Agreeing to disagree: a syntactic approach

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Abstract We develop a syntactic framework that allows us to emulate standard results from the "agreeing to disagree" literature with generalised decision functions (e.g. Bacharach (1985)) in a manner the avoids known incoherences pointed out by Moses and Nachum (1990). Avoiding the incoherences requires making some sacrifices: For example, we must require the decision functions to be independent of interactive information, and, the language in which the states are described must be "rich" - in some well-defined sense. Using weak additional assumptions, we also extend all previous results to allow agents to base their decisions on possibly false information. Finally, we provide agreement theorems in which the decision functions are not required to satisfy the Sure-Thing Principle (a central assumption in the standard results).

Keywords Agreeing to disagree, knowledge, common knowledge, belief, information, epistemic logic.
JEL classification D80, D83, D89.

1 Introduction

The agreement theorem of Aumann (1976) states that if agents have a common prior, then if their posteriors on some event are common knowledge, these posteriors must be equal, *even if the agents' updates are based on different information.* This was proved for posterior probabilities in the context of a *partitional* information structure.

Briefly, Ω is a finite set of states and any of its subsets E is an *event.* For each agent $i \in N$ there is an *information function* $I_i : \Omega \to 2^{\Omega}$; the *information cell* $I_i(\omega)$ is

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the set of states that i conceives as possible at state ω, and for each $i \in N$, it is assumed that (i) $\omega \in I_i(\omega)$, and (ii) $I_i(\omega)$ and $I_i(\omega')$ are either identical or disjoint, so the set $\mathcal{I}_i = \{I_i(\omega)|\omega \in \Omega\}$ partitions the state space. Furthermore, agent i is said to "know" event E at state ω if $\omega \in I_i(\omega) \subseteq E$; and an operator $\mathbf{K}_i(.)$ is defined, where "i knows event E" is the event $\mathbf{K}_i(E) = \{\omega \in \Omega | I_i(\omega) \subseteq E\}$. Informally, E is *common knowledge* for a group of agents $G \subseteq N$ if everyone knows that E, everyone knows that everyone knows it, everyone knows that everyone knows that everyone knows it, and so on ad infinitum. Note that in this framework, the knowledge operator inherits the following properties: (i) $\mathbf{K}_i(E \cap F) = \mathbf{K}_i(E) \cap \mathbf{K}_i(F)$, (ii) $\mathbf{K}_i(E) \subseteq E$, (iii) $\mathbf{K}_i(E) \subseteq \mathbf{K}_i(\mathbf{K}_i(E))$ and (iv) $\Omega \backslash \mathbf{K}_i(E) \subseteq \mathbf{K}_i(\Omega \backslash \mathbf{K}_i(E))$.

The robustness of the agreement result was tested through various generalisations. Still operating in a partitional information structure, Cave (1983) and Bacharach (1985) independently extended the probabilisitic result to general decision functions, $D_i : \mathcal{F} \rightarrow \mathcal{A}$, that map from a field $\mathcal{F}$ of subsets of Ω into an arbitrary set $\mathcal{A}$ of *actions*. To derive the result, it is assumed that agents have the same decision function (termed "like-mindedness"), and that the decision functions satisfy what we call the Disjoint Sure-Thing Principle ($DSTP$): $\forall E \in \mathcal{E}$, if $D_i(E) = x$ then $D_i(\cup_{E \in \mathcal{E}} E) = x$, where $\mathcal{E}$ is a set of disjoint events.[1] The following states their result.[2]

> *If agents i and j are "like-minded", decision functions satisfy $DSTP$, information is partitional, and it is common knowledge at some state ω that i takes action x and j takes action y, then $x = y$.*

Moses and Nachum (1990) criticise the result above on the grounds that defining decisions over unions of information cells, as required by the $DSTP$, does not a have clear meaning in the context of generalised decision functions. Bacharach's decision functions map from subsets of Ω to capture the idea that actions must be contingent upon the agent's *information* - in a similar manner to the way in which posterior probabilities are contingent upon the information function at a given state. And, $DSTP$ is intended to capture the intuition that if one chooses to do x in every case where one is "better informed" (e.g. $D_i(I_i(\omega)) = x$ and $D_i(I_i(\omega')) = x$), then one must also choose to do x when one is more "ignorant". However, one's decision when one is more ignorant in this case is taken to be $D_i(I_i(\omega) \cup I_i(\omega')) = x$. This is problematic because it is not clear that the informational content of $I_i(\omega) \cup I_i(\omega')$ captures "more ignorance". This point can be illustrated with the following example. Consider a scenario in which there is a coin

[1] The $DSTP$ is trivially satisfied when the decision functions are posterior probabilities.

[2] Note that Aumann (1976) can be derived as a corollary by defining a common prior probability distribution over the states, and by setting, for an event E, $D_i^E(I_i(\omega)) = \Pr(E|I_i(\omega))$.

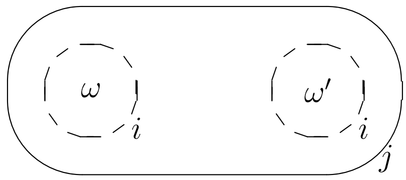

Figure 1: Coin in a box

in a box. Agent j cannot look into the box, but can see that i is looking into the box. This situation is represented in Figure 1.[3] We have that $I_i(\omega) = \{\omega\}$ and $I_i(\omega') = \{\omega'\}$, and $I_j(\omega) = \{\omega, \omega'\}$. Suppose that ω is the state in which the coin is facing heads up, whereas ω' is the state in which the coin is facing tails up. The set of states in which i knows which side is up is $\{\omega, \omega'\}$; and since $I_j(\omega) \subseteq \{\omega, \omega'\}$, we can interpret the event $E = \{\omega, \omega'\}$ as "Agent j knows that i knows which side is up". Note that at each state, i knows E. But now, suppose we take the union $I_i(\omega) \cup I_i(\omega')$. Now we may ask, what is the informational content of this set? Well, on the one hand, since $I_i(\omega) \cup I_i(\omega') \subseteq E$, it would appear that i knows E. That is, i knows that j knows that i knows which side is up. On the other hand, it is not possible that i knows E because now, it is no longer the case that i knows which side is up!

To be clear, this example does not show that there is anything *formally* wrong with Bacharach's result. Rather, there is a conceptual difficulty: The union of information cells - which is not itself an information cell - is intended to capture "more ignorance". This may be appropriate in a single-agent setting since indeed, $I_i(\omega) \cup I_i(\omega')$ does contain the information that i no longer knows which side is facing up. However, it is not clear that cell union captures "more ignorance" in a setting where there is *interactive* information - events of the type: i knows that j knows that E.

Note that information is *implicitly* modelled in Bacharach's framework, by being somehow contained in a set of states. Our solution will consist in *explicitly* modelling the syntactic information at each state, and the incoherence will be avoided by selecting only the information that is *not* interactive - and allowing decisions only to depend on non-interactive information.

Moses and Nachum (1990) propose their own solution to the generalised agreement theorem by defining a projection from states to an arbitrary set, intended to capture the information at each state that is relevant to the decision, and the decision functions map relevant information into actions. Now, relevant informa-

[3] Obviously, we could have created a more complicated model representing a situation where j does not see that i can see into the box. That is, a situation in which j does not know that i knows which side is facing up.

tion is defined over a variety of sets of states, so the above criticism is resolved. However they require a *stronger* version of the Sure-Thing Principle, which does not require the "disjointness" of the relevant information, which we term the Non-Disjoint Sure-Thing Principle, $NDSTP$.
More recently, Aumann and Hart (2006) use the framework developed in Aumann (1999) to reproduce the results of Bacharach and of Moses & Nachum in a coherent manner. They resort to the same idea, pursued here and in Moses and Nachum (1990), that the incoherence can be avoided by singling out the "right" kind of information at each state. Our approach has some similarities with theirs. However, there are also some important differences: Our framework allows us to give a very natural ranking of information (informativeness) - which turns out to be a useful concept -, and is more versatile in that it easily admits a generalisation to non-partitional information structures.

In an altogether different strand of the literature, Samet (1990) and Collins (1997) prove agreement theorems in a *non-partitional* information structure. This is an important line of investigation since partitional structures imply that agents can only know what is the case; in other words, agents cannot base their decisions on false information. But surely, it is perfectly plausible for rational agents to do so. The culprit is the assumption that for all $\omega \in \Omega$, $\omega \in I_i(\omega)$ since the "actual" state is always included in the set of states that the agent considers possible. Discarding this assumption, the operator $\mathbf{K}$ is then interpreted as a "belief" operator (since it is possible to believe what is false, but not to know it; in particular, it is now no longer necessary that $\mathbf{K}_i(E) \subseteq E$).
The results of Samet (1990) and Collins (1997) are derived in a probabilistic framework, and require further assumptions ("Consistency in Samet (1990), "Zero priors" in **?**). In a similar vein, Bonanno and Nehring (1998) also derive an agreement theorem in a non-partitional information structure, but their framework is more general. Their analogues of decision functions are required to satisfy a "properness" condition which in some cases implies the Disjoint Sure-Thing Principle, and in other cases, is equivalent to the non-disjoint version. Furthermore, they require an assumption of "quasi-coherence" over the state space - which we define later. However, it is not clear that any of those results are able to explicitly avoid the criticism of Moses and Nachum (1990).

In this paper, we use standard concepts from epistemic logic to derive agreement theorems with generalised decision functions in both partitional and non-partitional models, that are analogous to the results mentioned above, but that do not suffer from the incoherences pointed out by Moses and Nachum (1990).[4]

[4] Although still working with a partitional structure, Samet (2010) takes an altogether different

We set up a framework in which we can model the syntactic information at each state *explicitly*. This allows us to single out non-interactive information, and to allow decisions to be based only on such information. This is shown to resolve the conceptual problems of the Sure-Thing Principle used by Bacharach. Furthermore, we derive a result in which we use an analogous version of the Disjoint Sure-Thing Principle, which we see as an improvement to the solution proposed by Moses and Nachum (1990); but this requires us to assume that the language in which the states are described be "rich" - in some well-defined sense. It is important to note that our results in partitional models are not intended as generalisations of Bacharach's result. Rather, they are analogues of his result with sounder foundations.
Our framework is versatile in that it easily admits a natural extension to non-partitional models. This extension is important for two reasons. Firstly, since agents are restricted to base their decisions only on correct information in partitional models, there may be a sense in which they are already forced into agreement. Indeed, the restriction of the information base - to only correct information - may be a factor driving the agreement results. Therefore, it is important to verify whether similar results hold when agents are allowed to base their decisions on possibly false information. Secondly, we show that there are non-trivial examples that cannot be modelled with parititonal information structures. Nevertheless, we find that agreement results do hold in non-partitional models, given a weak additional assumption ("Heterogeneity") which essentially requires enough variation in the information of the agents.
Finally, we are also able to prove an agreement theorem in which less restrictions are imposed on the decision functions. Namely, we no longer require them to satisfy the Sure-Thing Principle (whether disjoint or not).

In section 2, we introduce the basic concepts that we use from epistemic logic. In section 3, we expand the standard epistemic logic framework to encompass decision functions, and we state our main assumptions. We derive our main results in partitional models in section 4, and in non-partitional models in section 5. The finial theorem with minimal restrictions on the decision functions is found in section 6. All proofs are in the appendix.

approach, deriving a generalised agreement theorem by assuming an "interpersonal" Sure-Thing Principle ($ISTP$), which is a condition imposed on actions *across* different agents. The generalisation of his result in our framework to non-partitional structures is the subject of a companion paper (Tarbush (2011)).

2 Epistemic Logic

This section introduces concepts from epistemic logic. All the definitions and results in this section are standard (e.g. see Chellas (1980) and van Benthem (2010) for general reference).

We must develop the *language* that our results will be stated in. This will consist in defining a *syntax* - which determines which symbols or chains of symbols are part of the language (e.g. "dog" is permissible in the English language, but "a@b6tt" is not) -, and in defining *semantics* - which assigns a meaning to the symbols and thus determines a grammar (e.g. "The dog is barking" is semantically permissible in the English language, but "Dog towards rain table" is not) -.

A *proposition* is a sentence, usually represented by a lower case letter. For example, "The dog is barking", and, "It is raining" can represented by p and q respectively. Propositions can be combined in various ways using the standard *Boolean operators*: not, and, or, if...then, if and only if, which are represented by the following symbols respectively $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$. An example of a combination of propositions is "The dog is barking *and* it is raining" (formally $p \wedge q$).
Finally, we will also allow for *modal* operators in our language. These are operators that qualify an entire proposition. For example, "I know that the dog is barking" is made up of the proposition "The dog is barking" and the modal operator "I know that". We will have two basic symbols for modal operators in our language, namely $\Box_i$ and C_G, although their exact interpretation will be developed later. Essentially, depending on the semantics, $\Box_i p$ will either stand for "Agent i knows that p", or "Agent i believes that p", whereas $C_G p$ will either stand for "It is common knowledge among the subset of agents G that p", or "It is common belief among the subset of agents G that p".
Propositions are *atomic* if they do not contain any operators (whether Boolean or modal), and are thus reduced to the most basic building block. For example, "The dog is barking" contains no operators, so cannot be made more basic, whereas "The dog is barking and it is raining" can be reduced to two propositions, so is not atomic.
A *formula* is any chain of symbols that is acceptable in the language. Formally, we construct the syntax, or the set of formulas in our language, as follows:

Definition 1 (Basic syntax)**.** Define a finite set of atomic *propositions*, $\mathcal{P}$, which consists of all propositions that cannot be further reduced. Let N denote the set of all agents. We then inductively create all the *formulas* in our language, $\mathcal{L}$, as follows:

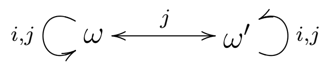

Figure 2: Example of a Kripke structure

(i) Every $p \in \mathcal{P}$ is a formula.
(ii) If ψ is a formula, so is $\neg\psi$.
(iii) If ψ and ϕ are formulas, then so is $\psi \circ \phi$, where $\circ$ is one of the following Boolean operators: $\wedge$, $\vee$, $\rightarrow$, or $\leftrightarrow$.
(iv) If ψ is a formula, then so is $\bullet\psi$, where $\bullet$ is one of the modal operators $\Box_{i\in N}$ or $C_{G\subseteq N}$.
(v) Nothing else is a formula.

So far, we have pure uninterpreted syntax. Indeed, "Agent i knows that it is raining and knows that it is not raining" is a formula of our language (represented as $\Box_i q \wedge \Box_i \neg q$), but surely it cannot be true. We therefore introduce the semantics of our language to determine the truth or falsity of formulas. To do this we use standard Kripke semantics.

