Overcoming the infeasibility of super-efficiency DEA model: a model with generalized orientation

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Abstract
The super-efficiency (SE) model is identical to the standard model, except that the unit under evaluation is excluded from the reference set. This model has been used in ranking efficient units, identifying outliers, sensitivity and stability analysis, measuring productivity changes, and solving two-player games. Under the assumption of variable, non-increasing and non-decreasing returns to scale (VRS, NIRS, NDRS), the SE model may be infeasible for some efficient DMUs. Based on the necessary and sufficient conditions for the infeasibility of SE, in the current paper, we have developed a DEA model with generalized orientation to overcome infeasibility issues. The DEA model with generalized orientation extends the orientation of the DEA model from the traditional input-orientation and output-orientation to the modified input-orientation, input-prioritized non-orientation, modified output-orientation, and output-prioritized non-orientation. All of the extended orientations are always feasible in the associated super-efficiency models. In addition, the modified input-oriented and the modified output-oriented approaches are developed to deal with the problem of infeasibility in super-efficiency models while keeping the concordance with the traditional oriented models. The newly developed model is illustrated with a real world dataset.

Keywords: Data envelopment analysis (DEA); Super-efficiency (SE); Infeasibility; Orientation

1. **Introduction**

Data envelopment analysis (DEA), originally developed by Charnes et al. (1978)[1], is a linear programming methodology for evaluating the relative technical efficiency for each member of a set of peer decision making units (DMUs) with multiple inputs and multiple outputs. A weakness of DEA is that, typically, more than one unit exists that can be evaluated as efficient when the number of DMUs is not enough relative to the number of inputs and outputs. Anderson and Petersen (1993)[2] developed the so-called super-efficiency (SE) model, which is an approach that is used to overcome this weakness. The SE model is very similar to the standard model, with the exception being the exclusion of the unit under evaluation from the reference set. SE score is a measure for efficiency assessment of an operational unit relative to the frontier created by the remaining units of the sample under evaluation. The SE model has been used in ranking efficient units (Andersen & Petersen, 1993[2]; Xue & Harker, 2002[3]; Chen, 2005[4]; Ray, 2008[5]), identifying outliers (Wilson, 1995[6]; Banker & Chang, 2006[7]), sensitivity and stability analysis(Charnes et al., 1992[8]; Seiford & Zhu, 1998[9, 10]; Zhu, 1996, 2001[11, 12]), measuring productivity changes (Färe et al., 1992[13]; Berg et al., 1992[14]), and solving two-player games (Rousseau & Semple, 1995[15]).

However broad the applicability of the SE model may be, it may not provide feasible solutions under certain conditions. Many efforts have been made to explore and justify the necessary and sufficient conditions for infeasibility in SE DEA models (Thrall, 1996; Zhu, 1996; Dula & Hickman, 1997; Seiford & Zhu, 1999; Xue & Harker, 2002). To be more precise, Thrall (1996) pointed out that the SE constant returns to scale or CCR model may be infeasible. Zhu (1996) showed that the SE CCR model is infeasible if, and only if, certain zero patterns appear in the dataset, and that other SE DEA models also may be infeasible, even when such zero patterns are not present in the input/output dataset. Seiford and Zhu (1999) provided necessary and sufficient conditions for infeasibility of the input-oriented and output-oriented super-efficiency DEA models. Lovell and Rouse (2003) [16] developed an equivalent standard DEA model to provide SE scores by scaling up the inputs (scaling down the outputs) of a DMU under evaluation. According to the equivalent standard DEA model, the SE score for an efficient unit that lacks feasible solutions in the standard SE model is equal to the user-defined scaling factor. In addition, Chen (2005)[4] determined that both the input-oriented and output-oriented super-efficiency models lead to failure when infeasibility occurs, especially, when both models are infeasible for an efficient unit. In order to deal with the infeasibility problem in SE models when the technology of variable returns to scale (VRS) prevails, Cook et al. (2009)[17] developed a novel formula for computing "feasible" SE scores. However, it is proved in following section of this study that the results yielded by applying Cook et al.'s formula are not flawless.

