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# Generalized type spaces\*

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## Abstract

Ordinary type spaces (Heifetz and Samet, 1998) are essential ingredients of incomplete information games. With ordinary type spaces one can grab the notions of beliefs, belief hierarchies and common prior etc. However, ordinary type spaces cannot handle the notions of finite belief hierarchy and unawareness among others.

In this paper we consider a generalization of ordinary type spaces, and introduce the so called generalized type spaces which can grab all notions ordinary type spaces can and more, finite belief hierarchies and unawareness among others. We also demonstrate that the universal generalized type space exists.

**Keywords and phrases:** type space; unawareness; finite belief hierarchy; generalized type space; generalized belief hierarchy; incomplete information games.

**JEL codes:** C72, D83

## 1 Introduction

Ordinary type spaces are widely used tools of models for incomplete information situations. Harsányi (1967-68) introduces the notion of ordinary types as the complete descriptions of the players' physical and mental characteristics. Ordinary types constitute the so called ordinary type space (Heifetz and Samet, 1998). These objects, the ordinary type spaces, can model the notions of beliefs, infinite belief hierarchies (Battigalli and Siniscalchi, 1999; Pintér, 2008) and common prior among others.

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However, there are other notions, mainly epistemic ones, which cannot be modeled by ordinary type spaces. Bounded rationality of players is an accepted idea in economics (see e.g. Rubinstein (1998)). One form of bounded rationality might be when the players, or at least one of the players, cannot take belief hierarchies in full length, but can consider only finite belief hierarchies, hierarchies of beliefs which are finitely deep only. This constrained ability of reasoning may have significant impact on solutions of games.

Rubinstein (1989)'s electronic mail game is a well-known example for the role and importance of bounded reasoning ability of the players in decision theory. His example also sheds light on the difference between arbitrary high, but finite and infinite belief hierarchies. An other example for that finite belief hierarchies, or for something very similar, are applied in economic theory is the notion of  $k$ -rationalizability introduced by Bernheim (1984).

Finite hierarchies of beliefs cannot be grabbed by ordinary type spaces, those contain only infinite belief hierarchies, so a new notion is needed. This basic observation is also recognized by Kets (2010), who introduces the notion of extended type spaces, which contain both finite and infinite belief hierarchies. We differ from Kets (2010) in many important points, e.g. we consider the purely measurable setting, while she applies a topological one, our model can handle unawareness too, her model cannot, and her model is different from ours in its setup as well.

Unawareness is a type of uncertainty where the decision maker ignores a fact (event) and ignores that she ignores that. Unawareness has a huge literature in decision sciences, see Rantala (1982); Fagin and Halpern (1988); Wansing (1990); Modica and Rustichini (1994); Dekel et al (1998); Modica and Rustichini (1999); Halpern (2001); Heifetz et al (2006); Sillarri (2006); Halpern and Rego (2008); Heifetz et al (2008); Sillarri (2008); Li (2009); Halpern and Rego (2009); Hill (2010); Heinsalu (2011) among others. However, up to our knowledge, there is no paper in the literature which incorporate the unawareness into the type space setting. Our paper is about this too.

Enormous literature is about the theory of belief hierarchies and type spaces, see e.g. Böge and Eisele (1979); Mertens and Zamir (1985); Heifetz (1993); Brandenburger and Dekel (1993); Mertens et al (1994); Heifetz and Samet (1999); Meier (2008); Pintér (2005) among others. Heifetz and Samet (1998) show that the universal ordinary type space, the ordinary type space which encompasses all ordinary types, exists (uniquely) in the setting where only measure theory notions applied. Pintér (2010) proves, however, that a similar result does not hold in the topological setting, that is, there is no universal topological ordinary type space.

In this paper we introduce the notion of generalized type space. Gener-

alized types constitute the generalized type spaces (see Definition 4). Each point in a generalized type space gives the players' generalized hierarchies of beliefs, in the same way as each point in an ordinary type space gives the players' (ordinary) belief hierarchies (Battigalli and Siniscalchi, 1999; Pintér, 2008).

Generalized types can describe finite belief hierarchies, unawareness and many other interesting (interactive) epistemic phenomena. The idea behind generalized type spaces is simple. Let  $(X, \mathcal{M})$  be a measurable space, where event  $A \in \mathcal{M}$  can “mean“ a proposition (formula). Then the decision maker's beliefs modeled by probability measures defined on any sub- $\sigma$ -field of  $\mathcal{M}$ , which is denoted by  $\mathfrak{M}$ , that is, contrary to ordinary models where the beliefs are probability measures on a fixed  $\sigma$ -field, in our model the beliefs can vary more.

Suppose that at event  $A \in \mathcal{M}$  the decision maker's belief is not defined, then we can interpret this as the decision maker ignores event (proposition, formula)  $A$ . In addition to this,  $\Delta(X, \mathfrak{M}) \setminus \{\mu \in \Delta(X, \mathfrak{M}) : \mu(A) \geq 0\}$  is for the beliefs (probability measures) which are not defined at event  $A$ . In other words,  $\Delta(X, \mathfrak{M}) \setminus \{\mu \in \Delta(X, \mathfrak{M}) : \mu(A) \geq 0\}$  is the event that the decision maker ignores event  $A$ .

For instance, if  $\Delta_i(X, \mathfrak{M})$  is the set of Player  $i$ 's first order beliefs, and at a certain state of the world Player  $j$ 's belief about Player  $i$ 's first order belief is the trivial probability measure, that is, the probability measure defined on the the trivial  $\sigma$ -field (consisting of only the empty set and its complement), then we can say that Player  $j$  has no second order belief with respect to Player  $i$ .