Definition 2 (Kripke semantics)**.** A *frame* is a pair $\langle \Omega, R_{i\in N}\rangle$, where Ω is a finite, non-empty set of *states* (or "possible worlds"), and $R_i \subseteq \Omega \times \Omega$ is a binary relation for each agent i, also called the *accessibility relation* for agent i. A *model* on a frame $\langle \Omega, R_{i\in N}\rangle$, is a triple $\mathcal{M} = \langle \Omega, R_{i\in N}, \mathcal{V}\rangle$, where $\mathcal{V} : \mathcal{P} \times \Omega \rightarrow \{0, 1\}$ is a *valuation map.*

Definition 3 (Truth)**.** We say that a proposition $p \in \mathcal{P}$ is *true* at state ω in model $\mathcal{M} = \langle \Omega, R_{i\in N}, \mathcal{V}\rangle$, denoted $\mathcal{M}, \omega \models p$, if and only if $\mathcal{V}(p, \omega) = 1$. Truth is then extended inductively to all other formulas ψ as follows:
(i) $\mathcal{M}, \omega \models \neg\psi$ if and only if it is not the case that $\mathcal{M}, \omega \models \psi$.
(ii) $\mathcal{M}, \omega \models (\psi \wedge \phi)$ if and only if $\mathcal{M}, \omega \models \psi$ and $\mathcal{M}, \omega \models \phi$.[5]
(iii) $\mathcal{M}, \omega \models \Box_i\psi$ if and only if $\forall \omega' \in \Omega$, if $\omega R_i \omega'$ then $\mathcal{M}, \omega' \models \psi$.
(iv) $\mathcal{M}, \omega \models C_G\psi$ if and only if $\forall \omega' \in \Omega_G(\omega)$, $\mathcal{M}, \omega' \models \psi$.
The *component* of ω, $\Omega_G(\omega)$, is the set of all states that are accessible from ω in a finite sequence of R_i $(i \in G)$ steps.

The above definitions can be illustrated by the model $\mathcal{M} = \langle \Omega, R_{i\in N}, \mathcal{V}\rangle$ represented in Figure 2. The state space is $\Omega = \{\omega, \omega'\}$. The accessibility relations for agents i and j are as represented in the figure. Namely, $R_i = \{(\omega, \omega), (\omega', \omega')\}$ and $R_j = \{(\omega, \omega), (\omega', \omega'), (\omega, \omega'), (\omega', \omega)\}$. Finally, we can let $\mathcal{P} = \{h, t\}$, $\mathcal{V}(h, \omega) = 1$, and $\mathcal{V}(t, \omega') = 1$. From this alone, we can generate several new formulas. For example, note that every state ω'' that is accessible from ω for agent i is such that

[5]The truth of formulas involving the other Boolean operators are similarly defined.

$\omega'' \models h$ (indeed, the only state that i can "access" from ω is ω itself, and h is true at ω). Therefore, by the definition of truth, we have that $\omega \models \Box_i h$. Similarly, we have $\omega' \models \Box_i t$. On the other hand, we have $\omega \models \neg\Box_j h$. This is because from ω, j can "access" the state ω' in which h is not true, but rather t is true.
We can even go further. One can verify that $\omega \models \Box_i h \vee \Box_i t$ and $\omega' \models \Box_i h \vee \Box_i t$; and therefore, $\omega \models \Box_j(\Box_i h \vee \Box_i t)$ and $\omega' \models \Box_j(\Box_i h \vee \Box_i t)$. In fact, since for any state accessible from ω in a finite sequence of R_k ($k \in \{i, j\}$) steps, it is the case that $\Box_i h \vee \Box_i t$, we can also conclude that $\omega \models C_{\{i,j\}}(\Box_i h \vee \Box_i t)$.

We can note that some formulas, such as $\Box_i h$ are only true at some state of the model, whereas others, like $\Box_i h \vee \Box_i t$ are true at *every* state in the model. The latter are said to be *valid in the model.* But there are more general levels of validity. For example, suppose we keep the same states and accessibility relations as the model in Figure 2, but modify the valuation map. Then, we obtain a set of new models, all with the same frame. The formulas that remain true at every state of each of these models are said to be *valid in the frame.* Even more generally, we can allow the frame itself to vary, but within a class of frames. For example, we could consider all the frames in which for every $\omega \in \Omega$, $(\omega, \omega) \in R_i$ for each $i \in N$. The formulas that remain true at every state of every model in every frame within this class are said to be *valid (within this class of frames).* Formally, we have the following definition.

Definition 4 (Validity)**.** Formula ψ is valid in a model $\mathcal{M}$, denoted $\mathcal{M} \models \psi$ if and only if $\forall\omega \in \Omega$ in $\mathcal{M}$, $\omega \models \psi$. Formula ψ is valid in a frame $\langle\Omega, R_{i\in N}\rangle$, denoted $\langle\Omega, R_{i\in N}\rangle \models \psi$, if and only if $\forall\mathcal{M}$ over $\langle\Omega, R_{i\in N}\rangle$, $\mathcal{M} \models \psi$. Formula ψ is $\mathcal{T}$-valid (or valid), denoted $\models \psi$, if and only if $\forall\langle\Omega, R_{i\in N}\rangle \in \mathcal{T}$ ($\mathcal{T}$, a class of frames), $\langle\Omega, R_{i\in N}\rangle \models \psi$.

The frame classes can be determined by the conditions that are imposed on the accessibility relations. The following gives a selection of conditions that are often used to classify frames.

Definition 5 (Conditions on frames)**.** We say that a frame $\langle\Omega, R_{i\in N}\rangle$ is,
(i) Reflexive if $\forall i \in N$, $\forall\omega \in \Omega, \omega R_i \omega$.
(ii) Symmetric if $\forall i \in N$, $\forall\omega, \omega' \in \Omega,$ if $\omega R_i \omega'$ then $\omega' R_i \omega$.
(iii) Transitive if $\forall i \in N$, $\forall\omega, \omega', \omega'' \in \Omega,$ if $\omega R_i \omega'$ and $\omega' R_i \omega''$ then $\omega R_i \omega''$.
(iv) Euclidean if $\forall i \in N$, $\forall\omega, \omega', \omega'' \in \Omega,$ if $\omega R_i \omega'$ and $\omega R_i \omega''$ then $\omega' R_i \omega''$.
(v) Serial if $\forall i \in N$, $\forall\omega \in \Omega, \exists\omega' \in \Omega, \omega R_i \omega'$.

We will be interested in two particular classes of frames. One of them is the $S5$ class, which consists of all frames that are reflexive, symmetric and transitive. The other class, known as $KD45$, is the class of all frames that are transitive, Euclidean and serial.

We have so far, in our example in Figure 2, been careful not to interpret the symbol $\Box$ as a knowledge operator. Indeed, to allow such an interpretation, we must guarantee that the operator possesses the properties that one might expect of knowledge. For example, one distinguishing characteristic of knowledge is that one cannot know what is false. So, we must at least impose the restriction that the formula $\Box_i\psi \to \psi$ for any agent i and any formula ψ, be *valid.*

It turns out that the following formulas are valid in $S5$ frames:
(i) Distribution: $\Box_i(\psi \to \phi) \to (\Box_i\psi \to \Box_i\phi)$.
(ii) Veracity: $\Box_i\psi \to \psi$.
(iii) Positive introspection: $\Box_i\psi \to \Box_i\Box_i\psi$.
(iv) Negative introspection: $\neg\Box_i\psi \to \Box_i\neg\Box_i\psi$.
In fact, the converse also holds: Namely, if we require (i) - (iv) to be validities, then the frame must be $S5$.

The formulas (i) - (iv) happen to be precisely the properties that are considered to be the defining characteristics of knowledge (Early formal philosophical underpinnings can be found in Hintikka (1962)). For example, veracity states that if i knows that ψ, then ψ must be true. In other words, one cannot know what is false. Positive introspection states that if i knows that ψ, then i knows that i knows that ψ. That is, if one knows something, then one knows that one knows it. Finally, negative introspection states that if i does not know that ψ then i knows that i does not know that ψ. That is, if one does not know something, then one knows that one does not know it. Admittedly these are properties of a very strong notion of knowledge. However, they are taken as standard, and we will not discuss their justification. In fact, they are completely analogous to the properties of the knowledge operator **K** mentioned in the introduction.

Given the above, we can return to the model given in Figure 2. One can verify that the model has an $S5$ frame, and the modal operators can thus be interpreted as knowledge and common knowledge.
In fact, the model can be seen as a representation of the coin in a box scenario described in the introduction, where h is the proposition "The coin is heads side up", and t is "The coin is tails side up".
It was shown previously that $\omega \models \Box_i h$, which means that in the state in which the coin is indeed heads side up, agent i knows this (since he can see it). Also, $\omega' \models \Box_i t$ means that in the state in which the coin is tails side up, i also knows this. Furthermore, we had that $\omega \models \neg\Box_j h$, so in the state in which the coin is

heads side up, j does not know that the coin is heads side up.
Note that the modal formulas in the above paragraph have a single modal operator, so are said to have a *modal depth* of 1. However, a formula such as $\Box_i \Box_j \psi$ has two nested modal operators, so has a modal depth of 2. In our example, the formula $\Box_j(\Box_i h \vee \Box_i t)$, interpreted as "$j$ knows that i knows which side of the coin is facing up", also has a modal depth of 2. Clearly, *interactive* knowledge - of the form "I know that you know..." - requires a modal depth of at least 2. A formal definition of this notion is given below.

Definition 6 (Modal depth). The modal depth $md(\psi)$ of a formula ψ is the maximal length of a nested sequence of modal operators. This can be defined by the following recursion on our syntax rules: (i) $md(p) = 0$ for any $p \in \mathcal{P}$, (ii) $md(\neg\psi) = md(\psi)$, (iii) $md(\psi \wedge \phi) = md(\psi \vee \phi) = md(\psi \rightarrow \phi) = md(\psi \leftrightarrow \phi) = \max(md(\psi), md(\phi))$, (iv) $md(\Box_i \psi) = 1 + md(\psi)$, (v) $md(C_G \psi) = 1 + md(\psi)$.

Finally, returning to our example one last time, we showed that $C_{\{i,j\}}(\Box_i h \vee \Box_i t)$ is valid in the model. This is interpreted as it being *common knowledge* among i and j that i knows which side is facing up, in the sense that they both know this, both know that they know it, both know that they know that they know it, and so on ad infinitum. This is the interpretation of the C_G operator because, completely generally, if $\mathcal{M}, \omega \models C_G \psi$, then one can generate any formula of finite modal depth of the form $\Box_i \Box_j ... \Box_r \psi$ with $i, j ... r \in G$, and this formula will be true at ω in model $\mathcal{M}$.[6]

3 Models with information and decisions

All the definitions in this section are completely general, so hold for arbitrary frame classes.

Let P be a finite set of atomic propositions. Since P is finite, its closure under the standard Boolean operators, denoted P^*, is tautologically finite.[7] So P^* is just the set of all possible inequivalent formulas that can be created out of the propositions in P and the Boolean operators. Let Ψ_0^r be the set of all possible modal formulas that can be generated from P^* with modal depth 0 up to r for an

[6]Note that the definition of the operator C_G is drawn from van Benthem (2010), where it is also mentioned that an alternative definition can be given: One can define a new accessibility relation R_G^* for the whole group G as the reflexive transitive closure of the union of all separate relations R_i $(i \in G)$, and then simply let $\mathcal{M}, \omega \models C_G \psi$ if and only if $\forall \omega' \in \Omega$, if $\omega R_G^* \omega'$ then $\mathcal{M}, \omega' \models \psi$.

[7]In the sense that there is only a finite number of inequivalent formulas (so p and $p \wedge p$ count as one).

arbitrary $r \in \mathbb{N}_0$. Again, since P^* is finite, so is Ψ_0^r, so $|\Psi_0^r| = m$, for some $m \in \mathbb{N}$; and note that $\Psi_0^0 = P^*$.[8]

Definition 7 (New operators)**.** For each agent $i \in N$ create a set of modal operators, $O_i = \{\Box_i, \hat{\Box}_i, \dot{\Box}_i\}$, where for every formula ψ, $\hat{\Box}_i\psi := \Box_i\neg\psi$ and $\dot{\Box}_i\psi := \neg(\Box_i\psi \vee \hat{\Box}_i\psi)$.
In $S5$, $\hat{\Box}_i\psi$ stands for "Agent i knows that it is not the case that ψ", and $\dot{\Box}_i\psi$ stands for "Agent i does not know whether it is the case that ψ".

Definition 8 (Kens)**.** Order the set Ψ_0^r into a vector of length m: $(\psi_1, \psi_2, ..., \psi_m)$, and for each agent $i \in N$, create the sets

$$U_i = \{(\nu_i^1\psi_1 \wedge \nu_i^2\psi_2 \wedge ... \wedge \nu_i^m\psi_m)|\forall n \in \{1, ..., m\}, \nu_i^n \in O_i\}$$

$$V_i = \{\nu_i \in U_i| \models \neg(\nu_i \leftrightarrow (p \wedge \neg p))\}$$

A *ken* ($\nu_i \in V_i$) for agent i, describes i's information concerning every formula in Ψ_0^r. So, calling $\nu_i^n\psi_n$ the n^{th} *entry* of i's ken, the formula $\nu_i^n\psi_n$ states - in $S5$ - whether i knows that the formula ψ_n is the case, or knows that it is not the case, or does not know whether it is the case.
Note that V_i is a restriction of U_i to the set of kens that are not logically equivalent to a contradiction; so only the logically consistent kens are considered.[9]

The following lemma shows that at each state, there exists a ken for each agent which holds at that state, and moreover, that any two different kens must be contradictory at any given state.

Lemma 1. *(i)* $\forall\omega \in \Omega, \exists\nu_i \in V_i, \omega \models \nu_i$, *(ii)* $\forall\omega \in \Omega, \forall\nu_i, \mu_i \in V_i$, *if* $\nu_i \neq \mu_i$ *then* $\omega \models \neg(\nu_i \wedge \mu_i)$.

By the above lemma, there is a unique ken in V_i that holds at a given state.