The objective of the research described in this paper was to develop a modified model with generalized orientation to resolve the infeasibility problem in SE models under VRS on the basis of Cook et al.'s idea. In addition, we discuss special cases of the generalized model in contrast with its equivalent, traditional-oriented models. It is shown that the new approach yields standard efficiency scores and SE scores that are identical to those determined by its counterpart traditional models, and it also yields optimal solutions and scores for efficient

DMUs that are regarded as infeasible by the traditional models. This study presents the modified SE model for VRS only, but it easily can be extended to non-increasing returns to scale (NIRS) and non-decreasing returns to scale (NDRS).

This paper unfolds as follows. In Section 2, we identify the problems associated with the traditional SE VRS models and analyze the properties of the new (general-oriented) model. In Section 3 we analyze the transformation of the non-linear programming models, initially developed to express the suggested general-oriented SE approach, into linear-programming models. Section 4 provides a comparison between the general-oriented SE model and Cook et al.'s approach for overcoming the infeasibility problem in SE variable returns to scale models. Finally, Section 5 concludes.

2. The new model with generalized orientation

2.1 Infeasibility for input-oriented SE-BCC model

The input-oriented VRS model, or, in brief, the input-oriented BCC model for the evaluated DMU_k can be formulated as shown:

$$\begin{aligned} &\min \theta \\ &\text{s.t.} \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{ik}, \quad i = 1,2,\dots,m \\ &\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{rk}, \quad r = 1,2,\dots,s \\ &\sum_{j=1}^{n} \lambda_j = 1 \\ &\lambda_j \ge 0, \quad j = 1,2,\dots,n \end{aligned} \tag{1}$$

For an efficient DMU_k, the SE-BCC model is:

$$\begin{aligned} &\min \theta \\ &\text{s.t.} \sum_{\substack{j=1 \\ j\ne k}}^{n} \lambda_j x_{ij} \le \theta x_{ik}, \quad i = 1,2,\dots,m \\ &\sum_{\substack{j=1 \\ j\ne k}}^{n} \lambda_j y_{rj} \ge y_{rk}, \quad r = 1,2,\dots,s \\ &\sum_{\substack{j=1 \\ j\ne k}}^{n} \lambda_j = 1 \\ &\lambda_j \ge 0, \quad j \quad 1,2,\dots,n \quad (j \ne k) \end{aligned} \tag{2}$$

In the input-oriented SE-BCC model, infeasibility occurs when an efficient DMU_k cannot reach the frontier formed by the remaining sample DMUs through increasing inputs. Since the constraint for inputs in (2) is always feasible, the necessary and sufficient condition for infeasibility is that the constraint for outputs in (2) be infeasible, i.e. $\sum_{\substack{j=1 \\ j\ne k}}^{n} \lambda_j y_{rj} \ge y_{rk}$ is infeasible.

A sufficient but not necessary condition for infeasibility of the input-oriented SE-BCC models is that there exists at least one output that has a value for the evaluated DMU_k greater than the values of any other DMUs.

Proof: If $y_{rk} > y_{rj}$ $(j = 1, 2, ..., n; j \neq k)$, then $\sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j y_{rj} < \sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j y_{rk}$. With $\sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j = 1$ and $\sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j y_{rk} = y_{rk}$, then $\sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j y_{rj} < y_{rk}$ always holds. As a result, $\sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j y_{rj} \geq y_{rk}$ is infeasible.

2.2 Infeasibility for output-oriented SE-BCC model

The output-oriented BCC model can be formulated as shown:

$$\begin{aligned} &\max \varphi \\ &s.t. \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{ik}, \quad i = 1, 2, ..., m \\ &\sum_{j=1}^{n} \lambda_j y_{rj} \geq \varphi y_{rk}, \quad r = 1, 2, ..., s \\ &\sum_{j=1}^{n} \lambda_j = 1 \\ &\lambda_j \geq 0, \quad j = 1, 2, ..., n \end{aligned} \tag{3}$$

For an efficient DMU_k, the SE-BCC model is:

$$\begin{aligned} &\max \varphi \\ &s.t. \sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j x_{ij} \leq x_{ik}, \quad i = 1, 2, ..., m \\ &\sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j y_{rj} \geq \varphi y_{rk}, \quad r = 1, 2, ..., s \\ &\sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j = 1 \\ &\lambda_j \geq 0, \quad j = 1, 2, ..., n \ (j \neq k) \end{aligned} \tag{4}$$

In the output-oriented SE-BCC model, infeasibility occurs when an efficient DMU_k cannot reach the frontier formed by the rest of the DMUs through decreasing outputs. Since the constraint for outputs in (4) is always feasible, the necessary and sufficient condition for infeasibility is that the constraint for inputs in (4) be infeasible, i.e. $\sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j x_{ij} \leq x_{ik}$ is infeasible.