If at a given state of the world Player  $j$  has no second or higher order beliefs with respect to any other player, then it means that Player  $i$  has only first order belief. In other words, if Player  $j$ 's belief does not catch any detail about the other players' beliefs, if Player  $j$  ignores the other players' beliefs, then Player  $j$  has only first order belief, she has a finite belief hierarchy. Our model grabs the notion of finite hierarchies of beliefs in this way.

If at a certain state of the world Player  $i$ 's belief on  $\Delta_i(X, \mathfrak{M})$  is not defined at event  $\Delta_i(X, \mathfrak{M}) \setminus \{\mu \in \Delta(X, \mathfrak{M}) : \mu(A) \geq 0\}$ , that is, Player  $i$  ignores that she ignores event  $A$ , then we say that Player  $i$  is unaware about event  $A$ . In other words, generalized types can describe unawareness too.

Beyond that in generalized type spaces the players' beliefs are probability measures defined on not a concrete  $\sigma$ -field but on one of a family of  $\sigma$ -fields (see above), the main characteristics of our notion of generalized type space are as follows.

Since Pintér (2010)'s result (who shows that there is no universal topological ordinary type space) we work in the purely measurable setting, that is,

our generalized type space is a generalization of Heifetz and Samet (1998)'s ordinary type space. We demonstrate that there exists a universal generalized type space in this setting.

Our generalized type space is not a Harsányi type space (Heifetz and Mongin, 2001), we do not recommend a player know or believe with probability 1 her own lower order beliefs. We apply this more general model of type spaces because modeling unawareness requires that a player be able to ignore own ignorance, so be capable of forming false beliefs about her lower order beliefs (see the discussion above).

Furthermore, we do not discuss it in this paper, but one can incorporate the knowledge into generalized type spaces, these enlarged objects called generalized knowledge-belief spaces, in the very same way as Meier (2008) does it for ordinary type spaces (Heifetz and Samet, 1998).

It is worth mentioning that the proof for the existence of universal generalized type space goes as Heifetz and Samet (1998)'s, Meier (2008)'s, Pintér (2008)'s proof, the construction of canonical model in modal logic go, that is, the same machinery lays behind all the above results. We do not go into the details of the common behind these results, only mention that the theory of coalgebras and final coalgebras is the common umbrella for these and other results, see (Moss and Viglizzo, 2004, 2006; Cirstea et al, 2011; Moss, 2011) among others.

The setup of the paper is as follows: In the following section we introduce the notion of generalized type space. In Section 3 we discuss the generalized belief hierarchies, Section 4 is devoted for introducing and characterizing the generalized type morphism. In Section 5 we prove that the universal generalized type space does exist. Moreover, a short appendix about inverse systems and inverse limits is attached to the paper.

## 2 Generalized type spaces

*Notations:* Let  $N$  be the set of the players, w.l.o.g. we can assume that  $0 \notin N$ , and let  $N_0 = N \cup \{0\}$ , where 0 is for the nature as a player.

Let  $\#A$  be the cardinality of set  $A$ . For any set system  $\mathcal{A} \subseteq \mathcal{P}(X)$ :  $\sigma(\mathcal{A})$  is the coarsest  $\sigma$ -field which contains  $\mathcal{A}$ . Let  $(X, \mathcal{M})$  and  $(Y, \mathcal{N})$  be measurable spaces, then  $(X \times Y, \mathcal{M} \otimes \mathcal{N})$  or briefly  $X \otimes Y$  is the measurable space on the set  $X \times Y$  equipped with the  $\sigma$ -field  $\sigma(\{A \times B : A \in \mathcal{M}, B \in \mathcal{N}\})$ .

Furthermore,  $\mathbb{N} = \{n \in \mathbb{Z} : n \geq 1\}$ .

In the following we introduce the notion of generalized type space. We generalize ordinary type spaces, and use terminologies, notions similar to

those Heifetz and Samet (1998) apply.

**Definition 1.** Let  $X$  be a space,  $\mathfrak{M}$  be a class of  $\sigma$ -fields on set  $X$  and  $\Delta(X, \mathfrak{M})$  be the class of probability measures on the  $\sigma$ -fields of  $\mathfrak{M}$ , formally  $\Delta(X, \mathfrak{M}) = \{\mu \in \Delta(X, \mathcal{M}) : \mathcal{M} \in \mathfrak{M}\}$ . Then the  $\sigma$ -field  $\mathcal{A}^*$  on  $\Delta(X, \mathfrak{M})$  is defined as follows:

$$\mathcal{A}^* = \sigma(\{\{\mu \in \Delta(X, \mathfrak{M}) : \mu(A) \geq p\}, A \in \mathcal{M}, p \in [0, 1]\}) .$$

In other words,  $\mathcal{A}^*$  is the smallest  $\sigma$ -field among the  $\sigma$ -fields which contain the sets  $\{\mu \in \Delta(X, \mathfrak{M}) : \mu(A) \geq p\}$ , where  $\mathcal{M} \in \mathfrak{M}$ ,  $A \in \mathcal{M}$  and  $p \in [0, 1]$  are arbitrarily chosen.

In models of incomplete information situations it is recommended the players be able to consider own and the other players' beliefs, that is, to reason about events like a player believes with probability at least  $p$  that a certain event occurs (beliefs operator see e.g. Aumann (1999b)). For this reason, for any  $\mathcal{M} \in \mathfrak{M}$ ,  $A \in \mathcal{M}$  and  $p \in [0, 1]$ :  $\{\mu \in \Delta(X, \mathfrak{M}) \mid \mu(A) \geq p\}$  must be an event (a measurable set).

