Inter and intra-group conflicts as a foundation for contest success functions

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2011

manuscript No.
(will be inserted by the editor)

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Received: date / Revised version: date

Abstract This paper introduces a notion of partitioned correlated equilibrium that extends Aumann's correlated equilibrium concept (1974, 1987). This concept captures the non-cooperative interactions arising simultaneously within and between groups. We build on this notion in order to provide a foundation for contest success functions (CSFs) in a game wherein contests arise endogenously. Our solution concept and analysis are general enough to give a foundation for any model of contest using standard equilibrium concepts like e.g., Nash, Bayesian-Nash or Perfect-Nash equilibria. In our environment, popular CSFs can be interpreted as a list of equilibrium conjectures held by players whenever they contemplate deviating from the "peaceful outcome" of the "group formation game". Our setup allows to relate the form of prominent CSFs with some textbook examples of (linear) utility functions, social utility functions in the spirit of Fehr and Schmidt (1999) and non-expected models of utility a la Quiggin (1981, 1982). We also show that our framework can accommodate situations in which agents cannot correlate their actions.
Keywords: Contest success functions · Correlated equilibrium · Inter and intra-group conflicts · Induced contests
JEL Classification Numbers: C72 · C73

1 Introduction

In a contest game agents exert irreversible effort to increase their probability of winning a prize. These situations cover phenomena ranging from

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I thank Luis Corchón for very helpful comments and suggestions.

litigation, conflict and appropriation, sport events, competition for promotion within a firm, influence activities and rent seeking, situations of war and peace, innovation and patent-race games to name just a few.[1]
Central to these studies, is the mechanism that determines final success or failure for each contestant. Most of the existing contest literature starts out by assuming a probabilistic choice function that translates an individual's effort into his probability of winning. Usually, in the literature this function is called "technology of conflict" or contest success function (CSF). An important question is whether it is possible to justify certain of the technologies of conflict used in the modeling of contests. The literature has addressed this question in different ways.
Following the seminal work of Skaperdas (1996), a strand of the literature provides some axiomatic foundations (see e.g. Clark and Riis, 1998). Hillman and Riley (1989) offer a model of the political process where the impact of effort is uncertain. Fullerton and McAfee (1999), Baye and Hoppe (2003) and Fu and Lu (2011) offer micro-foundations for certain CSFs for innovation tournaments and patent races. Another approach rationalizes CSFs by assuming some mediated or cooperative frameworks (Epstein and Nitzan 2006, Corchón and Dahm, 2009, 2010). However, CSFs may also arise as the result of equilibrium behavior, in a non-cooperative framework. This is the main purpose of this paper.
A formal description of a CSF is as follows. Given a vector of efforts, $\mathbf{G}$, each contestant $i \in N \equiv \{1, ..., n\}$, has a probability $p_i(\mathbf{G})$ of winning a prize such that $\sum_{i=1}^{n} p_i(\mathbf{G}) = 1$. As a specific instance of a CSF, Tullock (1980) proposed the following form, where, given a positive scalar σ,

$$p_i(\mathbf{G}) = \frac{G_i^\sigma}{\sum_{j=1}^{n} G_j^\sigma} \text{ for } i = 1, ..., n. \tag{1}$$

A generalization of this form is the (general) additive form,

$$p_i(\mathbf{G}) = \frac{f_i(G_i)}{\sum_{j=1}^{n} f_j(G_j)} \text{ for } i = 1, ..., n \tag{2}$$

where $f_i(\cdot)$ is a non-negative, increasing function called *effectivity function* which measures the merit of i in the contest.
The building block of our model is a two-stage game called the gun-butter game. In the gun-butter game, agents simultaneously choose an activity of production or appropriation, or any mixed combination of these two activities and then decides how much effort they exert. A natural interpretation is that individuals choose how much of their time and effort they spend on rent-seeking (resp. productive) activities.
Our foundation for CSFs builds on the analysis of the first-stage of this game, when players simultaneously consider the choice of their activities at the individual and coalitional level. In order to model these situations, we

[1] For thorough surveys of this literature see e.g. Corchón (2007) and Konrad (2009).

introduce the notion of partitioned correlated equilibrium. This notion is a natural extension of the correlated equilibrium concept (Aumann 1974, 1987). Aumann's correlated equilibrium captures important aspects of collective strategic behavior by allowing correlation between *all* players. However there are situations in which correlation takes place simultaneously, within several groups of players. In this paper we motivate and develop such a concept in our foundation of CSFs.

Roughly speaking, the partitioned correlated equilibrium concept aims at modeling the idea that subgroups of players may simultaneously form disjoint coalitions in order to exploit the strategic correlation opportunities specific to their own group while taking into account the correlation devices of other groups. More formally, take any non-cooperative game in normal form, and suppose that (disjoint) groups are formed. Then, in a partitioned correlated equilibrium, each group plays a correlated equilibrium given the strategies played within the other groups. In other words, a partitioned correlated equilibrium can be seen as an attempt to provide a solution concept that (i) requires self-enforcing agreements *within* each group of players (ii) imposes the resulting profile of non-binding agreements to be self-enforcing *between* all groups.

After proving general existence results for finite strategic-form games, we apply this concept to select some particular perfect Bayesian equilibria (PBEs) of the gun-butter game. In the context of our model, the notion of partitioned correlated equilibrium is a natural solution concept: a subset of individuals will form a group because by doing so its members have the possibility to exploit the strategic correlation opportunities specific to their group. Hence, an underlying aspect of the partitioned correlated equilibrium concept is to connect the formation of a coalition with its correlation devices. In particular, in our model, this suggests that the grand coalition may exploit its correlated signals in order to coordinate the activities of its members on the peaceful outcome, wherein no player devote their time on conflict. We show that popular CSFs emerge from such an environment, when each player contemplates the possibility to deviate from the peaceful outcome.

To understand the intuitions behind our derivations, let us consider the following scenario. Suppose players form the grand coalition in order to achieve the peaceful outcome of the gun-butter game, in a (trivial)[2] correlated equilibrium. Then, suppose a player contemplates seceding from the grand coalition to form a stand-alone coalition against the rest of players. In this case, the seceding player needs to form some equilibrium beliefs by taking into account the reactions of the rest of players, which would anticipate her own deviation. In our model, this is captured by requiring that agents play in a partitioned correlated equilibrium, in which the seceding agent play against the correlated (equilibrium) strategies of the rest of players. Our results show that popular CSFs can be understood as the lists of

[2] In our model, this correlated equilibrium boils down to a Nash equilibrium.

equilibrium beliefs generated by the set of seceding players.
Here, the sequential nature of the gun-butter game allows us to exploit the fact that players must hold interim beliefs about the others' choices of an activity, conditionally on any vector of effort **G** that might be chosen in the second-stage. This way we obtain a CSF as arising from the conjectures of players in a particular subset of the perfect Bayesian equilibria of the gun-butter game. This establishes a link between the form of win probabilities used in contests and the probabilistic beliefs of players in the gun-butter game. More specifically, in our model, a win probability for agent i, p_i, coincides with i's equilibrium belief to appropriate the goods produced by others when i is playing the first-stage of the gun-butter game in a stand-alone coalition while others correlate their choices on productive activities.
Note that the idea of modeling the interaction of agents in an initial state-of-nature, as a two-stage game is not new. This modeling choice was notably used by Muthoo (2004) in order to study the emergence of property rights. In our setup, this allows to separate the decision of appropriative or productive activity (war or peace) with the intensity of effort (how much should be allocated to war) in a clean way.
Our notion of partitioned correlated equilibrium allows to rationalize a large class of prominent CSFs for any number of contestants. But we also study an alternative notion of a rationalizability, when players are unable to use some correlated devices. In this case, our approach can still yield a foundation for CSFs by considering a modified version of the gun-butter game. In this version of the model, we notably show (see section 7) that CSFs can be supported by a set of subgame perfect Nash equilibria of the gun-butter game wherein players choose their *activities sequentially*.
A key aspect of the model and analysis is to determine the plausibility of a CSF in terms of the players' utility functions. We derive several prominent CSFs by assuming some textbook examples of utility functions. For instance, when players have preferences that exhibit perfect substitutes over the production of others, we can rationalize the standard Tullock CSF (2). Moving beyond the traditional expected utility model, we obtain a surprising connection between a prominent family of CSFs that relies on absolute effort differences (like difference-form CSFs) and\or relative effort differences and recent behavioral models where individuals exhibit other-regarding preferences (Fehr and Schmidt (1999)), or rank-dependent utility (RDU) (Quiggin (1981, 1982)).

Relationship with the literature

The main objective of this paper is closely related to Corchón and Dahm (2009) and Corchón and Dahm (2010). Our paper should be considered as complementary to their papers in the following sense. Corchón and Dahm (2010) adopt a cooperative framework in which CSFs are related to bargaining, claims and taxation problems. In two other approaches their derivations arise in some mediated environments, which explicitly require the presence of a planner.[3] Instead, we provide foundations for popular CSFs within a purely *non-cooperative* and *unmediated* environment.

Our approach is also related to the economics literature on conflict see e.g., Skaperdas (1992), Grossman and Kim (1995), Hirshleifer (1995), and Esteban and Ray (1999). In particular, our model draws its inspiration from the recent studies by Bloch et al. (2006) and Esteban and Ray (1999) and Jackson and Morelli (2009). Esteban and Ray (1999) introduce a general model of conflict. Their analysis focuses on the relation between distribution and the level of conflict. The analysis of Bloch et al. (2006) is geared toward the endogenous formation of groups in models of conflict. In their setup, when an agent contemplates breaking the universal agreement, she must anticipate the reaction of other players to this initial secession. We focus our attention on a different question, but the scenario underlying our notion of a rationalizable CSF is similar in spirit. In our model, one may imagine that players will form the grand coalition in order to coordinate their activities on the peaceful outcome in the first-stage of the gun-butter game. Moreover, in both models the focus is on the conjectures held by players when they deviate unilaterally from the grand coalition. In our case, the way a deviating player anticipates the reaction of others when he deviates from the initial agreement is determined by the concept of partitioned correlated equilibrium. By contrast, Bloch et al. (2006) model the reaction of external players by adopting a non-cooperative approach of games of coalition formation, as suggested by Hart and Kurz (1983). We will further discuss this point in Section 5.

Our paper is also related to the recent rationalist literature on war. Beviá and Corchón (2009) study which kind of agreements can prevent war. They analyze a two-stage game where two players can transfer some of their resources to the other players in a first-stage. Then in a second-stage, players are allowed to decide whether to declare war on the other player. Jackson and Morelli (2009) explore a model where two countries choose armament levels and then whether or not to go to war. Hence, a common assumption with the present analysis is that war-peace decisions are endogenous in both models. This contrast with most of the literature on war, which usually postulates the existence of a future conflict.[4]

The present paper shares also some aspects with Muthoo (2004). As in Muthoo, we assume that players does not have property rights over the fruits of their labor. But our focus is different. Muthoo explores the emergence of these property rights by assuming the existence of a contest and a class of conflict technologies is assumed rather than derived. By contrast, this paper seeks to understand the emergence of a contest, but remains silent on the determinants of the peaceful outcome. As already mentioned, we develop a similar two-stage game to model some agents in an initial state

[3] In Corchón and Dahm (2009), the planner can also be seen as a surrogate of what the system achieves by its own forces.

