On Variable Discounting in Dynamic Programming: Applications to Resource Extraction and Other Economic Models

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1 Introduction

The discounted utility with a constant discount rate was introduced by Samuelson (1937). Twenty years later Koopmans (1960) characterized axiomatically a class of recursive utilities, which also includes the classical model as a special case. A fairly large number of researchers have used the discounting rule to various sequential decision-making problems, see Blackwell (1965); Stokey et al. (1989); Hernández-Lerma and Lasserre (1996) and references therein. A common feature of the aforementioned works is the assumption that the discount factor is constant. This fact significantly simplifies the analysis. Namely, in order to obtain a solution to the Bellman equation, one can directly apply the Banach Contraction Principle, if the immediate return function is bounded. In case of unbounded instanteneous utility functions, one must combine this principle with a weighted norm approach, see Becker and Boyd III (1997); Boyd III (2006); Hernández-Lerma and Lasserre (1999). A construction of recursive utilities with the aid of the Banach Contraction Principle has been commenced by Denardo (1967), Lucas and Stokey (1984) and further developed in Becker and Boyd III (1997); Boyd III (2006); **?**); Marinacci and Montrucchio (2010) and references therein. Although the range of various results is pretty large nowadays, it appears that the literature does not cover an elementary case that is the subject of this paper.

Our idea is to construct recursive utilities in an infinite time horizon by a direct extension of the Samuelson's approach. Let us first introduce a *bounded from above* sequence of utilities $\{u_t\}_0^\infty$ received by the agent in a sequential decision-making process and a fixed real-valued function δ. Then, we shall study the following total utility [2]:

$$u_0 + \delta(u_1 + \delta(u_2 + \delta(u_3 + \cdots))) = \lim_{n\to\infty}[u_0 + \delta(u_1 + \delta(u_2 + \cdots + \delta(u_n))\cdots)]. \qquad (1)$$

Obviously, if $\delta(z) = \beta z$, with $0 < \beta < 1$, then (1) becomes

$$u_0 + \beta u_1 + \beta^2 u_2 + \beta^3 u_3 + \cdots$$

as in the seminal paper of Samuelson (1937). The total utility given in (1) belongs to the class of recursive utilities, because it is induced by the aggregator $W(y, r) = y + \delta(r)$. If δ is a Lipschitz function, with a positive constant less than 1, then utility (1) can be obtained from the results in Denardo (1967); Lucas and Stokey (1984); Boyd III (1990). However, neither of

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[1] A revised and extendede version of this work will be published in the *Annals of Operations Research*

[2] In Section 3, we show that the limit in (1) belongs to $[-\infty, \infty)$ under Assumption (A1) on the function δ given in Section 2.

these papers provides its elegant explicit form. Our approach also embraces a case of a Lipschitz function δ with a positive constant not necessarily less than 1 (e.g., δ can be differentiable and $\delta'(0+) = 1$). Then, the Banach Contraction Principle cannot be applied (see Remark 5). In other words, due to the terminology in Marinacci and Montrucchio (2010), the aggregators examined in this note need not be of the Blackwell type. Moreover, they do not belong to the class of Thompson aggregators either, since δ does not have to be concave (see Remark 5).

Our objective is to prove the existence of a solution to the Bellman equation in an infinite time horizon models with a fixed real-valued discount function δ. This is done by a virtue of an extension of the Banach Contraction Principle due to Matkowski (1975) that allows for a study of different nonlinear functions δ. Next, by iteration of the optimality equation, we obtain recursive utility of type (1). Our analysis permits unbounded from below and continuous return functions. It is worth mentioning that similar models, but with a constant discount factor, were considered by Strauch (1966), Hernández-Lerma and Lasserre (1996) or Stokey et al. (1989). For a further discussion see also Feinberg (2002).

The discount functions δ may possess different features, which are briefly discussed in Section 2. Namely, they are functions of future incomes and have their motivation in empirical studies, see Benzion et al. (1989); Frederick et al. (2002); Green et al. (1997); Kirby (1997); Thaler (1981). In particular, they might take into account such a phenomenon as "sign effect" or "magnitute effect". Finally, we emphasize that the results on the Bellman equations presented in Theorems 2 and 3 are *new* and cannot be deduced from previous works Boyd III (1990); Boyd III (2006); Lucas and Stokey (1984); Marinacci and Montrucchio (2010). The recursive utilities obtained in this way have very attractable form (1) and can further be used for deriving stationary optimal plans.