In our model, moreover, it is possible that for a certain event  $A \in \mathcal{M}$ ,  $\mathcal{M} \in \mathfrak{M}$ , and probability measure  $\mu \in \Delta(X, \mathfrak{M})$  proposition  $\mu(A) \geq 0$  does not hold, since  $\mu$  is not defined at event  $A$ . Therefore, for any event  $A$  such that  $A \neq X$  and  $A \neq \emptyset$ , that is,  $A$  is neither the sure nor the impossible event,  $\{\mu \in \Delta(X, \mathfrak{M}) : \mu(A) \geq 0\} \subset \Delta(X, \mathfrak{M})$  (proper subset). If probability measure  $\mu \in \Delta(X, \mathfrak{M})$  (belief of a player) is not defined at event  $A$ , then we say that the given player ignores event  $A$ . Moreover,  $\Delta(X, \mathfrak{M}) \setminus \{\mu \in \Delta(X, \mathfrak{M}) : \mu(A) \geq 0\}$  is for the event that the given player ignores event  $A$ . To keep the class of events as small (coarse) as possible, we take the  $\mathcal{A}^*$   $\sigma$ -field.

Notice also that  $\mathcal{A}^*$  is not a fixed  $\sigma$ -field, we mean, it depends on the measurable spaces on which the probability measures are defined. Therefore  $\mathcal{A}^*$  is similar to the *weak\** topology, which depends on the topology of the base (primal) space.

**Assumption 2.** Let the parameter space  $(S, \mathcal{A})$  be an arbitrary measurable space.

Henceforth we assume that  $(S, \mathcal{A})$  is a fixed parameter space which contains all states of the nature. We can think of  $S$  as a set which encompasses all the not commonly known parameters of the considered situation.

**Definition 3.** Let  $\Omega$  be the space of the states of the world and for each  $i \in N_0$ : let  $\mathcal{M}_i$  be a  $\sigma$ -field on  $\Omega$ . The  $\sigma$ -field  $\mathcal{M}_i$  represents Player  $i$ 's

information,  $\mathcal{M}_0$  is for the information available for the nature, hence it is the representative of  $\mathcal{A}$ , the  $\sigma$ -field of the parameter space  $S$ . Let  $\mathcal{M} = \sigma(\cup_{i \in N_0} \mathcal{M}_i)$ , the smallest  $\sigma$ -field which contains all  $\mathcal{M}_i$   $\sigma$ -fields.

Each point in  $\Omega$  provides a complete description of the actual state of the world. It includes both the state of nature and the players' states of the mind. The different  $\sigma$ -fields are for modeling the informedness of the players, they have the same role as e.g. the partitions in Aumann (1999a)'s paper have. Therefore, if  $\omega, \omega' \in \Omega$  are not distinguishable<sup>1</sup> in the  $\sigma$ -field  $\mathcal{M}_i$ , then Player  $i$  is not able to discern the difference between them, that is, she believes the same things and behaves in the same way at the two states  $\omega$  and  $\omega'$ .  $\mathcal{M}$  represents all information available in the model, it is the  $\sigma$ -field got by pooling the information of the players and the nature.

For the sake of brevity, henceforth – if it does not make confusion – we do not indicate the  $\sigma$ -fields. E.g. instead of  $(S, \mathcal{A})$  we write  $S$ , or  $\Delta(S)$  instead of  $(\Delta(S, \mathcal{A}), \mathcal{A}^*)$ . However, in some cases we refer to the non-written  $\sigma$ -field: e.g.  $A \in \Delta(X, \mathcal{M})$  is a set of  $\mathcal{A}^*$ , that is, it is a measurable set in the measurable space  $(\Delta(X, \mathcal{M}), \mathcal{A}^*)$ , but  $A \subseteq \Delta(X, \mathcal{M})$  keeps its original meaning:  $A$  is a subset of  $\Delta(X, \mathcal{M})$ .

**Definition 4.** Let  $(\Omega, \{\mathcal{M}_i\}_{i \in N_0})$  be a space of the states of the world (see Definition 3). The generalized type space based on the parameter space  $S$  is a tuple  $(S, \Omega, \{\mathcal{M}_i\}_{i \in N_0}, g, \{f_i\}_{i \in N})$ , where

1.  $g : \Omega \rightarrow S$  is  $\mathcal{M}_0$ -measurable,
2.  $f_i : \Omega \rightarrow \Delta(\Omega, \mathfrak{M})$  is  $\mathcal{M}_i$ -measurable,  $i \in N$ ,

where  $\mathfrak{M} = \{\mathcal{N} \text{ is a } \sigma\text{-field on } \Omega : \mathcal{N} \subseteq \mathcal{M}\}$ .

Put Definition 4 differently,  $S$  is the parameter space, it contains the "types" of the nature.  $\mathcal{M}_i$  represents the information available for Player  $i$ , hence it corresponds to the concept of types (see Harsányi (1967-68)).  $f_i$  is the type function of Player  $i$ , it assigns Player  $i$ 's (subjective) beliefs to her types. Furthermore, notice that if for each state of the world  $\omega \in \Omega$  and Player  $i \in N$  the type function is such that  $f_i(\omega)$  is defined on  $\mathcal{M}$ , then the generalized type space is an (non-Harsányi) ordinary type space, that is, each ordinary type space is a generalized type space.

