarelli**

*Dept. of Mathematics, Statistics, Computer Science and Applications,*

*University of Bergamo,* ***Italy***

email: gambarex@unibg.it

**Angelo Uristani**

*Doctorate on Computational Methods for Forecasting and Decision Making in Economics and Finance,*

*University of Bergamo,* ***Italy***

email: angeluri@yahoo.it

**Abstract.** Many decisions in different fields of application have to take into account the joined effects of two elements that can interfere with each other. For example, in Industrial Economics the demand of an asset can be influenced by the supply of another asset, with synergic or antagonistic effects. The same happens in Public Economics, where two differing economic policies can create mutual interference. Analogously in Medicine and Life Sciences with drugs whose combined administration can produce extra damages or synergies. Other examples occur in Agriculture, Zootechnics and so on. When it is necessary to intervene in such elements, there is sometimes a primary interest for one effect rather than another. For example, if the importance of the effect of an element is ten times greater than the importance of the effect of another, then it is convenient to take this importance into consideration in deciding to what extent it should be employed.

With this in mind, the model proposed here allows the optimal quantities of two elements that interfere with each other to be calculated, taking into account the minimum quantities to be allocated. Algorithms for determining solutions for continuous effects' functions are given, together with software specifically for the case of bilinear functions. It concludes with the presentation of applications particularly to economical problems.

***Keywords.*** *Antagonist Element; Interfering Elements; Optimal dosage; combination of drugs; Synergies; Balancing Interfering Productions*

## 1) Introduction

Many decisions in different fields of application have to take into account the joined effects of two elements that can interfere with each other. For example in Industrial Economics the demand of an asset can be influenced by the supply of another asset, with synergic or antagonistic effects. The same happens in Public Economics where two differing economic policies can create mutual interference. Analogously, in Medicine and Life Sciences with drugs whose combined administration can produce extra damages or synergies. Other examples occur in Agriculture, Zootechnics and so on. When it is necessary to intervene in such elements, there is sometimes a primary interest for one effect rather than another. For example, if the importance of the effect of an element is ten times greater than the importance of the effect of another, then it is convenient to take this importance into consideration in deciding to what extent it should be employed.

With this in mind, the model proposed here allows the optimal quantities of two elements that interfere with each other to be calculated, taking into account the minimum quantities to be allocated.

Algorithms for determining solutions for continuous effects' functions are given, together with software specifically for the case of bilinear functions. In the next two sections, the problem will be defined in general terms; in sections 4 and 5, the case of bilinear interference (free and truncated) will be dealt with; in the following section, an algorithm will be presented (the software for which is given in the Appendix) for the direct calculation of solutions for the above cases; in section 7, the study of more general cases will be examined, in which the effects are represented by continuous functions; in section 8, generalisation problems for more than two interfering elements will be looked at.

It concludes with observations concerning applications of the model, especially in the field of Economics and Medicine.

## 2) Definitions

Let N = $\{1, 2\}$ be the set of labels of the considered interfering elements (i.e. drugs, fungicides, commodities and so on) and related effects resulting from their use (e.g., curing diseases, killing parasites, commodity demand and so on). From here on, if not otherwise specified, use of the index "$i$" will imply "for all $i \in N$", with an analogous use of the index "$j$".

### 2.1) The quantities

We denote as follows the non-negative quantities of the $i$-th element:

- $Q_i$ is the quantity effectively used;
- $Q_i^{\max}$ is the optimal quantity if the $i$-th element is used alone;
- $Q_i^{\min}$ is the minimum necessary quantity if the $i$-th element is used alone;
- $q_i$ and $q_i^{\min}$ are the corresponding ratios with respect to $Q_i^{\max}$:

$$q_i = Q_i/Q_i^{\max},$$

$$q_i^{\min} = Q_i^{\min}/Q_i^{\max}.$$

It is assumed that $Q_i^{\min} < Q_i^{\max}$ and $Q_i^{\min} \leq Q_i \leq Q_i^{\max}$. Given such conditions, $q_i$ and $q_i^{\min}$ belong to the interval [0,1].
We call $Q$, $Q^{\max}$, $Q^{\min}$, $q$, and $q^{\min}$ the corresponding $n$-vectors.

## 2.2) The effects

Let $e_i(q)$ be a non-negative function expressing the level of the $i$-th effect when percent quantities $q$ are used. The space of the effects is the set of points $x = (x_1,..., x_n) = e(q)$ according to variations of $q$. This function should satisfy the following conditions (which should be present, given a suitable adjustment of scale).

If no elements are used, then all the effects are null. If a single element is employed in the optimal dose for use alone, then the level of the relative effect is 1, while the level of the effect for the other is null. Finally, if both elements are employed in the optimal doses for use alone, the resulting effects are given by the vector $\delta = (\delta_1, \delta_2)$ with real positive components. In formulae:

if $q_i = 0$ for all $i \in N$, then $e_i = 0$ for all $i \in N$;

if $q_i = 1$ and $q_j = 0$ for all $j \in N, j \neq i$, then $e_i = 1$ and $e_j = 0$;

if $q_i = 1$ for all $i \in N$, then $e_i = \delta_i$.

See Figure 0 as an example of an effects' function.

*Figure 0 about here.*

Without loss of generality, we may place the elements in order so that:

$$\delta_1 \leq \delta_2 \qquad (1).$$

## 2.3) Quantity and minimum effects

We use $e_i^{\min}$ to indicate the minimum necessary level of the $i$-th effect. This level is derived from the function $e_i(q)$ given $q_i = q_i^{\min}$ and $q_j = 0$ for the other component $j \neq i$. We use $e^{\min}$ to indicate the related $n$-dimensional vector.

We assume the minimum necessary level of the $i$-th effect should not exceed 1 (if $\delta_i \leq 1$) or $\delta_i$ (elsewhere). Thus

$$e_i^{\min} \leq \max\{1, \delta_i\} \quad (2)$$

## 2.4) The feasible Pareto optimal boundary

Importing a classical definition, we shall call every point $x$ of the codomain of $e$ which is not jointly improvable a *Pareto optimal effect*, in the sense that if we move from that point in this set to improve the $i$-th effect, then the other effect necessarily decreases. It is easy to prove that even here every Pareto optimal point is a boundary point of the set of effects; we shall therefore call the set of Pareto optimal effects the *Pareto optimal boundary*.

The term *feasible Pareto optimal boundary P* is given to the set of the points of the Pareto optimal boundary that respect the conditions $x_i \geq e_i^{\min}$ for all $i \in N$.

## 2.5) The required optimal ratios

We use $r$ to indicate the required optimal ratio between the effects $e_1$ and $e_2$.

We call $R$ the half-line centred on the origin, the inclination of which is defined by $r$.

For each point $x$ of the feasible set, we use $E$ to indicate the half-line centred on the origin, passing through $x$.

## 3) The optimization problem

### 3.1) The data

The input data of the model are $\delta$ , $e^{\min}$ and $r$.

In some applications we do not know directly the minimal effect $e_i^{\min}$ for some element $i$, while we know the necessary minimal and optimal quantities $Q_i^{\min}$ and $Q_i^{\max}$. From this relationship, it is thus possible to deduce $q_i^{\min}$ which, introduced into the equation $e_i(q)$, gives $e_i^{\min}$, as indicated in section 2.3.

### 3.2) The objective

The problem is to find the set of vectors $q^*$ such that the corresponding effects $e(q^*)$ belong to the feasible Pareto optimal boundary and are such that the half-lines that join them to the origin form a minimum angle with $R$.

### 3.3) Existence and uniqueness

If the necessary minimum effects are excessive as a whole, the feasible set might possibly be empty and therefore without a solution. However, for those cases where determining the minimum quantity is open to variations, we have introduced certain indications as to modifications to be used each time. Solution uniqueness should also be checked each time.

### 3.4) Solution methods

Determining the optimal combination of $q$ clearly depends on the form of the effects function $e(q)$. This function may be defined directly, according to the type of problem, or may be constructed on the basis of available cases, using statistical methods. In any case, for the vertices of the domain, the values determined in section 2 should be respected (to obtain suitable conditions, it would be possible to use, for example, an adjustment of scale).

In the following paragraphs, we shall present a complete study of all bilinear functions (subdivided into free and truncated) providing closed form formulas and geometrical descriptions of the solutions; we shall then present a search method for continuous functions.

The symbol $\chi$ will be used in the text to denote the indicator function, i.e.

$$\chi(condition) = \begin{cases} 1 & \text{if the condition is satisfied} \\ 0 & \text{if the condition is not satisfied} \end{cases}$$

## 4) Free Bilinear Case

In such cases the effects' function $e(q)$ is defined as follows:

$$e_1 = q_1(1 - q_2) + q_1 q_2 \delta_1$$

$$e_2 = (1 - q_1)q_2 + q_1 q_2 \delta_2$$

The problem of minimizing the angle between $R$ e $E$ is rendered as:

$$\min_{q_1,q_2}\left|\frac{e_2}{e_1}-r\right|$$

We shall examine the various types of interference separately, varying the values of $\delta$ under the constraint (1).

We shall represent such types as graphs, with the corresponding numbers. In each of these graphs the grey part indicates the area in which $\delta$ can vary, while the bold line indicates the feasible Pareto optimal boundary.

We shall then give the solutions with the relative steps for achieving them as tables (all to be understood implicitly as proof); these, too, with corresponding numbers.

### 4.1) Type 1 (independent or synergic elements)

This type will be either $\delta_1 = \delta_2 = 1$ (independent elements) or $\delta_1 > 1$, $\delta_2 \geq 1$ (synergic elements) and is illustrated in Figure 1.

*Figure 1 about here.*

The set of effects is represented by the quadrangle having vertices (0, 0), (0, 1), (1, 0) and ($\delta_1$, $\delta_2$). The feasible Pareto optimal boundary is made up of the single point $\delta$. The input condition (2) guarantees the existence of the solution, given in Table 1.

*Table 1 about here.*

### 4.2) Type 2 (partially synergic and partially antagonistic elements)

With this type we have $\delta_1 + \delta_2 > 1$, $\delta_1 \geq 1$, $\delta_2 < 1$. It is illustrated in Figure 2.