2 Discount functions

Let R be the set of real numbers. By $I \subset R$ we denote a set of outcomes in a decision process. Assume that $0 \in I$ and $\xi : I \to [0, \infty)$ is a function such that $\xi(0) = 0$. Let $y \in I$ be an outcome associated with a decision epoch t. Then, $z \in I$ is a counterpart of y at time $t+1$ if

$$z = \varphi(y) := (1 + \rho_0 + \xi(y))y. \tag{2}$$

Here $\rho_0 \geq 0$ is a constant *rate,* ξ is a *variable rate of return* in the period between t and $t+1$. We assume that φ is continuous and increasing. Then, φ has the inverse function φ^{-1}. It is reasonable to call $y = \delta(z) := \varphi^{-1}(z)$ a *discount function.* Clearly, δ is also increasing. The domain of δ is denoted by D. Let us define $\rho(z) := \frac{z}{\delta(z)} - 1$, for $z \neq 0$, and call ρ a *discount rate.* We have

$$\delta(z) = \frac{z}{1 + \rho(z)}, \tag{3}$$

and we shall refer to the quantity

$$\frac{\delta(z)}{z} = \frac{1}{1 + \rho(z)} \tag{4}$$

as a *variable discount factor.* Since $y = \delta(z)$, from (2) and (3), it follows that

$$\rho(z) = \rho_0 + \xi(y) = \rho_0 + \xi(\delta(z)). \tag{5}$$

We now make our basic assumption on the discount function δ.

(A1) There exists a continuous increasing function $\gamma : [0,\infty) \to [0,\infty)$ such that $\gamma(z) < z$ for each $z > 0$ and

$$|\delta(z_1) - \delta(z_2)| \leq \gamma(|z_1 - z_2|) \tag{6}$$

for all $z_1, z_2 \in D$.

Assumption (A1) implies that $\gamma(0) = 0$.

2.1 The "sign effect"

Quite a number of empirical studies show that gains are discounted more than losses, (see Frederick et al. (2002); Thaler (1981)). The discount function δ having the aforementioned property is given in the following example.

Example 1 Let

$$\delta(z) := \begin{cases} \beta_1 z, \ z \geq 0 \\ \beta_2 z, \ z < 0, \end{cases}$$

where $0 < \beta_1 < \beta_2 < 1$. It is easy to see that (6) holds with the linear function $\gamma(z) = \beta z$, where $\beta = \max\{\beta_1, \beta_2\}$.

2.2 The "magnitude effect" in average discount functions

According to recent empirical research psychologists and economists have led with the conclusion that larger outcomes are discounted at a lower rate than small ones, see Benzion et al. (1989); Frederick et al. (2002); Green et al. (1997); Kirby (1997); Thaler (1981). In other words, the variable discount factor defined in (4) is increasing. For example, Thaler (1981) demonstrated that respondents were, on *average*, indifferent between \$15 immediately and \$60 in a year, \$250 immediately and \$350 in a year, and \$3000 immediately and \$4000 in a year. These results yield that $\delta(60)/60 = 0.25$, $\delta(350)/350 \approx 0.71$, $\delta(4000)/4000 = 0.75$. We might call this type of discounting *socially acceptable,* because it is consistent with a common opinion on time preferences. The instance of such a function is given below. [3]

[3] Note that we do not claim that this is the exact function corresponding with the aforementioned empirical studies. We only wish to indicate that for this type of δ, functions γ are very often linear. More suitable functions δ sometimes cannot be expressed in such a simple analytically tractable form.

Example 2 Consider the discount function $\delta(z) := 3z/10$ for $z \in D_0 := [0, 50]$ and

$$\delta(z) := \frac{3z^2}{4z+300} = \frac{3z}{4} - \frac{225}{4} + \frac{16875}{4z+300} \quad \text{for} \quad z \in D^0 := (50, \infty).$$

Note that δ is convex and increasing on $D = D_0 \cup D^0 = [0, \infty)$. Moreover, the variable discount factor is increasing on D^0. This function gives similar numerical results to those observed by Thaler (1981). Indeed, $\delta(50) = 15$, $\delta(60) = 20$, $\delta(350) \approx 216.2$, $\delta(4000) \approx 2944.8$ and, consequently, $\delta(50)/50 = 0.3$, $\delta(60)/60 = 1/3$, $\delta(350)/350 \approx 0.618$, $\delta(4000)/4000 \approx 0.736$. Furthermore, it follows that $\delta'(z) = 3/10$ for $0 < z < 50$ and

$$0 < \delta'(z) = \frac{3}{4} - \frac{67500}{(4z+300)^2} < \frac{3}{4} \quad \text{for} \quad z > 50.$$