[4] Usually, this literature postulates a given relative military power and does not account for war-peace decisions.

of nature. Transported in our setting, his time line is opposite to ours: in Muthoo's model, players first decide how much effort they spend on two activities (work or leisure), and in a second-stage, they simultaneously choose the nature of this effort (fight or peace).
The rest of the paper goes as follows. We present the basic model in Section 2. In Section 3 we introduce the notion of partitioned correlated equilibrium and establish existence results. In Section 4-5 we present our notion of rationalizability. In section 6, we present our derivations. Section 7 presents an alternative notion of a rationalizable CSF and Section 8 concludes.

2 Model

Consider an environment populated by a finite set $N = \{1, 2, ..., n\}$ of individuals ($n \geq 2$). In the base model we consider a n-player two-stage game called the *gun-butter game*.
In the first-stage of the game, each player i simultaneously chooses an action θ_i interpreted as an *activity*. There are two kind of activities: *productive* activities, $\underline{\theta} = 0$ and *appropriative* activities, $\overline{\theta} = 1$, e.g., rent-seeking activities. We assume that players can allocate their time between these two sort of activities. [5] To model this situation, we allow each player to pick a *mixed* strategy defined over the set of pure activities, $\left\{\underline{\theta}, \overline{\theta}\right\} \equiv \Theta$. Thereafter, $\Delta(\Theta)$ denotes the set of mixed strategies of each player i in the first-stage of the gun-butter game.[6] In light of this formulation, the unit interval, $[0, 1]$, represents the set of possible activities and we will refer to $[0, 1]$ as the set of *mixed activities*. It is convenient to write $\theta_i \in [0, 1]$ for the mixed activity of player i that attaches probability θ_i to the appropriative activity, $\overline{\theta}$.[7] In a second-stage of the game, each player i chooses how much he expends effort (i.e., intensity), $G_i \in \mathbb{R}_+$, given the realization of his mixed strategy. For future references, let $\theta_N = (\theta_1, ..., \theta_n)$, and as usual, $\theta_{-i} = (\theta_1, ..., \theta_{i-1}, \theta_{i+1}, ...\theta_n)$ and $G_{-i} = (G_1, ..., G_{i-1}, G_{i+1}, ...G_n)$.
To summarize, we consider the following sequence of events.

1. Players simultaneously choose an activity θ_i that is a mixture of an appropriative and productive activity.
2. There is a chance move: given the mixed activity θ_i chosen by player i, nature chooses a pure activity in Θ, according to the mixed strategy of player i.
3. Each player privately observes the nature of his activity and players simultaneously choose an effort they apply to the resulting activity.

[5] The model is formulated in general terms such that different interpretations for the underlying structured environment are possible.

[6] The restriction that players have the same space of strategies is without loss of generality.

[7] From this viewpoint, appropriative and productive activities are thus just special, "extreme", mixed activities.

4. Players receive their payoffs. In particular, a player obtains a prize if he successfully appropriates outputs produced by others.

3 Partitioned correlated equilibrium

In our main approach, our foundation of CSFs requires that agents correlate their choice of an activity in the first stage of the gun-butter game. This is in line with Aumann's correlated equilibrium (1974, 1987). But, in our case correlation takes place within some *proper subset* of players only, whereas in a correlated equilibrium correlation is between *all* players.

3.1 definition

Attention in this section is focused on finite n-player normal-form games, $\Gamma = \langle N, (\Theta_i, U_i)_{i \in N} \rangle$, where Θ_i is a finite set of strategies available to player i, and $U_i : \times_{i \in N} \Theta_i \to \mathbb{R}$ is the payoff function of player i. A coalition structure, $\mathscr{P}(N)$, is a partition of N, and its elements are called coalitions (or groups). For any coalition $S \subseteq N$, let $\Theta_S = \times_{i \in S} \Theta_i$. For a set A let $\Delta(A)$ be the set of all probability measures. For any non-singleton coalition S, an element p_S of $\Delta(\Theta_S)$ is called a *correlated strategy distribution* for S. For any coalition structure $\mathscr{P}(N)$, let $p_{-S} \equiv (p_{S'})_{S' \in \mathscr{P}(N) \setminus \{S\}}$.
When a coalition S forms, the correlation device, $(\Omega_S, q_S, (\mathscr{P}^i_S)_{i \in S}) \equiv d_S$, of group S is described by a finite set of signals Ω_S, a probability distribution q_S over Ω_S and a partition $\mathscr{P}^i_S$ of Ω_S for every player $i \in S$. Since Ω_S is finite, the probability distribution q_S is just a real vector $q_S = (q_S(w))_{w \in \Omega_S}$.
From Γ and $(d_S)_{S \in \mathscr{P}(N)} \equiv d_{\mathscr{P}(N)}$, we define the extended game $\Gamma_{\mathscr{P}(N)}$ as follows:

- for each coalition $S \in \mathscr{P}(N)$, w is chosen in Ω_S according to q_S
- every player $i \in S$ is informed of the element $P^i_S(w)$ of $\mathscr{P}^i_S$ which contains w.
- Γ is played: every player i chooses a strategy θ_i in Θ_i and gets the utility $U_i(\theta_N)$ where $\theta_N = (\theta_i)_{i \in N}$.

A (pure) strategy for player $i \in S$ in $\Gamma_{\mathscr{P}(N)}$ is a mapping $\tau^i_S : \Omega_S \to \Theta_i$ which is $\mathscr{P}^i_S$-measurable.[8] Let $\tau_S = (\tau^i_S)_{i \in S}$ be a strategy profile in game $\Gamma_{\mathscr{P}(N)}$. The interpretation is that in, $\Gamma_{\mathscr{P}(N)}$, every player i in a coalition S chooses θ_i as a function of his private information on the random signal $w \in \Omega_S$ which is selected before the beginning of Γ.
For our purposes, what really matters in correlated equilibrium is the induced probability distribution over the action profiles. Therefore, with some abuse of terminology, we will directly define a correlated equilibrium as a

[8] In other words, $\tau^i_S(w') = \tau^i_S(w)$ if $w' \in P^i_S(w)$.

probability distribution over the action profiles. Let (d_S, τ_S) be a correlation device for coalition S. Its *induced* probability distribution over action profiles is given by the function $p_S : \Theta_S \to [0,1]$ defined by,

$$p_S(\theta_S) = q_S(\{w \in \Omega_S : \tau_S(w) = \theta_S\}) = \sum_{\{w \in \Omega_S : \tau_S(w) = \theta_S\}} q_S(w).$$

For any coalition $S \in \mathscr{P}(N)$, and any strategy profile, p_{-S}, we define, $U_i(\cdot, p_{-S}) : \Theta_S \to \mathbb{R}$, as the payoff function of player $i \in S$. This way, it is then natural to define the (parametrized) *intra-coalition game* of coalition S, $\Gamma_S(p_{-S}) = \langle S, (\Theta_i, U_i(\cdot, p_{-S}))_{i \in S} \rangle$, for every p_{-S}. The multi-linear extension of $U_i(\cdot, p_{-S})$ to $\Delta(\Theta_S)$ is still denoted by $U_i(\cdot, p_{-S})$.
In a partitioned correlated equilibrium, each coalition plays a correlated equilibrium of his own intra-coalition game.

Definition 1 *Let $\Gamma_S(p_{-S})$ be the intra-coalition game of S. A correlated distribution $p_S \in \Delta(\Theta_S)$ is a correlated equilibrium for coalition S in $\Gamma_S(p_{-S})$ if*

$$\sum_{\theta_{S \setminus i} \in \Theta_{S \setminus i}} p_S(\theta_S) U_i(\theta_S, p_{-S}) \geq \sum_{\theta_{S \setminus i} \in \Theta_{S \setminus i}} p_S(\theta_S) U_i(\theta_i^{'}, \theta_{S \setminus i}, p_{-S})$$

with $\theta_S = (\theta_i, \theta_{S \setminus i})$, $\forall i \in S$, $\forall \theta_i \in \Theta_i$,$\forall \theta_i^{'} \in \Theta_i$.

In a $\mathscr{P}(N)$-correlated equilibrium, each coalition $S \in \mathscr{P}(N)$ plays a correlated equilibrium given the strategies played within the other coalitions $S^{'} \neq S$. Formally:

Definition 2 *We say that a profile $(p_S)_{S \in \mathscr{P}(N)}$ is a partitioned correlated equilibrium with partition, $\mathscr{P}(N)$, ($\mathscr{P}(N)$-correlated equilibrium hereafter) for Γ if for any non-singleton coalition $S \in \mathscr{P}(N)$, distribution $p_S \in \Delta(\Theta_S)$ is a correlated equilibrium of $\Gamma_S(p_{-S})$ and, (ii) if player i is in a singleton coalition, $\{i\} \in \mathscr{P}(N)$, then i plays a best response to p_{-i}.*

Let us first discuss how the concept of partitioned correlated equilibrium is related to the correlated equilibrium concept of Aumann (1974, 1987). By definition, when $\mathscr{P}(N) = \{N\}$ i.e. the grand coalition forms, then a $\{N\}$-partitioned correlated equilibrium boils down to the usual correlated equilibrium concept. Thereafter we say that a $\mathscr{P}(N)$-correlated equilibrium is *trivial* if it coincides with a correlated equilibrium. Aumann's concept is a very powerful solution concept to model groups' behavior. Therefore, a partitioned correlated equilibrium seems to be a compelling solution concept when our aim is to simultaneously analyze the conflicts arising within and between the coalitions: in a partitioned correlated equilibrium, the play is indeed required to be self-enforcing *within* and *between* the coalitions.
A natural interpretation of the partitioned correlated equilibrium is that there is a mediator, one for each coalition $S \in \mathscr{P}(N)$, that selects a distribution of play for coalition S according to a distribution p_S and privately recommends an action to player $i \in S$, for every $S \in \mathscr{P}(N)$.
It is also instructive to relate the partitioned correlated equilibrium and

Nash equilibrium concept: a profile θ_N is a Nash equilibrium if and only if θ_N is a canonical $\{\{i\}\}_{i\in N}$-correlated equilibrium. Unlike the Nash equilibrium concept, one of the main feature of a partitioned correlated equilibrium is that each player must simultaneously consider the correlated distribution of his own coalition together with correlated distributions of other coalitions. However, notice that the definition does not imply that the resulting profile $(p_S)_{S\in\mathscr{P}(N)}$ must constitute a Nash equilibrium of some "coalitional" game where each coalition is treated as a single player. In particular, it is worth noting that the conditions for an "equilibrium" between the coalitions are weaker than what the Nash equilibrium concept requires: in a non-trivial partitioned correlated equilibrium, the distribution p_{-S} does not render each member i of a coalition S indifferent between his pure strategies θ_i whenever $p_S(\theta_{S\setminus\{i\}},\theta_i) > 0.$, The requirement is only that the distribution p_{-S} induced by the correlated equilibrium distributions of the other coalitions, $(p_{S'})_{S'\neq S}$, makes p_S a correlated equilibrium for the intra-group game of coalition S. Therefore, in a $\mathscr{P}(N)$-correlated equilibrium each distribution p_S is a correlated equilibrium of $\Gamma_S(p_{-S})$, which depends, in turns, on the profile of correlated equilibria $(p_{S'})_{S'\neq S}$ of the intra-coalition games, $\left\{\Gamma_{S'}(p_{-S'})\right\}_{S'\neq S}$. On the other hand, note that by definition, $\times_{j\in S}\Delta(\Theta_j)\subset\Delta(\times_{j\in S}\Theta_j)$. Thus, a non-trivial $\mathscr{P}(N)$-correlated equilibrium does not necessarily require the use of correlated mixtures within coalitions.
As it is well-known, a correlated equilibrium cannot be construed as a cooperative agreement externally enforced among coalition members. That is, members of a group *cannot* make agreements that are binding in the sense of providing some method (outside the given game) for punishing agents who violate the agreement.[9] Therefore, requiring the notion of partitioned correlated equilibrium does not imply that agents can write binding agreements within groups.
Finally, it is worth noting that our concept of partitioned correlated equilibrium is reminiscent of the first step of the coalitional concept developed in Ray and Vohra (1997). However, in our approach, a "coalitional strategy" p_S is just a surrogate of what the players in S can achieve non-cooperatively. By contrast, in Ray and Vohra, the best responses of a coalition are ordered by a Pareto criterion: their solution concept requires that an equilibrium between the coalitions is a strategy profile such that, given the strategic choice of the other coalitions, no coalition can improve all its members' utility.