Definition 9 (Informativeness)**.** Create an order $\succsim \subseteq V_i \times V_j$ for all $i, j \in N$. We say that the ken ν_i is *more informative* than the ken μ_j, denoted $\nu_i \succsim \mu_j$, if and only if whenever i knows that ψ then j either also knows that ψ or does not know whether ψ, and whenever i does not know whether ψ, then j also does not know whether ψ.[10]
Note that $\succsim$ is not a complete order on kens. For example, consider any two kens ν_i and μ_i for agent i, in which the n^{th} entry is $\nu_j^n\psi_n = \Box_i\psi_n$ and $\mu_j^n\psi_n = \hat{\Box}_i\psi_n$.

[8] If $P = \{p, q\}$, then one can generate 20 inequivalent formulas: 2 from p alone, 2 from q alone and 16 out of p and q together, so $|P^*| = 20$.

[9] An example of a logically inconsistent ken would be one containing $\Box_i p$, $\Box_i q$ and $\Box_i(p \to \neg q)$.

[10] Formally, (i) if $\nu_i^n\psi_n = \Box_i\psi_n$ then ($\mu_j^n\psi_n = \Box_j\psi_n$ or $\mu_j^n\psi_n = \dot{\Box}_j\psi_n$), (ii) if $\nu_i^n\psi_n = \hat{\Box}_i\psi_n$ then ($\mu_j^n\psi_n = \hat{\Box}_j\psi_n$ or $\mu_j^n\psi_n = \dot{\Box}_j\psi_n$), and (iii) if $\nu_i^n\psi_n = \dot{\Box}_i\psi_n$ then ($\mu_j^n\psi_n = \dot{\Box}_j\psi_n$).

These two kens would not be comparable with $\succsim$.
Finally, note that $\nu_i \sim \mu_j$ denotes $\nu_i \succsim \mu_j$ and $\mu_j \succsim \nu_i$; which is interpreted as ν_i and μ_j carrying the *same* information, but seen from the perspectives of agents i and j respectively.

The infimum of ν_i and μ_i, denoted $\inf\{\nu_i, \mu_i\}$, is the most informative ken that is less informative than ν_i and μ_i.

Lemma 2. *For any $\nu_i, \mu_i \in V_i$, $\inf\{\nu_i, \mu_i\}$ exists in V_i and is characterised by:*

$$\inf\{\nu_i, \mu_i\}^n \psi_n = \Box_i \psi_n \text{ iff } (\nu_i^n \psi_n = \mu_i^n \psi_n = \Box_i \psi_n)$$
$$\inf\{\nu_i, \mu_i\}^n \psi_n = \hat{\Box}_i \psi_n \text{ iff } (\nu_i^n \psi_n = \mu_i^n \psi_n = \hat{\Box}_i \psi_n)$$
$$\inf\{\nu_i, \mu_i\}^n \psi_n = \dot{\Box}_i \psi_n \text{ iff } (\nu_i^n \psi_n \neq \mu_i^n \psi_n \text{ or } \nu_i^n \psi_n = \mu_i^n \psi_n = \dot{\Box}_i \psi_n)$$

Definition 10 (Decision function)**.** For each $i \in N$, $D_i : V_i \rightarrow \mathcal{A}$, is the *decision function* of agent i, where $\mathcal{A}$ is a set of *actions*.

The decision function D_i determines what agent i *would do* given every possible ken. By Lemma 1, there is a unique ken that is actually true at a given state, so only one action actually ends up being performed at each state. So, if $\omega \models \nu_i$ and $D_i(\nu_i) = x$, we write $\omega \models d_i^x$, where $d_i^{D_i(\nu_i)}$ is a *formula* - added to the syntax - that states "Agent i performs action $D_i(\nu_i)$", where $D_i(\nu_i)$ is the decision that is taken over the ken, ν_i, that is true at that state.

Definition 11 (Richness)**.** The language in a component $\Omega_G(\omega)$ is *rich* if and only if for all $i \in G$, and any $\omega', \omega'' \in \Omega_G(\omega)$ such that $\omega' \models \nu_i$, $\omega'' \models \mu_i$ and $\nu_i \neq \mu_i$, there is $n \in \{1, ..., m\}$ such that $\nu_i^n = \Box_i$ and $\mu_i^n = \hat{\Box}_i$.

Essentially, the language in a component is rich if any two distinct kens in the component for agent i are incomparable via $\succsim$. In other words, any two distinct kens must be contradictory about *some* "fact" - i.e. formula - (so in one ken, the agent knows that the fact is true, whereas in the other ken, the agent knows that it is false). Richness is how we capture the idea of "disjointness" in our framework.[11]

3.1 Main assumptions

We will assume two distinct versions of the Sure-Thing Principle, and prove an agreement theorem with each respectively. The first version is the analogue of the "non-disjoint" version used by Moses and Nachum (1990):

[11] Note that richness is analogous to what we understand as the requirement that all knowledge be "elementary" in Aumann and Hart (2006); and is intended to capture the idea that the information be "disjoint" (in line with the Sure-Thing Principle of Bacharach (1985)).

$$
\begin{array}{ccc}
\nu_i & \mu_i & \inf\{\nu_i, \mu_i\} \\
\Box_i a & \hat{\Box}_i a & \dot{\Box}_i a \\
\hat{\Box}_i b & \Box_i b & \dot{\Box}_i b \\
\hat{\Box}_i c & \hat{\Box}_i c & \hat{\Box}_i c \\
\Box_i(a \vee b) & \Box_i(a \vee b) & \Box_i(a \vee b)
\end{array}
$$

Figure 3: Dinner party example

Assumption 1 (Non-Disjoint Sure-Thing Principle - $NDSTP$).
For all $i \in N$ and all $\nu_i, \mu_i \in V_i$, if $D_i(\nu_i) = D_i(\mu_i)$ then $D_i(\inf\{\nu_i, \mu_i\}) = D_i(\nu_i)$.

This states that whenever an agent would take the same decision given the information ν_i and μ_i, then the agent would take the same decision over the infimum of those kens - i.e. in the situation in which the agents is "just" less informed. The second version of the Sure-Thing Principle, which we call $DTSP$, is closer to the original one used by Bacharach (1985), because it requires disjointness. In our framework, $DSTP$ is simply $NDSTP$ but is only required to hold over kens that are expressed in a "rich" language.

Assumption 1' (Disjoint Sure-Thing Principle - $DSTP$).
Let $T_i = \{(\nu_i, \mu_i) \in V_i \times V_i | \exists n$ such that $\nu_i^n = \Box_i$ and $\mu_i^n = \hat{\Box}_i\}$.
For all $i \in N$ and all $\nu_i, \mu_i \in T_i$, if $D_i(\nu_i) = D_i(\mu_i)$ then $D_i(\inf\{\nu_i, \mu_i\}) = D_i(\nu_i)$.

The above versions of the Sure-Thing Principle can be illustrated by means of the following examples. Alice, Bob and Charlie are invited to dinner. Charlie eats anything, but Alice and Bob are vegetarian. Agent i cooks vegetarian if i knows that a vegetarian guest is coming. This can be represented in Figure 3. With ken ν_i, agent i knows that Alice is coming, and that Bob and Charlie are not coming; whereas with ken μ_i, agent i knows that Bob is coming, but that Alice and Charlie are not coming. In both cases, i knows that some vegetarian guest is coming, so i cooks vegetarian. By the Sure-Thing Principle, i must also cook vegetarian when her information is the infimum of those kens. This makes sense since with the infimum, i still knows that a vegetarian guest is coming. Indeed, i no longer know *which* vegetarian guest is coming ($\dot{\Box}_i a \wedge \dot{\Box}_i b$), but does know that at least one is indeed coming ($\Box_i(a \vee b)$).[12]

Note that the kens in the above example are expressed in a "rich" language. However, there are situations in which we ought to *require* disjointness. Consider the following example represented in Figure 4: A prize is behind door A, B or C.

[12]Incidentally, this shows the importance of basing kens on all possible formulas, not just on the atomic propositions.

$$\begin{array}{ccc} \nu_i & \mu_i & \inf\{\nu_i, \mu_i\} \\ \hat{\Box}_i a & \dot{\Box}_i a & \dot{\Box}_i a \\ \dot{\Box}_i b & \dot{\Box}_i b & \dot{\Box}_i b \\ \dot{\Box}_i c & \hat{\Box}_i c & \dot{\Box}_i c \end{array}$$

Figure 4: Prize behind doors example

Agent i is willing to make a bet about which door the prize is behind only if i knows that it is behind one of only two doors. With ken ν_i, agent i knows that the prize is not behind door A, so the agent bets. In the second case, i knows the prize is not behind door C, so the agent bets. The Non-Disjoint Sure-Thing Principle would require the agent to also make a bet when her ken is the infimum; but in this case, the agent no longer knows that the prize is behind only one of two doors![13]
Note that in Bacharach's framework, the richness of kens is implicitly assumed: Disjoint sets of states must have contradictory information. (E.g. in our case above, if we model the situation with three possible states - one for each possible location of the prize - then any two disjoint states do contain contradictory information; that is, the kens are "rich").

Assumption 2 (Like-mindedness)**.** *For all $\nu_i \in V_i$ and $\nu_j \in V_j$, if $\nu_i \sim \nu_j$ then $D_i(\nu_i) = D_j(\nu_j)$.*

The assumption of like-mindedness captures the idea that the agents would take the same decision if they had the same information.

4 Results in $S5$

In $S5$, the accessibility relation R_i of each agent i is reflexive, symmetric and transitive. So it is an equivalence relation for each $i \in N$. Let $I_i(\omega) = \{\omega' \in \Omega | \omega R_i \omega'\}$ be the *information cell* of i at ω. One can verify that the set $\mathcal{I}_i = \{I_i(\omega) | \omega \in \Omega\}$ is a partition of the state space Ω - we thus have a *partitional* model.

[13] Aumann et al. (2005) essentially argue that the Sure-Thing Principle is a reasonable assumption when all possible *signals* are taken into consideration. In our example, in ken ν_i the agent knows that he/she was not told that the prize is behind door A, and knows that he/she was told that the prize is behind doors B or C. Similarly for the ken μ_i. In this case, we can let the decision rule be: Make a bet if you know that you were told that the prize is behind only one of two doors. Now, the Sure-Thing Principle makes more sense, and the information has become disjoint because the signals must be disjoint.
Given this interpretation, the language is called "rich" because it also contains the atomic propositions describing the way in which the information was acquired - i.e. all possible signals.

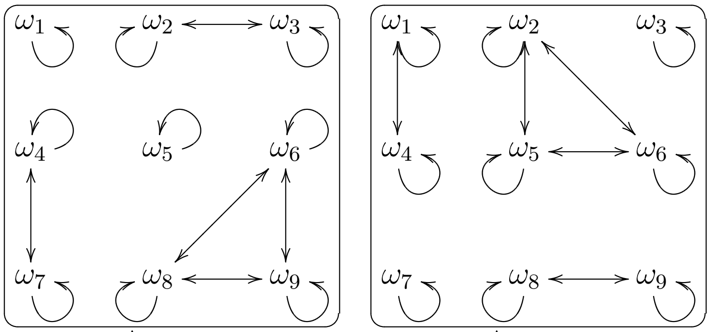

Figure 5: State space in $S5$

We provide a schematic representation of an $S5$ model in Figure 5. The state space is $\Omega = \{\omega_1, ..., \omega_9\}$. The partition for agent i is given by the set $\mathcal{I}_i = \{\{\omega_1\}, \{\omega_5\}, \{\omega_2, \omega_3\}, \{\omega_4, \omega_7\}, \{\omega_6, \omega_8, \omega_9\}\}$. Agent j's partition is found similarly. Furthermore, $\Omega_{\{i,j\}}(\omega_1) = \{\omega_1, \omega_4, \omega_7\}$, and $\Omega_{\{i,j\}}(\omega_2) = \Omega \backslash \Omega_{\{i,j\}}(\omega_1)$.

The following lemma states that in $S5$, the information cells of every agent exhaust any component.

Lemma 3. *$\forall i \in G$, $\bigcup_{\omega' \in \Omega_G(\omega)} I_i(\omega') = \Omega_G(\omega)$.*

The lemma below states that kens are identical across all the states that are in the same information cell.

Lemma 4. *If for some $\omega' \in I_i(\omega)$, $\omega' \models \nu_i$, then for all $\omega'' \in I_i(\omega)$, $\omega'' \models \nu_i$.*

It will now be useful to introduce a new definition which will eventually allow us to provide a semantic characterisation of $\inf\{\nu_i, \mu_i\}$ for any kens $\nu_i, \mu_i \in V_i$.

Definition 12 (Cell merge)**.** Consider a model in $S5$, $\mathcal{M} = \langle \Omega, R_{i \in N}, V \rangle$. Let $I_i(\omega) = \{\omega'' \in \Omega | \omega R_i \omega''\}$ and $I_i(\omega') = \{\omega'' \in \Omega | \omega' R_i \omega''\}$. Create a new model $\mathcal{M}(I_i(\omega), I_i(\omega')) = \langle \Omega', R'_{i \in N}, V' \rangle$ where,

$$\Omega' = \Omega$$

$$R'_i = R''_i \cup R_i|_{\Omega \backslash I_i(\omega) \cup I_i(\omega')}$$

where $R''_i = \{(\omega'', \omega''') \in \Omega \times \Omega | \omega'', \omega''' \in I_i(\omega) \cup I_i(\omega')\}$

and $R_i|_{\Omega \backslash I_i(\omega) \cup I_i(\omega')} = \{(\omega'', \omega''') \in R_i | \omega'', \omega''' \in \Omega \backslash I_i(\omega) \cup I_i(\omega')\}$

$$R'_j = R_j \text{ for all } j \neq i$$

$$V' = V$$

One can verify that the model $\mathcal{M}(I_i(\omega), I_i(\omega'))$ is itself a model in $S5$, but where the cells $I_i(\omega)$ and $I_i(\omega')$ are *merged* to form a single information cell (with all the accessibility relations appropriately "rewired"), yet leaving the rest of the original model, $\mathcal{M}$, unchanged.

For sake of illustration, let us return to the example given in Figure 5. Let the model represented be $\mathcal{M}$. We can, for example, create the "merged" model, $\mathcal{M}(I_i(\omega_4), I_i(\omega_5))$, in which j's information partition is unchanged, but i's partition is now $\{\{\omega_1\}, \{\omega_2, \omega_3\}, \{\omega_4, \omega_5, \omega_7\}, \{\omega_6, \omega_8, \omega_9\}\}$.

The following lemmas provides a semantic characterisation of $\inf\{\nu_i, \mu_i\}$ in $S5$, which turns out to be the ken that holds in a model in which the information cells at which ν_i and μ_i hold are merged (ignoring interactive information).