*Figure 2 about here.*

The set of effects is described by the quadrangle having vertices (0, 0), (0, 1), (1, 0) and ($\delta_1$, $\delta_2$).
In order to simplify the notations, in this section we define

$$a_1 = \max(0, e_1^{\min})$$

$$b_1 = \min(\delta_1, \frac{\delta_1}{\delta_2 - 1}(e_2^{\min} - 1))$$

The existence of a solution requires, besides (2), the additional condition

$$e_1^{\min} \leq b_1$$

This condition results in $a_1 \leq b_1$ and the feasible Pareto optimal boundary is not empty. This boundary is the set of points $(x_1, x_2)$ such that

$$x_1 \in [a_1, b_1]$$
$$x_2 = \frac{\delta_2 - 1}{\delta_1} x_1 + 1$$

In the event of no solution, the existence of one may be brought about by modifying $e_1^{\min}$ and/or $e_2^{\min}$ as follows:

- by fixing $e_2^{\min}$, we can use $e_1^{\min} = \frac{\delta_1}{\delta_2 - 1}(e_2^{\min} - 1)$;
- by fixing $e_1^{\min}$, we can use $e_2^{\min} = \frac{\delta_2 - 1}{\delta_1} e_1^{\min} + 1$.

Other ways are also open, if both $e_i^{\min}$ are modified.

The solution is given in the final row of Table 2.

*Table 2 about here.*

#### 4.3) Type 3 (weakly antagonistic elements)

With this type we have $\delta_1 + \delta_2 \geq 1$, $\delta_1 < 1$, $\delta_2 < 1$. It is illustrated in Figure 3.

*Figure 3 about here.*

The set of effects is represented by the quadrangle having vertices (0, 0), (0, 1), (1, 0) and ($\delta_1$, $\delta_2$).

In order to simplify the notations, in this section we define

$$a_1 = \max(0, e_1^{\min}),$$
$$b_1 = \min(\delta_1, \frac{\delta_1}{\delta_2 - 1}(e_2^{\min} - 1))$$
$$a_2 = \max(\delta_1, e_1^{\min}),$$
$$b_2 = \min(1, \frac{(\delta_1 - 1)}{\delta_2} e_2^{\min} + 1)$$

The existence of a solution requires, besides (2), the additional condition

$$e_1^{\min} \leq \max(b_1, b_2)$$

This condition results in $a_1 \leq b_1$ e $a_2 \leq b_2$ and the feasible Pareto optimal boundary is not empty. This boundary is the set of points $(x_1, x_2)$ given by $R_1 \cup R_2$, where:

$$R_1 = \begin{cases} \left\{ x = (x_1, x_2) \,\middle|\, \begin{array}{l} x_2 = \frac{(\delta_2 - 1)}{\delta_1} x_1 + 1 \\ x_1 \in [a_1, b_1] \end{array} \right\} & \text{if } e_1^{\min} \leq \delta_1 \\ \varnothing & \text{otherwise} \end{cases}$$

and

$$R_2 = \begin{cases} \left\{ x=(x_1,x_2) \,\middle|\, \begin{array}{l} x_2 = \dfrac{\delta_2}{(\delta_1 - 1)}(x_1 - 1) \\ x_1 \in [a_2, b_2] \end{array} \right\} & if\ e_2^{\min} \le \delta_2 \\ \varnothing & otherwise \end{cases}$$

In the event of no solution, the existence of one may be brought about by modifying $e_1^{\min}$ and/or $e_2^{\min}$ as follows:

- by fixing $e_2^{\min}$, we can use $e_1^{\min} = \max(\frac{\delta_1}{\delta_2 - 1}(e_2^{\min} - 1), \frac{\delta_1 - 1}{\delta_2} e_2^{\min} + 1)$;
- by fixing $e_1^{\min}$, we can use $e_2^{\min} = \min(\frac{\delta_2 - 1}{\delta_1} e_1^{\min} + 1, \frac{\delta_2}{\delta_1 - 1}(e_1^{\min} - 1))$;

Other ways are also open, if both $e_i^{\min}$ are modified.

The solution is given in the final row of Table 3.

*Table 3 about here.*

## 4.4) Type 4 (strongly antagonistic elements)

With this type we have $\delta_1 + \delta_2 < 1$. It is illustrated in Figure 4.

*Figure 4 about here.*

It may be deduced from Carfì (2009, pages 42-44) that the set of effects is the pseudo-triangle with vertices (0, 0), (0, 1) and (1, 0), delimited at North-East by the curve now to be defined. Having called $\delta_1' = 1 - \delta_1$ and $\delta_2' = 1 - \delta_2$, the resulting line is the union of:

- the segment of extremes (0, 1) and $H = (H_1, H_2) = (\delta_1^2/\delta_2', \delta_1')$,
- the segment of extremes (1, 0) and $K = (K_1, K_2) = (\delta_2', \delta_2^2/\delta_1')$,
- the section of the curve between $H$ and $K$, whose equation is

$$x_2 = (1 - \sqrt{\delta_2' x_1})^2 / \delta_1'$$

Note that $H$ belongs to the segment connecting (0, 1) and $(\delta_1, \delta_2)$, and $K$ belongs to the segment connecting (1, 0) and $(\delta_1, \delta_2)$; then $H_1 \le \delta_1$ and $H_2 \le \delta_2$.

In order to simplify the notations, in this section we define

$$a_1 = \max(0, e_1^{\min}),$$

$$b_1 = \min(H_1, \frac{\delta_1}{(\delta_2 - 1)}(e_2^{\min} - 1))$$

$$a_2 = \max(K_1, e_1^{\min}),$$

$$b_2 = \min(1, \frac{(\delta_1 - 1)}{\delta_2} e_2^{\min} + 1)$$

$$a_3 = \max(H_1, e_1^{\min}),$$

$$b_3 = \min(K_1, \frac{(1 - \sqrt{(1-\delta_1)e_2^{\min}})^2}{1 - \delta_2})$$

The existence of a solution requires, besides (2), the additional condition

$$e_1^{\min} \leq \max(b_1, b_2, b_3)$$

This condition results in $a_1 \leq b_1$, $a_2 \leq b_2$ e $a_3 \leq b_3$. In this case the feasible Pareto optimal boundary is not empty. This boundary is the set of points $(x_1, x_2)$ given by $R_1 \cup R_2 \cup R_3$, where:

$$R_1 = \begin{cases} \left\{ x = (x_1, x_2) \middle| \begin{array}{l} x_2 = \dfrac{(\delta_2 - 1)}{\delta_1} x_1 + 1 \\ x_1 \in [a_1, b_1] \end{array} \right\} & \text{if } e_1^{\min} \leq H_1 \\ \varnothing & \text{otherwise} \end{cases}$$

and

$$R_2 = \begin{cases} \left\{ x = (x_1, x_2) \middle| \begin{array}{l} x_2 = \dfrac{\delta_2}{(\delta_1 - 1)} (x_1 - 1) \\ x_1 \in [a_2, b_2] \end{array} \right\} & \text{if } e_2^{\min} \leq K_2 \\ \varnothing & \text{otherwise} \end{cases}$$

and

$$R_3 = \begin{cases} \left\{ x = (x_1, x_2) \middle| \begin{array}{l} x_2 = \dfrac{(1 - \sqrt{(1-\delta_2)x_1})^2}{1 - \delta_1} \\ x_1 \in [a_3, b_3] \end{array} \right\} & \begin{array}{l} \text{if } K_2 \leq e_2^{\min} \leq H_2 \\ \text{and } H_1 \leq e_1^{\min} \leq K_1 \end{array} \\ \varnothing & \text{otherwise} \end{cases}$$

In the event of no solution, the existence of one may be brought about by modifying $e_1^{\min}$ and/or $e_2^{\min}$ in a way analogous to the previous cases:

- by fixing $e_2^{\min}$, we can use

$$e_1^{\min} = \max\left( \frac{\delta_1}{\delta_2 - 1} (e_2^{\min} - 1), \frac{(\delta_1 - 1)}{\delta_2} e_2^{\min} + 1, \frac{(1 - \sqrt{(1-\delta_1)e_2^{\min}})^2}{1 - \delta_2} \right);$$

- by fixing $e_1^{\min}$, we can use

$$e_2^{\min} = \min\left(\frac{\delta_2 - 1}{\delta_1} e_1^{\min} + 1, \frac{\delta_2}{\delta_1 - 1}(e_1^{\min} - 1), \frac{(1 - \sqrt{(1-\delta_2)e_1^{\min}})^2}{1-\delta_1}\right);$$

Intermediate solutions are also possible, in which both $e_i^{\min}$ are modified.

The solution is given in the final row of Table 4.

*Table 4 about here.*

**5) Truncated Bilinear Case**

These cases involve situations in which the effects, beyond a certain maximum level, fall to zero. Here the effects' function $e(q)$ is defined by the following equations:

$$e_1 = \chi(q_1(1-q_2) + q_1 q_2 \delta_1 \leq 1)[q_1(1-q_2) + q_1 q_2 \delta_1]$$

$$e_2 = \chi(q_2(1-q_1) + q_1 q_2 \delta_2 \leq 1)[(1-q_1)q_2 + q_1 q_2 \delta_2]$$

**5.1) Type 1 truncated (independent or synergic elements)**

This type will be either ($\delta_1 = \delta_2 = 1$) or ($\delta_1 > 1, \delta_2 \geq 1$).
It is illustrated in Figure 1T.

*Figure 1T about here.*

The set of effects is the quadrangle having vertices (0, 0), (0, 1), (1, 0) and ($\delta_1$, $\delta_2$). The feasible Pareto optimal boundary is made up of the single point (1, 1). Therefore $x_1 = x_2 = 1$.

The input condition (2) guarantees the existence of the solution, which is given in Table 1T.

*Table 1T about here.*

**5.2) Type 2 truncated (partially synergic and partially antagonistic elements)**

Here we have $\delta_1 + \delta_2 > 1$, $\delta_1 \geq 1$, $\delta_2 < 1$.
This type is illustrated in Figure 2T.

*Figure 2T about here.*

The set of effects is the quadrangle having vertices (0, 0), (0, 1), (1, 0) and ($\delta_1$, $\delta_2$). Although it is analogous to Type 2 in the case given in the previous paragraph, in this case the effects cannot exceed the value of 1.

In order to simplify the notation, in this paragraph we define

$$a_1 = \max(0, e_1^{\min})$$

$$b_1 = \min(1, \frac{\delta_1}{\delta_2 - 1}(e_2^{\min} - 1))$$

Using $a_1$ and $b_1$ as defined immediately above, the conditions for the existence of a solution, related calculations and all considerations concerning the case under examination, are the same as those for section 4.1.2, to which section we would refer the reader.

The solution is given in the final row of Table 2T.

*Table 2T about here.*

### 5.3) Types 3 and 4 truncated

Types 3 and 4 truncated are the same as those for the bilinear free case. We therefore refer the reader to the definitions given in the related sections 4.3 and 4.4 above.

## 6) An algorithm for the bilinear cases

The algorithm we propose is focused exclusively on bilinear cases (free and truncated).