3.2 Existence

For arbitrary coalition structures, $\mathscr{P}(N)$, the existence of a $\mathscr{P}(N)$-correlated equilibrium of a finite game Γ is a priori not guaranteed. Nevertheless, as shown below, finite games, always admit a $\mathscr{P}(N)$-correlated equilibrium,

[9] For a thorough discussion of this issue, see Aumann (1974) .

for all $\mathscr{P}(N)$.
It is well-known that the set of correlated equilibria of a game contains at least the convex hull of its Nash equilibria. For our purpose, we will be mainly concerned to the case in which *every* distribution p_S of a $\mathscr{P}(N)$-correlated equilibrium profile, $(p_S)_{S\in\mathscr{P}(N)}$, is a distribution over some of the Nash equilibria of the intra-coalition game of S, $\Gamma_S(p_{-S})$. In the sequel, we will refer to this particular subset of partitioned correlated equilibria as the set of *canonical* $\mathscr{P}(N)$*-partitioned correlated equilibria* of Γ.

Theorem 1 *Fix a coalition structure* $\mathscr{P}(N)$*. For every finite game,* $\Gamma = \langle N, (\Theta_i, U_i)_{i\in N}\rangle$ *,*
(1) there exists a $\mathscr{P}(N)$*-correlated equilibrium;*
(2) if each intra-coalition game, $\Gamma_S(p_{-S}), S \in \mathscr{P}(N)$*, has at least two Nash equilibria, there exists a canonical* $\mathscr{P}(N)$*-correlated equilibrium in which each (non-singleton) coalition S randomizes over the set of Nash equilibria of his intra-coalition game,* $\Gamma_S(p_{-S})$*.*

Proof. See Appendix A. ■
Property (2) of Theorem 1 exhibits an appealing feature of the $\mathscr{P}(N)$-correlated equilibrium concept: players within the same coalition S randomize over the multiple Nash equilibria. Property (1) proves that the existence of an arbitrary $\mathscr{P}(N)$-correlated equilibrium is not a issue if one assumes that players can agree on a particular profile of actions that is self-enforcing. More precisely, this result asserts that for any coalition structure, $\mathscr{P}(N)$, every finite game will have at least one $\mathscr{P}(N)$-correlated equilibrium, with players randomizing independently or using some correlated mixed strategies within each coalition. Here, the need to extend the game to correlated mixtures within coalitions plays the same role as the use of mixed strategies to prove the existence of Nash equilibria in finite games. Notice in particular that the conditions of existence of a non-trivial $\mathscr{P}(N)$-correlated equilibrium are sufficient not necessary. As in the case of the Nash equilibrium concept, it is always possible to have some pure Nash equilibria (resp. $\mathscr{P}(N)$-correlated equilibria) without resorting to mixtures (resp. correlation within coalitions). Nevertheless, the statement is tight in the sense that the use of independent mixed (rather than correlated) strategies within coalitions leads in general to the failure of the existence of a $\mathscr{P}(N)$-correlated equilibrium.

4 Partitioned correlated equilibria in the gun-butter game

For the purpose of this paper, we need to apply the concept of partitioned correlated equilibrium to the first stage of the gun-butter game. We first give a characterization of the $\{\{N \setminus i\}, \{i\}\} \equiv \mathscr{P}(N,i)$-correlated equilibria, $i = 1, ..., n$, of this game when the class of payoff functions is admissible.

4.1 Admissible payoffs

In the gun-butter game, the overall utility of player i depends on the decisions that player i and its opponents make about the choice of an activity – production or appropriation – and its intensity $\mathbf{G} = (G_i, G_{-i})$. In the remainder of this paper, we assume that the payoffs have the following properties.
Whether the effort is of a welfare enhancing or appropriative nature, it is not costless. Therefore the choice of an activity along with $\mathbf{G}$ delivers a utility to each player i, $U_i(\theta_i, \theta_{-i}, \mathbf{G})$, given by

$$U_i(\theta_i, \theta_{-i}, \mathbf{G}) = W_i(\theta_i, \theta_{-i}, \mathbf{G}) - C_i(G_i)$$

where $W_i(\theta_i, \theta_{-i}, \mathbf{G})$ is player i's *gross revenue*, subject to the players' choice of a vector of activities, $\theta = (\theta_i, \theta_{-i})$, with players' effort intensity profile $\mathbf{G}$. $C_i(G_i)$ is the cost of expending effort intensity G_i borne by player i regardless of his choice of an activity θ_i. We consider the class of cost functions, $C_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$.
Each player i has a strictly increasing production function, $f_i(\cdot)$, that produces a good for consumption y_i as a function of player i's effort G_i.Thus, a vector of production, $(f_1(G_1), ..., f_n(G_n)) \equiv f(\mathbf{G})$, corresponds to a bundle of consumption goods $(y_1, ..., y_n)$. It seems reasonable to require that the class of gross output functions, W_i, fulfill the following intuitive properties:

1. $W_i(\overline{\theta}_i, \underline{\theta}_{-i}, \mathbf{G}) = V_i(\mathbf{G})$ for all $\mathbf{G}$.
2. $W_i(\overline{\theta}_i, \overline{\theta}_{-i}, \mathbf{G}) = 0$ for all $\mathbf{G}$.
3. $W_i(\underline{\theta}_i, \overline{\theta}_{-i}, \mathbf{G}) \equiv L_i(f(\mathbf{G}))$ with $L_i(f(\mathbf{G})) < 0$ for all $\mathbf{G}$.
4. $W_i(\underline{\theta}_i, \underline{\theta}_{-i}, \mathbf{G}) = \widehat{U}_i(f(\mathbf{G})) + \widehat{U}_0(\mathbf{G}) + V_i(\mathbf{G})$ where $\widehat{U}_i(f(\mathbf{G})) \geq 0$ and $\widehat{U}_0(\mathbf{G}) \geq 0$ for all $\mathbf{G}$.
5. Let $\theta(k)$ be the activity profile when the number of players choosing $\theta \in \Theta$ is exactly k. Then, $W_i(\underline{\theta}_i, \underline{\theta}(k), \mathbf{G})$ and $W_i(\overline{\theta}_i, \overline{\theta}(k), \mathbf{G})$ are weakly increasing in $k = 0, ..., n-1$, for all $\mathbf{G}$.

From properties 1-4, it follows that profiles $\underline{\theta}_N$ and $\overline{\theta}_N$ are two Nash equilibria of the game played in the first-stage of the gun-butter game, $\Gamma(\mathbf{G})$, for any continuation profile $\mathbf{G}$. Assumption 5 is for simplicity of exposition; it could be weakened in several ways and our results would still hold. Basically, we use this assumption in order to analyze the $\mathscr{P}(N, i)$-correlated equilibria of $\Gamma(\mathbf{G})$. In fact, 5 ensures that profiles $\underline{\theta}_{N\setminus i}$ and $\overline{\theta}_{N\setminus i}$ are two Nash equilibria of the intra-coalition game, $\Gamma_{N\setminus i}(\mathbf{G})(p_i)$, played in the first-stage of the gun-butter game by coalition $N \setminus i$, for any continuation profile $\mathbf{G}$, any p_i with $p_i \equiv p_i(\cdot \mid \mathbf{G})$, and any number of players. However, notice that since we focus attention on canonical $\mathscr{P}(N, i)$-correlated equilibria (in our main approach), it is enough to have $\underline{\theta}_{N\setminus i}$ and $\overline{\theta}_{N\setminus i}$ as two Nash equilibria for coalitions, $N \setminus i$, of size $n - 1$. Hence, in this paper, we do not exploit the fact that this property continues to hold for some coalitions of smaller size. In addition, the need to have $\underline{\theta}_{N\setminus i}$ and $\overline{\theta}_{N\setminus i}$ as two Nash equilibria of $\Gamma_{N\setminus i}(\mathbf{G})(p_i)$ is in fact required to hold for only one, arbitrary, nondegenerate mixed strategy of player i. The reason for this is given in Remark 1. Even if assumption 5 is not fully exploited, it allows to view the two Nash equilibria of $\Gamma_{N\setminus i}(\mathbf{G})(p_i)$ as arising from a more general property of the underlying game. In fact, assumption 5 indicates that the benefit that an individual derives from being a producer depends on how many other individuals spend their effort on production. This property is reminiscent of Shelling's (1978) *threshold model of collective action*, in which the participation of an individual in an action depends on the fraction of the population engaged in the action. In our case, this property captures the idea of production externalities that depends only on the size of the population of rent-seekers.

The intuitions behind the other properties are as follows. Property (1) means that if player i chooses to appropriate others' production with an effort G_i while other players exert G_{-i} in a joint production process, then player i obtains a prize, $V_i(\mathbf{G})$. Property (2) indicates that when all players choose the gun activity, all players bear the cost of conflict and there is no production to seize. Property (3) means that when player i chooses productive activities while others engage in appropriative activities, then he cannot defend what he himself has produced. In this case anarchy prevails and player i cannot prevent the rest of the players from seizing his output. Here, $L_i(f(\mathbf{G}))$ permits us to capture the disutility incurred by player i in case of a pillage. In particular, $L_i(f(\mathbf{G}))$ may refer to the fact that player i derives disutility from not consuming the bundle of goods, $f(\mathbf{G})$. This might explicitly involves loss aversion, as the disutility of loss may exceed the production value $f_i(G_i)$. Property (4) represents the "*peaceful outcome*" in which all players choose the peace activity, $\underline{\theta}$. Hence, $\widehat{U}_i(f(\mathbf{G}))$ reflects the fact that i may potentially consume the quantities of output, $(f_1(G_1), ..., f_n(G_n)) = f(\mathbf{G})$, produced by others. The prize $V_i(\mathbf{G})$ together with $\widehat{U}_0(\mathbf{G})$ have intuitive interpretations: they reflects the fact that in the peaceful outcome player i exercises some control over the fruits of his effort. The additional term, $\widehat{U}_0(\mathbf{G})$, is assumed to be identical across players. Thus, it can be thought of as a minimal lump-sum transfer of total output.