Lemma 5. *Consider Ψ_0^r with $r = 0$.*
If $\mathcal{M}, \omega \models \nu_i$ and $\mathcal{M}, \omega' \models \mu_i$, then for all $\omega''' \in I_i(\omega) \cup I_i(\omega')$, $\mathcal{M}(I_i(\omega), I_i(\omega')), \omega''' \models \inf\{\nu_i, \mu_i\}$.

Lemma 6. *Consider Ψ_0^r with $r = 0$ and let $G = \{i, j\}$.*
For any $\Omega_G(\omega)$, $\inf\{\nu_i | \omega' \models \nu_i \;\&\; \omega' \in \Omega_G(\omega)\} \sim \inf\{\nu_j | \omega' \models \nu_j \;\&\; \omega' \in \Omega_G(\omega)\}$.

Finally, we are in a position to state our agreement results in $S5$:

Theorem 1. *Consider Ψ_0^r with $r = 0$, suppose NDSTP holds, the agents are like-minded, and the system is $S5$. Let $G = \{i, j\} \subseteq N$. Then, $\models C_G(d_i^x \wedge d_j^y) \to (x = y)$.*

Theorem 2. *Consider Ψ_0^r with $r = 0$, suppose DSTP holds, the agents are like-minded, the language is rich in every component, and the system is $S5$. Let $G = \{i, j\} \subseteq N$. Then, $\models C_G(d_i^x \wedge d_j^y) \to (x = y)$.*

Note that there is a slight abuse of notation in the statement of the theorems above. Technically, "=" is not part of our syntax, so $x = y$ should not appear anywhere. However, we simply use it as shorthand. Our results should really be read as: $\models C_G(d_i^x \wedge d_j^y) \to (d_i^z \wedge d_j^z)$, and $z = x = y$.

4.1 Discussion

The intuition behind the results is that at each state, each agent has some ken, and performs some action, say d_i^x, based on it. However, the Sure-Thing Principle allows us to discover that i's decision *would* also be x *if i's information were* $\inf\{\nu_i | \omega' \models \nu_i \;\&\; \omega' \in \Omega_G(\omega)\}$. This is *not* the ken that i has at ω, so i's action is not taken based on this ken. However, over a similar ken, we find that j's decision would be y. But since this is the same uninformative ken, and agents are

like-minded, we conclude that $x = y$.
Note the role of the infimum of kens in the theorems: Effectively, it only preserves those propositions that *both* agents know. Any proposition p where i knows that p while j knows that $\neg p$, or where i knows that p and j does not know whether p, is discarded. That is, implicitly, the only information that becomes relevant for the decisions of the agents is the information on which they already agree.

Theorem 2 in particular, highlights an awkwardness of the agreement results: If we require the weaker version of the Sure-Thing Principle to hold ($DSTP$), then whether or not the agreement theorem holds depends on the richness of the language. In other words, it depends on the way in which the environment is described! (That is, if the language were not rich enough in every component, then agreement would not necessarily follow).

Note that both theorems rely on the restriction that only Ψ_0^r with $r = 0$ be considered (that is, $\Psi_0^0 = P^*$). This means that decisions cannot be based on formulas involving nested modal operators; that is, on *interactive* information.[14] This is analogous to the assumption made in Aumann and Hart (2006) that decisions be *substantive*: "Only knowledge of elementary facts matters, not knowledge about knowledge (i.e. interactive knowledge)". Formally, the reason for the restriction is that Lemma 5 does not hold for Ψ_0^r if $r > 0$. This is because the truth of formulas of a modal depth one or greater is fully determined by the accessibility relations of all agents. The trouble is that by moving from the model $\mathcal{M}$ to a merged model $\mathcal{M}(I_i(\omega), I_i(\omega'))$, we are modifying the accessibility relations, and there is no guarantee that truth of higher depth formulas will remain unchanged, so kens in the merged model may be incomparable (via $\succsim$) with the kens in the original model.
The coin in a box example, reproduced in Figure 6, provides a counter-example to Lemma 5 when $r > 0$: Recall that h is true at ω while t is true at ω'. One can verify that for all $\omega \in \Omega$, $\mathcal{M}, \omega \models \Box_i \Box_j (\Box_i h \vee \Box_i t)$, and $\mathcal{M}(I_i(\omega), I_i(\omega')), \omega \models \hat{\Box}_i \Box_j (\Box_i h \vee \Box_i t)$. Therefore, whatever ken i might have in the merged model, it is incomparable (via $\succsim$) with her kens in the original model, so the ken in the merged model is not the infimum of i's kens.
Conceptually, the restriction $r = 0$ avoids the incoherence presented in Bacharach (1985). Firstly, the infimum of kens is a ken, whereas the union of information cells is not an information cell. Secondly, we have just shown that in the original model, i knows that j knows that i knows which side is facing up, in each of i's kens. Suppose that i takes the same decision over both of those kens. Bacharach's

[14]Note: Tarbush (2011) finds that a distinguishing feature of the agreement result in Samet (2010) is that it holds for all $r \geq 0$.

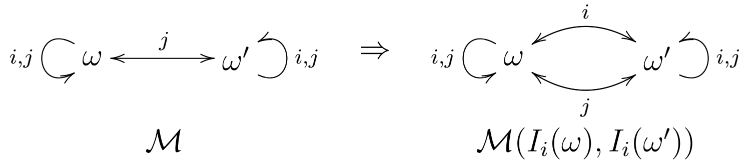

Figure 6: Merge with $r > 0$

Sure-Thing Principle would require i to take the same decision over her ken in the model where her information cells are merged. However, in the merged model, it is not the case that i knows that j knows that i knows which side is facing up. That is, we do not have a clear case of i being "more ignorant". In fact, i's information over the proposition "j knows that i knows which side is facing up" is the *opposite* in the merged model to what it was in the original model! With the restriction $r = 0$ however, Lemma 5 guarantees that the move to "more ignorance" is respected in merged models.

Of course, the upshot of this is that, given our assumptions, agents *could* agree to disagree if their decision functions are allowed to depend on interactive information. We can illustrate this point as follows: In the non-interactive case, suppose agent i performs action x when he knows heads is up, and when he knows tails is up. By the Sure-Thing Principle, i must also perform x when he does not know which side is up. Similarly, j must also perform y when she does not know which side is up. But this information is the same for both agents, so we must have $x = y$. Contrast this with the case where interactive information does matter for decisions. For example, suppose the agents decide to listen to the other agent's claim about which side is up only if they know that the other knows which side is up. In the state in which the coin is heads up, i knows that j does not know which side is facing up, and decides not to listen. Similarly in the state in which the coin is tails up. By the Sure-Thing Principle, at the infimum, i also knows that j does not know which side is up, and therefore also decides not to listen. However, at all her possible kens, and therefore also over the infimum (syntactic), j knows that i knows which side is up, so j decides to listen. No contradiction arises here, so the agents can agree to disagree!

The above demonstrates how our syntactic approach to explicitly model information allows us to avoid the known incoherence, because we can single out only the non-interactive information of kens. This cannot formally be done within Bacharach's framework. Note furthermore that our results so far are not generalisations of Bacharach (1985), but rather analogous results with sounder foundations.

It should be noted that the condition $r = 0$ can be seen as restrictive. Consider a simple game where i and j are required to write what side of the coin is facing up on a piece of paper. If they get it right, they earn a prize. Now, if j's decision can depend on the fact that she knows that i knows what side is facing up, j can write: "The side that is facing up is the one that i says is facing up". However, if this interactive information must be ignored, this strategy is, as far as j is concerned, just as good as simply guessing, since she might as well not know that i knows which side is facing up.

Both Theorems 1 and 2, and indeed all the results in this paper, are stated with *global* assumptions, however, *local* assumptions would have sufficed. Indeed, our framework is such that decision functions are global, in the sense that they somehow exist "outside" any given model. As a result, all our other assumptions involving decision functions, like the Sure-Thing Principle and like-mindedness, are also global. However, all the results in this paper would also hold in the following modified version of our framework: We could let a "decision rule", Δ_i, be a set of conditional formulas of the form $\nu_i \to d_i^x$, with $\nu_i \in V_i$, $x \in \mathcal{A}$, and the requirement that d_i^x is unique for each ν_i. So, a decision rule is a set of formulas specifying for each ken ν_i: "If agent i has ken ν_i, then agent i performs action x". This is a relaxation of our original setup because now, in principle, different decision rules can be true at different states in a model (That is, we *could* have $\omega \models \Delta_i$ and $\omega' \models \Delta_i'$ where $\Delta_i \neq \Delta_i'$). Nevertheless, if we impose the following assumptions *within a given component*, then we can recover all our results *locally* - within that component: (i) The decision rules of both agents must be invariant across all the states within a given component - which amounts to requiring the decision rules to be commonly known, and (ii) we can require syntactic analogues of the Sure-Thing Principles and like-mindedness assumptions to hold. Within any component in which these conditions hold, the agents cannot agree to disagree. Such results would be *local* in the sense that there may be components within the state space in which the assumptions hold, and others where they may not. That is, there may exist "pockets" of agreement and "pockets" of disagreement. On the other hand, we could recover the *global* results by requiring the assumptions to hold in *every* component in the state space.

Finally, to conclude this section, we can show that our framework in $S5$ can be mapped directly into that of Bacharach (1985), and to that of Aumann and Hart (2006) (see Appendix B). However, the framework developed here has some advantages: (i) The use of epistemic logic allows for a very transparent account of the conditions on the modal depth of formulas, (ii) the ordering $\succsim$ on kens gives a clear definition of informativeness, and hence of $\inf\{\nu_i, \mu_i\}$, (iii) explicitly modelling the accessibility relations between states allows us to easily consider extensions in a non-partitional state space - in the next section -, and finally (iv) our approach allows us to unify and compare the results of the literature in one methodological approach.

5 Results in $KD45$

We have so far derived all our results within partitional models - that is, in $S5$ frames. However, the $\Box$ operator has very strong properties in such frames. In particular, one cannot "know" what is false. There may be a sense in which the agreement theorems are driven by this property. Indeed, the information on which agents can base their decisions is restricted - to only correct information - and this de facto "coordination" on correct information may be a crucial factor in driving the results. That is, since their information must already "agree", maybe they are also forced into agreement over decisions.
Furthermore, there is nothing inherent to the notion of rationality that requires rational agents to base their decisions only on correct information. For this reason, we will now consider models in which agents can base their decisions on potentially *false* information.
We therefore consider $KD45$ frames, in which the accessibility relations are transitive, Euclidean and serial. The following formulas are valid in $KD45$ frames:
(i) Distribution: $\Box_i(\psi \to \phi) \to (\Box_i\psi \to \Box_i\phi)$.
(ii) Consistency: $\Box_i\psi \to \neg\Box_i\neg\psi$.
(iii) Positive introspection: $\Box_i\psi \to \Box_i\Box_i\psi$.
(iv) Negative introspection: $\neg\Box_i\psi \to \Box_i\neg\Box_i\psi$.
The converse also holds: Namely, if we require (i) - (iv) to be validities, then the frame must be $KD45$.
These validities describe the properties that we would require $\Box$ to satisfy in order to be interpreted as a *belief*, rather than a knowledge, operator (Again, see Hintikka (1962) for philosophical underpinnings). Essentially, the only difference is that unlike knowledge, belief is not *infallible*: By dropping reflexivity, it is possible to have $\Box_i p \wedge \neg p$ in a $KD45$ frame - that is, agents are allowed to believe what is false, and thus to base decision on *false* information. Note however, that agents are at least required to have consistent beliefs.
One can verify that all $S5$ frames are also $KD45$ frames, but the converse is not true. In fact, $S5 = KD45 +$ reflexivity.

We can provide a description of the links between states in a $KD45$ frame: Some sets of states within Ω are "completely connected", in the sense that the accessibility relation over states within such sets in an equivalence relation, so these sets have the same properties as information cells in $S5$; and, for each one of these

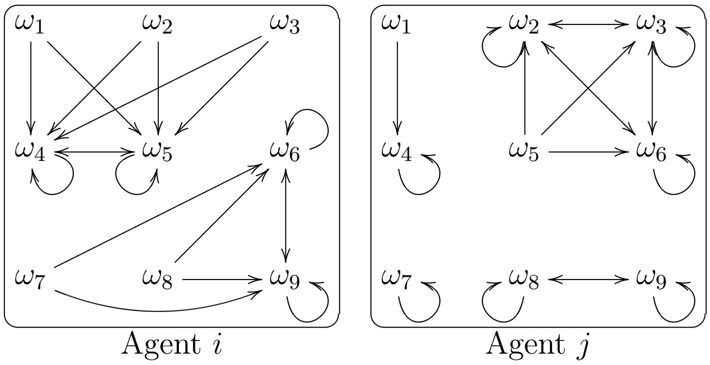

Figure 7: State space in $KD45$

completely connected sets there exists a (possibly empty) set of "associated" states that have arrows pointing from them to every state in the completely connected set, but with no arrow (by the same agent) pointing towards them. The set of all completely connected sets and their set of associated states exhaust the state space.

Formally, let $S_i(\omega) = \{\omega' \in \Omega | \omega E_i \omega'\}$, where E_i is an equivalence relation. We call this set of completely connected states the *information sink* of state ω for player i. The set S_i do not necessarily partition the state space, hence we have a *non-partitional* model. Note, that this way of defining the sink guarantees that if $S_i(\omega) \neq \emptyset$ then $\omega \in S_i(\omega)$. Furthermore, we define ω's *set of associated states* as $A_i(\omega) = \{\omega'' \in \Omega | \forall \omega''' \in S_i(\omega), \omega'' F_i \omega'''\}$, where F_i is now a simple arrow. So, note that now, for any agent i, we have that $R_i = E_i \cup F_i$. Finally, we can define $J_i(\omega) = S_i(\omega) \cup A_i(\omega)$, and note that $\mathcal{J}_i = \{J_i(\omega) | \omega \in \Omega\}$ exhausts the entire state space.

Proposition 1. *The above is a complete characterisation of the $KD45$ state space.*

We provide a schematic representation of a $KD45$ model in Figure 7. For example, i's information sink at state ω_4 is the set $S_i(\omega_4) = \{\omega_4, \omega_5\}$, and the set of associated states is $A_i(\omega_4) = \{\omega_1, \omega_2, \omega_3\}$. Furthermore, note for example that the component of state ω_1 is the set $\Omega_{\{i,j\}}(\omega_1) = \Omega \backslash \{\omega_1, \omega_7\}$, so it is now possible that $\omega \notin \Omega_G(\omega)$.