The generalized type spaces are not Harsányi type spaces (Heifetz and Mongin, 2001), that is, the players do not know their own types, more precisely, they do not believe with probability 1 their own types. This is because

<sup>1</sup>Let  $(X, \mathcal{T})$  be a measurable space and  $x, y \in X$  be two points.  $x$  and  $y$  are measurably indistinguishable if  $\forall A \in \mathcal{T}: (x \in A) \Leftrightarrow (y \in A)$ .

when we model unawareness, then we must allow the players to have “false“ beliefs about their own beliefs, that is, e.g. Player  $i$  ignores that she ignores event  $A$ .

The following examples illustrate the notion of generalized type space. The first example is for how finite belief hierarchies can be modeled by generalized type spaces.

*Example 5.* Let  $S = \{s_1, s_2\}$ ,  $\mathcal{A} = \mathcal{P}(S)$  and  $N = \{1, 2\}$ . Consider the following generalized type space:

$$(S, \Omega, \{\mathcal{M}_i\}_{i=0,1,2}, g, \{f_i\}_{i=1,2}) , \quad (1)$$

where

- $\Omega = S \times \Delta(S)^1 \times \Delta(S)^2$ ,
- $g : \Omega \rightarrow S$  is the coordinate projection,
- $\mathcal{M}_0$  is induced by  $g$ ,
- $f_i = pr_i$ , where  $pr_i : \Omega \rightarrow \Delta(S)^i$  is the coordinate projection, that is, for any state of the world  $\omega \in \Omega$ :  $f_i(\omega)$  is defined on the  $\sigma$ -field  $\{g^{-1}(A) : A \in S\}$ ,  $i = 1, 2$ ,
- $\mathcal{M}_i$  is induced by  $f_i$ ,  $i = 1, 2$ .

In this type space both players have all possible first order beliefs (elements of  $\Delta(S)$ ), however, no player has second order belief.

Player  $i$ 's first order belief at state of the world  $\omega \in \Omega$   $v_1^i(\omega)$  is the probability measure defined as follows for all  $A \in S$ :

$$v_1^i(\omega)(A) = f_i(\omega)(g^{-1}(A)) .$$

Player  $i$ 's second order belief at state of the world  $\omega \in \Omega$   $v_2^i(\omega)$  is the probability measure defined as follows for all  $A \in S \otimes (\Delta(S) \cup \{\mu\})^1 \otimes (\Delta(S) \cup \{\mu\})^2$  such that  $f_i(\omega)$  is defined at event  $(g, v_1^1, v_1^2)^{-1}(A)$ , where  $\mu$  is the trivial probability measure<sup>2</sup> on  $S$ :

$$v_2^i(\omega)(A) = f_i(\omega)((g, v_1^1, v_1^2)^{-1}(A)) .$$

Notice that  $v_2^i(\omega)$  is defined on the  $\sigma$ -field  $\{g^{-1}(A) : A \in S\}$ , which  $\sigma$ -field represents the events about Player  $i$  can form second order beliefs. Therefore,

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<sup>2</sup>The trivial probability measure is a measure such that it is defined only on the empty and the universal set.



Player  $i$ 's second order belief at state of the world  $\omega \in \Omega$  is nothing more than her first order belief, that is, Player  $i$  has no second order belief.

In a similar way one can see that no player has higher than first order beliefs in this model.

The following very simple example demonstrates that by generalized type spaces we can model unawareness too.

*Example 6.* Let  $S = \{s_1, s_2\}$ ,  $\mathcal{A} = \mathcal{P}(S)$  and  $N = \{1, 2\}$ . Consider the following generalized type space:

$$(S, \Omega, \{\mathcal{M}_i\}_{i=0,1,2}, g, \{f_i\}_{i=1,2}), \quad (2)$$

where

- $\Omega = S \times T_1 \times T_2$ , where  $T_1 = \{t_1^1, t_1^2\}$ ,  $T_2 = \{t_2\}$ ,
- $g : \Omega \rightarrow S$  is the coordinate projection,
- $\mathcal{M}_0$  is induced by  $g$ ,
- $f_1$  depends only on the component of  $T_1$ , so does  $f_2$  on the component of  $T_2$ , so that  $f_1(s_1, t_1^1, t_2) = f_1(s_2, t_1^1, t_2)$  is defined on  $\mathcal{P}(\Omega)$  such that  $f_1(s_1, t_1^1, t_2) = f_1(s_2, t_1^1, t_2) = \delta_{(s_1, t_1^1, t_2)}^3$ ,  $f_1(s_1, t_1^2, t_2) = f_1(s_2, t_1^2, t_2)$  and  $f_2(s_1, t_1^1, t_2) = f_2(s_2, t_1^1, t_2) = f_2(s_1, t_1^2, t_2) = f_2(s_2, t_1^2, t_2)$  are defined on the trivial  $\sigma$ -field, that is, on the  $\sigma$ -field consisting of only two sets: the empty and the universal set,
- $\mathcal{M}_i$  is induced by coordinate projection  $pr_i : \Omega \rightarrow T_i$ ,  $i = 1, 2$ .