Assume now that we have a list of payoff functions $\{U_i\}_{i\in N}$ fulfilling conditions 1-5. When payoffs are admissible, Theorem 1 (2) guarantees the existence of a *canonical* $\{\{N \setminus i\}\{i\}\} \equiv \mathscr{P}(N,i)$-correlated equilibrium, $i = 1, ..., n$, of $\Gamma(\mathbf{G})$ (the game played in the first-stage of the gun-butter game conditionally on the continuation strategy profile $\mathbf{G}$), in which players in coalition $N \setminus i$ randomize over $\underline{\theta}_{N\setminus i}$ and $\overline{\theta}_{N\setminus i}$. In order to render our notion of partitioned correlated equilibrium more familiar to the reader, the following Lemma provides a direct proof of this fact.

Lemma 1 *Fix a profile of effort* $\boldsymbol{G}$. *If* $\{U_i\}_{i\in N}$ *is a list of admissible payoff functions, then for all* $i \in N$, *the game* $\Gamma(\boldsymbol{G})$ *has a non-trivial canonical* $\mathscr{P}(N,i)$*-correlated equilibrium,* $(p_i(\cdot \mid \boldsymbol{G}), p_{N\setminus i}(\cdot \mid \boldsymbol{G}))$, *in which players in* $N \setminus \{i\}$ *randomize over the two pure Nash equilibria of* $\Gamma_{N\setminus i}(p_i)(\boldsymbol{G})$, $\underline{\theta}_{N\setminus i}$ *and* $\overline{\theta}_{N\setminus i}$.

Proof. It suffices to apply Theorem 1 (2). Otherwise, we can also check this result directly as follows. To economize on notations, let $p_i = p_i(\cdot \mid \mathbf{G})$. Let $\Gamma_{N\setminus i}(p_i)(\mathbf{G})$, be the parametrized intra-coalitional game of coalition $N \setminus i$, under a profile of effort $\mathbf{G}$ when player i plays $p_i \in \Delta(\Theta)$. It is well-known that *any* convex combination of Nash equilibria of a normal form game defines a (canonical) correlated equilibrium of this game. From the admissible payoff conditions (see condition 5), $\underline{\theta}_{N\setminus i}$ and $\overline{\theta}_{\setminus i}$ are two pure Nash equilibria of $\Gamma_{N\setminus i}(p_i)(\mathbf{G})$, regardless of the distribution p_i. Therefore, we can show that a non-trivial canonical $\mathscr{P}(N,i)$-correlated equilibrium exists by picking the distribution $p_{N\setminus i}$ which renders player i indifferent between his two pure activities, $\underline{\theta}$ and $\overline{\theta}$ (here we apply condition (ii) of the definition of a $\mathscr{P}(N)$-correlated equilibrium). Since the resulting distribution, $p_{N\setminus i}$, is a canonical correlated equilibrium of $\Gamma_{N\setminus i}(p_i)(\mathbf{G})$, the profile $(p_{N\setminus i}, p_i)$ constitutes a $\mathscr{P}(N,i)$-correlated equilibrium whenever p_i is i's best response to $p_{N\setminus i}$. This completes the proof. ■

Remark 1 Suppose assumption 5 is relaxed and there exists only *certain* non-degenerate mixed strategies, $\widetilde{p}_i(\cdot \mid \mathbf{G})$, such that $\underline{\theta}_{N\setminus i}$ and $\overline{\theta}_{\setminus i}$ are two pure Nash equilibria of $\Gamma_{N\setminus i}(\widetilde{p}_i(\cdot \mid \mathbf{G}))(\mathbf{G})$, for every $\mathbf{G}$. In a $\mathscr{P}(N,i)$-correlated equilibrium, player i is indifferent between $\underline{\theta}$ and $\overline{\theta}$. Hence, in particular, we can choose the distribution $\widetilde{p}_i(\cdot \mid \mathbf{G})$ as the mixed strategy used by player i in the canonical $\mathscr{P}(N,i)$-correlated equilibrium of $\Gamma(\mathbf{G})$. Clearly, this shows that assumption 5 is much stronger than necessary.

5 $\mathscr{P}(N)$-correlated PBE

We will now use our notion of $\mathscr{P}(N)$-correlated equilibrium as a refinement criterion to select certain perfect Bayesian equilibria (PBEs) of the gun-butter game.

5.1 Strategies and beliefs in the gun-butter game

In the gun-butter game, player i's behavioral strategy specifies a (possibly degenerate) probability distribution, $p_i \in \Delta(\Theta)$, in stage 1 and a pure effort level $\widehat{G}_i(\theta_i) \in \mathbb{R}_+$ for each outcome $\theta_i \in \Theta$ of the mixed strategy p_i in stage 2. Formally, this means that $\widehat{G}_i$ must specify an effort level G_i for each information set, $\{(\theta_i, \theta_{-i})\}_{\theta_{-i}\in\Theta_{-i}} \equiv I_i(\theta_i)$, with $\theta_i \in \Theta$ and $\theta_{-i} \in \Theta_{-i}$, in stage 2 of the gun-butter game.
We want to analyze the gun-butter game for its PBEs. More specifically, we are interested to select those PBEs in which players play a non-trivial canonical partitioned correlated equilibrium in the first-stage of the game. Clearly, this requires that we specify the players' strategies and beliefs conditionally on a coalition structure. It is therefore useful to introduce the following notations.

In the case of the coalition structure, $\mathscr{P}(N,i)$, in which all players $N \setminus \{i\}$ form a coalition against i, we shall denote by $(p_i, \widehat{G}_i)_{|\mathscr{P}(N,i)} \equiv x^i_{|\mathscr{P}(N,i)}$ the strategy-beliefs pair of i. Given $\mathscr{P}(N,i)$, define player i's interim beliefs when player i forms a stand-alone coalition against the others, in $\Gamma(\mathbf{G})$, as a probability distribution on Θ_{-i}, $\mu^i_{\mathscr{P}(N,i)}(\cdot \mid \theta_i, \mathbf{G})$, conditionally on i's pure activity $\theta_i \in \Theta$ and a continuation strategy profile $\mathbf{G}$. Let $(x,\mu)^j_{|\mathscr{P}(N)}$ be a strategy-belief for player j under a coalition structure $\mathscr{P}(N)$ and $((x,\mu)^j_{|\mathscr{P}(N)})_{j\in N} \equiv (x,\mu)_{|\mathscr{P}(N)}$ denotes a strategy-belief profile under coalition structure $\mathscr{P}(N)$.

Definition 3 *Let $\mathscr{P}(N)$ be an arbitrary coalition structure. We say that a strategy-belief profile, $(x,\mu)_{|\mathscr{P}(N)}$, is a $\mathscr{P}(N)$-correlated PBE of the gun-butter game if it is a PBE, such that for any continuation strategy $\boldsymbol{G}$, we have that $\{p_S(\cdot \mid \boldsymbol{G})\}_{S\in\mathscr{P}(N)}$ is a canonical $\mathscr{P}(N)$-correlated equilibrium of $\Gamma(\boldsymbol{G})$.*

In the discussion below, we argue that in a $\mathscr{P}(N)$-correlated PBE, all the desiderata of a usual PBE are indeed verified.
By definition, in a canonical $\mathscr{P}(N)$-correlated PBE, any player i in a coalition S cannot benefit from deviating unilaterally from the recommendation made by the mediator of his coalition S given the recommendations made by the other mediators of other coalitions $S' \neq S$, in the first stage of the gun-butter game. A a result, in a $\mathscr{P}(N)$-correlated PBE, players must be *sequentially rational* as in any standard PBE. Moreover, in a $\mathscr{P}(N)$-correlated equilibrium, the play of stage 1 of the gun-butter game implies that conditional on θ_j, the interim equilibrium beliefs of players $j \in S$,

$$\mu^{*j}_{\mathscr{P}(N)}(\theta_{-j} \mid \theta_j, \mathbf{G}) = p(\theta_{S\setminus j} \mid \theta_j, \mathbf{G}) \times \prod_{S' \in \mathscr{P}(N)\setminus\{S\}} p_{S'}(\theta_{S'} \mid \mathbf{G}),$$

are correct, for any continuation strategy $\mathbf{G}$. In a $\mathscr{P}(N)$-correlated PBE, these conditions must hold for any player $j \in N$. Hence, our notion of $\mathscr{P}(N)$-correlated PBE captures all the standard requirements of a PBE: beliefs $\mu^j_{\mathscr{P}(N)}(\cdot \mid \theta_j, \mathbf{G})$ are consistent with the strategies, which are optimal given the beliefs. Therefore, the $\mathscr{P}(N)$-correlated PBE constitutes a standard PBE.
Next, we use the concept of partitioned correlated equilibrium in order to select a particular subset of PBEs. In fact, our notion of a rationalizable CSF requires that we use an even stronger form of refinement, since we shall focus our analysis on PBEs in which agents play a *canonical* $\mathscr{P}(N,i)$-correlated equilibrium in the first stage of the gun-butter game.

5.2 Rationalizable CSFs

A (general) contest is a n-player strategic-form game, $\langle N, (\mathcal{G}_i, \Pi_i)_{i\in N}\rangle$, with $\mathcal{G}_i \subseteq \mathbb{R}_+$ the set of actions available to player i, and $\Pi_i : \times_{i\in N}\mathcal{G}_i \to \mathbb{R}$ the payoff function of player i defined by $\Pi_i(G_i, G_{-i}) = p_i(G_i, G_{-i})V_i(\mathbf{G}) - C_i(G_i)$ with player i's valuations for winning and $p_i(G_i, G_{-i})$ the contest success function defined such that $p_i(G_i, G_{-i}) \geq 0$ and $\sum_{i \in N} p_i(G_i, G_{-i}) = 1$ for all (G_i, G_{-i}).

The definition below says that a canonical partitioned correlated PBE of the gun-butter game induces a contest for player i whenever i's (equilibrium) interim payoff derived in a stand-alone coalition against all others coincides with his expected payoff obtained in a contest.

Definition 4 *We say that a canonical $\mathscr{P}(N,i)$-correlated PBE $(x,\mu)_{\mid \mathscr{P}(N,i)}$ of the gun-butter game induces the contest, $\langle N, (\mathcal{G}_i, \Pi_i)_{i \in N} \rangle$, for player i if his interim equilibrium belief, $\mu^{*i}_{\mathscr{P}(N,i)}(\cdot \mid \overline{\theta}_i, \boldsymbol{G})$, induces in the game $\Gamma(\boldsymbol{G})$ starting at information set, $I_i(\overline{\theta}_i) = \left\{(\overline{\theta}_i, \theta_{-i})\right\}_{\theta_{-i} \in \Theta_{-i}}$, a conditional expected payoff*

$$\sum_{\theta_{-i} \in \Theta_{-i}} \mu^{*i}_{\mathscr{P}(N,i)}(\theta_{-i} \mid \overline{\theta}_i, \boldsymbol{G}) U_i(\overline{\theta}_i, \theta_{-i}, \boldsymbol{G}) = p_i(\boldsymbol{G}) V_i(\boldsymbol{G}) - C_i(G_i) \equiv \Pi_i(\boldsymbol{G})$$

for all $\boldsymbol{G}$.