We can see how having false beliefs is possible in such frames: For example, let $\omega_1 \models \neg p$ and $\omega_4 \models p$. Then, $\omega_1 \models \Box_j p \wedge \neg p$. This also shows how it is only in the sets A_i that the agent could potentially hold false beliefs.

We will need to add the following assumptions to derive the main results:

Assumption 3 (Heterogeneity)**.** *For all $i \in G$, if for all $\omega' \in \Omega_G(\omega)$, we have $\omega \models \nu_i$, $\omega' \models \mu_i$ and $\nu_i = \mu_i$, then for all $\omega' \in \Omega_G(\omega)$, with $\omega' \models \nu'_i \wedge \nu'_j$ we have $\nu'_i \sim \nu'_j$.*

This assumption is termed "heterogeneity" because it is equivalent to the statement: In any component, either the agents have the same information, or at least one of the agents has different information at different states within the component.

The following lemmas are generalisations of the ones found for $S5$.

Lemma 7. *$\forall i \in G$, $\bigcup_{\omega' \in \Omega_G(\omega)} S_i(\omega') \subseteq \Omega_G(\omega) \subseteq \bigcup_{\omega' \in \Omega_G(\omega)} J_i(\omega')$.*

Lemma 8. *If for some $\omega' \in J_i(\omega)$, $\omega' \models \nu_i$, then for all $\omega'' \in J_i(\omega)$, $\omega'' \models \nu_i$.*

Definition 13 (Sink merge)**.** Consider a model in $KD45$, $\mathcal{M} = \langle \Omega, R_{i \in N}, V \rangle$. Let $J_i(\omega) = S_i(\omega) \cup A_i(\omega)$ and $J_i(\omega') = S_i(\omega') \cup A_i(\omega')$. Create a new model $\mathcal{M}(J_i(\omega), J_i(\omega')) = \langle \Omega', R'_{i \in N}, V' \rangle$ where,

$$
\begin{aligned}
&\Omega' = \Omega \\
&R'_i = E'_i \cup F'_i \\
&E'_i = E''_i \cup E_i|_{\Omega \backslash S_i(\omega) \cup S_i(\omega')} \\
&\quad \text{where } E''_i = \{(\omega'', \omega''') \in \Omega \times \Omega | \omega'', \omega''' \in S_i(\omega) \cup S_i(\omega')\} \\
&\quad \text{and } E_i|_{\Omega \backslash S_i(\omega) \cup S_i(\omega')} = \{(\omega'', \omega''') \in E_i | \omega'', \omega''' \in \Omega \backslash S_i(\omega) \cup S_i(\omega')\} \\
&F'_i = F''_i \cup F_i|_{\Omega \backslash A_i(\omega) \cup A_i(\omega')} \\
&\quad \text{where } F''_i = \{(\omega'', \omega''') \in \Omega \times \Omega | \omega'' \in A_i(\omega) \cup A_i(\omega'), \omega''' \in S_i(\omega) \cup S_i(\omega')\} \\
&\quad \text{and } F_i|_{\Omega \backslash A_i(\omega) \cup A_i(\omega')} = \{(\omega'', \omega''') \in F_i | \omega'', \omega''' \in \Omega \backslash A_i(\omega) \cup A_i(\omega')\} \\
&R'_j = R_j \text{ for all } j \neq i \\
&V' = V
\end{aligned}
$$

One can verify that the model $\mathcal{M}(J_i(\omega), J_i(\omega'))$ is itself a model in $KD45$, but where $J_i(\omega)$ and $J_i(\omega')$ are *merged* to form a new information sink with a set of associated states, yet leaving the rest of the original model, $\mathcal{M}$, unchanged.

For sake of illustration, let us return to the example given in Figure 7. Let the model represented be $\mathcal{M}$. We can, for example, create the "merged" model, $\mathcal{M}(J_j(\omega_1), J_j(\omega_8))$, in which i's accessibility relation is unchanged, but j now has a sink $S_i(\omega_8) = \{\omega_4, \omega_8, \omega_9\}$ and a set of associated states $A_j(\omega_8) = \{\omega_1\}$. That is, there is an equivalence relation over the states in $S_i(\omega_8)$, and there are arrows from ω_1 pointing to each of the states in $S_i(\omega_8)$; and, the relations between the rest of the states remain as they were in the original model for j.

Lemma 9. *Consider Ψ_0^r with $r = 0$.*
If $\mathcal{M}, \omega \models \nu_i$ and $\mathcal{M}, \omega' \models \mu_i$, then for all $\omega'' \in J_i(\omega) \cup J_i(\omega')$, $\mathcal{M}(J_i(\omega), J_i(\omega')), \omega'' \models \inf\{\nu_i, \mu_i\}$.

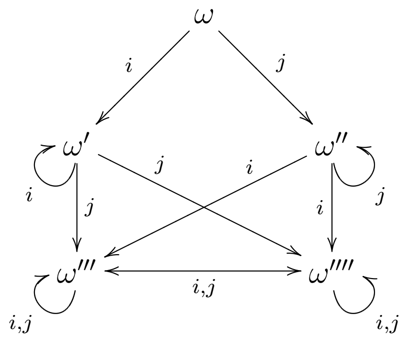

Figure 8: Trading example

Lemma 10. *Consider Ψ_0^r with $r = 0$, and heterogeneity holds. Let $G = \{i, j\}$. For any $\Omega_G(\omega)$, $\inf\{\nu_i | \omega' \models \nu_i \;\&\; \omega' \in \Omega_G(\omega)\} \sim \inf\{\nu_j | \omega' \models \nu_j \;\&\; \omega' \in \Omega_G(\omega)\}$.*

We can now state our generalised agreement results in $KD45$.

Theorem 3. *Consider Ψ_0^r with $r = 0$, suppose $NDSTP$ holds, the agents are like-minded, heterogeneity holds, and the system is $KD45$. Let $G = \{i, j\} \subseteq N$. Then, $\models C_G(d_i^x \wedge d_j^y) \to (x = y)$.*

Theorem 4. *Consider Ψ_0^r with $r = 0$, suppose $DSTP$ holds, the agents are like-minded, heterogeneity holds, the language is rich in every component, and the system is $KD45$. Let $G = \{i, j\} \subseteq N$. Then, $\models C_G(d_i^x \wedge d_j^y) \to (x = y)$.*

5.1 Discussion

We can provide an example of a situation that cannot be dealt with within an $S5$ system, but for which our agreement results hold. It should be noted that due to veracity in $S5$, the following must always be true at any state: $\Box_i p \to \neg\hat{\Box}_j p$. So, it is impossible for agents to completely be at odds about any fact. That is, $\Box_i p \wedge \hat{\Box}_j p$ cannot be true at any state in $S5$. However, it is possible in $KD45$. Consider the $KD45$ model represented in Figure 8, where p is true at ω' and ω''' while $\neg p$ is true at ω'' and ω''''. Note that we have $\omega \models \Box_i p \wedge \hat{\Box}_j p$, and $\omega \models \Box_i \dot{\Box}_j p \wedge \Box_j \dot{\Box}_i p$. We can imagine this as representing a situation where a trader i believes that p, "the price share will go up", while trader j believes that the price share will not go up, and each trader thinks the other one has no particular information about the share pointing in either direction. Our theorem applies here, so the agents cannot agree to disagree: If it becomes common knowledge among them what each will do given his/her information, they must do the same thing.
This can be seen as a strong version of "no-trade" results (see Milgrom and Stokey

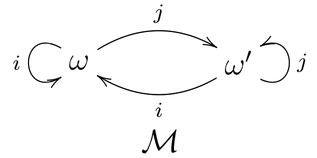

Figure 9: Example of disagreement when heterogeneity does not hold

(1982)) because, (i) in contrast with the standard results, the agents are not assumed to be maximising expected utilities. In fact, the do not even have subjective probabilities, so are behaving in a situation of complete uncertainty about the relevant outcomes; and, (ii) the standard results can only model situations in which one trader may have some correct information about a share while others are uncertain about the potential share price, whereas in our case, trader can have outright *contradictory* information about the share price.

Heterogeneity requires that at least some agent has some variation in her information in the set $\Omega^S(\omega)$ (or that the agents' information be the same). Note that the assumption is *always* satisfied in an $S5$ model:

Proposition 2. *Heterogeneity holds in any $S5$ model.*

We construct a model in which heterogeneity fails (where both agents have no variation in their information), and show that the agents can agree to disagree. Consider the model represented in Figure 9 and suppose that $\omega \models p$, and $\omega' \models \neg p$. In this model, at *every* state, i believes that p is the case, whereas j believes that $\neg p$ is the case. So we can let i's decision at every state be x while letting j's be y.
An interpretation of this example is that the agents are *systematically biased in the way they acquire new information.* For example, suppose Alice and Bob have a decision function whereby they leave the country if they believe that taxes will rise after the election, and stay if they believe that taxes stay the same. Now, suppose that in state ω, Alice consults one expert, and in ω', she consults another, but both experts tell her that taxes will rise; so Alice would always come to believe that taxes will rise, so she decides to leave the country. On the other hand, in state ω, Bob consults one expert, and another in ω', but in both cases, he is told that taxes will not rise, so he always comes to believe that they will not rise, and thus decides to stay.
Now, even though it is the case that Bob knows that Alice will leave the country, and he knows that she has the same decision function as he does, he cannot "update" his decision when he is given the information about her decision, because there is simply no other information that he deems it is possible to acquire.

5.1.1 A taxonomy of conditions

In this section, we contrast and compare various conditions that have been used in the literature in relation to agreement theorems. This will allow us to place heterogeneity in relation to more familiar conditions, and also to provide a discussion of the richness assumption in $KD45$.
Each condition will be given semantically (a), and essentially syntactically (b).[15]

Definition 14 (Condition 1)**.** Condition (1.a): For all $\omega \in \Omega$ and $i, j \in G$, there exists an $\omega' \in \Omega_G(\omega)$ such that $S_i(\omega') = S_j(\omega')$. Condition (1.b): For all $\omega \in \Omega$ and $i, j \in G$, there exists an $\omega' \in \Omega_G(\omega)$ such that $\omega' \models \nu_i \wedge \nu_j$ for some ν_i, ν_j, such that $\nu_i \sim \nu_j$.

Definition 15 (Condition 2)**.** Condition (2.a): For all $\omega \in \Omega$ and $i, j \in G$, there exists an $\omega' \in \Omega_G(\omega)$ such that $S_i(\omega') \subseteq S_j(\omega')$. Condition (2.b): For all $\omega \in \Omega$ and $i, j \in G$, there exists an $\omega' \in \Omega_G(\omega)$ such that $\omega' \models \nu_i \wedge \nu_j$ for some ν_i, ν_j, such that $\nu_i \succsim \nu_j$, and there exists an $\omega'' \in \Omega_G(\omega)$ such that $\omega'' \models \mu_i \wedge \mu_j$ for some μ_i, μ_j, such that $\mu_j \succsim \mu_i$.

Condition 1 states that in any component, there must exist a state in which both agents have the same sink. Syntactically: It must not be commonly believed, among the agents, that they do not have the same information. This condition can be seen as a requirement that there be a "grain of agreement" among the agents - in the sense that there must exist some state within each component in which the agents have the same information.
Condition 2 states that in any component, a state must exist in which i's ken is more informative than j's, and a state must exist in which j's ken is more informative than i's.

Definition 16 (Condition 3)**.** Condition (3.a): For all $\omega \in \Omega$ and $i, j \in G$, there exists an $\omega' \in \Omega_G(\omega)$ such that $\cup_{\omega'' \in \Omega_G(\omega')} S_i(\omega'') = \cup_{\omega'' \in \Omega_G(\omega')} S_j(\omega'')$. Condition (3.b): For all $\omega \in \Omega$ and $i, j \in G$, there exists an $\omega' \in \Omega_G(\omega)$ such that for all $\omega'' \in \Omega_G(\omega')$, $\omega'' \models (\bigwedge_{n \in \{1,\ldots,m\}} \bigwedge_{i \in G} \Box_i \psi_n \rightarrow \psi_n)$.

Definition 17 (Condition 4)**.** Condition (4.a): For all $\omega \in \Omega$ and $i, j \in G$, there exists an $\omega' \in \Omega_G(\omega)$ such that $S_i(\omega') \subseteq \cup_{\omega'' \in \Omega_G(\omega)} S_j(\omega'')$. Condition (4.b): Heterogeneity.

Condition 3 and 4 are clearly weaker counterparts of conditions 1 and 2 respectively. Their direct interpretation is not obvious. However, their syntactic implications are interpretable: Condition (3.b) is what Bonanno and Nehring (1998) term

[15] The (b) conditions are syntactic in the sense that they could be stated purely in our syntax, but they are complicated formulas so stating them explicitly might obscure their meaning.

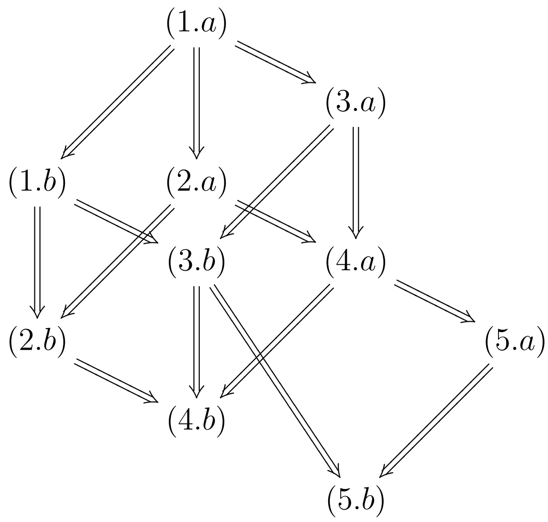

Figure 10: Taxonomy of conditions

quasi-coherence: "agents consider it jointly possible that they commonly believe that what they believe is true". They show that it is equivalent to the impossibility of unbounded gains from betting (with moderately risk averse agents), which gives it normative appeal. Condition (4.b) is simply heterogeneity: "if agents' beliefs (kens) are commonly believed, then their beliefs (kens) must be the same".

Definition 18 (Condition 5)**.** Condition (5.a): For all $\omega \in \Omega$ and $i, j \in G$, there exists an $\omega' \in \Omega_G(\omega)$ such that $S_i(\omega') \cap S_j(\omega') \neq \emptyset$. Condition (5.b): For all $\omega \in \Omega$ and $i, j \in G$, there exists an $\omega' \in \Omega_G(\omega)$ such that $\omega' \models (\bigwedge_{n\in\{1,\ldots,m\}} \neg(\Box_i\psi_n \wedge \hat{\Box}_j\psi_n))$.