A sufficient but not necessary condition for infeasibility of the output-oriented SE-BCC models is that there exists at least one input that has a value for the evaluated DMU_k smaller than the values of any other DMUs.

Proof: If $x_{ik} < x_{ij}$ $(j = 1, 2, ..., n;\ j \neq k)$, then $\sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j x_{ij} > \sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j x_{ik}$. Respecting the constraints $\sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j = 1$ and $\sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j x_{ik} = x_{ik}$, then $\sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j x_{ij} > x_{ik}$ always holds. As a result, $\sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j x_{ij} \le x_{ik}$ is infeasible.

2.3 Solutions for infeasibility of SE-BCC model

In case of infeasibility, both input-increasing and output-decreasing decisions must be made for the efficient DMU_k to reach the frontier. In order to tackle the problem of infeasibility, a non-oriented model should be developed in which both input-increasing and output-decreasing options are allowed for an efficient DMU to reach the frontier constructed by the remaining sample DMUs.

In this context, consider the following model for DMU_k:

$$\begin{aligned} &\min \ \frac{1-\alpha}{1+\beta} \\ &s.t. \sum_{j=1}^{n} \lambda_j x_{ij} \le (1-\alpha) x_{ik}, \quad (i = 1, 2, ..., m) \\ &\sum_{j=1}^{n} \lambda_j y_{rj} \ge (1+\beta) y_{rk}, \quad (r = 1, 2, ..., s) \\ &\sum_{j=1}^{n} \lambda_j = 1 \\ &\lambda_j \ge 0, \ j = 1, 2, ..., n \end{aligned} \tag{5}$$

where both input-decreasing and output-increasing strategies are permitted for an inefficient DMU to reach the frontier, and the efficiency score is defined as $(1-\alpha^*)/(1+\beta^*)$, with the numerator $(1-\alpha^*)$ indicating the degree of input shrinkage and the denominator $(1+\beta^*)$ denoting the output expansion.

If DMU_k is efficient in model (5), its SE model is expressed as:

$$
\begin{aligned}
&\min \ \frac{1-\alpha}{1+\beta} \\
&s.t. \ \sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j x_{ij} \leq (1-\alpha) x_{ik}, \quad (i = 1,2,...,m) \\
&\sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j y_{rj} \geq (1+\beta) y_{rk}, \quad (r = 1,2,...,s) \\
&\sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j = 1 \\
&\alpha \leq 0, \beta \leq 0, \lambda_j \geq 0, \ j = 1,2,...,n \ \ (j \neq k)
\end{aligned}
\tag{6}
$$

where both input increasing *($\alpha \leq 0$)* and output decreasing *($\beta \leq 0$)* are allowed for an efficient DMU to reach the frontier formed by the rest of the DMUs, and the SE score is defined as the ratio: $(1-\alpha^*)/(1+\beta^*)$, in which the numerator $(1-\alpha^*)$ indicates the degree of input expansion and the denominator $(1+\beta^*)$ indicates the output shrinkage.

In model (6), it is possible that an efficient DMU increases its inputs and decreases its outputs simultaneously to reach the reference set determined by the rest of the sample DMUs under evaluation. Consequently, model (6) is always feasible.

Proof: Acknowledging the constraint $\alpha \leq 0$, $\sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j x_{ij} \leq (1-a) x_{ik}$ is always feasible, and respecting the constraint $\beta \leq 0$, $\sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j y_{rj} \geq (1+\beta) y_{rk}$ is always feasible as well. As a result, model (6) is always feasible.

By taking into account the solutions provided for the infeasibility problem of the SE-BCC model we extend models (1) and (2) to a generalized form:

i.e. for DMU_k

$$
\begin{aligned}
\min \quad & \frac{1 - w^I \alpha}{1 + w^O \beta} \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \le (1-\alpha) x_{ik}, \quad i = 1,2,...,m \ (\text{if } w^I > 0) \\
& \sum_{j=1}^{n} \lambda_j x_{ij} \le x_{ik}, \quad i = 1,2,...,m \ (\text{if } w^I = 0) \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \ge (1+\beta) y_{rk}, \quad r = 1,2,...,s \ (\text{if } w^O > 0) \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{rk}, \quad r = 1,2,...,s \ (\text{if } w^O = 0) \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j \ge 0, \ j = 1,2,...,n
\end{aligned} \tag{7}
$$

where w^I and w^O are user-defined non-negative numbers, with at least one of them being positive. The superscripts I and O denote the distinction between the user-defined values attached to the inputs and the outputs respectively. The efficiency score is defined by the ratio: $(1-\alpha)/(1+\beta)$.