In this model at state of the world  $(s_1, t_1^2, t_2)$ , Player 1 ignores event  $\{s_1\}$ , since  $f_1(s_1, t_1^2, t_2)$  is not defined at  $\{s_1\}$ . Moreover, Player 1's second order belief (see Example 5) is not defined at event  $(\{S\} \times \{t_1^2\} \times \{t_2\}) = (g, v_1^1, v_1^2)^{-1}(\{S\} \times ((\Delta(S) \cup \{\mu\})^1 \setminus \{\nu \in (\Delta(S) \cup \{\mu\})^1 : \nu(\{s_1\}) \geq 0\}) \times (\Delta(S) \cup \{\mu\})^2)$ , this is the event of Player 1 ignores event  $\{s_1\}$ , therefore Player 1 at state of the world  $(s_1, t_1^2, t_2)$  ignores that she ignores event  $\{s_1\}$ , that is, Player 1 is unaware about event  $\{s_1\}$ .

We do not provide more examples, but we remark that generalized type spaces can be models also for situations where a player can form opinion (belief) about e.g. the degree of finiteness of the other players' and own belief hierarchies, and situations where a player has beliefs about only certain, arbitrary events.

Summing up the above discussion, generalized type spaces can be models for many epistemic phenomena.

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<sup>3</sup>The Dirac measure concentrated on point  $(s_1, t_1^1, t_2)$

### 3 Generalized belief hierarchies

In this section we formally introduce the generalized belief hierarchies, and show that each state of the world in a generalized type space determines the players' generalized belief hierarchies.

First we introduce the notion of generalized belief space. This notion is the generalization of that Mertens et al (1994) use.

**Definition 7.** *In Diagram (3)*

$$\begin{array}{ccc}
 \Theta^i & & \Delta(S \times \Theta, \mathfrak{M}) \\
 \downarrow p_{n+1}^i & & \downarrow \text{id}_S \quad \downarrow p_n \\
 \Theta_{n+1}^i & = & \Delta(S \times \Theta_n, \mathfrak{M}_n) \\
 \downarrow q_{nn+1}^i & & \downarrow \text{id}_S \quad \downarrow q_{n-1n} \\
 \Theta_n^i & = & \Delta(S \times \Theta_{n-1}, \mathfrak{M}_{n-1})
 \end{array} \quad (3)$$

- $i \in N$  is a player,
- $n \in \mathbb{N}$ ,
- $S$  is the parameter space (see Assumption 2),
- $\Theta_{n-1} = \times_{i \in N} \Theta_{n-1}^i$ ,
- $q_{mn} = \times_{i \in N} q_{mn}^i$ , that is,  $q_{mn}$  is the product of mappings  $q_{mn}^i$ ,  $i \in N$ ,
- $\#\Theta_0^i = 1$ ,
- $\mathfrak{M}_n = \{\mathcal{M} \text{ is a } \sigma\text{-field on } S \times \Theta_{n-1} : \mathcal{M} \subseteq S \otimes \Theta_{n-1}\}$ ,  $n \in \mathbb{N}$ ,
- $q_{01}^i : \Theta_1^i \rightarrow \Theta_0^i$ ,
- for all  $m, n \in \mathbb{N}$  such that  $2 \leq m \leq n$ ,  $\mu \in \Theta_n^i$ :  $q_{mn}^i(\mu) \in \Delta(S \times \Theta_{m-1}, \mathfrak{M}_{m-1})$ :

$$q_{mn}^i(\mu) = \mu|_{(S \times \Theta_{m-1}, \mathfrak{M}_{m-1})} ,$$

that is,  $q_{mn}^i(\mu)$  is the restriction of  $\mu$  on the  $\sigma$ -field  $\{A \in S \otimes \Theta_{m-1} : \mu \text{ is defined at } (\text{id}_S, q_{m-1n-1})^{-1}(A)\}$ , where  $(\text{id}_S, q_{m-1n-1})$  is the product of mappings  $\text{id}_S$  and  $q_{m-1n-1}$ . Therefore,  $q_{mn}^i : \Theta_n^i \rightarrow \Theta_m^i$  is a measurable mapping.

- $\Theta^i = \varprojlim (\Theta_n^i, \mathbb{N}, q_{mn}^i)$ , that is,  $\Theta^i$  is the inverse limit of the inverse system  $(\Theta_n^i, \mathbb{N}, q_{mn}^i)$ ,
- $p_n^i : \Theta^i \rightarrow \Theta_n^i$  is the canonical projection,  $n \in \mathbb{N}$ .

Then  $T = S \times \Theta$  is called generalized belief space, where  $\Theta = \times_{i \in N} \Theta^i$ .

The interpretation of generalized belief space is the following. For each point  $\theta^i \in \Theta^i$ :  $\theta^i = (\mu_1^i, \mu_2^i, \dots)$ , where  $\mu_n^i \in \Theta_n^i$  is Player  $i$ 's  $n$ th order generalized belief, that is, each point in  $T$  gives a complete description of the state of the nature (a point in  $S$ ) and the players' hierarchies of generalized beliefs.

If for Player  $i$  at "type"  $\theta^i \in \Theta^i$  and level  $n$  it holds that for each  $m \geq n$   $\mu_n^i = \mu_m^i$ , we mean  $\mu_n^i$  is defined at event  $A \in S \otimes \Theta_{n-1}$  if and only if  $\mu_m^i$  is defined at event  $(\text{id}_S, q_{n-1m-1})^{-1}(A)$ , then we say that Player  $i$  has only  $n$ th order (generalized) beliefs at "type"  $\theta^i$ , that is, Player  $i$  has a finite belief hierarchy.

Next we formally give how one can deduce the players' generalized belief hierarchies in generalized type spaces. The same property, we mean the belief hierarchies can be deduced from types, is well-known for ordinary type spaces (see e.g. Battigalli and Siniscalchi (1999), Pintér (2008)), that is, this is not a special generalized type space feature.