This condition states that it cannot be the case that all the information sinks are disjoint across agents. (Syntactically: The agents must jointly consider it possible that they are not completely at odds about *every* "fact" - i.e. one believes that it is the case while the other believes that it is not the case). Obviously, imposing such a condition would rule out the scenario represented in Figure 9.

Proposition 3. *The arrows ($\Rightarrow$) represent logical implication in Figure 10.*

Notably, it is shown that quasi-coherence implies heterogeneity. However, the converse does not hold, as shown in Figure 11. Suppose that $\omega \models p$, $\omega' \models p$ and $\omega'' \models \neg p$. Clearly, there is a state, namely ω'' in $\Omega_G(\omega)$ at which $(\omega'', \omega'') \notin R_i$ so quasi-coherence fails. However, at $\omega \models \Box_j p$ whereas $\omega'' \models \dot{\Box}_j p$ so heterogeneity holds.

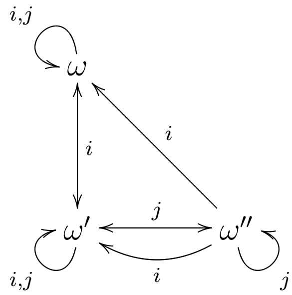

Figure 11: Quasi-coherence fails but heterogeneity holds

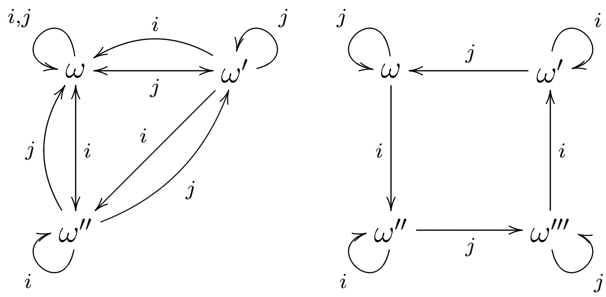

Figure 12: Condition (5.b) holds and heterogeneity fails (left); Heterogeneity holds and (5.b) fails (right)

On the other hand, there is no implication in either direction between heterogeneity and condition (5.b). In the model on the left in Figure 12, let $\omega \models p$, $\omega' \models \neg p$ and $\omega'' \models p$. It is easy to see that condition (5.b) holds since the sinks intersect at ω. However, $\Box_i p$ holds at every state while $\dot{\Box}_j p$ holds at every state, so heterogeneity fails. However, in the model on the right, let $\omega \models p \wedge q$, $\omega' \models \neg q \wedge \neg p$, $\omega'' \models q \wedge \neg p$ and $\omega''' \models p \wedge \neg q$. One can verify that at every state, there exists a proposition ψ such that $\Box_i \psi \wedge \hat{\Box}_j \psi$, so (5.b) fails. However, there is variation in the agents' kens across states, so heterogeneity holds.

Finally, it is important to note that (1.a) is in fact strictly stronger than (1.b): If ω' is such that $S_i(\omega') = S_j(\omega')$, then at that state, not only do the agent have the same information regarding all the atomic propositions, it is also common knowledge among them that they have the same information about the atomic propositions. On the other hand, (1.b) does not have this strong implication. More generally, agents having the same information sink (or cell) is not *equivalent* to them having the same information!

6 Agreement without the Sure-Thing Principle

In this section, we present a theorem that does not restrict the decision functions to satisfy the Sure-Thing Principle; so the following result applies even when the principle is violated behaviourally (which is common, as surveyed in Shafir (1994)).

Theorem 5. *Suppose agents are like-minded and condition (1.b) hold, and the system is* $S5$ *or* $KD45$*. Let* $G = \{i, j\} \subseteq N$*. Then,* $\models C_G(d_i^x \wedge d_j^y) \rightarrow (x = y)$*.*

This result has several striking features: Firstly, it does not assume anything about the decision functions, other than the requirement of like-mindedness. Therefore, this theorem applies to all decision functions, including the ones that do not satisfy the Sure-Thing Principle. Secondly, it makes no requirement on the richness of the language. Thirdly, it does not require any restriction on r, the modal depth of formulas. This means that decisions can be based on interactive information. That is, formulas of the form: i believes that j believes that p.
Furthermore, recall that in our discussion of the results in $S5$, we mentioned that our "global" decision functions, D_i, could have been replaced by "decision rules", Δ_i, on which local conditions could have been imposed. One requirement that would have been needed for all the results in this paper is that decision rules are invariant across all the states within a component. This invariance requirement essentially means that the decision rules of the agents must be commonly known among them. However, for this result, the invariance requirement would not be needed. That is, how the agents "reason" - i.e. their decision rules - need not be common knowledge among them!
Of course, the main driver of the result is condition (1.b) - the "grain of agreement" condition - which states that it must not be commonly believed among the agents that they do not have the same information.

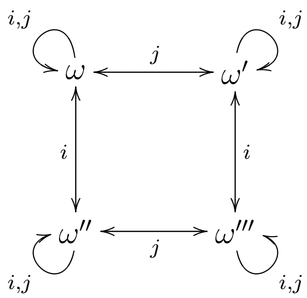

Figure 13: Application of Theorem 5 in $S5$

The model with the traders represented in Figure 8 is an example where Theorem 5 applies in $KD45$. The model represented in Figure 13 provides an example in $S5$. Let p be false at ω' and true everywhere else. At ω, the agents have different information, however, they do have the same information at some state within the component, namely at ω''. Therefore, by Theorem 5, they cannot agree to disagree.[16]

[16]Note that in this model, the agents do not have the same information cells, however condition (1.b) does hold.

Appendix A

Proof of Lemma 1 (i) Consider an arbitrary $i \in N$ and $\omega \in \Omega$, and suppose that $\omega \models \psi$, for some formula $\psi \in \Psi_0^r$. It must be the case that either (i.a) $\forall \omega' \in \Omega$, if $\omega R_i \omega'$ then $\omega' \models \psi$, or (i.b) $\forall \omega' \in \Omega$, if $\omega R_i \omega'$ then $\omega' \models \neg\psi$, or (i.c) $\exists \omega', \omega'' \in \Omega$, such that $\omega R_i \omega'$ and $\omega R_i \omega''$, and $\omega' \models \psi$ and $\omega'' \models \neg\psi$ (i.e. neither (i.a) nor (i.b)). If (i.a) is the case, then $\omega \models \Box_i \psi$. If (i.b) is the case, then $\omega \models \hat{\Box}_i \psi$, and finally, if (i.c) is the case, then $\omega \models \dot{\Box}_i \psi$. Therefore, in all cases, the operator over ψ belongs to the set O_i, and since this holds for any $\psi \in \Psi_0^r$, it holds for each entry of a ken. Furthermore, $\models$ can only generate consistent lists of formulas, so kens cannot be inconsistent. This implies that a ken must exist that belongs to V_i.
(ii) Consider an arbitrary $i \in N$ and $\omega \in \Omega$. Let $\nu_i, \mu_i \in V_i$, and consider the n^{th} entry of each ken such that $\nu_i^n \psi_n \neq \mu_i^n \psi_n$. Case (ii.a): Suppose $\omega \models \nu_i^n \psi_n = \Box_i \psi_n$. So, $\forall \omega' \in \Omega$, if $\omega R_i \omega'$, then $\omega' \models \psi_n$. By definition, this rules out the possibility that also, $\omega \models \hat{\Box}_i \psi_n$, or $\omega \models \dot{\Box}_i \psi_n$. For cases (ii.b), $\omega \models \nu_i^n \psi_n = \hat{\Box}_i \psi_n$, and (ii.c), $\omega \models \nu_i^n \psi_n = \dot{\Box}_i \psi_n$, proceed analogously to (ii.a).

Proof of Lemma 2 For ease of notation, let $\inf\{\nu_i, \mu_i\} = \eta_i$.
(a) Suppose $\nu_i^n \psi_n = \mu_i^n \psi_n = \Box_i \psi_n$. Then, if $\nu_i \succsim \eta_i$ and $\mu_i \succsim \eta_i$, it must be the case that $\eta_i^n \psi_n = \Box_i \psi_n$ or $\eta_i^n \psi_n = \dot{\Box}_i \psi_n$. However, if the latter, then η_i would not be maximal in the set $\{\eta_i \in V_i | \nu_i \succsim \eta_i \text{ and } \mu_i \succsim \eta_i\}$. Therefore, $\eta_i^n \psi_n = \Box_i \psi_n$. Conversely, suppose $\eta_i^n \psi_n = \Box_i \psi_n$. Furthermore, suppose, without loss of generality that $\mu_i^n \psi_n = \hat{\Box}_i \psi_n$ or $\mu_i^n \psi_n = \dot{\Box}_i \psi_n$. In the former case, η_i and μ_i would not be comparable, and in the latter case, η_i would be more informative than μ_i on that entry. Therefore, in either case, η_i would not belong to the set $\{\eta_i \in V_i | \nu_i \succsim \eta_i \text{ and } \mu_i \succsim \eta_i\}$. Therefore, $\nu_i^n \psi_n = \mu_i^n \psi_n = \Box_i \psi_n$. Proving cases (b), $\eta_i^n \psi_n = \hat{\Box}_i \psi_n$ iff ($\nu_i^n \psi_n = \mu_i^n \psi_n = \hat{\Box}_i \psi_n$) and (c), $\eta_i^n \psi_n = \dot{\Box}_i \psi_n$ iff ($\nu_i^n \psi_n \neq \mu_i^n \psi_n$ or $\nu_i^n \psi_n = \mu_i^n \psi_n = \dot{\Box}_i \psi_n$) can be done analogously to case (a).
Finally, suppose $\models \eta_i \leftrightarrow (p \wedge \neg p)$. Then, there exist n and n' such that $\eta_i^n \psi_n \leftrightarrow \neg \eta_i^{n'} \psi_{n'}$. But η_i^n is essentially generated by the conjunction of ν_i^n and μ_i^n. So, we have $(\nu_i^n \psi_n \wedge \mu_i^n \psi_n) \leftrightarrow \neg(\nu_i^{n'} \psi_{n'} \wedge \mu_i^{n'} \psi_{n'})$. But this implies that $\nu_i^n \psi_n \leftrightarrow \neg \nu_i^{n'} \psi_{n'}$ or $\mu_i^n \psi_n \leftrightarrow \neg \mu_i^{n'} \psi_{n'}$. That is, η_i is not in V_i if ν_i or μ_i are not in V_i. Therefore, $\eta_i \in V_i$.

Proof of Lemma 3 Suppose $\omega'' \in \bigcup_{\omega' \in \Omega_G(\omega)} I_i(\omega')$. So, $\omega'' \in I_i(\omega')$ for some $\omega' \in \Omega_G(\omega)$. But, $\omega' R_i \omega''$, and there exists a sequence of R_i ($i \in G$) steps such that ω' is reachable from ω. Therefore, there exists a sequence, one step longer, such that ω'' is reachable from ω. So, $\omega'' \in \Omega_G(\omega)$. (And, note that $I_i(\omega'') \subseteq \Omega_G(\omega)$).
Suppose $\omega'' \in \Omega_G(\omega)$. Reflexivity guarantees that $\omega'' \in I_i(\omega'')$. So, for some $\omega^* \in \Omega_G(\omega)$, $\omega'' \in I_i(\omega^*)$, so $\omega'' \in \bigcup_{\omega' \in \Omega_G(\omega)} I_i(\omega')$.

Proof of Lemma 4 Suppose $\omega' \models \nu_i$ for some $\omega' \in I_i(\omega)$. Consider the n^{th}

entry of the ken, namely, $\nu_i^n \psi_n$.
(a) Suppose $\omega' \models \nu_i^n \psi_n = \Box_i \psi_n$. Then, for all $\omega'' \in \Omega$, $\omega' R_i \omega''$ implies $\omega'' \models \psi_n$. So, for all $\omega'' \in I_i(\omega')$, $\omega'' \models \psi_n$. But since R_i is an equivalence relation, and $\omega' \in I_i(\omega)$, it follows that $I_i(\omega') = I_i(\omega)$. So, for all $\omega'' \in I_i(\omega)$, $\omega'' \models \psi_n$, from which it follows that for all $\omega'' \in I_i(\omega)$, $\omega'' \models \Box_i \psi_n$.
Case (b), $\omega' \models \nu_i^n \psi_n = \hat{\Box}_i \psi_n$ and (c), $\omega' \models \nu_i^n \psi_n = \dot{\Box}_i \psi_n$ are analogous to case (a).

Proof of Lemma 5 Suppose that for all $\omega' \in I_i(\omega)$, $\mathcal{M}, \omega' \models \nu_i$ and for all $\omega'' \in I_i(\omega')$, $\mathcal{M}, \omega'' \models \mu_i$. Consider the n^{th} entry of each of these kens, which are only defined for formulas in Ψ_0^0.
Case (a): Suppose that $\nu_i^n p_n = \mu_i^n p_n = \Box_i p_n$, then for all $\omega''' \in I_i(\omega) \cup I_i(\omega')$, $\omega''' \models p_n$, and therefore, for all $\omega''' \in I_i(\omega) \cup I_i(\omega')$, $\mathcal{M}(I_i(\omega), I_i(\omega')), \omega''' \models \inf\{\nu_i, \mu_i\}^n p_n = \Box_i p_n$.
Case (b), $\nu_i^n p_n = \mu_i^n \psi_n = \hat{\Box}_i p_n$, and (c) ($\nu_i^n p_n \neq \mu_i^n p_n$ or $\nu_i^n p_n = \mu_i^n p_n = \dot{\Box}_i p_n$) are treated analogously to case (a).