Using the Law of Mean Value, one can easily show that inequality (6) holds with $\gamma(z) = 3z/4$. Since $y = \delta(z) = \varphi^{-1}(z)$, we obtain $z = \varphi(y) = y(1 + \xi(y))$ with $\xi(0) = 0$, $\xi(z) = 7/3$ for $y \in [0, 15]$ and

$$\xi(y) = \frac{-1 + 2\sqrt{1 + \frac{225}{y}}}{3} \quad \text{for} \quad y \in (15, \infty).$$

The facts that δ is increasing, ξ is decreasing on $(15, \infty)$ and (5) imply that the discount rate $\rho(z) = \xi(\delta(z))$ is decreasing D^0.

2.3 The "magnitude effect" in subjective discount functions

Consider an economic agent who is able to achieve a higher rate of return for larger capital. For instance, he can realize more (often complementary) investment projects or negotiate higher rates for bank deposits. Thus, such an agent is characterized by an increasing function ξ and, consequently, by an increasing *rate of return* $\rho(z) = \rho_0 + \xi(\delta(z))$. Clearly, the variable discount factor defined in (4) is decreasing. The next example illustrates this type of discounting.

Example 3 Let the function ξ in (2) be defined as follows $\xi(y) := \frac{\epsilon y}{y+1}$ for $y \in I = [0, \infty)$ and some constant $\epsilon > 0$. Then $\rho_0 + \xi(y)$ (which can be called a *requested rate of return*) is increasing. We have

$$z = \varphi(y) = y\left(1 + \rho_0 + \frac{\epsilon y}{y+1}\right) = y\left(1 + \rho_0 + \epsilon - \frac{\epsilon}{y+1}\right) = (1 + \rho_0 + \epsilon)y - \epsilon + \frac{\epsilon}{y+1}.$$

This function is strictly convex and increasing. [4] Moreover, $\varphi(0) = 0$. Therefore, $y = \delta(z) = \varphi^{-1}(z)$ is strictly concave, increasing and $\delta(0) = 0$. Simple calculations yield

$$\delta(z) = \frac{z - 1 - \rho_0 + \sqrt{(z - 1 - \rho_0)^2 + 4z(1 + \rho_0 + \epsilon)}}{2(1 + \rho_0 + \epsilon)}, \quad z \in D = [0, \infty).$$

[4] The function φ is chosen to show that the corresponding discount function δ may be subadditive and such that $\delta'(0+) = 1$. In fact, in many applications φ can be a more complicated function.

Since δ is concave, we have $\delta'(z) \le \delta'(0+) = \frac{1}{1+\rho_0}$. If $\rho_0 > 0$, then, by the Law of Mean Value, inequality (6) holds with $\gamma(z) = \frac{z}{1+\rho_0}$. Assume now that $\rho_0 = 0$. In this case $\delta'(z) \le 1$ and the Law of Mean Value does not conclude that (A1) is satisfied. However, we can show that $\left(\frac{\delta(z)}{z}\right)' < 0$ for $z > 0$. Hence, $\frac{\delta(z)}{z}$ is decreasing on $(0, \infty)$. Consequently, for any $z_1, z_2 \in (0, \infty)$, $\frac{\delta(z_1)}{z_1} \ge \frac{\delta(z_1+z_2)}{z_1+z_2}$ and $\frac{\delta(z_2)}{z_2} \ge \frac{\delta(z_1+z_2)}{z_1+z_2}$. This implies that δ is *subadditive*, i.e.,

$$\delta(z_1 + z_2) \le \delta(z_1) + \delta(z_2) \tag{7}$$

for all $z_1, z_2 \in (0, \infty)$. Clearly, (7) holds for all $z_1, z_2 \in D$. Now, let $z_1 \ge z_2$ and note that

$$|\delta(z_1) - \delta(z_2)| = \delta(z_1) - \delta(z_2) \le \delta(z_1 - z_2 + z_2) - \delta(z_2) \le \delta(z_1 - z_2) = \delta(|z_1 - z_2|).$$

This property and the fact that δ is increasing imply that (6) holds with $\gamma = \delta$.