*Demonstration 8.* Take generalized type space  $(S, \Omega, \{\mathcal{M}_i\}_{i \in N_0}, g, \{f_i\}_{i \in N})$ , state of the world  $\omega \in \Omega$  and Player  $i \in N$ .

Player  $i$ 's first order generalized belief at state of the world  $\omega \in \Omega$   $v_1^i(\omega)$  is the probability measure defined as follows for all  $A \in S$  such that  $f_i(\omega)$  is defined at  $g^{-1}(A)$ :

$$v_1^i(\omega)(A) = f_i(\omega)(g^{-1}(A)) .$$

$f_i$  is  $\mathcal{M}_i$ -measurable, hence  $v_1^i$  is also  $\mathcal{M}_i$ -measurable.

Player  $i$ 's second order generalized belief at state of the world  $\omega \in \Omega$   $v_2^i(\omega)$  is the probability measure defined as follows for all  $A \in S \otimes \Theta_1$  such that  $f_i(\omega)$  is defined at event  $(g, v_1)^{-1}(A)$ , where  $v_1$  is the product of mappings  $v_1^i$ ,  $i \in N$ , and so is  $(g, v_1)$  of mappings  $g$  and  $v_1$ :

$$v_2^i(\omega)(A) = f_i(\omega)((g, v_1)^{-1}(A)) ,$$

Since  $f_i$  is  $\mathcal{M}_i$ -measurable, so is  $v_2^i$ .

Generally, Player  $i$ 's  $n$ th order generalized belief at state of the world  $\omega \in \Omega$   $v_n^i(\omega)$  is the probability measure defined as follows for all  $A \in \mathcal{S} \otimes \Theta_{n-1}$  such that  $f_i(\omega)$  is defined at event  $(g, v_{n-1})^{-1}(A)$ , where  $v_{n-1}$  is the product of mappings  $v_{n-1}^i$ ,  $i \in N$ , and so is  $(g, v_{n-1})$  of mappings  $g$  and  $v_{n-1}$ :

$$v_n^i(\omega)(A) = f_i(\omega)((g, v_{n-1})^{-1}(A)) .$$

Again, since  $f_i$  is  $\mathcal{M}_i$ -measurable, so is  $v_n^i$ .

## 4 Generalized type morphisms

In this section we introduce the notion of generalized type morphism. By generalized type morphisms we can compare generalized type spaces to each other, and we can say that a generalized type space is "bigger" than an other. Our concept is closely related to that Heifetz and Samet (1998) introduce to compare ordinary type spaces.

**Definition 9.** Mapping  $\varphi : \Omega \rightarrow \Omega'$  is a generalized type morphism between generalized type spaces  $(S, \Omega, \{\mathcal{M}_i\}_{i \in N_0}, g, \{f_i\}_{i \in N})$  and  $(S, \Omega', \{\mathcal{M}'_i\}_{i \in N_0}, g', \{f'_i\}_{i \in N})$ , if

1.  $\varphi$  is a  $(\mathcal{M}_i, \mathcal{M}'_i)$ -measurable mapping,  $i \in N_0$ ,
2. Diagram (4) is commutative, that is, for all  $\omega \in \Omega$ :  $g' \circ \varphi(\omega) = g(\omega)$ ,

$$\begin{array}{ccc}
 \Omega & & \\
 \varphi \downarrow & \searrow g & \\
 \Omega' & \xrightarrow{g'} & S
 \end{array} \tag{4}$$

3. for each  $i \in N$ : Diagram (5) is commutative, that is, for all  $\omega \in \Omega$ :  $f'_i \circ \varphi(\omega) = \hat{\varphi} \circ f_i(\omega)$ ,

$$\begin{array}{ccc}
 \Omega & \xrightarrow{f_i} & \Delta(\Omega, \mathfrak{M}) \\
 \varphi \downarrow & & \hat{\varphi} \downarrow \\
 \Omega' & \xrightarrow{f'_i} & \Delta(\Omega', \mathfrak{M}')
 \end{array} \tag{5}$$

where  $\hat{\varphi} : \Delta(\Omega, \mathfrak{M}) \rightarrow \Delta(\Omega', \mathfrak{M}')$  is defined as for each  $\mu \in \Delta(\Omega, \mathfrak{M})$ ,  $A \in \mathfrak{M}'$ :  $\mu(\varphi^{-1}(A)) = \hat{\varphi}(\mu)(A)$ , we mean if the left hand side is not defined, then neither is the right hand side and vice versa.

Generalized type morphism  $\varphi$  is a generalized type isomorphism, if  $\varphi$  is a bijection and  $\varphi^{-1}$  is also a generalized type morphism.

As we have already mentioned the above definition is an adaptation of Heifetz and Samet (1998)'s notion of ordinary type morphisms (Pintér, 2008). Generalized type morphisms preserve generalized belief hierarchies (see Proposition 11), so, in this context, a generalized type morphism maps a state of the world of a generalized type space into a state of the world of an other generalized type space in such a way that the players' epistemic characteristics<sup>4</sup> are the same in the two states.

The following result is a direct corollary of Definitions 4 and 9.

**Corollary 10.** *The generalized type spaces based on the parameter space  $S$  as objects and the generalized type morphisms form a category. Let  $\mathcal{C}^S$  denote this category.*

By applying the notions of category theory one can introduce and present the notions of generalized type spaces in a clear and handy way, in other words, the language of category theory fits both ordinary and generalized type spaces.