If DMU_k is efficient in model (3), its generalized SE model is expressed as:

$$
\begin{aligned}
\min \quad & \frac{1 - w^I \alpha}{1 + w^O \beta} \\
s.t. \quad & \sum_{\substack{j=1 \\ j \ne k}}^{n} \lambda_j x_{ij} \le (1-\alpha) x_{ik}, \quad i = 1,2,...,m \ (\text{if } w^I > 0) \\
& \sum_{\substack{j=1 \\ j \ne k}}^{n} \lambda_j x_{ij} \le x_{ik}, \quad i = 1,2,...,m \ (\text{if } w^I = 0) \\
& \sum_{\substack{j=1 \\ j \ne k}}^{n} \lambda_j y_{rj} \ge (1+\beta) y_{rk}, \quad r = 1,2,...,s \ (\text{if } w^O > 0) \\
& \sum_{\substack{j=1 \\ j \ne k}}^{n} \lambda_j y_{rj} \ge y_{rk}, \quad r = 1,2,...,s \ (\text{if } w^O = 0) \\
& \sum_{\substack{j=1 \\ j \ne k}}^{n} \lambda_j = 1 \\
& \alpha \le 0, \beta \le 0, \lambda_j \ge 0, \ j = 1,2,...,n \ (j \ne k)
\end{aligned} \tag{8}
$$

where w^I and w^O are user-defined non-negative numbers imputed to the inputs and outputs, respectively, and at least one of the user-defined values is positive. The SE score is defined as

(1-α)/(1+β).

In model (7) and model (8), w^I and w^O denote the priority of orientation. The generalized SE model (8) is always feasible when both w^I and w^O have non-zero values. The proof of the feasibility statement for model (8) is based on the proof of all-time feasibility for model (6).

Table 1 lists seven special cases of the generalized model and their definitions for (super-) efficiency score.

Table 1 Special cases of the standard and the generalized super-efficiency models and their definitions for computing efficiency scores

Case	Model	Standard efficiency model			Super-efficiency model		
		w^I	w^O	Score	w^I	w^O	Score
1	Input-oriented	1	0	$1-\alpha^*$	1	0	$1-\alpha^*$
2	Output-oriented	0	1	$\frac{1}{1+\beta^*}$	0	1	$\frac{1}{1+\beta^*}$
3	Non-oriented	1	1	$\frac{1-\alpha^*}{1+\beta^*}$	1	1	$\frac{1-\alpha^*}{1+\beta^*}$
4	Input-oriented (modified)	1	ε	$1-\alpha^*$	ε	1	$1-\alpha^*$
5	Non-oriented (input-prioritized)	1	ε	$\frac{1-\alpha^*}{1+\beta^*}$	ε	1	$\frac{1-\alpha^*}{1+\beta^*}$
6	Output-oriented (modified)	ε	1	$\frac{1}{1+\beta^*}$	1	ε	$\frac{1}{1+\beta^*}$
7	Non-oriented (output-prioritized)	ε	1	$\frac{1-\alpha^*}{1+\beta^*}$	1	ε	$\frac{1-\alpha^*}{1+\beta^*}$

ε is the non-Archimedean infinitesimal

Case 1: The generalized model is equivalent to the traditional input-oriented model, so the (super-) efficiency score $1-\alpha^*$ in the generalized model is equal to θ^* in the traditional input-oriented model.

Proof: In case 1, $w^I = 1$ and $w^O = 0$, the standard efficiency model can be expressed as:

$$
\begin{aligned}
&\min\ (1-\alpha) \\
&s.t.\ \sum_{j=1}^{n} \lambda_j x_{ij} \le (1-\alpha) x_{ik},\ \ i = 1,2,...,m \\
&\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{rk},\ \ r = 1,2,...,s \\
&\sum_{j=1}^{n} \lambda_j = 1 \\
&\lambda_j \ge 0,\ \ j = 1,2,...,n
\end{aligned} \tag{9}
$$

If $(1\text{-}\alpha)$ is replaced by θ, model (9) becomes the traditional input-oriented BCC model (1).