A different type of function for which (6) holds is $\gamma(z) = \delta(z) = \log(1 + z)$, $z \ge 0$.

2.4 Other discount functions

In Examples 1 and 2, γ is linear, whereas $\gamma = \delta$ is subadditive in Example 3. However, the class of discount functions satisfying Assumption (A1) is fairly large. There exist non-subadditive functions δ for which (A1) cannot be concluded from the Law of Mean Value Theorem either. The next example is devoted to such a construction of δ.

Example 4 Consider the domain $D = [0, \infty)$ and $\delta(z) = \gamma(\eta(z))$, where

$$q = \eta(z) = \frac{z + z^3}{1 + z + z^2}, \quad z \ge 0, \quad \text{and} \quad \gamma(q) = \frac{q}{3(q+1)} + \frac{2q}{3}, \quad q \ge 0.$$

Note that $\gamma(q)/q$ is decreasing in $(0, \infty)$, so γ is subadditive. Therefore, for any $q_1, q_2 \ge 0$, we have

$$|\gamma(q_1) - \gamma(q_2)| \le \gamma(|q_1 - q_2|).$$

Moreover, γ is increasing. Observe that $0 < \eta(z) < z$ for $z > 0$, $\eta(0) = 0$. Hence, $\delta(0) = 0$ and $0 < \delta(z) < z$ for $z > 0$. In addition, note that $\eta'(0+) = 1$ and

$$0 < \eta'(z) = \frac{1 + 2z^2 + 2z^3 + z^4}{(1 + z + z^2)^2} < 1$$

for all $z > 0$, $\eta'(z) \to \eta'(0+) = 1$ as $z \to 0+$. Thus, η is a Lipschitz function with constant 1. We show that δ satisfies Assumption (A1). Indeed, for any $z_1, z_2 \in D$, we have

$$|\delta(z_1) - \delta(z_2)| = |\gamma(\eta(z_1)) - \gamma(\eta(z_2))| \le |\gamma(\eta(z_1) - \eta(z_2))| \le \gamma(|z_1 - z_2|).$$

Finally, we point out that

$$\delta(z) = \frac{z(z^2 + 1)(2z^3 + 3z^2 + 5z + 3)}{3(z^3 + z^2 + 2z + 1)(z^2 + z + 1)}.$$

Since

$$\delta(3) - \delta(2) - \delta(1) = \frac{990}{559} - \frac{410}{357} - \frac{26}{45} = \frac{134054}{2993445} > 0,$$

the function δ is not subadditive. It is easy to see that $\delta'(z) \to \delta'(0+) = 1$ as $z \to 0+$. Therefore, δ has no Lipschitz constant less than 1.

The main results in this paper are based on the following extension of the Banach Contraction Principle given (in a slightly more general form) by Matkowski (1975), which is also stated as Theorem 5.2 in Dugundji and Granas (2003).

Proposition 1 *Let (Y, ρ) be a complete metric space, $\gamma : [0, \infty) \mapsto [0, \infty)$ be a continuous increasing function such that $\gamma(y) < y$ for every $y \in (0, \infty)$. If an operator $T : Y \mapsto Y$ satisfies the inequality*

$$\rho(Ty_1, Ty_2) \le \gamma(\rho(y_1, y_2))$$

for any $y_1, y_2 \in Y$, then T has a unique fixed point $y^* \in Y$ and

$$\lim_{n\to\infty} \rho(T^{(n)}y, y^*) = 0,$$

for each $y \in Y$, where $T^{(n)}$ is the nth composition of T with itself.

Remark 1 The continuity condition, imposed on γ in Proposition 1, can be weakened. Namely, we may assume that γ is non-decreasing and the nth composition of γ with itself $\gamma^{(n)}(z)$ tends to zero for any $z > 0$ as $n \to \infty$. This requirement, in turn, implies that $\gamma(z) < z$ for $z > 0$.

We also emphasize that if, for example, $\gamma'(0+) = 1$, then Proposition 1 cannot be deduced from the Banach Contraction Principle.

3 The model and results

We shall consider a dynamical system specified by the following objects $\{X, A, \Psi, f, u, \delta\}$, where:

- X denotes the state space with generic element x; X is a metric space;
- A is a set of all actions of the agent; A is a metric space;
- $\Psi : X \mapsto A$ is a set-valued mapping; for each x, the *non-empty* set $\Psi(x) \subset A$ describes the set of all feasible actions to the agent in state x;
- $f : X \times A \mapsto X$ is the law of transition for the system;
- $u : X \times A \mapsto R$ is an immediate return function (one-period utility);
- $\delta : D \mapsto R$ is a discount function.