Proof of Lemma 6 By Lemma 1, for each $\omega' \in \Omega$, there is a ken that holds at ω'. That is, $\omega' \models \nu_i$ for some $\nu_i \in V_i$. By Lemma 4, we have that for all $\omega'' \in I_i(\omega')$, $\omega'' \models \nu_i$. Now, consider the set of kens $\{\nu_i | \omega' \models \nu_i \ \& \ \omega' \in \bigcup_{\omega' \in \Omega_G(\omega)} I_i(\omega')\}$. By Lemma 5, it follows that for all $\omega'' \in \bigcup_{\omega' \in \Omega_G(\omega)} I_i(\omega')$, $\mathcal{M}(\{I_i(\omega') | \omega' \in \Omega_G(\omega)\}), \omega'' \models \inf\{\nu_i | \omega' \models \nu_i \ \& \ \omega' \in \bigcup_{\omega' \in \Omega_G(\omega)} I_i(\omega')\}$. By Lemma 3, for all $\omega'' \in \Omega_G(\omega)$, $\mathcal{M}(\{I_i(\omega') | \omega' \in \Omega_G(\omega)\}), \omega'' \models \inf\{\nu_i | \omega' \models \nu_i \ \& \ \omega' \in \Omega_G(\omega)\}$. So, in the model in which i's information cell is equal to $\Omega_G(\omega)$, leaving j's partition unchanged, $\inf\{\nu_i | \omega' \models \nu_i \ \& \ \omega' \in \Omega_G(\omega)\}$ holds at every state in $\Omega_G(\omega)$. Reasoning similarly for agent j, in the model in which j's information cell is equal to $\Omega_G(\omega)$, leaving i's partition unchanged, $\inf\{\nu_j | \omega' \models \nu_j \ \& \ \omega' \in \Omega_G(\omega)\}$ holds at every state in $\Omega_G(\omega)$. However, since $r = 0$, an agent i's ken only depends on i's accessibility relation (higher depth nested formulas are ignored). So, $\inf\{\nu_i | \omega' \models \nu_i \ \& \ \omega' \in \Omega_G(\omega)\}$ and $\inf\{\nu_j | \omega' \models \nu_j \ \& \ \omega' \in \Omega_G(\omega)\}$ hold at every state $\omega' \in \Omega_G(\omega)$ of a model $\mathcal{M}^*$ in which all the set $I_i(\omega')$ are "merged" *and* all the sets $I_j(\omega')$ are merged. But $\cup_{\omega' \in \Omega_G(\omega)} I_i(\omega') = \cup_{\omega' \in \Omega_G(\omega)} I_j(\omega') = \Omega_G(\omega)$. That is, the agents have the same information cell in $\mathcal{M}^*$. Trivially, it follows that $\inf\{\nu_i | \omega' \models \nu_i \ \& \ \omega' \in \Omega_G(\omega)\} \sim \inf\{\nu_j | \omega' \models \nu_j \ \& \ \omega' \in \Omega_G(\omega)\}$.

Proof of Theorem 1 Suppose $r = 0$, so $\Psi_0^0 = P^*$ and that $NDSTP$ holds, the agents are like-minded, and the system is $S5$. Arbitrarily choose $\omega \in \Omega$, and consider the set $\Omega_G(\omega)$. By Lemma 1 part (i), we have that at every state, $\omega \models \nu_i \wedge \nu_j$ for some ν_i and ν_j. Since the decision function is defined over those kens, we have that $\omega \models d_i^{D_i(\nu_i)} \wedge d_j^{D_j(\nu_j)}$. Now, suppose that $\omega \models C_G(d_i^x \wedge d_j^y)$. By definition, $\forall \omega'' \in \Omega_G(\omega)$, $\omega'' \models d_i^x \wedge d_j^y$. In particular, since $\omega \in \Omega_G(\omega)$, $\omega \models d_i^x \wedge d_j^y$.

Therefore, we have that $D_i(\nu_i) = x$ and $D_j(\nu_j) = y$ for all kens that are true at the states within the component . It remains to show that $x = y$.
By $NDSTP$ we obtain that $D_i(\inf\{\nu_i | \omega' \models \nu_i \ \& \ \omega' \in \Omega_G(\omega)\}) = x$.
By a similar argument, we have that that $D_i(\inf\{\nu_j | \omega' \models \nu_j \ \& \ \omega' \in \Omega_G(\omega)\}) = y$.
By Lemma 6, $\inf\{\nu_i | \omega' \models \nu_i \ \& \ \omega' \in \Omega_G(\omega)\} \sim \inf\{\nu_j | \omega' \models \nu_j \ \& \ \omega' \in \Omega_G(\omega)\}$. So, by like-mindedness, it follows that $x = y$.

Proof of Theorem 2 Repeat the proof of Theorem 1, replacing Assumption $NDSTP$ with $DSTP$ and the assumption that the language is rich in every component.

Proof of Proposition 1 Let "i-arrow" refer to an arrow of i's accessibility relation. Firstly, we can show that $R_i = E_i \cup F_i$. An arbitrary $\omega \in \Omega$ either has an i-arrow pointing to it or it does not. If it does not, by seriality, it points to another state. If it does, then there exists a state ω' that points to ω which itself points to some state ω'' by seriality. Transitivity implies that ω' points to ω'' and Euclideaness implies that ω'' points to ω. From here it is easy to prove that ω, ω' and ω'' are in an equivalence class.
Secondly, we show that if $J_i(\omega') \neq J_i(\omega'')$ then $J_i(\omega') \cap J_i(\omega'') = \emptyset$. Suppose $\omega \in J_i(\omega') \cap J_i(\omega'')$. If $\omega \in S_i(\omega') \cap S_i(\omega'')$ then $S_i(\omega')$ and $S_i(\omega'')$ are indistinguishable, and one can verify that $J_i(\omega') = J_i(\omega'')$. If $\omega \in S_i(\omega') \cap A_i(\omega'')$ then ω both does have and does not have an i-arrow pointing to it. Finally, if $\omega \in A_i(\omega') \cap A_i(\omega'')$ then by Euclideaness, ω' and ω'' are indistinguishable, and $J_i(\omega') = J_i(\omega'')$.
Thirdly, we can show that $\cup_{\omega \in \Omega} J_i(\omega) = \Omega$. Suppose $\omega' \in \cup_{\omega \in \Omega} J_i(\omega)$, then by the definitions of S_i and A_i, $\omega' \in \Omega$. On the other hand, suppose $\omega \in \Omega$. Then if there is an i-arrow pointing to ω, $\omega \in S_i(\omega) \subseteq J_i(\omega)$. If there is no i-arrow pointing to it, then by seriality, there is an ω' that ω points to, so $\omega \in A_i(\omega') \subseteq J_i(\omega')$. So, $\omega \in \cup_{\omega \in \Omega} J_i(\omega)$.

Proof of Lemma 7 Suppose $\omega'' \in \bigcup_{\omega' \in \Omega_G(\omega)} S_i(\omega')$. So, $\omega'' \in S_i(\omega')$ for some $\omega' \in \Omega_G(\omega)$. But, $\omega' E_i \omega''$, and there exists a sequence of R_i ($i \in G$) steps such that ω' is reachable from ω. Therefore, there exists a sequence, one step longer, such that ω'' is reachable from ω. So, $\omega'' \in \Omega_G(\omega)$.
Suppose $\omega'' \in \Omega_G(\omega)$. Either ω'' has an i-arrow pointing towards it, in which case $\omega'' \in S_i(\omega'')$. So, $\omega'' \in S_i(\omega'') \cup A_i(\omega'') = J_i(\omega'')$, or, ω'' has no i-arrow pointing towards it, in which case, by seriality, there exists some ω''' such that $\omega'' \in A_i(\omega''')$. Note that ω''' must be in $\Omega_G(\omega)$ since it is reachable from ω''. So, $\omega'' \in S_i(\omega''') \cup A_i(\omega''') = J_i(\omega''')$. In either case, for some $\omega^* \in \Omega_G(\omega)$, $\omega'' \in J_i(\omega^*)$, so $\omega'' \in \bigcup_{\omega' \in \Omega_G(\omega)} J_i(\omega')$.

Proof of Lemma 8 Suppose $\omega' \models \nu_i$ for some $\omega' \in J_i(\omega)$. Firstly, suppose $\omega' \in S_i(\omega)$, and consider the n^{th} entry of the ken, namely, $\nu_i^n \psi_n$.
(a) Suppose $\omega' \models \nu_i^n \psi_n = \Box_i \psi_n$. Then, for all $\omega'' \in \Omega$, $\omega' E_i \omega''$ implies $\omega'' \models \psi_n$. So, for all $\omega'' \in S_i(\omega')$, $\omega'' \models \psi_n$. But since E_i is an equivalence relation, and $\omega' \in S_i(\omega)$, it follows that $S_i(\omega') = S_i(\omega)$. So, for all $\omega'' \in S_i(\omega)$, $\omega'' \models \psi_n$, from which it follows that for all $\omega'' \in S_i(\omega)$, $\omega'' \models \Box_i \psi_n$. Also, each $\omega''' \in A_i(\omega)$ has an arrow pointing to each state in $S_i(\omega)$, so for all $\omega^* \in S_i(\omega)$, if $\omega''' F_i \omega^*$, $\omega^* \models \psi_n$. So, for all $\omega''' \in A_i(\omega)$, $\omega''' \models \Box_i \psi_n$. It follows that for all $\omega'' \in J_i(\omega)$, $\omega'' \models \Box_i \psi_n$.
Case (b), $\omega' \models \nu_i^n \psi_n = \hat{\Box}_i \psi_n$ and (c), $\omega' \models \nu_i^n \psi_n = \dot{\Box}_i \psi_n$ are analogous to case (a).
Now, suppose $\omega' \in A_i(\omega)$, and consider the n^{th} entry of the ken, namely, $\nu_i^n \psi_n$.
(d) Suppose $\omega' \models \nu_i^n \psi_n = \Box_i \psi_n$. Then, for all $\omega'' \in \Omega$, $\omega' F_i \omega''$ implies $\omega'' \models \psi_n$. So, for all $\omega'' \in S_i(\omega')$, $\omega'' \models \psi_n$. This implies that $\omega'' \models \Box_i \psi_n$ for all $\omega'' \in S_i(\omega)$, and $\omega''' \models \Box_i \psi_n$ for all other states $\omega''' \in A_i(\omega)$. It follows that for all $\omega'' \in J_i(\omega)$, $\omega'' \models \Box_i \psi_n$.
Case (e), $\omega' \models \nu_i^n \psi_n = \hat{\Box}_i \psi_n$ and (f), $\omega' \models \nu_i^n \psi_n = \dot{\Box}_i \psi_n$ are analogous to case (d).

Proof of Lemma 9 Suppose that for all $\omega \in J_i(\omega)$, $\mathcal{M}, \omega \models \nu_i$ and for all $\omega' \in J_i(\omega')$, $\mathcal{M}, \omega' \models \mu_i$. Consider the n^{th} entry of each of these kens, defined only for formulas in Ψ_0^0.
Case (a): Suppose that $\nu_i^n p_n = \mu_i^n p_n = \Box_i p_n$, then for all $\omega'' \in S_i(\omega) \cup S_i(\omega')$, $\omega'' \models p_n$, and therefore, following the proof of Lemma 8, for all $\omega'' \in J_i(\omega) \cup J_i(\omega')$, $\mathcal{M}(J_i(\omega), J_i(\omega')), \omega'' \models \inf\{\nu_i, \mu_i\}^n p_n = \Box_i p_n$.
Case (b), $\nu_i^n p_n = \mu_i^n \psi_n = \hat{\Box}_i p_n$, and (c) ($\nu_i^n p_n \neq \mu_i^n p_n$ or $\nu_i^n p_n = \mu_i^n p_n = \dot{\Box}_i p_n$) are treated analogously to case (a).

Proof of Lemma 10 By Lemma 1, for each $\omega' \in \Omega$, there is a ken that holds at ω'. That is, $\omega' \models \nu_i$ for some $\nu_i \in V_i$. By Lemma 8, we have that for all $\omega'' \in J_i(\omega')$, $\omega'' \models \nu_i$. Now, consider the set of kens $\{\nu_i | \omega' \models \nu_i \ \& \ \omega' \in \bigcup_{\omega' \in \Omega_G(\omega)} J_i(\omega')\}$. By Lemma 9, it follows that for all $\omega'' \in \bigcup_{\omega' \in \Omega_G(\omega)} J_i(\omega')$, $\mathcal{M}(\{J_i(\omega') | \omega' \in \Omega_G(\omega)\}), \omega'' \models \inf\{\nu_i | \omega' \models \nu_i \ \& \ \omega' \in \bigcup_{\omega' \in \Omega_G(\omega)} J_i(\omega')\}$.
By Lemma 7, since $\Omega_G(\omega) \subseteq \bigcup_{\omega' \in \Omega_G(\omega)} J_i(\omega')$, it follows that for all $\omega'' \in \Omega_G(\omega)$, we have that $\mathcal{M}(\{J_i(\omega') | \omega' \in \Omega_G(\omega)\}), \omega'' \models \inf\{\nu_i | \omega' \models \nu_i \ \& \ \omega' \in \bigcup_{\omega' \in \Omega_G(\omega)} J_i(\omega')\}$. Furthermore, the kens that hold in states $(\bigcup_{\omega' \in \Omega_G(\omega)} J_i(\omega')) \setminus \Omega_G(\omega)$ must be identical to the ones that hold at the states in $\Omega_G(\omega)$, because all the states in the former set must be associated states, and thus the information that holds at them must be the same as the information that holds true in their respective information sinks, which are contained in $\Omega_G(\omega)$. Therefore, $\inf\{\nu_i | \omega' \models \nu_i \ \& \ \omega' \in \bigcup_{\omega' \in \Omega_G(\omega)} J_i(\omega')\} = \inf\{\nu_i | \omega' \models \nu_i \ \& \ \omega' \in \Omega_G(\omega)\}$. It follows there-

fore, that for all $\omega'' \in \Omega_G(\omega)$, we have that $\mathcal{M}(\{J_i(\omega')|\omega' \in \Omega_G(\omega)\}), \omega'' \models \inf\{\nu_i|\omega' \models \nu_i \ \& \ \omega' \in \Omega_G(\omega)\}$. So, in the model in which i's information sink plus associated states is equal to $\Omega_G(\omega)$, leaving j's accessibility relation unchanged, $\inf\{\nu_i|\omega' \models \nu_i \ \& \ \omega' \in \Omega_G(\omega)\}$ holds at every state in $\Omega_G(\omega)$. Reasoning similarly for agent j, in the model in which j's information sink plus associated states is equal to $\Omega_G(\omega)$, leaving i's accessibility relation unchanged, $\inf\{\nu_j|\omega' \models \nu_j \ \& \ \omega' \in \Omega_G(\omega)\}$ holds at every state in $\Omega_G(\omega)$.[17]
Now, since $r = 0$, an agent i's ken only depends on i's accessibility relation (higher depth nested formulas are ignored). So, $\inf\{\nu_i|\omega' \models \nu_i \ \& \ \omega' \in \Omega_G(\omega)\}$ and $\inf\{\nu_j|\omega' \models \nu_j \ \& \ \omega' \in \Omega_G(\omega)\}$ hold at every state $\omega' \in \Omega_G(\omega)$ of a model $\mathcal{M}^*$ in which all the set $S_i(\omega')$ are "merged" *and* all the sets $S_j(\omega')$ are merged. Now, by heterogeneity, it follows that $\inf\{\nu_i|\omega' \models \nu_i \ \& \ \omega' \in \Omega_G(\omega)\} \sim \inf\{\nu_j|\omega' \models \nu_j \ \& \ \omega' \in \Omega_G(\omega)\}$.[18]