We begin by acquiring the input data $\delta$ and $e^{\min}$ with the free-truncated option and check they display the conditions given in section 2. With regard to *r*, it is quite possible that the user is unable to determine this *a priori* and it is therefore useful to supply the user with an interval of variability *r_int* to allow this parameter to be established.

Having asked the user to enter the required value of *r*, data processing may proceed using the tables given in sections 4 e 5. If a feasible solution is reached, processing stops. Otherwise inform the user that $e_1^{\min}$ and/or $e_2^{\min}$ are too binding and ask the user to modify them, giving suitable indications for doing this.

A definitive calculation can now be made and the results communicated.

A listing in Matlab language is supplied in the Appendix.

## 7) Cases of continuous functions

If continuous functions are to be dealt with, the following approach may be used. This approach can obviously also be used for bilinear functions but, as has been said, the technique presented above offers greater advantages in the case of such functions. The approach we are going to propose allows, where possible, explicit equations for the Pareto optimal boundary to be obtained, thereby avoiding any need to resort to numeric methods, which are unable to guarantee precision in results.

The reliability of the method we are about to present will be demonstrated later.

Thus let $e(q)$ be a continuous function that respects the constraint (1).

We begin by examining the function in the interior of the domain. Here the "critical zone" must be determined, that is, the set of points $(q_1, q_2)$ in which the function may be differentiated and which render to zero the determinant of the matrix of the first partial derivatives (Jacobian) of $e(q)$. Let $I$ indicate the image of $e(q)$ in the critical zone. If $I$ is not empty, it is characterised by $h$ functions $\underline{I}_s$ $(s = 1, \ldots, h)$ defined on the space of the effects. In the event of $I$ being empty, there are no functions to characterise it and we shall see how to proceed further on.

Let $B$ be the image of the function on the boundary, that is, on the four sides of the quadrangle that makes up the domain. $B$ is characterised by four functions $\underline{B}_s$ ($s = 1, \ldots, 4$) defined on the space of the effects.

Let us now consider the set of points in the interior of the domain, in which the function may not be differentiable; we use $A$ to indicate this set. If the set $e(A)$ can be characterised by a finite number $w$ of functions defined on the space of the effects, let us call these functions $\underline{W}_s$ ($s = 1, \ldots, w$); otherwise, we shall use the approach described below at point b.

In the first case, it can be proved (see below) that the Pareto optimal boundary belongs to $I \cup B \cup e(A)$.

Let us use $g$ to denote an indexed family of the functions $\underline{I}_s$, $\underline{B}_s$ and $\underline{W}_s$ (if, for instance, the functions are $\underline{I}_1$, $\underline{I}_2$, $\underline{B}_1$, $\underline{B}_2$, $\underline{B}_3$, $\underline{B}_4$, $\underline{W}_1$, then a family $g$ can be made up of: $g_1 = \underline{B}_1$, $g_2 = \underline{I}_1$, $g_3 = \underline{W}_1$, $g_4 = \underline{B}_2$, $g_5 = \underline{I}_2$, $g_6 = \underline{B}_3$, $g_7 = \underline{B}_4$).
For each function $g_i$ we use $D(g_i)$ to indicate the relative domain.

There are two possibilities:

a) $I$ is not empty and all functions $\underline{I}_s$, $\underline{B}_s$ and $\underline{W}_s$ can be solved for $x_2$;

b) other cases.

In case (a) the Pareto optimal boundary is the set of $x = (x_1, x_2(x_1))$ such that:

$$\begin{cases} x_2(x_1) = \max\limits_{i \in U(x_1)} (g_i(x_1)) \\ U(x_1) = \{i : x_1 \in D(g_i)\} \end{cases}$$

In cases (b) the Pareto optimal boundary can be identified by means of a graph representing the set $I \cup B \cup e(A)$ (this will be illustrated below with an example).

Let us now calculate the intersection between the half-line $R$ and the feasible Pareto optimal boundary $P$. If such an intersection exists, it is unique (for reasons of Pareto-optimality) and in this case the solution to the problem is the pair $(q_1, q_2)$ that correspond to this point.

If such an intersection does not exist, then we need to solve the optimization problem:

$$\begin{aligned} &\min_{q_1, q_2} \left| \frac{e_2(q)}{e_1(q)} - r \right| \\ &s.t. \ (e_1(q), e_2(q)) \in P \end{aligned}$$

The problem can be solved using optimization software. More precisely, the constraint that the effects must take values belonging to $P$ is added. The solution to the problem is thus made up of the couples $(q_1, q_2)$, whose corresponding optimum values are $e_1(q)$ e $e_2(q)$.

*Example.* Consider the function

$$\begin{aligned} e_1 &= q_1 + (\delta_1 - 1) q_1 q_2 \\ e_2 &= (q_2 + (\sqrt{\delta_2} - 1) q_1 q_2)^2 \end{aligned}$$

where $\delta_1, \delta_2 \in [0, 1]$ and $e_1^{\min} = e_2^{\min} = 0$.

In order to simplify the notation, we call $a = 1 - \delta_1$ and $b = 1 - \sqrt{\delta_2}$ . Thus

$$e_1(q_1, q_2) = q_1 - aq_1q_2$$
$$e_2(q_1, q_2) = (q_2 - bq_1q_2)^2.$$

Note that $a, b \in [0, 1]$. The cases $a, b \in \{0, 1\}$ are obvious, so we will study only the cases $a, b \in (0, 1)$.
The Jacobian is the determinant of the matrix

$$\begin{pmatrix} 1 - aq_2 & -aq_1 \\ -2bq_2(q_2 - bq_1q_2) & 2(1 - bq_1)(q_2 - bq_1q_2) \end{pmatrix}$$

that is

$$2q_2(1 - bq_1)(1 - aq_2 - bq_1)$$

The Jacobian is null if

$$q_2 = 0 \vee q_1 = \frac{1}{b} \vee q_2 = \frac{1 - bq_1}{a}$$

The first equation identifies the set of points ($q_1$, 0) for all $q_1 \in (0, 1)$. However this set is not included in the interior of the domain, so we eliminate these points.

The second equation, considering that $b \in (0, 1)$ and $q_1 \in (0, 1)$, identifies the points (1, $q_2$) for all $q_2 \in (0, 1)$. In this case, too, we eliminate these points for the same reason as before.

Finally we study the third equation. Since $q_2 \in (0, 1)$, then

$$0 \le \frac{1 - q_1}{a} \le 1 \Leftrightarrow \begin{cases} 1 - bq_1 \ge 0 \\ 1 - bq_1 \le a \\ q_1 \in (0,1) \end{cases} \Leftrightarrow \begin{cases} bq_1 \le 1 \\ 1 - bq_1 \le a \\ q_1 \in (0,1) \end{cases}$$

Hence it must be $bq_1 \ge 1 - a$. Thus the points identified by the third equation are all pairs ($q_1$, $q_2$) for which $q_2 \in (0, 1)$ and $q_1 \in ((1-a)/b, 1)$.

If $\frac{(1-a)}{b} < 1$, then these points identify the critical zone. Otherwise the critical zone is empty.

Let $Z$ indicate the boundary of the square of vertices (0, 0), (1, 0), (0, 1) and (1,1).
We need to find the image of the points belonging to the critical zone (i.e. $I$)
and the image of those belonging to $Z$ (i.e. $B$).

In order to obtain $I$, we interrelate the values of $e$ in the critical zone with the space of the effects:

$$e\left(q_1, \frac{1 - bq_1}{a}\right) = \left(\sqrt{\frac{x_1}{b}}, \frac{\left(1 - \sqrt{bx_1}\right)^4}{a^2}\right) \qquad if\ \frac{(1-a)}{b} < 1$$

Thus the image $I$ is

$$I:\begin{cases} x_2 = \dfrac{\left(1-\sqrt{bx_1}\right)^4}{a^2} \\ x_1 \in [\dfrac{1-a}{b},\, b] \end{cases} \qquad if\ \frac{(1-a)}{b} < 1$$

We must now determine *B* (i.e., the image of *Z*).

We interrelate the values of *e* in the points that belong to *Z* with the space of the effects:

$$e(q_1,0) = (x_1,0)$$
$$e(0,q_2) = (0,x_2)$$
$$e(q_1,1) = \left(\frac{x_1}{1-a}, \left(1-\frac{b}{1-a}x_1\right)^2\right)$$
$$e(1,q_2) = \left(\frac{1-x_1}{a}, \left(\frac{(1-b)}{a}\right)^2 (1-x_1)^2\right)$$

The image *B* is the union of $B_1$, $B_2$, $B_3$ and $B_4$ defined as follows:

$$B_1:\begin{cases} x_2 = 0 \\ x_1 \in [0,\, 1] \end{cases}$$

$$B_2:\begin{cases} x_1 = 0 \\ x_2 \in [0,\, 1] \end{cases}$$

$$B_3:\begin{cases} x_2 = \left(1-\dfrac{b}{1-a}x_1\right)^2 \\ x_1 \in [0,\, 1-a] \end{cases}$$

$$B_4:\begin{cases} x_2 = \left(\dfrac{(1-b)}{a}\right)^2 (1-x_1)^2 \\ x_1 \in [1-a, 1] \end{cases}$$

Note that $e(q)$ is differentiable for all the domain and therefore the set *A* is empty.

Given that for all functions that characterize *B* and *I* it is possible to solve for $x_2$, we identify the Pareto optimal boundary as the set of points $x = (x_1, x_2)$ such that:

$$\begin{cases} x_2 = \max\begin{pmatrix} 0\cdot\chi(x_1 \in [0,1]), \\ 1\cdot\chi(x_1 = 0), \\ \left(1-\dfrac{b}{1-a}x_1\right)^2 \chi(x_1 \in [0,\, 1-a]), \\ \left(\dfrac{(1-b)}{a}\right)^2 (1-x_1)^2 \chi(x_1 \in [1-a, 1]), \\ \dfrac{\left(1-\sqrt{bx_1}\right)^4}{a^2} \chi\left(x_1 \in [\dfrac{1-a}{b},\, b]\right) \end{pmatrix} \\ x_1 \in [0,\, 1] \end{cases}$$

We note that the Pareto optimal boundary may also be identified through a graphical analysis (see Figure 5). To do this, we state, for example, $\delta_1 = \delta_2 = 1/8$, from which $a = 7/8$ and $b = 1-\sqrt{1/8}$.