In case 1, the SE model can be written as:

$$
\begin{aligned}
&\min\ (1-\alpha) \\
&s.t.\ \sum_{\substack{j=1 \\ j\ne k}}^{n} \lambda_j x_{ij} \le (1-\alpha) x_{ik},\ \ i = 1,2,...,m \\
&\sum_{\substack{j=1 \\ j\ne k}}^{n} \lambda_j y_{rj} \ge y_{rk},\ \ r = 1,2,...,s \\
&\sum_{\substack{j=1 \\ j\ne k}}^{n} \lambda_j = 1 \\
&\alpha \le 0, \beta \le 0, \lambda_j \ge 0,\ \ j = 1,2,...,n\ \ (j \ne k)
\end{aligned} \tag{10}
$$

If $(1\text{-}\alpha)$ is replaced by θ, model (10) becomes the traditional input-oriented SE-BCC model (2).

Case 2: The generalized model is equivalent to the traditional output-oriented model, and the (super-) efficiency score $1/(1 + \beta^*)$ in the generalized model is equal to $1/\varphi^*$ in the traditional output-oriented model. This can be proved as case 1.

Case 4: Input-orientation is given priority with output-orientation retained. This means that the output is allowed to decrease, and furthermore, this decrease should be minimized for the evaluated DMU_k to reach the frontier formed by the rest of the DMUs. The (super-) efficiency score is defined as $(1 - \alpha^*)$ omitting the denominator $(1 + \beta^*)$, which means that the efficiency score is measured by movements of inputs only. Case 4 expresses a modified input-orientation with the following properties:

1) the standard efficiency score $(1 - \alpha^*)$ is equal to θ^* in the traditional input-oriented model;
2) the SE score is equal to θ^* in the traditional input-oriented SE-BCC model when the traditional input-oriented SE-BCC model is feasible; and
3) when the traditional input-oriented SE model is infeasible, this modified input-oriented model will still yield an optimal solution.

These properties assume that the modified input-oriented model overcomes the problem of infeasibility while keeping the concordance with the traditional input-oriented model.

Proof: In case 4, the SE model is written as:

$$\min \frac{1-\alpha}{1+\varepsilon\beta}$$

$$s.t. \sum_{j=1}^{n} \lambda_j x_{ij} \leq (1-\alpha) x_{ik}, \quad i = 1,2,...,m$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} \geq (1+\beta) y_{rk}, \quad r = 1,2,...,s$$

$$\sum_{j=1}^{n} \lambda_j = 1$$

$$\lambda_j \geq 0, \quad j = 1,2,...,n \tag{11}$$

In model (11), the objective function is minimizing *(1-α)* and maximizing *(1+εβ)*. In its solutions, a larger β will result in a smaller α. Increasing β makes essentially no contribution to the reduction of the objective function, because of the effect of the coefficient ε (Non-Archimedean infinitesimal), so β will be minimized to zero in the optimal solution. When this occurs, model (11) is equivalent to the traditional input-oriented BCC model and property 1 is proven.

In case 4, the SE model can be expressed alternatively as:

$$\min \frac{1-\varepsilon\alpha}{1+\beta}$$

$$s.t. \sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j x_{ij} \leq (1-\alpha) x_{ik}, \quad i = 1,2,...,m$$

$$\sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j y_{rj} \geq (1+\beta) y_{rk}, \quad r = 1,2,...,s$$

$$\sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j = 1$$

$$\alpha \leq 0, \beta \leq 0, \lambda_j \geq 0, \quad j \quad 1,2,...,n \quad (j \neq k) \tag{12}$$

In model (12), the objective function is to minimize *(1-εα)* and to maximize *(1+β)*. In its solutions, both α and β have non-positive values, and the larger the absolute value of β, the smaller the absolute value of α. An increase in the absolute value of β will result in an increase in the objective function, but an increase in the absolute value of α will not result in an increase in the objective function because of the effect of the coefficient ε (Non-Archimedean infinitesimal). So, the absolute value of β will be minimized to zero in the optimal solution if β=0 is a solution. If β=0, model (12) can be formulated as:

$$\min\ 1-\varepsilon\alpha$$

$$s.t.\ \sum_{\substack{j=1\\ j\neq k}}^{n}\lambda_j x_{ij}\leq(1-\alpha)x_{ik},\ \ i=1,2,...,m$$