In the context of the present model, an appealing feature underlying the partioned correlated equilibrium concept is that player will form a coalition $S \subseteq N$ because by doing so, members of S have the possibility to exploit the correlation device, d_S, in order to correlate their activities in the first-stage of the gun-butter game. Hence, a correlation device, d_S, is a characteristic which can be used to describe the subset of individuals $S \subseteq N$. For instance, if all players decide to form a singleton coalition, then the signals are independent across all players, thereby restricting the possible outcomes to those of Nash equilibria. By contrast, if agents decide to form the grand coalition, $\{N\}$, then all players can correlate their activities in a correlated equilibrium. All other coalition structures will lead agents to play in a partitioned correlated equilibrium.

In this context, one may imagine that interactions across individuals occur in two main steps. In the first step, agents form groups in order to exploit the correlation devices specific to their own group. In the second stage, they engage in the gun-butter game.[10] With this scenario in mind, it is therefore natural to couch our notion of rationalizable CSF in terms of the following coalition formation process.

Assume that when players decide to form the grand coalition, they exploit the correlation device d_N in order to achieve the peaceful outcome by correlating their actions on the profile of activities, $\underline{\theta}_N$, in the first-stage of the gun-butter game. Under admissible payoff functions, this peaceful outcome can be achieved in a (trivial) canonical $\{N\}$-correlated PBE of the

[10] Note that this time line is the same as in the games of coalition formation developed by Hart and Kurz 1983, Bloch 1996 and Ray and Vohra 1997.

gun-butter game.[11] In particular, any coalition structure, $\mathscr{P}(N,i)$ corresponds to a situation where player i does not correlate his activities with the rest of players, whereas all players within coalition $N\setminus\{i\}$ exploit correlation device $d_{N\setminus\{i\}}$, "against" player i. This scenario allows to understand a CSF as the result of an underlying coalition formation process, where the possible payoffs obtained in a coalition structure $\mathscr{P}(N,i)$ and the grand coalition $\{N\}$ are obtained from the canonical $\mathscr{P}(N,i)$-correlated PBE and a $\{N\}$-correlated PBE , respectively.[12] Formally:

Definition 5 *Assume that agents play $\underline{\theta}_N$ i.e. the peaceful outcome, when they form the grand coalition $\{N\}$. Then, a CSF $\{p_1(\boldsymbol{G}), p_2(\boldsymbol{G}), ..., p_n(\boldsymbol{G})\}$ is rationalizable if there exists a list of admissible payoff functions $\{U_i(\theta, \boldsymbol{G})\}_{i\in N} = U$, and a set of canonical $\mathscr{P}(N,i)$-correlated PBEs $i = 1, ..., n$, of the gun-butter game such that when player i secedes from the grand coalition to form coalition, $\{i\} \in \mathscr{P}(N,i)$, she has*

(i) for all $\boldsymbol{G}$, an interim equilibrium belief $\mu^{*i}_{\mathscr{P}(N,i)}(\theta_{-i} = \underline{\theta}_{-i} \mid \overline{\theta}_i, \boldsymbol{G}) = p_i(\boldsymbol{G})$ at $I_i(\overline{\theta}_i)$ and;
(ii) a contest, $\langle N, (\mathcal{G}_i, \Pi_i)_{i\in N}\rangle$, is induced for i.

In this view, a CSF is thus a list of interim equilibrium beliefs that is induced when each player i contemplates playing the first-stage of the gun-butter in a stand-alone coalition rather than correlating his peaceful activity with the other agents, anticipating that the rest of players will correlate their activities accordingly.
Note that condition (i) is very much in line with the classical interpretation of CSFs as win probabilities. It tells us that a rationalizable win probability for player i must coincide with i's equilibrium belief that all other players have chosen to devote their effort to usefully productive activities, when i has chosen to grab others' output. Thus, in our setup, a CSF arises as the probability that each player i successfully appropriates others' output whenever i contemplates to play the gun-butter game against the rest of players. This interpretation of a CSF is thus also in line with the traditional interpretation of a "winner-take-all-contest", whereby a player is able to claim the entire production of others as his prize, leaving nothing. Condition (ii) implies that one can think of a contest as the conditional expected payoffs induced on the equilibrium path of some certain PBEs of the gun-butter game.
As already mentioned, our formulation is reminiscent of the noncooperative games of coalition formation initially proposed by Hart and Kurtz (1983) – called the δ model – and used by Bloch et al. (2006) and Sanchez-Pagés (2008), in which each player contemplates unilaterally his deviation from the grand coalition. In these models, the external players remain together

[11] Recall that admissible payoff functions imply that $\underline{\theta}_N$ is a Nash equilibrium of $\Gamma(\mathbf{G})$ for all continuation profile $\mathbf{G}$.
[12] By definition, a pure Nash equilibrium induces a canonical correlated equilibrium.

following the secession, so that the seceding agent faces a coalition of all the other players, as it is the case in a $\mathscr{P}(N,i)$-correlated equilibrium. In the present paper, this is modeled by the fact that each player breaking the peaceful agreement anticipates that the rest of players will correlate their activities in the first-stage of the gun butter game.

6 Results

Our first application of the concept of a rationalizable CSF concerns (2). In fact, many papers dealing with contest models assume (2) in which the outcome of contests depends on the ratio of efforts (see e.g., Nitzan 1994 and Konrad 2007).
In our model, $f_j(G_j)$ represents player j's own output when he commits to the productive activity. Hence, if player i successfully seizes the output of others, $\sum_{j\in N\setminus\{i\}} f_j(G_j)$ represents the total amount of goods available for consumption in the economy for player i. By contrast, $-f_i(G_i)$ corresponds to the loss in consumption incurred by agent i in case of an act of pillage i.e., player i losses his control over his own output. Our first result shows that when players' preferences capture this simple (endogenous) (re)allocation of outputs among players viz. players have linear utility functions, Tullock CSFs are rationalizable.

Proposition 1 *When the list, $\{U_i(\theta,\boldsymbol{G})\}_{i\in N} = U$, is admissible and the goods produced by others are viewed as perfect substitutes for each player in the peaceful outcome i.e., $\widehat{U}_i(f(\boldsymbol{G})) = \sum_{j\in N\setminus\{i\}} f_j(G_j)$, with $\widehat{U}_0(\boldsymbol{G}) = 0$, and the disutility in case of pillage is $L_i(f(\boldsymbol{G})) = -f_i(G_i)$, then the (general) additive CSF (2) is rationalizable.*

Proof. Consider $\mathscr{P}(N,i) = \{\{i\},\{N\setminus\{i\}\}\}$ i.e., player i forms a stand-alone coalition. Conditional on continuation strategies $\mathbf{G}$, let $p_i(\cdot\mid\mathbf{G}) \in \Delta(\Theta)$ and $p_{N\setminus\{i\}}(\cdot\mid\mathbf{G}) \in \Delta(\Theta_{-i})$ be the conditional probability distributions of player i and coalition $N\setminus\{i\}$, in the first stage of the gun-butter game. We will analyze the canonical $\mathscr{P}(N,i)$-correlated PBE of the gun-butter game. In a canonical $\mathscr{P}(N,i)$-correlated PBE, sequential rationality requires – assuming that the play continues according to $\mathbf{G}$ –, that profile $(p_i(\cdot\mid\mathbf{G}), p_{N\setminus\{i\}}(\cdot\mid\mathbf{G}))$ is a canonical $\mathscr{P}(N,i)$-correlated equilibrium of the game, $\Gamma(\mathbf{G})$, played in the first stage of the gun-butter game under continuation profile $\mathbf{G}$.
Let us first concentrate on condition (i) of rationalizability. Recall that admissibility implies that $\Gamma_{N\setminus\{i\}}(p_i(\cdot\mid\mathbf{G}),\mathbf{G})$ has two pure Nash equilibria, $(\underline{\theta}_j)_{j\neq i}$ and $(\overline{\theta}_j)_{j\neq i}$. We first need to examine the (canonical) correlated equilibria of $\Gamma_{N\setminus\{i\}}(p_i(\cdot\mid\mathbf{G}),\mathbf{G})$ where players in coalition $N\setminus i$ randomize over $(\underline{\theta}_j)_{j\neq i}$ and $(\overline{\theta}_j)_{j\neq i}$. In such a canonical $\mathscr{P}(N,i)$-correlated equilibrium, the indifference condition for player i implies that distribution $p_{N\setminus\{i\}}$ verifies $p^*_{N\setminus\{i\}}(\underline{\theta}_{-i}\mid\mathbf{G}) = \frac{U_i((\overline{\theta}_i,\overline{\theta}_{-i}),\mathbf{G}) - U_i((\underline{\theta}_i,\overline{\theta}_{-i}),\mathbf{G})}{U_i((\overline{\theta}_i,\overline{\theta}_{-i}),\mathbf{G}) - U_i((\underline{\theta}_i,\overline{\theta}_{-i}),\mathbf{G}) + U_i((\underline{\theta}_i,\underline{\theta}_{-i}),\mathbf{G}) - U_i((\overline{\theta}_i,\underline{\theta}_{-i}),\mathbf{G})}$. Using the payoff conditions given in Proposition 1, it is then easy to see that

$p^*_{N\setminus\{i\}}(\underline{\theta}_{-i}\,|\,\mathbf{G}) = \frac{f_i(G_i)}{\sum_{j\in N} f(G_j)}$. Moreover, at a PBE, the Bayesian updating requires that beliefs are correct, thereby inducing that i's interim beliefs verify $\mu^{*i}_{\mathscr{P}(N,i)}(\theta_{-i} = \underline{\theta}_{-i}\,|\,\theta_i, \mathbf{G}) = p^*_{N\setminus\{i\}}(\underline{\theta}_{-i}\,|\,\mathbf{G})$. Hence condition (i) for rationalizability is met. Last we check (ii). In the canonical $\mathscr{P}(N,i)$-correlated PBE, player i is indifferent between $\underline{\theta}$ and $\overline{\theta}$. Hence, when he holds belief, $\mu^{*i}_{\mathscr{P}(N,i)}(\cdot\,|\,\overline{\theta}_i, \mathbf{G})$, player i's conditional expected payoff,

$$U_i(\overline{\theta}_i, \mu^{*i}_{\mathscr{P}(N,i)}\,|\,\mathbf{G}) \equiv \sum_{\theta_{-i}\in\Theta_{-i}} \mu^{*i}_{\mathscr{P}(N,i)}(\theta_{-i}\,|\,\overline{\theta}_i, \mathbf{G}) U_i(\overline{\theta}_i, \theta_{-i}, \mathbf{G}),$$

boils down to $\frac{f_i(G_i)}{\sum_{j\in N} f_j(G_j)}(V_i(\mathbf{G}) - C_i(G_i)) + (1 - \frac{f_i(G_i)}{\sum_{j\in N} f_j(G_j)})(-C_i(G_i))$. This expression readily simplifies as $\frac{f_i(G_i)}{\sum_{j\in N} f_j(G_j)} V_i(\mathbf{G}) - C_i(G_i)$. These equilibrium conditions hold for any arbitrary agent i playing $\Gamma(\mathbf{G})$ in a canonical $\mathscr{P}(N,i)$-correlated PBE. Thus, we have generated a collection of probabilities, $\mu^{*i}_{\mathscr{P}(N,i)}(\underline{\theta}_{-i}\,|\,\overline{\theta}_i, \mathbf{G}) = \frac{f_i(G_i)}{\sum_{j\in N} f_j(G_j)} \equiv p_i(\mathbf{G})$ for $i = 1, ..., n$ that forms a probability distribution for every $\mathbf{G}$. This completes the proof. ■

Example 1 Suppose that each player i has the Cobb-Douglas production function, $f_i(G_i) = aG_i^\sigma$ where a and σ are two positive scalar so that aG_i^σ corresponds to the number of units of good y_i. Thus, we are considering production functions such as the one originally studied by Tullock (1980). The payoff conditions of proposition 1 entail that each player i's utility function has perfect substitution over the goods produced by others in the peaceful outcome i.e., $\widehat{U}_i(f(\mathbf{G})) = \sum_{j\neq i} aG_j^\sigma$. Thus, in light of proposition 1 we conclude that (1) is rationalizable.