Proof of Theorem 3 Suppose we restrict ourselves to Ψ_0^0 and that $NDSTP$ holds, the agents are like-minded, heterogeneity holds, and the system is $KD45$. Arbitrarily choose $\omega \in \Omega$, and consider the set $\Omega_G(\omega)$. By Lemma 1 part (i), we have that at every state, $\omega \models \nu_i \wedge \nu_j$ for some ν_i and ν_j. Since the decision function is defined over those kens, we have that $\omega \models d_i^{D_i(\nu_i)} \wedge d_j^{D_j(\nu_j)}$. Now, suppose that $\omega \models C_G(d_i^x \wedge d_j^y)$. By definition, $\forall\omega'' \in \Omega_G(\omega)$, $\omega'' \models d_i^x \wedge d_j^y$. By Lemma 8, the kens are uniform across the sets J, so even if $\omega \notin \Omega_G(\omega)$ - which is possible in $KD45$ - the actions that are performed at ω must be the same as the action that is performed in the set J that ω is a member of, for each agent. Thus in particular, $\omega \models d_i^x \wedge d_j^y$. Therefore, for any kens ν_i, ν_j that are true at any state in the set $\Omega_G(\omega) \cup \{\omega\}$, it is the case that $D_i(\nu_i) = x$ and $D_j(\nu_j) = y$. It remains to show that $x = y$.
By $NDSTP$ we obtain that $D_i(\inf\{\nu_i|\omega' \models \nu_i \ \& \ \omega' \in \Omega_G(\omega)\}) = x$.
By a similar argument, we have that that $D_i(\inf\{\nu_j|\omega' \models \nu_j \ \& \ \omega' \in \Omega_G(\omega)\}) = y$.
By Lemma 10, $\inf\{\nu_i|\omega' \models \nu_i \ \& \ \omega' \in \Omega_G(\omega)\} \sim \inf\{\nu_j|\omega' \models \nu_j \ \& \ \omega' \in \Omega_G(\omega)\}$.
So, by like-mindedness, it follows that $x = y$.

Proof of Theorem 4 Repeat proof of Theorem 3, replacing $NDSTP$ with $DSTP$ and the assumption that the language is rich in every component.

Proof of Proposition 2 Suppose that for some $\nu_i \in V_i$, $\omega' \models \nu_i$ for every $\omega' \in \Omega_G(\omega)$. Consider the n^{th} entry of the ken.

[17] Note that the set $\Omega_G(\omega)$ does not change as a result of the sink merge operation: No state in $\Omega_G(\omega)$ becomes connected to a state outside the set, and states within the set can only gain connections, never lose any.

[18] We require heterogeneity since there is no guarantee that $\cup_{\omega' \in \Omega_G(\omega)} S_i(\omega') = \cup_{\omega' \in \Omega_G(\omega)} S_j(\omega')$, and an agent i's ken essentially depends only on the sets S_i.

Case (a): Suppose that $\forall \omega' \in \Omega_G(\omega)$, $\omega' \models \nu_i^n \psi_n = \Box_i \psi_n$. Then, for all $\omega'' \in I_i(\omega')$, $\omega'' \models \psi_n$. But, by Lemma 3, since $\bigcup_{\omega' \in \Omega_G(\omega)} I_i(\omega') = \Omega_G(\omega)$, it follows that for all $\omega' \in \Omega_G(\omega)$, $\omega' \models \psi_n$.
Furthermore, by Lemma 3, $\bigcup_{\omega' \in \Omega_G(\omega)} I_j(\omega') = \Omega_G(\omega)$. Therefore, no matter what information cell j might be in, ψ_n will be true at each state in that information cell. Therefore $\forall \omega' \in \Omega_G(\omega)$, $\omega' \models \Box_j \psi_n$. That is, the n^{th} entry of the kens carry the same information.
Case (b): $\forall \omega' \in \Omega_G(\omega)$, $\omega' \models \nu_i^n \psi_n = \hat{\Box}_i \psi_n$ is treated analogously to case (a).
Case (c): Suppose that $\forall \omega' \in \Omega_G(\omega)$, $\omega' \models \nu_i^n \psi_n = \dot{\Box}_i \psi_n$. Then, there exists ω'' and ω''' with $\omega' R_i \omega''$ and $\omega' R_i \omega'''$, such that $\omega'' \models \psi_n$ and $\omega''' \models \neg\psi_n$. It follows that there exists $\omega'', \omega''' \in \Omega_G(\omega)$ such that $\omega'' \models \psi_n$ and $\omega''' \models \neg\psi_n$. Now, suppose that for all $\omega' \in \Omega_G(\omega)$, $\omega' \models \nu_j^n \psi_n = \Box_j \psi_n$ or that for all $\omega' \in \Omega_G(\omega)$, $\omega' \models \nu_j^n \psi_n = \hat{\Box}_j \psi_n$. If for all $\omega' \in \Omega_G(\omega)$, $\omega' \models \Box_j \psi_n$, then (as above) for all $\omega' \in \Omega_G(\omega)$, $\omega' \models \psi_n$, which contradicts the fact that $\omega''' \models \neg\psi_n$. Similarly, if for all $\omega' \in \Omega_G(\omega)$, $\omega' \models \hat{\Box}_j \psi_n$, then (as above) for all $\omega' \in \Omega_G(\omega)$, $\omega' \models \neg\psi_n$, which contradicts the fact that $\omega'' \models \psi_n$. Therefore, $\forall \omega' \in \Omega_G(\omega)$, $\omega' \models \dot{\Box}_j \psi_n$.
Since the above cases exhaust every possibility of an entry in a ken, and since the entry was chosen arbitrarily, it follows that for all $\omega' \in \Omega_G(\omega)$, such that $\omega' \models \nu_i \wedge \nu_j$, we have $\nu_i \sim \nu_j$.

Proof of Proposition 3 The implications among the conditions expressed semantically are simple.
Now, we can show that for any $\omega \in \Omega$ such that $S_i(\omega) \subseteq S_j(\omega)$, if $\omega \models \nu_i \wedge \nu_j$ then $\nu_i \succsim \nu_j$; which would establish the semantic to syntactic implications for conditions 1 and 2. Consider some arbitrary state $\omega \in \Omega$. Suppose $S_i(\omega) \subseteq S_j(\omega)$ and $\omega \models \nu_i \wedge \nu_j$. Consider the nth entry of these kens. (a) Suppose $\omega \models \nu_i^n \psi_n = \Box_i \psi_n$, and suppose that $\omega \models \nu_j^n \psi_n = \hat{\Box}_j \psi_n$. Then, $\forall \omega' \in S_j(\omega)$, $\omega' \models \neg\psi_n$. But if $S_i(\omega) \subseteq S_j(\omega)$, then $\forall \omega' \in S_i(\omega)$, $\omega' \models \neg\psi_n$, which contradicts the statement that $\omega \models \Box_i \psi_n$. Therefore, $\omega \models (\nu_j^n \psi_n = \Box_j \psi_n \vee \nu_j^n \psi_n = \dot{\Box}_j \psi_n)$. Cases (b), $\omega \models \nu_i^n \psi_n = \hat{\Box}_i \psi_n$ and (c) $\omega \models \nu_i^n \psi_n = \dot{\Box}_i \psi_n$ can dealt with analogously to case (a).
Now we can show that (3.a) implies (3.b): By the definition of (3.a), there is a state $\omega' \in \Omega_G(\omega)$ such that every state reachable from ω' is reflexive in both R_i and R_j. So, at each one of those states, $\Box_i \psi_n \rightarrow \psi_n$ for all formulas and all agents.
We can show that (4.a) implies (4.b): Suppose that (4.a) holds, but not (4.b). So suppose that for some ν_i and μ_j, $\omega' \models \nu_i \wedge \mu_j$ for all $\omega' \in \Omega_G(\omega)$, and yet, it is not the case that $\nu_i \sim \mu_j$. Case (a): At $\omega' \in S_i(\omega')$, for some ψ_n, we have $\omega' \models \Box_i \psi_n \wedge \hat{\Box}_j \psi_n$. But if $S_i(\omega') \subseteq S_j(\omega'')$ for some ω'', then $\omega'' \models \psi_n$, in which case we cannot have $\hat{\Box}_j \psi_n$ at every state in the component. Case (b): At $\omega' \in S_j(\omega')$, for some ψ_n, we have $\omega' \models \Box_i \psi_n \wedge \dot{\Box}_j \psi_n$. But if $S_j(\omega') \subseteq S_i(\omega'')$ for some ω'', then

$\omega''' \models \neg\psi_n$ for some $\omega''' \in S_i(\omega'')$, in which case we cannot have $\Box_i\psi_n$ at every state in the component. All other cases are trivial, or resemble one of the above.
We can show that (5.a) implies (5.b): Suppose $\omega' \in S_i(\omega') \cap S_j(\omega')$. Suppose that for some ψ_n, $\omega \models \Box_i\psi_n$. By reflexivity of R_i, $\omega \models \psi_n$. Now, suppose $\omega \models \hat{\Box}_j\psi_n$. By reflexivity of R_j at ω', $\omega' \models \neg\psi_n$, a contradiction.
Finally, that (1.b) implies (2.b) implies (4.b) is trivial. Also, that (1.b) implies (3.b) is trivial.
We can show that (3.b) implies (4.b): Suppose (3.b) holds and that (4.b) does not hold. (3.b) implies that there is a state $\omega' \in \Omega_G(\omega)$ such that every state reachable from ω' is reflexive in both R_i and R_j. Suppose that for some ν_i and μ_j, $\omega' \models \nu_i \wedge \mu_j$ for all $\omega' \in \Omega_G(\omega)$, and yet, it is not the case that $\nu_i \sim \mu_j$. Let ω'' be reachable from ω'. Case (a): suppose that at ω'', for some ψ_n, we have $\omega'' \models \Box_i\psi_n \wedge \hat{\Box}_j\psi_n$. By reflexivity of both R_i and R_j, $\omega'' \models \psi_n \wedge \neg\psi_n$, a contradiction. Case (b): $\omega'' \models \Box_i\psi_n \wedge \dot{\Box}_j\psi_n$. Then, for some reachable ω''', $\omega''' \models \neg\psi_n$. Since R_i is reflexive at ω''', it cannot be the case that $\omega''' \models \Box_i\psi_n$, thus contradicting the assumption that i's ken is the same across each state in the component.
Finally, we can show that (3.b) implies (5.b): (3.b), implies that there is a state $\omega' \in \Omega_G(\omega)$ such that every state reachable from ω' is reflexive in both R_i and R_j. Let ω'' be reachable from ω'. Suppose that at ω'', for some ψ_n, we have $\omega'' \models \Box_i\psi_n \wedge \hat{\Box}_j\psi_n$. By reflexivity of both R_i and R_j, $\omega'' \models \psi_n \wedge \neg\psi_n$, a contradiction.

Proof of Theorem 5 Suppose that there is some $\omega \in \Omega$ such that $\omega \models C_G(d_i^x \wedge d_j^y) \wedge (d_i^x \wedge d_j^y)$ and $x \neq y$. Then, for any ν_i and μ_j that are true at states within $\Omega_G(\omega)$, $D_i(\nu_i) = x \neq y = D_j(\mu_j)$. By like-mindedness, it follows that it is not the case that $\nu_i \sim \mu_j$ for any of those kens. But this contradicts (1.b). Namely, that there exists a state $\omega'' \in \Omega_G(\omega)$ such that $\omega'' \models \nu_i' \wedge \mu_j'$ such that $\nu_i' \sim \mu_j'$.

Appendix B

Map to Bacharach (1985)

Note that by Lemma 4, if for some $\omega \in \Omega$, $\omega \models \nu_i$, then for all $\omega' \in I_i(\omega)$, $\omega' \models \nu_i$; so decision are invariant across information cells. So, for each agent $i \in N$, we can define a function $H_i : 2^\Omega \to \mathcal{A}$, where $H_i(I_i(\omega)) = D_i(\nu_i)$ whenever $\omega \models \nu_i$. Furthermore, we can define another function $h_i : \Omega \to \mathcal{A}$ such that for all $\omega' \in I_i(\omega)$, $h_i(\omega') = H_i(I_i(\omega))$. This thus defines a map from our decision functions into Bacharach's framework.
Finally, we can also define Bacharach's Sure-Thing Principle: If $H_i(I_i(\omega)) = H_i(I_i(\omega'))$ and clearly $I_i(\omega) \cap I_i(\omega') = \emptyset$, then $H_i(m(I_i(\omega), I_i(\omega'))) = H_i(I_i(\omega))$.

Now, $m(I_i(\omega), I_i(\omega'))$, in Bacharach (1985) would simply be equal to $I_i(\omega) \cup I_i(\omega')$.

Map to Aumann & Hart (2006)

Aumann and Hart (2006) derive their agreement theorem using the framework developed in Aumann (1999) for the analysis of interactive knowledge in a partitional state space. Essentially, restricting ourselves to Ψ_0^0, we can define a mapping $\nu_i \mapsto e \in P^*$, where,

$$e := \bigwedge_{x \in \{p_n | \nu_i^n p_n = \Box_i p_n\}} x \wedge \bigwedge_{y \in \{\neg p_n | \nu_i^n p_n = \hat{\Box}_i p_n\}} y$$

That is, e is the conjunction of all the propositions (or their negation) that i knows, and all the propositions p_n for which $\nu_i^n p_n = \dot{\Box}_i p_n$ are ignored.
Given this, if we have $\omega \models \nu_i$, then e is the "minimal" formula that i knows at ω, in the sense that if $\Box_i e'$ then $e \to e'$. Note furthermore, that given our "richness" assumption on Ψ_0^0, we have that if $e \neq e'$ then $\neg(e \wedge e')$, and if $\nu_i \mapsto e'$ and $\mu_i \mapsto e''$, then $\inf\{\nu_i, \mu_i\} \mapsto (e' \vee e'')$. Given this map, our decision functions $D_i : V_i \to \mathcal{A}$ become $H_i : P^* \to \mathcal{A}$.
The Disjoint Sure-Thing Principle now becomes,

$$\models \bigwedge_{i \in N} \bigwedge_{e, e' \in P^*} [H_i(e) = H_i(e') \wedge \neg(e \wedge e') \to H_i(e \vee e') = H_i(e)]$$

which is the formulation given in Aumann and Hart (2006).

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