The sequential decision-making process is described in a usual way. At an initial state x_0 the agent chooses an action $a_0 \in \Psi(x_0)$. Then, the immediate return $u(x_0, a_0)$ is generated and the system moves to a next state $x_1 = f(x_0, a_0)$. In state x_1 again two things happen: the agent selects an action $a_1 \in \Psi(x_1)$ and the return $u(x_1, a_1)$ is incurred. This procedure repeats itself yielding the history of the system $(x_0, a_0, x_1, a_1, \ldots)$. Let H be the set of all feasible histories, i.e.,

$$h_{x_0} = (x_0, a_0, x_1, a_1, \ldots) \in H \quad \text{iff} \quad a_t \in \Psi(x_t),\ x_{t+1} = f(x_t, a_t),$$

for $t = 0, 1, \ldots$. Endow H with the product topology which is metrizable. Let $\Pi(x_0)$ be the set of all sequences $\mathbf{a} = \{a_t\}_0^\infty$ such that $a_0 \in \Psi(x_0)$ and $a_t \in \Psi(x_t)$ where $x_t = f(x_{t-1}, a_{t-1})$, $t = 1, 2, \ldots$. $\Pi(x_0)$ is a set of all *feasible action sequences* from the initial point x_0. Any sequence $\mathbf{a} = \{a_t\}_0^\infty \in \Pi(x_0)$ is called a *plan.*

We shall assume that δ satisfies (A1) and

(A2) δ is continuous and non-decreasing.
(A3) The functions u and f are continuous on $X \times A$.
(A4) The correspondence Ψ is continuous and $\Psi(x)$ is compact for each $x \in X$.

For a bounded from above function u and feasible history $h_{x_0} \in H$, we shall consider the following utility

$$\begin{aligned} & u(x_0, a_0) + \delta(u(x_1, a_1) + \delta(u(x_2, a_2) + \ldots)) \\ = & \lim_{n\to\infty} [u(x_0, a_0) + \delta(u(x_1, a_1) + \delta(u(x_2, a_2) + \cdots + \delta(u(x_n, a_n))) \cdots)]. \end{aligned} \tag{8}$$

In particular, we shall prove in Theorem 1 that for a bounded function u, the limit in (8) exists and is finite, and the aforementioned utility is a unique recursive utility for a specific aggregator.

3.1 Bounded returns

In this subsection we shall assume that u is *bounded.* Let Z be a metric space. Then, $C(Z)$ describes the set of all real-valued bounded continuous functions on Z. It is well-known that $C(Z)$ is complete metric space, if it is equipped with the supremum metric $\rho_Z(g_1, g_2) := \sup_{z\in Z} |g_1(z) - g_2(z)|$. Let

$$W(x, a, r) := u(x, a) + \delta(r), \tag{9}$$

where $x \in X$, $a \in \Psi(x)$ and $r \in R$. W is a specific *aggregator* function. One might think of $W(x, a, r)$ as the total payoff while starting at x, selecting a with the prospect of receiving r. Let $\mathbf{a} = \{a_t\}_0^\infty \in \Pi(x_0)$ be fixed. A function $U : H \mapsto R$ is called a *recursive utility,* if, for every history $h_{x_0} = (x_0, a_0, x_1, a_1, \ldots)$ and any $n \geq 0$, it holds

$$U(x_n, a_n, x_{n+1}, a_{n+1}, \ldots) = W(x_n, a_n, U(x_{n+1}, a_{n+1}, \ldots)).$$

This definition comes from Koopmans (1960), see also Becker and Boyd III (1997).