*Figure 5 about here.*

The feasible Pareto optimal boundary is therefore:

$$P:\begin{cases} \left(1-\dfrac{b}{1-a}x_1\right)^2 & x_1 \in [0,\, 1/8] \\ \left(\dfrac{(1-b)}{a}\right)^2 (1-x_1)^2 & x_1 \in [1/8, \approx 0.19] \\ \dfrac{\left(1-\sqrt{bx_1}\right)^4}{a^2} & x_1 \in [\approx 0.19, 1] \end{cases}$$

The solution to the problem is obtained by solving this system with the additional equation $x_2 = r\,x_1$. If there is no solution, we must resort to the optimization problem presented above.

***Remark.***

The method proposed above was obtained by adapting, to these situations, the method given by Carfì (2009, pages 38-42), which deals only with differentiable functions. The complete proof for cases of continuous functions is given by the following

***Theorem.*** *Let f be a function defined on a compact subset K of the Euclidean plane and taking values into the same plane. Let ∂K be the topological boundary of K; let C be the set of all interior points of K where the function f is differentiable and the Jacobian matrix of f is not invertible; let H be the set of all the interior points of K in which f is not differentiable (note that this set must contain the set of all interior points of K in which f is not continuous). Then, the part of the boundary of the image of the compact K which is contained in the image f(K), that is the set ∂f(K) ∩ f(K), is contained into the union of the following three sets: the image of the boundary of the compact K, that is the set f(∂K); the image of the interior critical zone C of the function f, that is the image f(C); the image of the non-differentiable zone H, that is f(H). In particular, the Pareto Optimal boundary of the image f(K) is contained in the above union.*

*Proof.* The intersection $\partial f(K) \cap f(K)$ is the image $f(K)$ minus the interior part of $f(K)$. In other words, a point $x$ of the image $f(K)$ is a boundary point if and only if it is not an interior point. Moreover, the intersection $\partial f(K) \cap f(K)$ is obviously contained into the image $f(K)$. So the intersection $\partial f(K) \cap f(K)$ is contained into $f(K°) \cup f(\partial K)$, where $K°$ is the interior part of the compact $K$. Moreover the image $f(K°)$ is the union $f(H) \cup f(K° \setminus H)$. So the intersection $\partial f(K) \cap f(K)$ is contained into the union $f(\partial K) \cup f(H) \cup f(K° \setminus H)$.

More specifically the part $f(K° \setminus H)$ is contained into the union of the two parts $f(C)$ and $f(K° \setminus (H \cup C))$; but the part $f(K° \setminus (H \cup C))$ contains only interior points of $f(K)$, which cannot be boundary points. So we can conclude that the intersection $\partial f(K) \cap f(K)$ is contained into the union $f(\partial K) \cup f(H) \cup f(C)$.

Notice that the part $f(K°\backslash(H \cup C))$ contains only interior points since the function $f$ is a local homeomorphism at every point $x$ belonging to the subset $K°\backslash(H \cup C)$. Indeed, this latter difference set is the set of all interior points of $K$ in which the function $f$ is differentiable and with invertible Jacobian matrix; hence $f$ is a local diffeomorphism at these points. As we already know, a local diffeomorphism at a point $x$ is also a local homeomorphism at that point, so that it sends a neighbourhood of $x$ into a neighbourhood of $f(x)$; consequently $f(x)$ is also an internal point.

Q.E.D.

## 8) Interference between more than two elements

*Figure 6 about here*

Figure 6 shows a graph corresponding to Figure 0 for the case $n$ = 3. It is easy to see that working with graphic methods is very difficult in the particular case of multilinear functions. Therefore, if more exact solutions are to be obtained, the method presented in section 7 should also be used here. This method may additionally be applied to cases of $n$>2, although it requires calculations that become extremely complicated. However, there is no reason to despair of finding an analogous approach that allows us to determine, whenever possible, the equations necessary for a solution.

## 9) Some applications

### 9.1) Some applications to Economics

The problem of finding the optimal quantities of goods to be produced is well-known. The fact that the demand for certain goods can be influenced by an interaction with the demand for other goods often plays a part in this problem. In some cases a firm has to decide the production quantities of a product that can partially or completely substitute other products ("cannibalization"). In other cases the effects of two products can be synergic (complementary).

Let us consider, as an example, the case of a company producing a particular commodity (denoted by A), but which has just developed a new commodity (denoted by B), the demand for which might negatively influence the demand for A. We assume that the company does not want to produce in order to create warehouse stock.

In the first place, the company has to calculate the optimal quantity it would sell when marketing only A ($Q_A^{\max}$), and similarly only B ($Q_B^{\max}$ ). Obviously, it could decide to sell no products at all, thereby rendering the quantities $Q_A^{\min}$ and $Q_B^{\min}$ equal to zero.

Let $e_A(q_A, q_B)$ be the projected market demand for product A, given the hypothesis in which percentage quantities $q_A$ for A and $q_B$ for B are sold. Thus $\delta_A$ and $\delta_B$ are measured by the respective demand for products $A$ and $B$ in the case where both products are sold in the quantities $Q_A^{\max}$ and $Q_B^{\max}$.

The decision regarding the quantities of B to sell depends on a willingness to sacrifice part of the demand for A. This willingness to cannibalize product A depends on various factors, examples being the future market situation of the two products and a company desire to place itself at a strategic advantage in an emerging market (the one for B); for a detailed analysis of the factors influencing the willingness to cannibalize see Chandy et al. (1998), Nijssen et al. (2004) and Battaggion et. al. (2009).

Thus the willingness to cannibalize is represented by $r$, the desired trade-off between demand for one product and demand for the other.

With the problem defined in these terms, the company can calculate the optimal quantities to produce, applying the methods provided in the previous sections.

### 9.2) Some other applications

The model can be used analogously in Public Economics to calibrate two differing economic policies that are interfering with each other. There are also other applications outside economics.

In Medicine, the balance of interfering drugs is usually performed by successive approximations, keeping the patient monitored. Thus the decision on the first dose is particularly delicate. Using this model, it is possible to establish the optimal dosages in relation to the desired ratios between improvements in the patient's health with respect to two diseases, taking into account the minimal needed quantity for each medicine.

In Veterinarian practice, as well as in Zootechnics to optimize diets, in Agriculture to calculate dosages of parasiticides or additives, so as to increase production, and so on.

## 10) Further developments

As can be seen above, there are many concrete applications of this model to real life. Further outstanding problems are, besides the generalisation mentioned in section 7, methods for non-continuous cases, which have not been resolved here, and new interpretations shifting from a Decision Theory viewpoint to that of Game Theory in various forms (see Gambarelli (2007)).

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**References**

Battaggion M. R., Grieco D. (2009). R&D Competition with Radical and Incremental Innovation, *Rivista Italiana degli Economisti*, 2.

Carfi, D. (2009) Payoff space in C1-games, *Applied Sciences (APPS)*, vol.11, pp.1-16, ISSN 1454-5101 can be retrieved from http://www.mathem.pub.ro/apps/

Chandy R. K., Tellis G. J. (1998). Organizing for Radical Product Innovation: The Overlooked Role of Willingness to Cannibalize, *Journal of Marketing Research*, Vol. XXXV, 474-487

Gambarelli, G. (2007). Transforming Games from Characteristic into Normal Form, *International Game Theory Review*, Vol. 9, Number 1, 87-104

Nijssen E. J., Hillebrand B., Vermeulen P. A. M., Kemp R. (2004). Understanding the Role of Willingness to Cannibalize in New Service Development, *Scales Research Reports H200308*, EIM Business and Policy Research.

## Appendix: A software for the bilinear cases

Here we present a program written in Matlab relative all'algorithm presented in section 4.

**main.m**

```
clear all;

% INPUT
trunc = input('Insert 0 if Free Case, 1 if Truncated Case: ');
d1 = input('Insert delta1: ');
d2 = input('Insert delta2: ');

condi=1;
while condi==1

    %INPUT CHECK
    cond=1;
    while cond ==1;
        e1min = input('Insert the value of e1min: ');
        e2min = input('Insert the value of e2min: ');
        if (e1min<=max(1,d1))
            if (e2min<=max(1,d2))
                cond=0;
                break;
            else
                disp(sprintf('e2min must be less than or equal to max(1, delta2)'));
            end
        else
            disp(sprintf('e1min must be less than or equal to max(1, delta1)'));
        end
    end

    [r_int]= compute_r_int(d1,d2,e1min,e2min,trunc);

    disp(sprintf('r should be included in r_int = [ %f , %f ]', r_int(1) , r_int(2)));

    %INPUT r
    r = input('Insert the value of r: ');

    [e,q, err, max_e_i_min]= compute_optimal(d1,d2,r,e1min,e2min,trunc);

    if err == 0
        disp (sprintf('----------------------------'));
        if trunc
            disp (sprintf('Bilinear Truncated Case'));
        else
            disp (sprintf('Bilinear Free Case'));
        end
        disp (sprintf('----------------------------'));
        disp (sprintf('INPUT'));
        disp (sprintf('----------------------------'));
        disp (sprintf('delta1: %4.4f', d1));
```

```
        disp (sprintf('delta2: %4.4f', d2));
        disp (sprintf('r     : %4.4f', r));
        disp (sprintf('----------------------------'));
        disp (sprintf(' '));
        disp (sprintf('----------------------------'));
        disp (sprintf('OUTPUT'));
        disp (sprintf('----------------------------'));
        disp (sprintf('Optimal q1: %4.4f', q(1)));
        disp (sprintf('Optimal q2: %4.4f', q(2)));
        disp (sprintf('----------------------------'));
        disp (sprintf('Optimal e1: %4.4f', e(1)));
        disp (sprintf('Optimal e2: %4.4f', e(2)));
        disp (sprintf('----------------------------'));
        disp (sprintf('----------------------------'));
        condi=0;
    else
        disp('e_i_min are too binding!');
        disp(sprintf('e_1_min should be less than or equal to %f', max_e_i_min));
    end
end
```