In the following proposition we demonstrate that generalized type morphisms preserve generalized belief hierarchies.

**Proposition 11.** *Generalized type morphisms preserve generalized belief hierarchies.*

*Proof.* Consider generalized type spaces  $(S, \Omega, \{\mathcal{M}_i\}_{i \in N_0}, g, \{f_i\}_{i \in N})$ ,  $(S, \Omega', \{\mathcal{M}'_i\}_{i \in N_0}, g', \{f'_i\}_{i \in N})$  and generalized type morphism  $\varphi : \Omega \rightarrow \Omega'$ . Take state of the world  $\omega \in \Omega$  and Player  $i \in N$ .

Points 1 and 2 of Definition 9 implies  $g^{-1} = (g' \circ \varphi)^{-1}$ , so from Point 3  $v_1^i = v_1'^i \circ \varphi$ , that is, Player  $i$ ' first order beliefs at sates of the world  $\omega$  and  $\varphi(\omega)$  coincide.

From Points 1, 3 and the previous paragraph ( $v_1^i = v_1'^i \circ \varphi$ ) we get that  $v_2^i = v_2'^i \circ \varphi$ , that is, Player  $i$ ' second order beliefs at sates of the world  $\omega$  and  $\varphi(\omega)$  coincide.

Generally for any  $n$ , by induction: from Points 1, 3 and that  $v_{n-1}^i = v_{n-1}'^i \circ \varphi$  we get  $v_n^i = v_n'^i \circ \varphi$ , that is, Player  $i$ '  $n$ th order beliefs at sates of the world  $\omega$  and  $\varphi(\omega)$  coincide.  $\square$

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<sup>4</sup>In this model the epistemic characteristics are the generalized belief hierarchies, that is, what the players believe and so on.

It is worth noticing that even if two generalized type spaces represent the same generalized belief hierarchies those might be not equal by generalized type morphisms, that is, generalized type morphisms preserve not only the generalized belief hierarchies, but something more. For further discussion on this topic for ordinary type spaces see Ely and Peski (2006) and Friedenberg and Meier (2011).

## 5 Universal generalized type space

Heifetz and Samet (1998) introduce the concept of universal ordinary type space, here we adapt their notion for generalized type spaces.

**Definition 12.** *Generalized type space  $(S, \Omega^*, \{\mathcal{M}_i^*\}_{i \in N_0}, g^*, \{f_i^*\}_{i \in N})$  is a universal generalized type space, if for each generalized type space  $(S, \Omega, \{\mathcal{M}_i\}_{i \in N_0}, g, \{f_i\}_{i \in N})$  there exists a unique generalized type morphism  $\varphi : \Omega \rightarrow \Omega^*$ .*

In other words, the universal generalized type space is the "biggest" generalized type space among generalized type spaces. It contains all generalized types, that is, those which appear in any of the generalized type spaces.

In the language of category theory Definition 12 means the following:

**Corollary 13.** *The universal generalized type space is a terminal (final) object in category  $\mathcal{C}^S$ .*

From the viewpoint of category theory the uniqueness of a universal generalized type space is a straightforward statement.

**Corollary 14.** *The universal generalized type space is unique up to generalized type isomorphism.*

*Proof.* Every terminal object is unique up to isomorphism. □

The only question is the existence of the universal generalized type space. Heifetz and Samet (1998) show that in the category of ordinary type spaces there exists the universal type space, in the following we show this is also the case in the category of generalized type spaces.

**Theorem 15.** *There exists a universal generalized type space, that is, there is a terminal object in the category of generalized type spaces  $\mathcal{C}^S$ .*

The strategy of the proof is the following: we take the subspace of the generalized belief space which contains all the generalized hierarchies of beliefs appearing in a generalized type space, then we show that the considered

subspace of the generalized belief space “is“ the universal generalized type space. This strategy is not new in the literature, Heifetz and Samet (1998), Meier (2008), Pintér (2008) apply this too, and from a more abstract view-point canonical models in modal logic constructed in the same way (Moss, 2011).

*The proof of Theorem 15.* As we have already showed in Demonstration 8, each point in a generalized type space ”consists of” a state of the nature and the players’ generalized belief hierarchies. Since, each point in the generalized belief space  $T$  also consists of a state of the nature and the players’ generalized belief hierarchies, we can say that for any Player  $i$  let

$$\Theta^{*i} = \{\theta^i \in \Theta^i : \exists (S, \Omega, \{\mathcal{M}_i\}_{i \in N_0}, g, \{f_i\}_{i \in N}) \in \mathcal{C}^S, \omega \in \Omega \text{ such that } \theta^i \text{ and } \omega \text{ induce the same generalized belief hierarchy for Player } i\} .$$

Let  $\Omega^* = S \otimes \Theta^*$ , where  $\Theta^* = \times_{i \in N} \Theta^{*i}$ .

Take generalized type space  $(S, \Omega^*, \{\mathcal{M}_i^*\}_{i \in N_0}, g^*, \{f_i^*\}_{i \in N})$ , where

- $g^* = pr_S$ ,
- $\mathcal{M}_0^*$  is induced by  $g^*$ ,
- $f_i^* = pr_{\Theta^{*i}}$ <sup>5</sup>,  $i \in N$ ,
- $\mathcal{M}_i^*$  is induced by  $f_i^*$ ,  $i \in N$ .

Then  $(S, \Omega^*, \{\mathcal{M}_i^*\}_{i \in N_0}, g^*, \{f_i^*\}_{i \in N}) \in \mathcal{C}^S$ , that is, it is a generalized type space.