Example 2 Next we further illustrate how our framework may be used to examine the implications of behavioral assumptions on the interpretation of CSFs. Assume that there are only two players and suppose that every player $i(=1,2)$ exhibits risk aversion (over consumption bundles) in the peaceful outcome (i.e. the state in which all players choose a productive activity), in that $\widehat{U}_i(f(\mathbf{G})) = \frac{G_j^{1-r}}{1-r}$ for r chosen so that $r > 0, r \neq 1$. On the other hand, assume that the disutility incurred in case of pillage is given by $\widehat{L}_i(f(\mathbf{G})) = -\widehat{U}_i(f(\mathbf{G}))$. Then, using proposition 1, it is straightforward to show that the power form $p_i^* = p_i(\mathbf{G}) = \frac{G_i^{\sigma(r)}}{G_1^{\sigma(r)} + G_2^{\sigma(r)}}$ where $\sigma(r) \equiv 1 - r$ is rationalizable. It is well-known that r equals the degree of relative risk aversion. Thus, an interpretation of the power $\sigma(r)$ is that it represents the degree of relative risk aversion behavior in consumption of players in the peaceful outcome.

6.1 Pride and status in the peaceful outcome

The second important popular class of CSFs builds on the idea that only differences in effort matter. It has been proposed by Hirshleifer (1989) and further studied in Skaperdas (1996), Baik (1998) and Che and Gale (2000). In particular, Che and Gale postulate the following piecewise linear difference-form

$$p_1 = \max\left\{\min\left\{\frac{1}{2} + \sigma(G_1 - G_2), 1\right\}, 0\right\} \text{ for } p_1 = 1 - p_2 \tag{3}$$

where σ is a positive scalar. Corchón and Dahm (2008) consider the following extension of (3) to three contestants. Suppose without loss of generality that $G_1 \geq G_2 \geq G_3$ and,

$$p_1 = \min\left\{\frac{1}{2} + \frac{1}{s}(G_1^\sigma - G_2^\sigma), 1\right\} \text{ for, } p_1 = 1 - p_2, p_3 = 0;$$

whenever $G_1^\sigma - G_3^\sigma \geq \frac{s}{3}$ and otherwise,

$$p_i = \frac{1}{3} + \frac{1}{2s}(2G_i^\sigma - G_j^\sigma - G_k^\sigma) \text{ for } i = 1, 2, 3 \text{ and } i, \neq j, k.$$

where s and σ are two positive reals. To rationalize this class of CSFs, we will examine the implications of the presence of relative concerns in individuals' preferences. As shown in the proposition below, parameter s may be interpreted as the *propensity for pride*. Individual i is "proud" in the sense that his utility in the peaceful outcome is reduced when the production level of others increase. More specifically, our foundation for this family of CSFs imports a utility function similar to that of Fehr-Schmidt (1999). However, in our context, players' utility in the peaceful outcome arises largely from how their output – instead of income – ranks relative to those produced by others. The idea is that players strive to produce more than others in the peaceful outcome for the sake of higher status.

For notational convenience we will state the next result for the three player-case. Nevertheless, it is straightforward to extend the following result to more than three contestants.

Let $N = \{1, 2, 3\}$ and assume without loss of generality that $G_1 \geq G_2 \geq G_3$. Suppose that individuals who produce the biggest outputs in the peaceful outcome are lauded for enhancing social welfare. This confers on them the pride of status, (at the same time as it inflicts envy on the others). We capture this idea by supposing that the prize $V_i(\mathbf{G})$ represents the pride that i feels in the peaceful outcome. Formally, we will assume a prize of the form,

$$V_i(\mathbf{G}) = \begin{cases} \alpha(2G_i^\sigma - G_j^\sigma - G_k^\sigma) \text{ if } G_i^\sigma - G_k^\sigma < K; \\ \alpha'(G_i^\sigma - G_j^\sigma) \text{ if } \; G_i^\sigma - G_k^\sigma \geq K, \end{cases}$$

where parameters α, α' and K are strictly positive.

In this setting, cardinal comparisons affect i's utility $\widehat{U}_i$ in the peaceful outcome with the following interpretation: individual i cares about his production level vis-a-vis others, feeling pride when his production is larger relative others by a long shot. Here, the number K can be thought of as a threshold that roughly measures the perceived relative effort of individual i into enhancing social welfare. If i feels he is relatively deserving, then he compares his performance with the next-best player only. In the following proposition, we show that with these preferences we can rationalize the family of difference-form CSFs. Hence, in our setup, the family of difference-form CSFs arises from the fact that, in the peaceful outcome, a player's well-being depends on how his production compares to that of others.

Proposition 2 *Suppose that* $N = \{1, 2, 3\}$. *Assume without loss of generality that* $G_1^\sigma \geq G_2^\sigma \geq G_3^\sigma$ *where* $\sigma > 0$. *When the list,* $\{U_i(\theta, \boldsymbol{G})\}_{i \in N} = U$, *is admissible with* $\widehat{U}_0(\boldsymbol{G}) = 0$ *for all* $\boldsymbol{G}$, *and ,*

$$\widehat{U}_i(f(\boldsymbol{G})) = \frac{1}{2} + V_i(\boldsymbol{G}) \text{ for } i = 1, 2 \text{ and } \widehat{U}_3(f(\boldsymbol{G})) = 0 \text{ whenever } G_1^\sigma - G_3^\sigma \geq \frac{s}{3}$$

where $V_i(\boldsymbol{G}) = \frac{1}{s}(G_i^\sigma - G_j^\sigma)$ *for* $s > 0$ *and otherwise*

$$\widehat{U}_i(f(\boldsymbol{G})) = \frac{2}{3} + V_i(\boldsymbol{G}) \text{ for } i = 1, 2, 3$$

with $V_i(\boldsymbol{G}) = \frac{1}{2s}(2G_i^\sigma - G_j^\sigma - G_k^\sigma)$, *for* $i \neq j, k$, *and the loss in case of pillage is* $L_i(\widehat{U}_i(f(\boldsymbol{G}))) = -\frac{1}{2} - V_i(\boldsymbol{G})$ *for* $i = 1, 2$ *where* $V_i(\boldsymbol{G}) = \frac{1}{s}(G_i^\sigma - G_j^\sigma)$ *and* $L_3(\widehat{U}_3(f(\boldsymbol{G}))) = 0$ *whenever* $G_1^\sigma - G_3^\sigma \geq \frac{s}{3}$ *and otherwise*

$$L_i(\widehat{U}_i(f(\boldsymbol{G}))) = -\frac{1}{3} - V_i(\boldsymbol{G}) \text{ for } i = 1, 2, 3$$

with $V_i(\boldsymbol{G}) = \frac{1}{2s}(2G_i^\sigma - G_j^\sigma - G_k^\sigma)$, *for* $i \neq j, k$, *then the above extension of (3) is rationalizable.*

Proof. See Appendix B. ■

6.2 Rank-dependent utility in the peaceful outcome

Alcade and Dahm (2007) stress the importance of CSFs incorporating simultaneously an absolute and relative criterion. In this section, we propose to rationalize the following extension of their *serial* CSF,

$$p_i = \sum_{j=i}^{n} \frac{f(G_j) - f(G_{j+1})}{j f_h(\mathbf{G})} \text{ where } f(G_{n+1}) = 0, \tag{4}$$

for $i = 1, ..., n$ where $f_h(\mathbf{G}) = \max\{f(G_1), f(G_2), ..., f(G_n)\}$ is the highest production level under a vector of effort $\mathbf{G}$. We will show that in order to derive this form of CSF one needs to assume that players exhibit rank-dependent preferences as in Quiggin (1981, 1982).[13]

[13] This class of non-expected utility functions is the basis of prospect theory (Tversky and Kahneman 1992)

Imagine that i's utility in the peaceful outcome depends on his (subjective) prior distribution of output. Formally, suppose that each player $i \in N$ has a *subjective prior* probability distribution, $(\widetilde{\alpha}_1^i, ..., \widetilde{\alpha}_n^i)$, over an ordered vector of outputs, $f(\mathbf{G}) = (f(G_1), f(G_2), ..., f(G_n))$ where $f(G_1) \leq f(G_2), ..., \leq f(G_n)$ for all $\mathbf{G}$. Consider i's (subjective) lottery $(G_1, \widetilde{\alpha}_1^i; G_2, \widetilde{\alpha}_2^i; ...; G_n, \widetilde{\alpha}_n^i)$ where $\widetilde{\alpha}_j^i$ means that i assigns probability $\widetilde{\alpha}_j^i$ to effort level G_j. It will also prove useful to consider the lottery $(G_1, \alpha_1^i; ..., G_i, \alpha_i^i; ...; G_n, \alpha_n^i) \equiv L_i$ with $\alpha_j^i = \sum_{l=j}^n \widetilde{\alpha}_l$ and $\alpha_j^i = 0$ for all $j < i$. Thus, α_j^i denotes i's prior probability of getting a consumption level of at least $f(G_j)$.
Assume further that an individual's utility in the peaceful outcome depends on his production relative to that of others. Suppose in particular that i's utility derived in the peaceful outcome arises solely from the minimal *positive increments* in production levels i.e., output levels less than or equal to $f(G_i)$ do not enter i's utility in the peaceful outcome as i's attention is only focused on the potential gains in productions $f(G_{i+1}), ..., f(G_n)$ above his own output. Such preferences can be modeled by assuming that each individual i has preferences that are "*rank dependent*" in the peaceful outcome.[14] Formally, suppose that $\widehat{U}_i(f(\mathbf{G}))$ is equal to the following welfare functional

$$F_i(L_i) = \sum_{j=1}^{n} w_i(\alpha_j^i)\left(f(G_j) - f(G_{j-1})\right),$$

where $w_i : [0,1] \to [0,1]$ is i's *weighting function* which is strictly increasing and continuous with $w_i(0) = 0$ and $w_i(1) = 1$. These preferences are referred to as rank dependent because a change in the ranking of individuals' output, $f(\mathbf{G})$, would affect the functional. Intuitively, this utility function entails that the attention given by individual i to the production level of a good y_j depends not only on the probability of the outcome, $f(G_j)$, it also depends on how much larger $f(G_j)$ is in comparison to $f(G_{j-1})$. Such preferences reflect the fact that in the peaceful outcome, each individual i enjoys outputs that are larger than i's own production level, whereas the cost is borne privately by others. The next proposition says that if each individual has rank dependent preferences over the production of others in the peaceful outcome then the serial CSF is rationalizable.