Let $h_{y_0} = (y_0, c_0, y_1, c_1, ...) \in H$. The *shift operator* $s : H \mapsto H$ is $s(h_{y_0}) := (y_1, c_1, y_2, c_2, ...)$. For any $v \in C(H)$, we define an operator T as follows

$$Tv(h_{y_0}) := W(y_0, c_0, v(s(h_{y_0}))). \tag{10}$$

By (A1), for every $y_0 \in X$, $c_0 \in \Psi(y_0)$, $v_1, v_2 \in C(H)$, we have

$$\begin{aligned} |W(y_0, c_0, v_1(s(h_{y_0}))) - W(y_0, c_0, v_2(s(h_{y_0})))| &\le |\delta(v_1(s(h_{y_0}))) - \delta(v_2(s(h_{y_0})))| \\ &\le \gamma(|v_1(s(h_{y_0})) - v_2(s(h_{y_0}))|) \\ &\le \gamma(\rho_H(v_1, v_2)). \end{aligned}$$

Hence,

$$\rho_H(Tv_1, Tv_2) \le \gamma(\rho_H(v_1, v_2)).$$

Moreover, since u, δ are continuous and u, v are bounded, then $Tv \in C(H)$. From Proposition 1 there exists a unique $U \in C(H)$ such that

$$U(h_{y_0}) = W(y_0, c_0, U(s(h_{y_0}))), \tag{11}$$

for each $h_{y_0} = (y_0, c_0, y_1, c_1, ...) \in H$. Now, by putting $y_k = x_{n+k}$, $c_k = a_{n+k}$ for $k \ge 0$, we conclude from (11) the following result.

Theorem 1 *If assumptions (A1)-(A3) hold, then there exists a unique recursive utility $U \in C(H)$. Moreover,*

$$\lim_{n\to\infty} \rho_H(U, T^{(n)}v) = 0 \tag{12}$$

for any $v \in C(H)$.

Remark 2 Theorem 1 shows that the simple aggregator, defined in (9), induces a non-additive utility when δ is not linear. Note that under (A1), for any $x \in X$, $a \in \Psi(x)$, $r_1, r_2 \in R$, we have

$$|W(x, a, r_1) - W(x, a, r_2)| \le \gamma(|r_1 - r_2|). \tag{13}$$

Other non-additive utilities were studied by Uzawa (1968) and Epstein and Hynes (1983) who considered aggregators of the form

$$\widetilde{W}(x, a, r) := [u(x, a) + r]e^{-w(x)}$$

with $u \le 0$ and $w > 0$. It is easy to see that

$$|\widetilde{W}(x, a, r_1) - \widetilde{W}(x, a, r_2)| \le |r_1 - r_2|e^{-w(x)} \tag{14}$$

for all $x \in X$, $a \in \Psi(x)$, $r_1, r_2 \in R$. Clearly, inequalities (13) and (14) are different.

Theorem 1 implies that $T^{(n)}0(h)$ converges to (8) for all $h \in H$. Therefore, (8) is a unique utility U described in Theorem 1.

Definition 1 *An optimal return function is defined as $\widehat{V}(x) = \sup_{h_x\in H} U(h_x)$. If there exists a history $\widehat{h}_x := (x, \widehat{a}_0, x_1, \widehat{a}_1, \ldots)$ such that $\widehat{V}(x) = U(\widehat{h}_x)$, then a plan $\widehat{\boldsymbol{a}} := \{\widehat{a}_t\}_0^\infty \in \Pi(x)$ generating this history $\widehat{h}_x$ is called optimal.*

For any $w \in C(X)$ we also define the maximum operator M as follows:

$$(Mw)(x) := \sup_{a\in\Psi(x)} W(x,a,w(f(x,a))) = \sup_{a\in\Psi(x)} [u(x,a) + \delta(w(f(x,a)))]. \tag{15}$$

We notice that assumptions (A2)-(A3) imply that the function $W(\cdot,\cdot,w(f(\cdot,\cdot)))$ is bounded and continuous. If, in addition, (A4) is satisfied, then, by the Maximum Theorem in Berge (1963), $Mw \in C(X)$. Moreover, we may replace 'sup' in (15) by 'max'.

We have the following result.

Theorem 2 *Assume (A1)-(A4) and that u is bounded.*

(a) *There exists a unique function $v^* \in C(X)$ such that it holds*

$$v^*(x) = \max_{a\in\Psi(x)} [u(x,a) + \delta(v^*(f(x,a)))] \tag{16}$$

for all $x \in X$. Moreover, v^* is an optimal return function.

(b) *A plan $\boldsymbol{a}^* = \{a_t^*\}_0^\infty \in \Pi(x)$ satisfying (16), i.e.,*

$$v^*(x_t) = u(x_t, a_t^*) + \delta(v^*(f(x_t, a_t^*))), \quad t = 0, 1, \dots \tag{17}$$

is optimal.