$$\sum_{\substack{j=1\\ j\neq k}}^{n}\lambda_j y_{rj}\geq y_{rk},\ \ r=1,2,...,s$$

$$\sum_{\substack{j=1\\ j\neq k}}^{n}\lambda_j=1$$

$$\alpha\leq 0,\beta\leq 0,\lambda_j\geq 0,\ j=1,2,...,n\ \ (j\neq k) \tag{13}$$

Because ε is a small positive number, the optimal solution will not change after the replacement of the objective function with "*min (1-α)*". After replacement, the model is equivalent to the traditional input-oriented SE-BCC model. If $\beta=0$ is a critical value in model (12), the traditional input-oriented SE-BCC model is feasible. Then, property 2 is proved.

When the traditional input-oriented SE-BCC model is infeasible, it means that β cannot be 0 in model (12). In such cases, model (12) still can yield an optimal solution in which the absolute value of β is minimized. Property 3 is proved.

Case 6: Output-orientation is given priority while input-orientation is retained. This assumption means that increase in the input is allowed, but it must be minimized for the evaluated DMU_k to reach the frontier constructed by the remaining sample DMUs. The (super-) efficiency score is defined as $1/(1+\beta^*)$ omitting the numerator $(1+\alpha^*)$, which means that the efficiency score is measured by the movements of outputs only. This is a modified output-orientated model with the following properties:

1) the standard efficiency score, $1/(1+\beta^*)$, is equal to $1/\varphi^*$ in the traditional output-oriented model;
2) the SE score is equal to $1/\varphi^*$ in the traditional output-oriented model as well when it is feasible; and
3) when the traditional output-oriented SE model is infeasible, this modified output-oriented model will still yield an optimal solution.

These properties can be proved by the same procedure that was used in case 4. They mean that the modified output-oriented model overcomes the problem of infeasibility, and, at the same time, it keeps the concordance with the traditional output-oriented model.

3. Solving the new model

3.1 Transformation of non-linear programming

The equations of the DEA model with generalized orientation are based on non-linear programming. However, they can be transformed into linear programming equations by applying a method similar to the Charnes et al. (1978) transformation.

Let us reformulate the objective function of model (8) as:

$$\min \ (1 - w^I \alpha) \times \frac{1}{1 + w^O \beta}$$

and replace the second term with a positive scalar variable

$$t = \frac{1}{1 + w^O \beta}$$

Let us multiply both the left-hand side and the right-hand side of the constraints by t in model (8). The multiplication does not affect the inequalities of the constraints as long as $t > 0$. Then, let us define $\Lambda = t\lambda$, $A = t\alpha$ and $B = t\beta$ in order to transform the non-linear programming model (8) into the following linear programming model:

$$\begin{aligned}
&\min \ t - w^I \mathrm{A} \\
&s.t. \sum_{\substack{j=1 \\ j \neq k}}^{n} \Lambda_j x_{ij} \leq (t - \mathrm{A}) x_{ik}, \ i = 1,2,...,m \ (\text{if } w^I > 0) \\
&\sum_{\substack{j=1 \\ j \neq k}}^{n} \Lambda_j x_{ij} \leq t x_{ik}, \ i = 1,2,...,m \ (\text{if } w^I = 0) \\
&\sum_{\substack{j=1 \\ j \neq k}}^{n} \Lambda_j y_{rj} \geq (t + \mathrm{B}) y_{rk}, \ r = 1,2,...,s \ (\text{if } w^O > 0) \\
&\sum_{\substack{j=1 \\ j \neq k}}^{n} \Lambda_j y_{rj} \geq t y_{rk}, \ r = 1,2,...,s \ (\text{if } w^O = 0) \\
&\sum_{\substack{j=1 \\ j \neq k}}^{n} \Lambda_j = t \\
&\mathrm{A} \leq 0, \mathrm{B} \leq 0, \Lambda_j \geq 0, j = 1,2,...,n \ \ (j \neq k)
\end{aligned} \qquad (14)$$

Let an optimal solution of model (14) be (t^*, A^*, B^*, Λ^*).Then, the optimal solution of model (8) is defined by:

$$\alpha = A/t^*, \beta = B/t^*, \lambda^* = \Lambda^*/t^*$$

3.2 Non-Archimedean infinitesimal

To deal with infeasibility both output decrease and input increase are permitted. However, the objective towards the attainment of the reference set by the rest of the sample DMUs is the minimization of the output decrease and of the input increase for the evaluated DMU_k in the input-oriented and the output-oriented SE-BCC model, respectively. To achieve such an idea, the non-Archimedean infinitesimal is used in the programming (cases 4, 5, 6 and 7). In practice, a small positive number, for example 10^{-5}, can be used instead.