Proposition 3 *When the list* $\{U_i(\theta, \boldsymbol{G})\}_{i \in N} = U$ *is admissible, and the utility of consumption over the goods produced by others in the peaceful outcome is given by the rank-dependent utility (RDU),* $F_i(L_i) = \widehat{U}_i(f(\boldsymbol{G}))$ *with the weighting function,* $w_i(\alpha_j^i) = \frac{1}{j}$ *whenever* $j \geq i$, $\widehat{U}_0(\boldsymbol{G}) = f_h(\boldsymbol{G})$, $V_i(\boldsymbol{G}) = \frac{\widehat{U}_i(f(G_1),...,f(G_j))}{2}$, *and the loss of consumption in case of pillage is,* $L_i(\widehat{U}_i(\theta, \boldsymbol{G})) = -\widehat{U}_i(f(G_1), ..., f(G_j))$, *then (4) is rationalizable.*

Proof. The proof uses the same arguments as in the previous propositions. Conditional on continuation strategies $\mathbf{G}$, we consider the game $\Gamma(\mathbf{G})$ played under the coalition structure, $\mathscr{P}(N, i) = \{\{i\}, \{N \setminus \{i\}\}\}$, wherein

[14] Recall that $f(\cdot)$ is any increasing function.

player i forms a stand-alone coalition. Straightforward algebraic manipulation shows that (using the standard formula given in the proof of Proposition 1), $p^*_{-i}(\underline{\theta}_{-i} \mid \mathbf{G}) = \frac{\widehat{U}_i(f(\mathbf{G}))}{f_h(\mathbf{G})}$ which amounts to $p^*_{-i}(\underline{\theta}_{-i} \mid \mathbf{G}) = \sum_{j=i}^{n} \frac{f(G_j)-f(G_{j+1})}{jf_h(\mathbf{G})}$ with $f(G_{n+1}) = 0$. Therefore, in a canonical $\mathscr{P}(N,i)$-correlated PBE, player i's interim (correlated) equilibrium beliefs are $\mu^{*i}_{\mathscr{P}(N,i)}(\theta_{-i} = \underline{\theta}_{-i} \mid \overline{\theta}_i, \mathbf{G}) = p^*_{-i}(\underline{\theta}_{-i} \mid \mathbf{G})$ for all $i = 1, ..., n$ at $I_i(\overline{\theta}_i)$ on the equilibrium path. Condition (ii) of rationalizability yields

$$\mu^{*i}_{\mathscr{P}(N,i)}(\theta_{-i} = \underline{\theta}_{-i} \mid \overline{\theta}_i, \mathbf{G}) \left[2\widehat{U}_i(f(G_1), ..., f(G_j)) - C_i(G_i)\right] + (1 - \mu^{*i}_{\mathscr{P}(N,i)}(\theta_{-i} = \underline{\theta} \mid \overline{\theta}_i, \mathbf{G}))(-C_i(G_i)),$$

which readily simplifies as $\mu^{*i}_{\mathscr{P}(N,i)}(\theta_{-i} = \underline{\theta} \mid \overline{\theta}_i, \mathbf{G})V_i(\mathbf{G}) - C_i(G_i)$. ■

7 Rationalizable CSFs without correlation

The difficulty to derive CSFs for more than two contestants has already been stressed in Corchón and Dahm (2008). In their mediated frameworks, these authors derive CSFs by assuming the existence of a contest administrator who allocates the prize to one of the contestants. In the present unmediated setting, we overcome this problem by taking into account the possibility that groups or coalitions of players may be willing to coordinate their moves, in order to achieve mutually beneficial outcomes even if no binding agreements are made. To our thinking, the notion of $\mathscr{P}(N)$-correlated equilibrium leads to a coherent model. But, in fact, this is not crucial to our analysis. We can derive the same class of contest success functions without relying on any extraneous correlation device by analyzing the following version of the gun-butter game.

In this alternative approach, players cannot form a coalition $S \subseteq N$ in order to use some correlation devices within their coalition. However, they can choose the order of moves. More precisely, assume that prior to choosing the effort profile $\mathbf{G}$, players pick their activity in a sequential-move game where the order of moves in the first-stage of the gun-butter game is endogeneously determined: player i can choose to select his mixed activity before the other players. Then player $j \neq i$, observing the realization of player i's mixed strategy picks θ_j etc.[15] As a result, the game that takes place before players choose (simultaneously) their effort is a sequential game which depends on the ordering of players.

In the sequel, $\widehat{\Gamma}_i(\mathbf{G})$, denotes the sequential-move game that is played in the first-stage of this version of the gun-butter game, when player i is the first mover and continuation strategy is $\mathbf{G}$. For each such game, $\widehat{\Gamma}_i(\mathbf{G})$, payoffs, $\left\{\widehat{U}_i\right\}$, are thus also dependent on the identity of the first mover and are determined at the end of this version of the gun-butter game in the obvious

[15] For players $j \neq i$, the order of players' moves is irrelevant.

manner. We will use a notion of rationalizability based on the subgame perfect Nash equilibria (SPNE) of this new version of the gun-butter game and show how this may allow to construct a list of beliefs for the first-movers (via the behavioral strategy of a player $j \neq i$) corresponding to a CSF. Below we present an example yielding a special case of the additive CSF (2) for three contestants. This example can be extended to more agents and to other CSFs.

Example 3 We construct a SPNE, one for each gun-butter game played under a first mover, $i = 1, 2, 3$, in the first-stage. For the sake of exposition, consider the case where player 1 is the first to move, player 2 the second and player 3 the third-mover. Let the payoffs $\left\{\widehat{U}_i\right\}$ of this version of the gun-butter game be admissible as in Proposition 1 for player 1 and 3. For player 2, we consider the following modifications: $W_2((\overline{\theta}_1, \underline{\theta}_2, \underline{\theta}_3), \mathbf{G}) = 0$ and $W_2((\underline{\theta}_1, \overline{\theta}_2, \overline{\theta}_3), \mathbf{G}) = \sum_{j \neq 2} f_j(\mathbf{G}_j)$, for any continuation strategy $\mathbf{G}$ (the other conditions remain unchanged). We assume that players' payoffs depend on the order in which players choose their *activity* in $\widehat{\Gamma}_i(\mathbf{G})$. Then, we proceed by backward induction and use the payoff conditions of Proposition 1. Player 3 plays the following optimal strategy:

$$\theta_3^* = \begin{cases} \overline{\theta} \text{ if } \ \theta_{12} = (\overline{\theta}_1, \overline{\theta}_2) \text{ or if } \theta_{12} = (\underline{\theta}_1, \overline{\theta}_2); \\ \underline{\theta} \text{ otherwise.} \end{cases}$$

Notice that the new payoff conditions for player 2 imply that,

$$U_2(\underline{\theta}_1, \overline{\theta}_2, \overline{\theta}_3, \mathbf{G}) = U_2(\underline{\theta}_1, \underline{\theta}_2, \underline{\theta}_3, \mathbf{G}), \text{ and } U_2(\overline{\theta}_1, \overline{\theta}_2, \overline{\theta}_3, \mathbf{G}) = U_2(\overline{\theta}_1, \underline{\theta}_2, \underline{\theta}_3, \mathbf{G}).$$

In particular, suppose 2 plays $\overline{\theta}$ with probability p and $\underline{\theta}$ with $1 - p$. From the first-mover's perspective, by construction, p corresponds to the probability that 2 and 3 will play $\overline{\theta}_{23}$, while $1 - p$ is the probability that they will coordinate on $\underline{\theta}_{23}$ in a SPNE. In particular, in a SPNE player 2 can randomize with some probability distribution, so that player 1 (the first-mover) becomes indifferent:

$$U_1(\overline{\theta}_{123}, \mathbf{G})(1-p) + U_1(\overline{\theta}_1, \underline{\theta}_{23}, \mathbf{G})p = U_1(\underline{\theta}_1, \overline{\theta}_{23}, \mathbf{G})(1-p) + U_1(\underline{\theta}_{123}, \mathbf{G})p.$$

Solving for p yields a behavioral strategy of 2, $p^* = b_2^*(\underline{\theta})$, such that:

$$b_2^*(\underline{\theta}) = \frac{U_1(\overline{\theta}_{123}, \mathbf{G}) - U_1((\underline{\theta}_1, \overline{\theta}_{23}), \mathbf{G})}{U_1((\overline{\theta}_{123}, \mathbf{G}) - U_1((\underline{\theta}_1, \overline{\theta}_{23}), \mathbf{G}) + U_1(\underline{\theta}_{123}, \mathbf{G}) - U_1((\overline{\theta}_1, \underline{\theta}_{23}), \mathbf{G})}.$$

By using the payoff conditions of proposition 1, and by proceeding similarly for the two other coalition structures, we obtain a list of probabilities forming the additive CSF (2) in the case of three players.

This alternative notion of rationalizability can be thought of as the strip-down version of a richer setup that would allow for endogenous order of moves: before they play the gun-butter game, players have the opportunity to choose the order in which they choose their activity. Hence, in this view, a CSF results from the fact that each player i contemplates the perspective of playing the gun-butter game as the first-mover.

8 Concluding remarks

This paper has shown how the extension of Aumann's correlated equilibrium could account for popular contest success functions (CSFs) used in standard models of contests. Our partitioned correlated equilibrium concept is defined as follows. Take any non-cooperative game in normal form, and suppose that (disjoint) groups are formed. In a partitioned correlated equilibrium, each group plays a correlated equilibrium given the strategies played within the other groups. We have established the existence of such an equilibrium for finite games. In addition, we showed that this concept can be used to refine the set of perfect Bayesian equilibria of certain multi-stage games.

This refinement criterion provides a natural way to justify prominent CSFs used in contest models, in an unmediated non-cooperative environment and allows to establish new connections with various recent behavioral models. In our setup, popular CSFs have a clear preference-based meaning: the specific form of a CSF reflects the preferences of the individuals. For example, we have seen that Tullock CSFs reflect the simple endogenous (re)allocation of the goods produced by players choosing a productive activity, social utility functions a la Fehr and Schmidt (1999) produce difference-form CSFs, while the class of serial CSFs introduced by Alcade and Dahm (2007) involves the rank-dependent utility model introduced by Quiggin (1981, 1982). A partitioned correlated equilibrium allows to model non-cooperative situations where players can correlate their actions within their group, whereas correlation is usually imposed to all players. The application of this concept for some other strategic environments seems to be a fruitful approach left for further research.