**compute_optimal.m**

```
function [e,q,err, max_e_i_min]= calcola_optimal(d1,d2,r,e1min,e2min,trunc)
err = 0;
max_e_i_min=0;
q=[];
e=[];

%CASE 1
if d1>=1 && d2>=1
    if trunc
        if d1==1 && d2==1
            q(2)=1;
            q(1)=1;
            e(1)=1;
            e(2)=1;
        else
            if d1>1 && d2==1
                q(2)=1;
                q(1)=1/d1;
                e(1)=1;
                e(2)=1;
            else
                q(2)=(-(1-(d1-1)+(d2-1))+((1-(d1-1)+(d2-1))^2+4*(d1-1))^(1/2))
                          /(2*(d1-1));
                q(1)=1/(1+q(2)*(d2-1));
                e(1)=1;
                e(2)=1;
            end
        end
    else
        q(1)=1;
        q(2)=1;
        e(1)=d1;
        e(2)=d2;
    end
end

%CASE 2
```

```
if d1>=1 && d2<1
    if trunc
        if e1min<=min(1,(d1/(d2-1))*(e2min-1))
            L(1)=e1min;
            L(2)=((d2-1)/d1)*e1min+1;
            R(1)=(d1/(d2-1))*(max(((d2-1)/d1+1),e2min)-1);
            R(2)=max(((d2-1)/d1+1),e2min);
            if r > L(2)/L(1)
                e(1)=L(1);
                e(2)=L(2);
                q(1)=e1min/d1;
                q(2)=1;
            end
            if r <= L(2)/L(1) && r >= R(2)/R(1)
                e(1)=d1/(r*d1-d2+1);
                e(2)=r*e(1);
                q(1)=1/(r*d1-d2+1);
                q(2)=1;
            end
            if r < R(2)/R(1)
                e(1)=R(1);
                e(2)=R(2);
                if d1==1
                    q(1)=e(1);
                    q(2)=1;
                else
                    M=max(((d2-1)/d1+1),e2min);
                    q(2)=((-M+d1-1)+((d1-1-M)^2+4*(d1-1)*M)^(1/2))/(2*(d1-1));
                    q(1)=((d1/(d2-1))*(max(((d2-1)/d1+1),e2min)-1))/(1+q(2)*(d1-1));
                end
            end
        else
            err=1;
            max_e_i_min=min(1,(d1/(d2-1))*(e2min-1));
        end
    else
        if e1min<=min(d1,(d1/(d2-1))*(e2min-1))
            L(1)=e1min;
            L(2)=((d2-1)/d1)*e1min+1;
            R(1)=(d1/(d2-1))*(max(d2,e2min)-1);
            R(2)=max(d2,e2min);
            if r > L(2)/L(1)
                e(1)=L(1);
                e(2)=L(2);
                q(1)=e1min/d1;
                q(2)=1;
            end
            if r <= L(2)/L(1) && r >= R(2)/R(1)
                e(1)=d1/(r*d1-d2+1);
                e(2)=r*e(1);
                q(1)=1/(r*d1-d2+1);
                q(2)=1;
            end
            if r < R(2)/R(1)
                e(1)=R(1);
                e(2)=R(2);
                q(1)=(max(d2,e2min)-1)/(d2-1);
                q(2)=1;
            end
```

```
        else
            err=1;
            max_e_i_min=min(d1,(d1/(d2-1))*(e2min-1));
        end
    end
end


%CASE 3
if d1+d2>=1 && d1<1 && d2<1
    if e1min<=max(min(d1,(d1/(d2-1))*(e2min-1)),min(1,((d1-1)/d2)*e2min+1))
        L(1)=e1min;
        L(2)=(((d2-1)/d1)*e1min+1)*(e1min<=d1)+(((d2/(d1-1))*(e1min-1))
                *(e1min>d1));
        R(1)=(((d1-1)/d2)*e2min+1)*(e2min<=d2)+(((d1/(d2-1))*(e2min-1))
                *(e2min>d2));
        R(2)=e2min;
        if r > L(2)/L(1) %&& e1min>0
            e(1)=L(1);
            e(2)=L(2);
            q(1)=(e1min/d1)*(e1min<=d1)+(e1min>d1);
            q(2)=((e1min-1)/(d1-1))*(e1min>d1)+(e1min<=d1);
        end
        if r < R(2)/R(1) %&& e2min>0
            e(1)=R(1);
            e(2)=R(2);
            q(1)=1;
            q(2)=(e2min/d2)*(e2min<=d2)+ (e2min/(1-d1))*(e2min>d2);
        end
        if r <= L(2)/L(1) &&  r >= d2/d1
            e(1)=d1/(r*d1-d2+1);
            e(2)=r*e(1);
            q(1)=(1/(r*d1-d2+1));
            q(2)=1;
        end
        if r <= d2/d1 && r >= R(2)/R(1)
            e(1)=-d2/(r*d1-d2-r);
            e(2)=r*e(1);
            q(1)=1;
            q(2)=-(r/(r*d1-d2-r));
        end
    else
        err = 1;
        max_e_i_min = max(min(d1,(d1/(d2-1))*(e2min-1)),min(1,((d1-1)/d2)
                                *e2min+1));
    end
end

%CASE 4
if d1+d2<=1
    H=[d1^2/(1-d2) (1-d1)];
    K=[(1-d2) d2^2/(1-d1)];

    if e1min<=max([min(H(1),(d1/(d2-1))*(e2min-1))    min(1,((d1-1)/d2)
                    *e2min+1)   min(K(1),(1-(((1-d1)*e2min)^(1/2)))^(2)/(1-d2))]);
        L(1)=e1min;
        L(2)=(((d2-1)/d1)*e1min+1)*(e1min<=H(1))+((d2/(d1-1))*(e1min-1))*
                (e1min>=K(1))+(((1-(((1-d2)*e1min)^(1/2)))^(2))/(1-d1))
                *(e1min>K(1))*(e1min<H(1));
```

```
        R(1)=(((d1-1)/d2)*e2min+1)*(e2min<=K(2))+((d1/(d2-1))*(e2min-1))
                *(e2min>=H(2))+(((1-(((1-d1)*e2min)^(1/2)))^(2))/(1-d2))
                *(e2min>K(2))*(e2min<H(2));
        R(2)=e2min;

        if r >= L(2)/L(1)
           e(1)=L(1);
           e(2)=L(2);
           t=(e1min*(1-d2))^(1/2);
           q(1)=(e1min/d1)*(e1min<=H(1))+(e1min>=K(1))+(-(e1min*(d2-1)+t)/((t)
                  *(d1-1)))*(e1min<K(1))*(e1min>H(1));
           q(2)=(e1min<=H(1))+((e1min-1)/(d1-1))*(e1min>=K(1))+(t/(1-d2))*
                  (e1min<K(1))*(e1min>H(1));
        end
        if r <= R(2)/R(1)
           e(1)=R(1);
           e(2)=R(2);
           t=(e2min*(1-d1))^(1/2);
            q(1)=((e2min-1)/(d2-1))*(e2min>=H(2))+(e2min<=K(2))-
                  ((t+e2min*(d1-1))/(t*(d2-1)))*(e2min<H(2))*(e2min>K(2));
           q(2)=(e2min>=H(2))+(e2min/d2)*(e2min<=K(2))+(t/(1-d1))
                  *(e2min<H(2))*(e2min>K(2));
        end
        if r >= H(2)/H(1) && r < L(2)/L(1) && r > R(2)/R(1)
           e(1)=d1/(r*d1-d2+1);
           e(2)=r*e(1);
           q(1)=d1/(1-d2);
           q(2)=1;
        end
        if r < H(2)/H(1) && r > K(2)/K(1)  && r < L(2)/L(1) && r > R(2)/R(1)
           t=(r*(d1-1)*(d2-1))^(1/2);
           e(1)=(2*((1-d2)+r*(1-d1))-2*(r*(1-d2)*(1-d1))^(1/2))/(2*((1-d2)
                 +r*(1-d1))^2);
           e(2)=r*e(1);
           t=((d1-1)*(d2-1))^(1/2);
           if d1==d2
              q(2)=-(1/2)*((d2-1)/(((d2-1)/(d1-1))^(1/2)*(d1-1)^2));
              q(1)=-(1/2)*((d1-1)/(((d1-1)/(d2-1))^(1/2)*(d2-1)^2));
           else
              q(2)=-(-1+d1+t)/((d1-1)*(d1-d2));
              q(1)=-(d2-1+t)/((d2-1)*(d1-d2));
           end

        end
        if r <= K(2)/K(1) && r < L(2)/L(1) && r > R(2)/R(1)
           e(1)=d2/(r*(1-d1)+d2);
           e(2)=r*e(1);
           q(1)=1;
           q(2)=-(r/(r*d1-d2-r));
        end
     else
        err = 1;
        max_e_i_min = max([min(H(1),(d1/(d2-1))*(e2min-1))    min(1,((d1-1)/d2)
                         *e2min+1)   min(K(1),(1-(((1-d1)*e2min)^(1/2)))^(2)/(1-d2))]);
     end
  end

  end
```

**compute_r_int.m**

```
function [r_int]= compute_r_int(d1,d2,e1min,e2min,trunc)

r_int=[];

%CASE 1
if d1==1 && d2==1
   r_int=[1 1];
end

if d1>1 && d2>=1
   if trunc
      r_int=[1 1];
   else
      r_int=[d2/d1 d2/d1];
   end
end

%CASE 2
if d1>=1 && d2<1
   if trunc
      L1=e1min;
      L2=((d2-1)/d1)*e1min+1;
      R1=(d1/(d2-1))*(max(((d2-1)/d1+1),e2min)-1);
      R2=max(((d2-1)/d1+1),e2min);
      r_int=[L2/L1 R2/R1];
   else
      L1=e1min;
      L2=((d2-1)/d1)*e1min+1;
      R1=(d1/(d2-1))*(max(d2,e2min)-1);
      R2=max(d2,e2min);
      r_int=[L2/L1 R2/R1];
   end
end

%CASE 3
if d1+d2>=1 && d1<1 && d2<1
   L1=e1min;
   L2=(((d2-1)/d1)*e1min+1)*(e1min<=d1)+(((d2/(d1-1))*(e1min-1))
          *(e1min>d1));
   R1=(((d1-1)/d2)*e2min+1)*(e2min<=d2)+(((d1/(d2-1))*(e2min-1))
          *(e2min>d2));
   R2=e2min;
   r_int=[L2/L1 R2/R1];
end

%CASE 4
if d1+d2<=1
   H=[d1^2/(1-d2) (1-d1)];
   K=[(1-d2) d2^2/(1-d1)];
   L1=e1min;
   L2=(((d2-1)/d1)*e1min+1)*(e1min<=H(1))+((d2/(d1-1))*(e1min-1))
          *(e1min>=K(1))+(((1-(((1-d2)*e1min)^(1/2)))^(2))/(1-d1))
          *(e1min>K(1))*(e1min<H(1));
   R1=(((d1-1)/d2)*e2min+1)*(e2min<=K(2))+((d1/(d2-1))*(e2min-1))
          *(e2min>=H(2))+(((1-(((1-d1)*e2min)^(1/2)))^(2))/(1-d2))
          *(e2min>K(2))*(e2min<H(2));
   R2=e2min;
   r_int=[L2/L1 R2/R1];
```

end

r_int=[min(r_int) max(r_int)];