The four special cases presented in sub-section 2.3 also can be resolved through a two-stage method as follows. To be more precise, by taking the SE programming model developed for case 4, i.e. model (12), as an example, at stage 1, the first programming is resolved to get a maximal non-positive β^*, namely:

$$
\begin{aligned}
&\min\ \beta \\
&s.t.\ \sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j y_{rj} \geq (1+\beta) y_{rk},\quad r = 1,2,\ldots,s \\
&\sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j = 1 \\
&\alpha \leq 0, \beta \leq 0, \lambda_j \geq 0,\ j = 1,2,\ldots,n\ \ (j \neq k)
\end{aligned} \tag{15}
$$

And at stage 2, a second programming is solved

$$
\begin{aligned}
&\min\ \alpha \\
&s.t.\ \sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j x_{ij} \leq (1-\alpha) x_{ik},\quad i = 1,2,\ldots,m \\
&\sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j y_{rj} \geq (1+\beta^*) y_{rk},\quad r = 1,2,\ldots,s \\
&\sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j = 1 \\
&\alpha \leq 0, \lambda_j \geq 0,\ j = 1,2,\ldots,n\ \ (j \neq k)
\end{aligned} \tag{16}
$$

The two-stage method has two advantages i.e. 1) it can avoid numerical error caused by approximation of the non-Archimedean infinitesimal, and 2) the problems in both stages relate to linear programming, hence the transformation is not needed.

4. Comparison of the generalized-oriented SE-BCC model with the approach proposed by Cook et al. (2009)

Cook et al. (2009) proposed an approach to deal with the problem of infeasibility in the SE-BCC. Model (17) describes the concept of the approach put forth by Cook et al. to deal with infeasibility yielded by the input-oriented SE-BCC model.

Consider the following model for an efficient DMU_k:

$$
\begin{aligned}
&\min\ \tau + M \times \beta \\
&s.t.\ \sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j x_{ij} \leq (1+\tau) x_{ik},\quad i = 1,2,\ldots,m \\
&\sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j y_{rj} \geq (1-\beta) y_{rk},\quad r = 1,2,\ldots,s \\
&\sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j = 1 \\
&\beta \geq 0, \lambda_j \geq 0,\ j = 1,2,\ldots,n\ \ (j \neq k)
\end{aligned} \tag{17}
$$

where M is a user-defined large positive number, and the SE score is defined by Cook et al. as: $1+\tau^*+1/(1-\beta^*)$. Model (17) is always feasible. The proof of this statement relies on the steps followed in model (6) for proving its feasibility. Regardless of the feasibility of model (17), two defects have been identified:

Defect 1: The definition of SE score $1+\tau^*+1/(1-\beta^*)$ may result in contradictions.

Let's discuss two special instances first.

Instance 1: $\tau^*=\beta^*=0$

In model (17), τ indicates SE attributed to input increase, and β indicates SE attributed to output decrease. If both τ and β are null, which indicates that neither inputs nor outputs show SE, the super-efficiency score should be equal to unity. Obviously, the super-efficiency score calculated by applying the definition: $1+\tau^*+1/(1-\beta^*)$ is not equal to unity in the case of $\tau^*=\beta^*=0$, which contradicts the above statement. Such a case easily can be easily constructed by adding a DMU to the sample that has exactly the same input and output values as DMU_k.

Let's assume that the absolute value of both τ^* and β^* are very small numbers slightly greater than zero, like DMU 1 (MITSUI & CO., Table 3 in Cook et al.'s paper) with $\tau^*=0.0104$ and $\beta^*=0.0057$, alternatively, Hospital A in Table 3 in the current paper with $\alpha^*=-0.051$ and $\beta^*=-0.099$. Then, the SE score is expected to be a number slightly greater than unity. However, the calculated SE score, according to the definition of SE associated with model (17), is a much larger number, i.e. 2.0161. This result also contradicts with the general statement of Instance 1.