Appendix A

Proof of Theorem 1. (1) Let us start by proving the existence of $\mathscr{P}(N)$-correlated equilibria when $\mathscr{P}(N)$ does *not* contain a singleton coalition. Fix such a coalition structure $\mathscr{P}(N)$. For each $p_{-S} \in \times_{S' \in \mathscr{P}(N)\setminus\{S\}} \Delta(\Theta_{S'})$, let $\Delta_S^{CE}(p_{-S}) \subseteq \Delta(\Theta_S)$ be the set of probability distributions over action profiles induced by the correlated equilibria of $\Gamma_S(p_{-S})$. Define Δ_S^{CE} as the set of all distributions that are contained in a set $\Delta_S^{CE}(p_{-S})$ with $p_{-S} \in \times_{S' \in \mathscr{P}(N)\setminus\{S\}} \Delta(\Theta_{S'})$. We need to show that Δ_S^{CE} is compact and convex.

For finite games, the set of correlated equilibrium distributions is a non-empty convex set (see Aumann, 1974). In fact, the set of correlated equilibria of a game Γ is a non-empty polytope which contains the convex hull of all the Nash equilibria of Γ. Therefore, $\Delta_S^{CE}(p_{-S})$ is a non-empty and convex and compact set for all product probability measures $p_{-S} \in \times_{S' \in \mathscr{P}(N)\setminus\{S\}} \Delta(\Theta_{S'})$. The set Δ_S^{CE} is convex because if p_S and $p_S^{'}$ are in Δ_S^{CE}, then for any $p_S^\lambda = \lambda p_S + (1-\lambda)p_S^{'}$ with $\lambda \in [0,1]$, there exists $p_{-S}^\lambda = \lambda p_{-S} + (1-\lambda)p_{-S}^{'} \in \times_{S' \in \mathscr{P}(N)\setminus\{S\}} \Delta(\Theta_{S'})$ (this set is convex) so that $p_S^\lambda \in \Delta_S^{CE}(p_{-S}^\lambda)$. Δ_S^{CE} is also compact by the continuity of payoffs. Let $\Delta^{CE} := \times_{S' \in \mathscr{P}(N)} \Delta_{S'}^{CE}$. This set is compact and convex as it is the Carte-

sian product of the sets, $\Delta^{CE}_{S'}$, each of which being compact and convex.
For each non-singleton coalition $S \in \mathscr{P}(N)$, define the correspondences $\Lambda^{CE}_S : \times_{S' \in \mathscr{P}(N)} \Delta^{CE}_{S'} \rightrightarrows \Delta^{CE}_S$. Each $\Lambda^{CE}_S(p_{-S})$ is convex-valued since $\Lambda^{CE}_S(p_{-S}) = \Delta^{CE}(p_{-S})$. Let $p = (p_S)_{S \in \mathscr{P}(N)}$ and define $\Lambda^{CE}(p) := \times_{S' \in \mathscr{P}(N)} \Lambda^{CE}_{S'}(p_{-S'})$. This implies that $\Lambda^{CE}(p)$ is also convex for each p and therefore Λ^{CE} is convex-valued. Observe that if $p \in \Lambda^{CE}(p)$, then $p_S \in \Lambda^{CE}_S(p_{-S})$ for all $S \in \mathscr{P}(N)$, and hence p is a $\mathscr{P}(N)$-correlated equilibrium of Γ.
By continuity of payoff functions, $\Lambda^{CE}_S(p_{-S})$ has a closed graph. Since this holds for each S, we conclude that Λ^{CE} has a closed graph. Under all these properties Kakutani's fixed point Theorem applies which completes the proof for coalition structures not containing some singleton coalitions. The above arguments extend easily to $\mathscr{P}(N)$-correlated equilibria when $\mathscr{P}(N)$ contains some singleton coalitions by defining the usual best response correspondences, $BR_i : \times_{S' \in \mathscr{P}(N)} \Delta(\Theta_{S'}) \rightrightarrows \Delta(\Theta_i)$. This proves the existence of non-trivial $\mathscr{P}(N)$-correlated equilibria.
Next we prove (2). Let us denote the set of Nash equilibria of $\Gamma_S(p_{-S})$ by $NE(p_{-S})$. By taking the convex hull of Nash equilibrium distributions of each game $\Gamma_S(p_{-S})$, we obtain the convex and compact subset, $\Delta^{NE}_S(p_{-S})$ of the set of all correlated equilibria of $\Gamma_S(p_{-S})$. Observe that the set of all sets $\Delta^{NE}_S(p_{-S})$ is finite since Γ is finite. Denote this set by Δ^{NE}_S. Being a finite union of compact sets it is also compact. Using the same arguments as above, this set is also convex. Define the correspondence, $\Lambda^{NE}_S : \times_{S' \in \mathscr{P}(N)} \Delta^{NE}_{S'} \rightrightarrows \Delta^{NE}_S$ with $\Lambda^{NE}_S(p) = \Delta^{NE}_S(p)$. Let $p = (p_S)_{S \in \mathscr{P}(N)}$ and define $\Lambda^{NE}(p) := \times_{S \in \mathscr{P}(N)} \Lambda^{NE}_S(p_{-S})$. Using the arguments above, it is clear that this correspondence verifies all the properties of Kakutani's fixed point theorem. This completes the proof of existence (2). ■

Appendix B

Proof of proposition 2. Let the payoffs satisfy the payoff conditions of Proposition 2. For the sake of exposition, let $N = \{1, 2, 3\}$. We will analyze the belief of player i in the $\mathscr{P}(N, i)$-correlated PBEs of $\Gamma(\mathbf{G})$, for $i = 1, 2, 3$. Let $p^*_{\{j,k\}}(\underline{\theta}_{\{j,k\}} \mid \mathbf{G})$ be the probability that players j and k coordinate on the productive activities, $\underline{\theta}_{\{j,k\}}$, and $1 - p^*_{\{j,k\}}(\underline{\theta}_{\{j,k\}} \mid \mathbf{G})$ the probability that they coordinate on the appropriative activities, $\overline{\theta}_{\{j,k\}}$. Assume, without loss of generality, that $G^\sigma_1 \geq G^\sigma_2 \geq G^\sigma_3$.
We first start by considering the canonical $\mathscr{P}(N, 1)$-correlated equilibrium of the first-stage of the game. In this partitioned correlated equilibrium we compute the equilibrium belief of player 1 when player 2 and 3 correlate their activities in a correlated equilibrium of their intra-coalition game. When $G^\sigma_1 \geq G^\sigma_2 \geq G^\sigma_3$, the indifference condition for player 1 yields the distribution $p^*_{2,3} \in \Delta(\left\{\underline{\theta}_{\{2,3\}}, \overline{\theta}_{\{2,3\}}\right\})$ for 1 such that, if $G^\sigma_1 - G^\sigma_3 \geq \frac{s}{3}$,

$$p^*_{2,3}(\underline{\theta}_{2,3} \mid \mathbf{G}) = \begin{cases} 1 \text{ if } \mathbf{G} \text{ is such that } U_1((\overline{\theta}_1, \overline{\theta}_{2,3}), \mathbf{G}) - U_1((\underline{\theta}_1, \overline{\theta}_{2,3}), \mathbf{G}) \geq 1; \\ \frac{1}{2} + \frac{1}{s}(G^\sigma_1 - G^\sigma_2) \text{ otherwise.} \end{cases}$$

and, when $G_1^\sigma - G_3^\sigma < \frac{s}{3}$,

$$p^*_{2,3}(\underline{\theta}_{2,3} \mid \mathbf{G}) = \begin{cases} 1 \text{ if } \mathbf{G} \text{ is such that } U_1((\overline{\theta}_1, \overline{\theta}_{2,3}), \mathbf{G}) - U_1((\underline{\theta}_1, \overline{\theta}_{2,3}), \mathbf{G}) \geq 1; \\ \frac{1}{3} + \frac{1}{2s}(2G_1^\sigma - G_2^\sigma - G_3^\sigma) \text{ otherwise.} \end{cases}$$

Hence, at a canonical $\mathscr{P}(N,1)$-correlated PBE, these conditions are equivalent to have player 1's beliefs in the second-stage of the gun-butter game such that,

$$\mu^{*1}_{\mathscr{P}(N,1)}(\underline{\theta}_{2,3} \mid \overline{\theta}_1, \mathbf{G}) = \begin{cases} \min\left\{\frac{1}{2} + \frac{1}{s}(G_1^\sigma - G_2^\sigma), 1\right\} \text{ if } G_1^\sigma - G_3^\sigma \geq \frac{s}{3}; \\ \frac{1}{3} + \frac{1}{2s}(2G_1^\sigma - G_2^\sigma - G_3^\sigma) \text{ otherwise.} \end{cases}$$

Let us repeat the same computations for the case of a canonical $\mathscr{P}(N,2)$-correlated PBE. In this case, we have look at the $\mathscr{P}(N,2)$-correlated equilibrium of $\Gamma(\mathbf{G})$. Here, player 2's beliefs are given by,

$$\mu^{*2}_{\mathscr{P}(N,2)}(\underline{\theta}_{1,3} \mid \overline{\theta}_2, \mathbf{G}) = \begin{cases} \frac{1}{2} + \frac{1}{2}(G_2^\sigma - G_1^\sigma) \text{ if } \mathbf{G} \text{ s.t } U_2((\overline{\theta}_2, \overline{\theta}_{1,3}), \mathbf{G}) - U_2((\underline{\theta}_2, \overline{\theta}_{1,3}), \mathbf{G}) \geq 0; \\ 0 \text{ otherwise.} \end{cases}$$

Otherwise, when $G_1^\sigma - G_3^\sigma < \frac{s}{3}$,

$$\mu^{*2}_{\mathscr{P}(N,2)}(\underline{\theta}_{1,3} \mid \overline{\theta}_2, \mathbf{G}) = \frac{1}{3} + \frac{1}{2s}(2G_2^\sigma - G_1^\sigma - G_3^\sigma).$$

Finally, we compute player 3's beliefs in a canonical $\mathscr{P}(N,3)$-correlated PBE. In this case, we have look at the $\mathscr{P}(N,3)$-correlated equilibrium of $\Gamma(\mathbf{G})$. When $G_1^\sigma - G_3^\sigma \geq \frac{s}{3}$, player 3's beliefs are such that $\mu^{*3}_{\mathscr{P}(N,3)}(\underline{\theta}_{1,2} \mid \overline{\theta}_3, \mathbf{G}) = 0$ since $U_3((\overline{\theta}_3, \overline{\theta}_{1,2}), \mathbf{G}) - U_3((\underline{\theta}_3, \overline{\theta}_{1,2}), \mathbf{G}) = 0$, and

$$\mu^{*3}_{\mathscr{P}(N,3)}(\underline{\theta}_{1,2} \mid \overline{\theta}_3, \mathbf{G}) = \frac{1}{3} + \frac{1}{2s}(2G_3^\sigma - G_1^\sigma - G_2^\sigma) \text{ otherwise .}$$

The rest of the proof is similar to proposition 1. ■

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