end

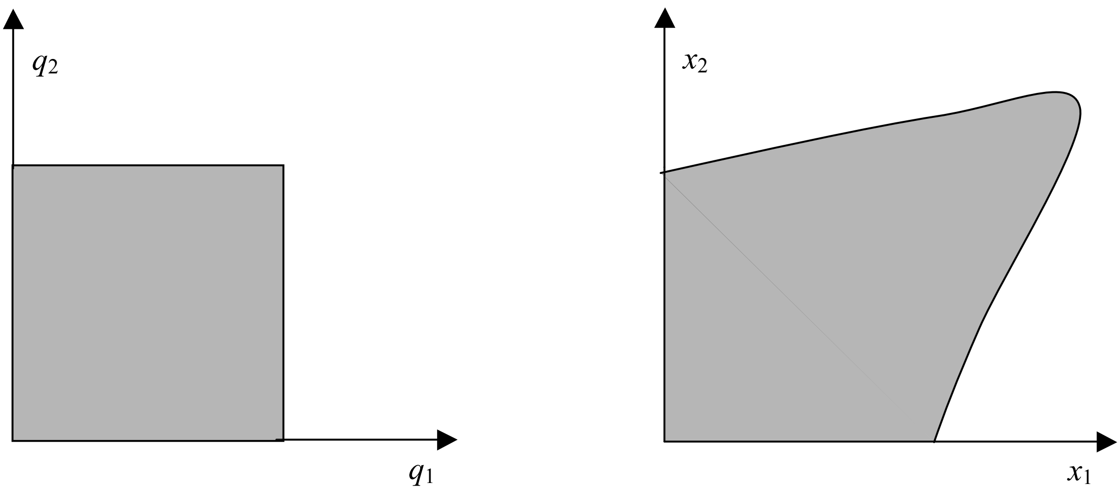


Fig. 0: $n$=2

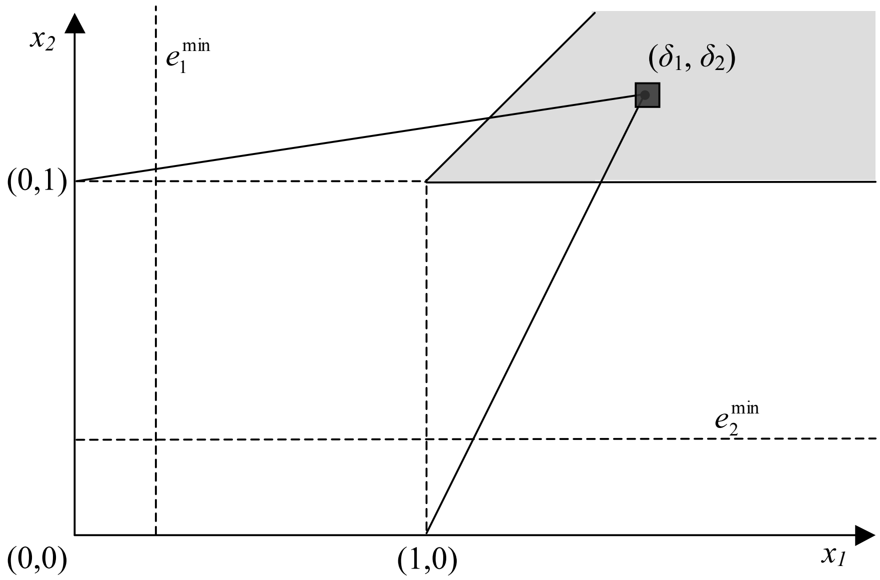


Fig. 1: $n$=2, case 1 (independent or synergic elements)

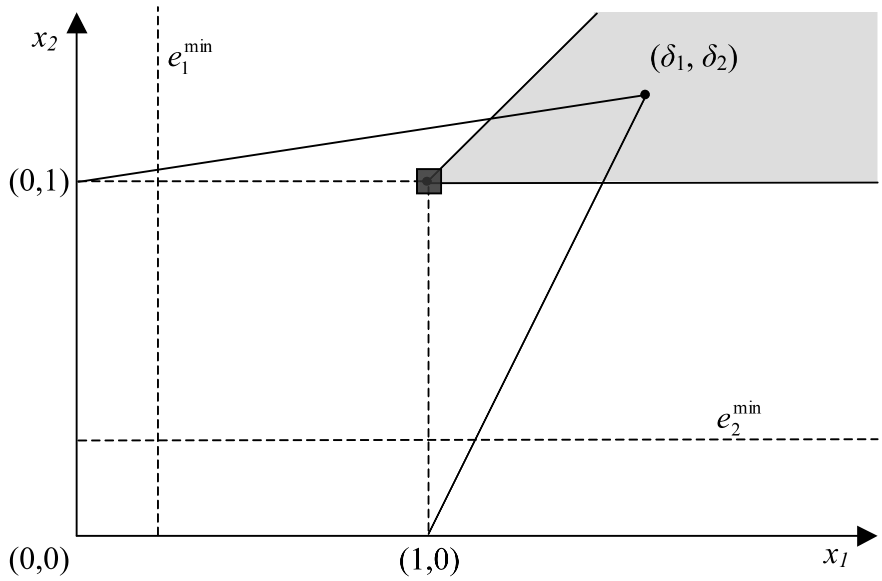


Fig. 1T: $n$=2, case 1 (independent or synergic elements)

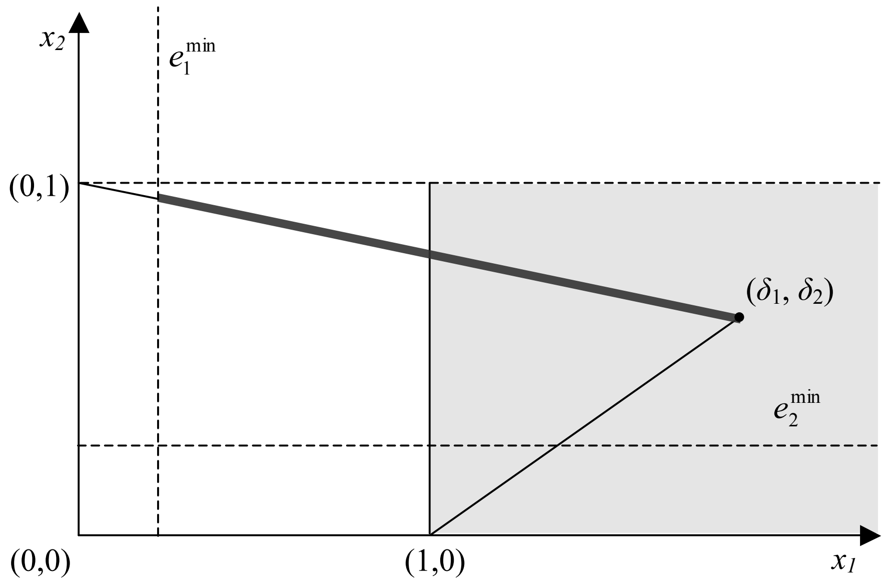


Fig. 2: $n$=2, case 2 (partially synergic and partially antagonistic elements)

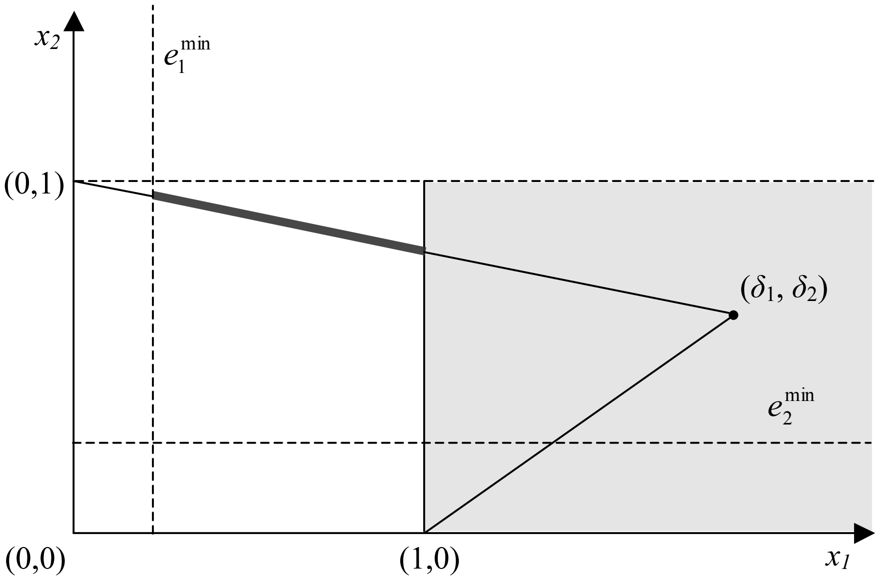


Fig. 2T: $n$=2, case 2 (partially synergic and partially antagonistic elements)

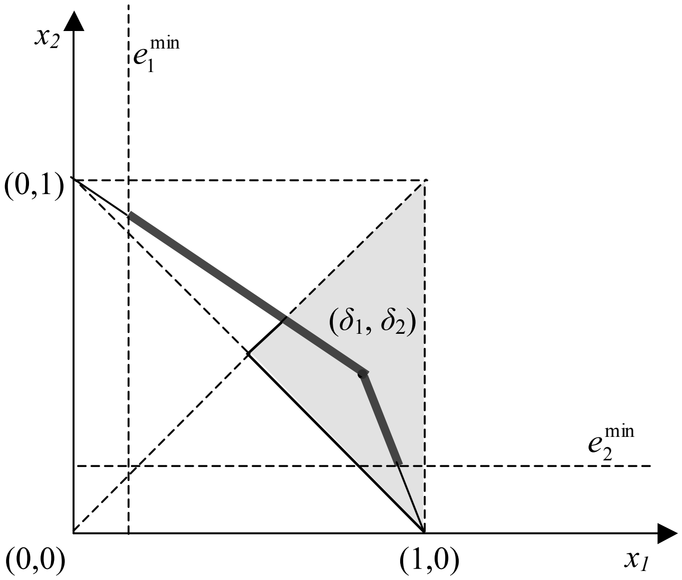


Fig. 3: $n$=2, case 3 (weakly antagonist elements)