Instance 2: $\tau^*>0$ and $\beta^*=0$

If τ^* is a positive number and β^* is zero, there a feasible optimal solution must be yielded by the traditional SE model (according to theorem 1 in Cook et al.'s paper), and the SE score calculated with the new model should be equal to that calculated with the traditional SE model, just as stated in Cook et al.'s paper. However, the super-efficiency score in this case calculated with (17) is $1+\tau^*+1/(1-\beta^*)=1+\tau^*+1$. It should be noted that $1 + \tau^*$ is the SE score in the traditional SE model (i.e. θ^*). This paradox should have been uncovered by the comparison of the SE scores associated with feasible DMUs calculated using the two models. However, the SE scores cited in Cook et al.'s paper were calculated with the traditional model rather than with the new model, so the flaw that was previously pointed out failed to become explicit.

Even if the definition of SE score, i.e. $1+\tau^*+1/(1-\beta^*)$, is used solely when the standard SE-BCC model is infeasible, it still can produce unreasonable results. Namely, let both τ^* and β^* to be very small numbers, slightly greater than 0; for example, $\tau^*=0.0104$ and $\beta^*= 0.0057$ (this case can be found in Table 3 of Cook et al.'s paper). Since τ and β indicate the input "super part" and the output "super part", respectively, and since both "super parts" are small enough, it is obvious that the calculated SE score should be slightly greater than unity. Nevertheless, the SE score determined by the definition formula put forth by Cook et al. is equal to: 1+0.0104+1/(1-0.0057) = 2.0161, which is a much larger number than was expected. Furthermore, we elaborate on an extreme example by assuming that $\tau^*=0.00000001$ and $\beta^*= 0.00000001$. In solving the problem, although both τ and β are rounded to zero and the SE score should be equal to unity, the calculated score based on the definition formula is about 2.

In fact, the relationship between the two components in the model for the determination of the SE score should be multiplicative, not additive. In other words, the SE score should be defined as: $(1 + \tau^*)/(1 - \beta^*)$. By applying this definition the aforementioned inconsistencies disappear.

Defect 2: In model (17), both τ and β should have positive values.

Since DMU_k is an efficient unit, and τ indicates the input increase needed for DMU_k to reach the frontier, τ must be a non-negative value. If τ<0, there will exist $\sum_{j=1}^{n} \lambda_j x_{ij} \le \sum_{\substack{j=1 \\ j \ne k}}^{n} \lambda_j x_{ij} \le (1+\tau) x_{ik}$, which contradicts the precondition that DMU_k is an efficient unit. Therefore, $\tau > 0$ must be added to model (17) as a constraint. Using the multiplicative definition $(1 + \tau^*)/(1 - \beta^*)$, if the constraint $\tau > 0$ is omitted, the SE score of an efficient DMU may be less than unity.

Similar defects exist in Cook et al.'s approach for output-oriented SE-BCC models.

The essence of Cook et al.'s approach for dealing with infeasibility was that output decrease is permitted but must be minimized for the evaluated DMU_k to reach the frontier formed by the rest of the DMUs involved in the input-oriented SE-BCC model.

The model that we developed, described in this paper, mainly uses Cook et al.'s idea for dealing with infeasibility, and we introduced some modifications in order to overcome the drawbacks associated with Cook et al.'s approach. Cases 5 and 7 in Table 1 are equivalent to Cook et al.'s approach for input- and output-oriented SE-BCC models, respectively, provided that the previously discussed defects are rectified. We recommend that case 4 be used for infeasibility in the input-oriented SE-BCC model, because the efficiency measure doesn't incorporate output decrease, which is in accordance with the meaning of input-orientation; and, for same reason, case 5 should be used for infeasibility in the output-oriented SE-BCC model.

5. Conclusions

Based on the necessary and sufficient conditions for the infeasibility issue of SE and the study of Cook et al. (2009) for dealing with infeasibility in SE-BCC models, in the current paper, we develop a DEA model with generalized orientation to overcome infeasibility problems. The DEA model with generalized orientation extends the orientation of the DEA model from the traditional input-orientation and output-orientation to modified input-orientation, input-prioritized non-orientation, modified output-orientation and output-prioritized non-orientation. All the extended orientations always are feasible in their SE models. In addition, the modified input- and output-oriented models address the problem of infeasibility in SE models, while keeping the concordance with the traditional-oriented models.

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