Proof By the above-mentioned remark, the operator M in (15) maps the space $C(X)$ into itself. Moreover, for any $w_1, w_2 \in C(X)$, we get

$$\begin{aligned}
|(Mw_1)(x) - (Mw_2)(x)| &\le \sup_{a\in\Psi(x)} |\delta(w_1(f(x,a))) - \delta(w_2(f(x,a)))| \\
&\le \sup_{y\in X} |\delta(w_1(y)) - \delta(w_2(y))| \\
&\le \sup_{y\in X} \gamma(|w_1(y) - w_2(y)|) \le \gamma(\sup_{y\in X} |w_1(y) - w_2(y)|) \quad \text{(by (A1))}
\end{aligned}$$

and, consequently,

$$\rho_X(Mw_1, Mw_2) = \sup_{x\in X} |(Mw_1)(x) - (Mw_2)(x)| \le \gamma(\rho_X(w_1, w_2)).$$

Hence, from Proposition 1, there exists a unique fixed point $v^* \in C(X)$ of the operator M.

We show that v^* is an optimal return function. Let $\mathbf{a} = \{a_t\}_0^\infty \in \Pi(x)$ be any plan. Then, by (16) we have

$$v^*(x) \ge u(x, a_0) + \delta(v^*(x_1)) \quad \text{with } x_1 = f(x, a_0).$$

Iterating this inequality n times and taking into account that δ is non-decreasing, we obtain

$$v^*(x) \ge u(x,a_0) + \delta(u(x_1,a_1) + \delta(u(x_2,a_2) + \ldots + \delta(v^*(x_{n+1})))) = (T^{(n)}v^*)(x),$$

where T is defined in (10). By virtue of this inequality and (12) (with $v(h_y) := v^*(y)$), we deduce that

$$v^*(x) \geq U(h_x) \quad \text{for any } h_x \in H. \tag{18}$$

On the other hand, by our assumptions (A2)-(A4), there exists a plan $\mathbf{a}^* = \{a_t^*\}_0^\infty \in \Pi(x)$ such that (17) holds. Let h_x^* be a history of the system generated by a plan $\mathbf{a}^* \in \Pi(x)$. Proceeding along similar lines as above, i.e., iterating the following equation

$$v^*(x) = u(x, a_0^*) + \delta(v^*(x_1))$$

n times and making use of (12), we conclude that

$$v^*(x) = U(h_x^*). \tag{19}$$

Thus, (18) and (19) complete the proof. □

3.2 Unbounded returns

In this subsection, we assume that u is continuous and bounded from above. Then the total utility U is by definition given as in (8). Let $\underline{R} := R \cup \{-\infty\}$. By $\mathcal{U}(X)$ we denote the set of all extended real-valued bounded from above upper semicontinuous functions on X. We start with a useful lemma (for a proof see Schäl (1975)).

Lemma 1 *If S is a metric space and $\{w_n\}$ is a non-increasing sequence of upper semicontinuous functions, bounded from above and $w_n : S \mapsto \underline{R}$, then*

(a) *$w_\infty = \lim_{n\to\infty} w_n$ exists and w_∞ is upper semicontinuous;*
(b) *if additionally S is compact, then $\sup_{s\in S} \lim_{n\to\infty} w_n(s) = \lim_{n\to\infty} \sup_{s\in S} w_n(s)$.*

Now we are ready to present our third result.

Theorem 3 *Assume (A1)-(A4) and that $u : X \times A \mapsto \underline{R}$ is bounded from above.*

(a) *There exists $V^* \in \mathcal{U}(X)$ such that it holds*

$$V^*(x) = \max_{a\in\Psi(x)} [u(x, a) + \delta(V^*(f(x, a)))] \tag{20}$$

for all $x \in X$. Moreover, V^* is an optimal return function.

(b) *A plan $\boldsymbol{b} = \{b_t\}_0^\infty \in \Pi(x)$ satisfying (20), i.e.,*

$$V^*(x_t) = u(x_t, b_t) + \delta(V^*(f(x_t, b_t))), \quad t = 0, 1, \ldots \tag{21}$$

is optimal.

The proof of this result makes use of Theorem 2, Lemma 1 and an approximation technique by models with truncated utility functions.