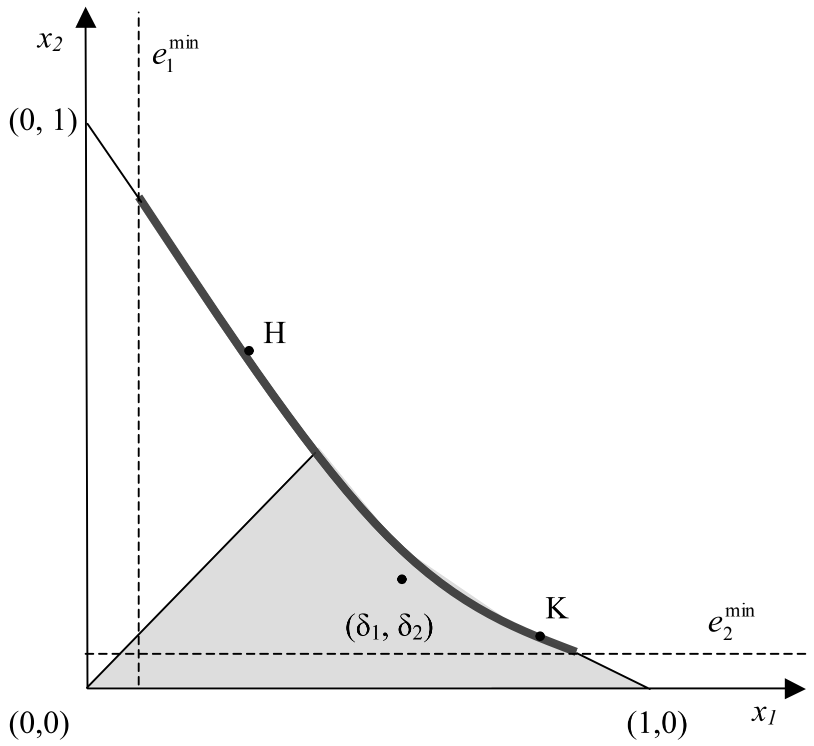


Fig. 4: $n$=2, case 4 (strongly antagonist elements)

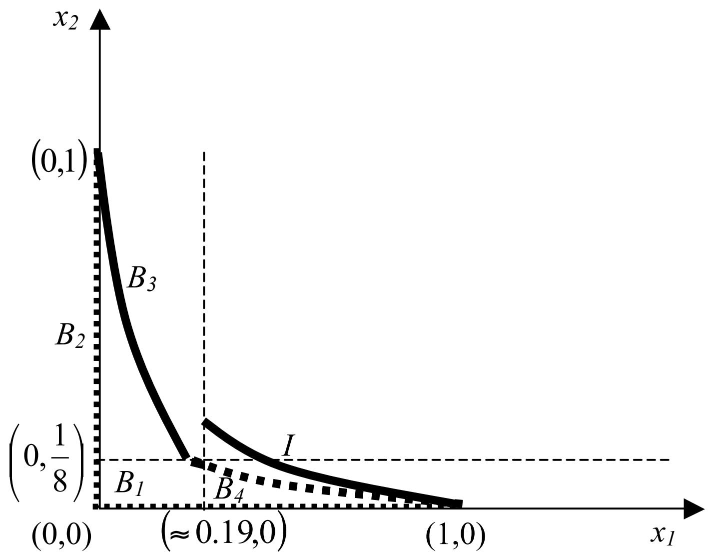


Fig. 5: Example (Section 7)

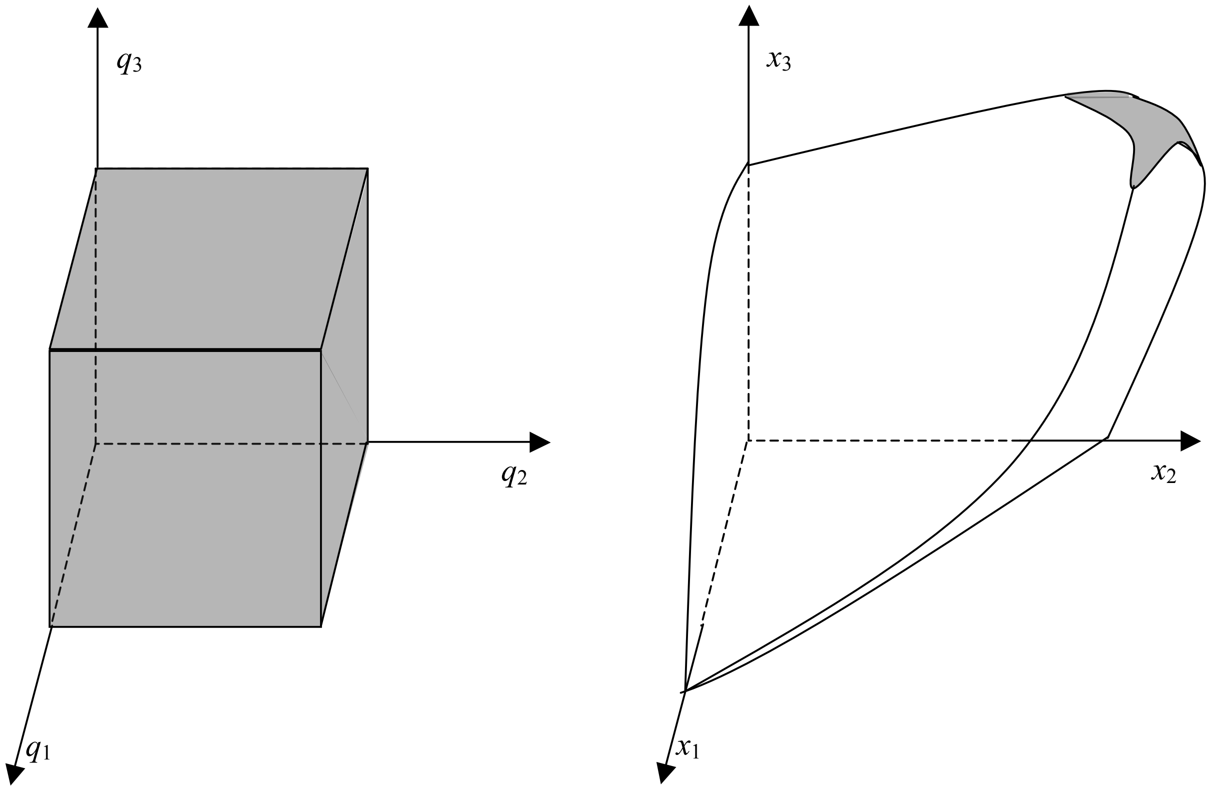


Fig. 6: $n$=3

Table 1: the optimal solution in type 1

| | values |
|---|---|
| optimal effects | $x^* = (\delta_1, \delta_2)$ |
| optimal quantities | $q_1 = 1$<br>$q_2 = 1$ |

Table 1T: the optimal solution in type 1T

| | $\delta_1 = \delta_2 = 1$ | $\delta_1 > 1 \ \ \delta_2 = 1$ | otherwise |
|---|---|---|---|
| optimal effects | $x^* = (1, 1)$ | $x^* = (1, 1)$ | $x^* = (1, 1)$ |
| optimal quantities | $q_1 = \dfrac{1}{\delta_1}$<br>$q_2 = 1$ | $q_1 = 1$<br>$q_2 = 1$ | $q_1 = \dfrac{1}{1 + q_2(\delta_2 - 1)}$<br>$q_2 = \dfrac{\sqrt{\kappa^2 - \kappa + 4(\delta_1 - 1))}}{2(\delta_1 - 1)}$<br>$\kappa = (1 - (\delta_1 - 1) + (\delta_2 - 1))$ |

Table 2: the optimal solution in type 2

| | | |
|---|---|---|
| existence condition | $e_1^{\min} \le \min\left(\delta_1, \frac{\delta_1}{\delta_2 - 1}(e_2^{\min} - 1)\right)$ | |
| Extremes of the feasible P.O. boundary | $\mathrm{L} = (L_1, L_2) = \left(e_1^{\min}, \frac{\delta_2 - 1}{\delta_1} e_1^{\min} + 1\right)$<br>$\mathrm{R} = (R_1, R_2) = \left(\frac{\delta_1}{\delta_2 - 1}(\max(\delta_2, e_2^{\min}) - 1), \max(\delta_2, e_2^{\min})\right)$ | |
| optimal effects | $L_2/L_1 \le r \le R_2/R_1$ | $x^* = (w_1, w_2)$<br>$w_1 = \delta_1/(r\,\delta_1 - \delta_2 + 1)$<br>$w_2 = r\,w_1$ |
| | $r > L_2/L_1$ | $x^* = L$ |
| | $r < R_2/R_1$ | $x^* = R$ |
| optimal solution | $L_2/L_1 \le r \le R_2/R_1$ | $q_1^* = 1/(r\delta_1 - \delta_2 + 1)$<br>$q_2^* = 1$ |
| | $r > L_2/L_1$ | $q_1^* = e_1^{\min}/\delta_1$<br>$q_2^* = 1$ |
| | $r < R_2/R_1$ | $q_1^* = \frac{\max(\delta_2, e_2^{\min}) - 1}{\delta_2 - 1}$<br>$q_2^* = 1$ |

Table 2T: the optimal solution in type 2T

| | | |
|---|---|---|
| existence condition | $e_1^{\min} \le \min\left(1, \frac{\delta_1}{\delta_2 - 1}(e_2^{\min} - 1)\right)$ | |
| Extremes of the feasible P.O. boundary | $\mathrm{L} = (L_1, L_2) = \left(e_1^{\min}, \frac{\delta_2 - 1}{\delta_1} e_1^{\min} + 1\right)$<br>$\mathrm{R} = (R_1, R_2) = \left(\frac{\delta_1}{\delta_2 - 1}\left(\max\left(\frac{\delta_2 - 1}{\delta_1} + 1, e_2^{\min}\right) - 1\right), \max\left(\frac{\delta_2 - 1}{\delta_1} + 1, e_2^{mi}\right.\right.$ | |
| optimal effects | $L_2/L_1 \le r \le R_2/R_1$ | $x^* = (w_1, w_2)$<br>$w_1 = \delta_1/(r\,\delta_1 - \delta_2 + 1)$<br>$w_2 = r\,w_1$ |
| | $r > L_2/L_1$ | $x^* = L$ |
| | $r < R_2/R_1$ | $x^* = R$ |
| optimal solution | $L_2/L_1 \le r \le R_2/R_1$ | $q_1^* = 1/(r\delta_1 - \delta_2 + 1)$<br>$q_2^* = 1$ |

| | | | |
|---|---|---|---|
| | $r > L_2/L_1$ | | $q_1^* = e_1^{\min}/\delta_1$<br>$q_2^* = 1$ |
| | $r < R_2/R_1$ | $\delta_1 = 1$ | $q_1^* = R_1$<br>$q_2^* = 1$ |
| | | $\delta_1 > 1$ | $q_1^* = \dfrac{\dfrac{\delta_1}{\delta_2 - 1}\max\left(\dfrac{\delta_2 - 1}{\delta_1} + 1, e_2^{\min}\right) - 1}{1 + q_2(\delta_1 - 1)}$<br>$q_2^* = \dfrac{(\delta_1 - \vartheta - 1) + \sqrt{(\delta_1 - \vartheta - 1)^2 + 4\vartheta(\delta_1 - 1)}}{2(\delta_1 - 1)}$<br>$\vartheta = \max\left(\dfrac{\delta_2 - 1}{\delta_1} + 1, e_2^{\min}\right)$ |