Let  $(S, \Omega, \{\mathcal{M}_i\}_{i \in N_0}, g, \{f_i\}_{i \in N})$  be a generalized type space, and  $\varphi : \Omega \rightarrow \Omega^*$  be defined as for each  $\omega \in \Omega$ :  $\varphi(\omega) = (g(\omega), \{v_1^i(\omega), v_2^i(\omega), \dots\}_{i \in N})$ , where  $v_n^i$  is Player  $i$ ’s  $n$ th order belief, and  $\varphi$  is the product of the involved mappings.

- Since  $\Omega^*$  consists of different generalized belief hierarchies  $\varphi$  is well-defined.
- $\varphi$  is  $(\mathcal{M}_i, \mathcal{M}_i^*)$ -measurable,  $i \in N_0$ : It directly comes from that  $g$  is  $\mathcal{M}_0$ -measurable,  $v_n^i$  is  $\mathcal{M}_i$ -measurable,  $n \in \mathbb{N}$ ,  $i \in N$ .
- Diagrams 4 and 5 are commutative, and  $\varphi$  is unique: It is a direct corollary of the definition of  $\varphi$ .

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<sup>5</sup>Notice that by definition each point in  $\Theta^{*i}$  gives a probability measure on a sub- $\sigma$ -field of  $\mathcal{M}^*$  (see Diagram (3)).

□

Theorem 15 says that there exists a universal generalized type space, and it contains every finite belief hierarchy (it comes from the definition of generalized belief space  $T$ ) and some infinite belief hierarchies. However, Pintér (2008) shows that the universal ordinary type space encompasses all infinite belief hierarchies, therefore, since the category of generalized type spaces contains the universal ordinary type space (we have discussed it after Definition 4), that is, the universal ordinary type space is a generalized type space, the universal generalized type space encompasses all finite and infinite belief hierarchies.

**Corollary 16.** *The universal generalized type space contains all finite and infinite belief hierarchies.*

## A Inverse systems, inverse limits

In this section we introduce the basic notions of inverse systems and inverse limits.

**Definition 17.** *Let  $(I, \leq)$  be a preordered set,  $(X_i)_{i \in I}$  be a family of nonvoid sets, and for all  $i, j \in I$  such that  $i \leq j$ ,  $f_{ij} : X_j \rightarrow X_i$ . The system  $(X_i, (I, \leq), f_{ij})$  is an inverse system if it meets the following points:*

- $f_{ii} = id_{X_i}$ ,
- $f_{ik} = f_{ij} \circ f_{jk}$ ,

*$i, j, k \in I$  such that  $i \leq j$  and  $j \leq k$ .*

The inverse system, also called projective system, is a family of sets connected in a certain way.

**Definition 18.** *Let  $((X_i, \mathcal{A}_i, \mu_i), (I, \leq), f_{ij})$  be an inverse system such that for all  $i \in I$ ,  $(X_i, \mathcal{A}_i, \mu_i)$  is a measure space. The inverse system  $((X_i, \mathcal{A}_i, \mu_i), (I, \leq), f_{ij})$  is an inverse system of measure spaces if it meets the following points:*

- $f_{ij}$  is a  $(\mathcal{A}_j, \mathcal{A}_i)$ -measurable function,
- $\mu_i = \mu_j \circ f_{ij}^{-1}$ ,

*$i, j \in I$  such that  $i \leq j$ .*



Next we introduce the notion of inverse limit.

**Definition 19.** Let  $(X_i, (I, \leq), f_{ij})$  be an inverse system,  $X = \times_{i \in I} X_i$  and  $P = \{x \in X : \text{for all } i, j \text{ such that } i \leq j, pr_i(x) = f_{ij} \circ pr_j(x)\}$ , where for all  $i \in I$ ,  $pr_i$  is the coordinate projection from  $X$  to  $X_i$ . Then  $P$  is called the inverse limit of the inverse system  $(X_i, (I, \leq), f_{ij})$ , and it is denoted by  $\varprojlim (X_i, (I, \leq), f_{ij})$ .

Moreover, let  $p_i = pr_i|_P$ , so for all  $i, j \in I$  such that  $i \leq j$ ,  $p_i = f_{ij} \circ p_j$ .  $p_i$  is called canonical mapping,  $i \in I$ .

In other words, the inverse limit is a generalization of the Cartesian product. If  $(I, \leq)$  is such that every element of  $I$  is related only to itself, that is, for all  $i, j \in I$ ,  $(i \leq j) \Rightarrow (i = j)$ , then the inverse limit is the Cartesian product.

**Definition 20.** Let  $((X_i, \mathcal{A}_i, \mu_i), (I, \leq), f_{ij})$  be an inverse system of measure spaces and  $P = \varprojlim (X_i, (I, \leq), f_{ij})$ . Then the measure space  $(P, \mathcal{A}, \mu)$  is the inverse limit of the inverse system of measure spaces  $((X_i, \mathcal{A}_i, \mu_i), (I, \leq), f_{ij})$  denoted by  $(P, \mathcal{A}, \mu) = \varprojlim ((X_i, \mathcal{A}_i, \mu_i), (I, \leq), f_{ij})$ , if it meets the following points:

1.  $\mathcal{A}$  is the coarsest  $\sigma$ -field for which the canonical projections  $p_i$  are  $(\mathcal{A}, \mathcal{A}_i)$ -measurable,  $i \in I$ ,
2.  $\mu$  is a measure such that  $\mu \circ p_i^{-1} = \mu_i$ ,  $i \in I$ .

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