Table 3: the optimal solution in type 3

| | | |
|---|---|---|
| existence condition | $e_1^{\min} \le \max\left(\min\left(\delta_1, \dfrac{\delta_1}{(\delta_2 - 1)}(e_2^{\min} - 1)\right), \min\left(1, \dfrac{(\delta_1 - 1)}{\delta_2} e_2^{\min} + 1\right)\right)$ | |
| Extremes of the feasible P.O. boundary | $\mathrm{L} = (L_1, L_2) = \left(e_1^{\min}, \left(\dfrac{\delta_2 - 1}{\delta_1} e_1^{\min} + 1\right)\chi\left(e_1^{\min} \le \delta_1\right) + \left(\dfrac{\delta_2}{\delta_1 - 1}(e_1^{\min} - 1)\right)\chi\left(e_1^{\min} > \delta_1\right)\right)$<br>$\mathrm{R} = (R_1, R_2) = \left(\left(\dfrac{\delta_1 - 1}{\delta_2} e_2^{\min} + 1\right)\chi\left(e_2^{\min} \le \delta_2\right) + \left(\dfrac{\delta_1}{\delta_2 - 1}(e_2^{\min} - 1)\right)\chi\left(e_2^{\min} > \delta_2\right), e_2^{\min}\right)$ | |
| optimal effects | $r > L_2/L_1$ | $x^* = L$ |
| | $r < R_2/R_1$ | $x^* = R$ |
| | $\delta_2/\delta_1 \le r \le L_2/L_1$ | $x^* = (w_1, w_2)$<br>$w_1 = \delta_1/(r\delta_1 - \delta_2 + 1)$<br>$w_2 = r w_1$ |
| | $R_2/R_1 \le r \le \delta_2/\delta_1$ | $x^* = (w_1, w_2)$<br>$w_1 = -\delta_2/(r\delta_1 - r - \delta_2)$<br>$w_2 = r w_1$ |
| optimal quantities | $r > L_2/L_1$ | $q_1^* = \dfrac{e_1^{\min}}{\delta_1}\chi\left(e_1^{\min} \le \delta_1\right) + \chi\left(e_1^{\min} > \delta_1\right)$<br>$q_2^* = \chi\left(e_1^{\min} \le \delta_1\right) + \dfrac{e_1^{\min} - 1}{\delta_1 - 1}\chi\left(e_1^{\min} > \delta_1\right)$ |

| | | |
|---|---|---|
| | $r < R_2/R_1$ | $q_1^* = 1$<br>$q_2^* = \frac{e_2^{\min}}{\delta_2}\chi\left(e_2^{\min} \le \delta_2\right) + \frac{e_2^{\min}}{1-\delta_1}\chi\left(e_2^{\min} > \delta_2\right)$ |
| | $\delta_2/\delta_1 \le r \le L_2/L_1$ | $q_1^* = \frac{1}{r\delta_1 + 1 - \delta_2}$<br>$q_2^* = 1$ |
| | $R_2/R_1 \le r \le \delta_2/\delta_1$ | $q_1^* = 1$<br>$q_2^* = -\frac{r}{r\delta_1 - \delta_2 - r}$ |

Table 4: The optimal solution in type 4

| | | |
|---|---|---|
| existence condition | $e_1^{\min} \le \max\left(\begin{array}{l}\min\left(H_1, \frac{\delta_1}{(\delta_2-1)}(e_2^{\min}-1)\right),\\ \min\left(1, \frac{(\delta_1-1)}{\delta_2}e_2^{\min}+1\right), \min\left(K_1, \frac{(1-\sqrt{(1-\delta_1)e_2^{\min}})^2}{1-\delta_2}\right)\end{array}\right)$ | |
| Extremes of the feasible P.O. boundary | $L = (L_1, L_2) = \left(e_1^{\min}, \left(\frac{\delta_2-1}{\delta_1}e_1^{\min}+1\right)\chi\left(e_1^{\min} \le H_1\right) + \left(\frac{\delta_2}{\delta_1-1}(e_1^{\min}-1)\right)\chi\left(e_1^{\min} \ge K_1\right) + \left(\frac{(1-\sqrt{(1-\delta_2)e_1^{\min}})^2}{1-\delta_1}\right)\chi\left(K_1 < e_1^{\min} < H_1\right)\right)$<br>$R = (R_1, R_2) = \left(\left(\frac{\delta_1-1}{\delta_2}e_2^{\min}+1\right)\chi\left(e_2^{\min} \le K_2\right) + \left(\frac{\delta_1}{\delta_2-1}(e_2^{\min}-1)\right)\chi\left(e_2^{\min} \ge H_2\right) + \left(\frac{(1-\sqrt{(1-\delta_1)e_2^{\min}})^2}{1-\delta_2}\right)\chi\left(K_2 < e_2^{\min} < H_2\right), e_2^{\min}\right)$ | |
| optimal effects | $r \ge L_2/L_1$ | $x^* = L$ |
| | $r \le R_2/R_1$ | $x^* = R$ |
| | $r \ge H_2/H_1$<br>$r < L_2/L_1$<br>$r > R_2/R_1$ | $x^* = (w_1, w_2)$<br>$w_1 = \delta_1/(r\,\delta_1 - \delta_2 + 1)$<br>$w_2 = r\,w_1$ |

| | | |
|---|---|---|
| | $H_2/H_1 \le r \le K_2/K_1$<br>$r < L_2/L_1$<br>$r > R_2/R_1$ | $x^* = (w_1, w_2)$<br>$w_1 = \left( \dfrac{2((1-\delta_2) + r(1-\delta_1)) - 2\sqrt{\xi}}{2((1-\delta_2) + r(1-\delta_1))^2} \right)$<br>$w_2 = rw_1$<br>where<br>$\xi = \sqrt{r(\delta_1 - 1)(\delta_2 - 1)}$ |
| | $r \le K_2/K_1$<br>$r < L_2/L_1$<br>$r > R_2/R_1$ | $x^* = (w_1, w_2)$<br>$w_1 = \left( \dfrac{\delta_2}{\delta_2 + r(1-\delta_1)} \right)$<br>$w_2 = r\, w_1$ |
| optimal quantities | $r \ge L_2/L_1$ | $q_1^* = \left( \dfrac{e_1^{\min}}{\delta_1} \right) \chi\left(e_1^{\min} \le H_1\right) + \chi\left(e_1^{\min} \ge K_1\right) +$<br>$- \left( \dfrac{(e_1^{\min}(\delta_2 - 1) + \eta}{\eta(\delta_1 - 1)} \right) \chi\left(H_1 < e_1^{\min} < K_1\right)$<br>$q_2^* = \chi\left(e_1^{\min} \le H_1\right) + \left( \dfrac{e_1^{\min} - 1}{\delta_1 - 1} \right) \chi\left(e_1^{\min} \ge K_1\right) +$<br>$+ \left( \dfrac{\eta}{1 - \delta_2} \right) \chi\left(H_1 < e_1^{\min} < K_1\right)$<br>where<br>$\eta = \sqrt{e_1^{\min}(1 - \delta_2)}$ |
| | $r \le R_2/R_1$ | $q_1^* = \chi\left(e_2^{\min} \le K_2\right) + \left( \dfrac{e_2^{\min} - 1}{\delta_2 - 1} \right) \chi\left(e_2^{\min} \ge H_2\right) +$<br>$- \left( \dfrac{\theta + e_2^{\min}(\delta_1 - 1)}{\theta(\delta_2 - 1)} \right) \chi\left(K_2 < e_2^{\min} < H_2\right)$<br>$q_2^* = \left( \dfrac{e_2^{\min}}{\delta_2} \right) \chi\left(e_2^{\min} \le K_2\right) + \chi\left(e_2^{\min} \ge H_2\right) +$<br>$+ \left( \dfrac{\theta}{1 - \delta_1} \right) \chi\left(K_2 < e_2^{\min} < H_2\right)$<br>where<br>$\theta = \sqrt{e_2^{\min}(1 - \delta_1)}$ |
| | $r \ge H_2/H_1$<br>$r < L_2/L_1$<br>$r > R_2/R_1$ | $q_1^* = -\dfrac{\delta_1 - 1}{2(\delta_2 - 1)^2 \sqrt{(\delta_1 - 1)/(\delta_2 - 1)}}$<br>$q_2^* = -\dfrac{\delta_2 - 1}{2(\delta_1 - 1)^2 \sqrt{(\delta_2 - 1)/(\delta_1 - 1)}}$ |

| | | |
|---|---|---|
| | $H_2/H_1 \le r \le K_2/K_1$<br>$r < L_2/L_1$<br>$r > R_2/R_1$ | If $\delta_1 = \delta_2$<br>$q_1^* = -\left(\frac{1}{2}\frac{1}{\sqrt{(\delta_2-1)/(\delta_1-1)}}\frac{\delta_2-1}{(\delta_1-1)^2}\right)$<br>$q_2^* = -\left(\frac{1}{2}\frac{1}{\sqrt{(\delta_1-1)/(\delta_2-1)}}\frac{\delta_1-1}{(\delta_2-1)^2}\right)$<br>otherwise<br>$q_1^* = -\left(\frac{\delta_1-1+\xi}{(\delta_1-1)(\delta_1-\delta_2)}\right)$<br>$q_2^* = -\left(\frac{\delta_2-1+\xi}{(\delta_2-1)(\delta_1-\delta_2)}\right)$<br>where<br>$\xi = \sqrt{(\delta_1-1)(\delta_2-1)}$ |
| | $r \le K_2/K_1$<br>$r < L_2/L_1$<br>$r > R_2/R_1$ | $q_1^* = 1$<br>$q_2^* = -\frac{r}{r\delta_1 - \delta_2 - r}$ |

Table 5: The input data of the example

| | symbol | Unit | | |
|---|---|---|---|---|
| Minimum quantities | $Q^{\min}$ | € | | |
| Maximum quantities | $Q^{\max}$ | € | | |
| effects if both $Q^{\max}$ | $\delta$ | % | | |
| required effects' ratio | $r$ | - | | |

Table 6: The solution of the example

| | symbol | unit | | |
|---|---|---|---|---|
| optimal quantities | $Q^*$ | € | | |
| resulting effects | $x^*$ | % | | |