Remark 3 We have assumed that the immediate return function u and the set-valued mapping $x \mapsto \Psi(x)$ are continuous. In fact, we may relax these restrictions and presume that u and $x \mapsto \Psi(x)$ are upper semicontinuous. Then, the space $C(X)$ in Subsection 3.1 should be replaced by the space of all bounded upper semicontinuous functions $\mathcal{U}_b(X)$. This space is a closed subset of all bounded real-valued functions and, thus, is complete, when equipped with a supremum metric. Then, each function U_m is upper semicontinuous on H. Moreover, by Theorem 6.3.2 in Berge (1963), we conclude that $v_m^* \in \mathcal{U}_b(X)$ for $m \geq 1$. Hence, Theorem 3 still holds by virtue of Lemma 1.

Remark 4 It is well-known that the function $V^* \in \mathcal{U}(X)$ satisfying (20) need not be unique even if u is bounded and $\delta(z) = \beta z$ with $\beta \in (0, 1)$, see Example 6.4 in Feinberg (2002). However, the uniqueness of a solution to the optimality equation can be shown, if we restrict ourselves to certain classes of models with Euclidean state and action spaces, strictly concave utility functions, see Boyd III (2006); Le Van and Vailakis (2005).

Remark 5 Boyd III (1990) proposes a modification of the Banach Contraction Principle (see also Theorem 9.2.1 Boyd III (2006)) and gives sufficient conditions for the existence of a unique recursive utility within pretty general framework. Namely, his approach requires a definition of an operator T that meets certain conditions. In our set-up, T is the operator defined in (10) with the aggregator introduced in (9). More precisely, for a fixed feasible history $h_{x_0} = (x_0, a_0, \ldots)$ and any $v \in C(H)$, we set

$$(Tv)(h_{x_0}) = W(x_0, a_0, v(s(h_{x_0}))) = u(x_0, a_0) + \delta(v(s(h_{x_0}))).$$

If u is bounded and continuous on $X \times A$, then T maps the space of bounded continuous functions into itself. In order to apply Theorem 9.2.1 in Boyd III (2006), one needs to prove that

$$T(\xi + \alpha\varphi) \leq T\xi + \alpha\theta\varphi, \quad \text{for some } \theta < 1 \quad \text{and all } \alpha > 0.$$

In our case, this condition reduces to the inequality

$$T\alpha \leq u(x_0, a_0) + \alpha\theta \quad \text{if} \quad \xi = 0. \tag{22}$$

If we assume that δ is of the form

$$\delta(z) = \frac{1}{2}\left(z + \frac{z}{z^2+1}\right), \quad z \in [0, \infty), \tag{23}$$

then (22) leads to $\frac{1}{2}\left(\alpha + \frac{\alpha}{\alpha^2+1}\right) \leq \alpha\theta$ for some $\theta < 1$ and all $\alpha > 0$. However, it is easy to see that this condition is not satisfied. Therefore, Theorems 1 and 2 can be deduced neither from Theorem 9.2.1 in Boyd III (2006) nor from Lucas and Stokey (1984) (where the Banach Contraction Principle was directly applied). However, observe that δ given in (23) is subadditive and (A1) is thus satisfied with $\gamma = \delta$. Consequently, by Proposition 1, we are able to find a bounded continuous function $U \in C(H)$ that is a fixed point of the operator T.

In the theory of recursive utilities, it is also common to assume that $W(x, a, r)$ is concave with respect to r, see Boyd III (2006); Marinacci and Montrucchio (2010). We notice that this condition is not satisfied for the above-mentioned δ, since δ is convex on $(\sqrt{3}, \infty)$ and concave on $[0, \sqrt{3})$. Hence, our aggregator W belongs neither to the class of aggregators studied by Boyd III (2006) nor to the class of Thompson aggregators examined in Marinacci and Montrucchio (2010).

4 Examples

Example 5 The inventory model (compare Section 5.11 in Stokey et al. (1989)). A manager wants to sell up to 1 unit of a certain product each period at price p. If he has $x \geq 0$ units in stock, he can sell $\min\{x, 1\}$. He can also order any amount a of new goods to be delivered at the beginning of next period at a cost $c_0 + c_1 a$ paid now ($c_0, c_1 > 0$). Hence, the system equation is of the form:

$$x_{t+1} = x_t - \min\{x_t, 1\} + a_t, \quad \text{for } t = 0, 1, \ldots.$$

The manager discounts his revenues according to a function δ satisfying (A1).

This model can be viewed as a dynamical system, in which

- $X := [0, \infty)$ is the state space, the set of possible levels of stock,
- $A = \Psi(x) := [0, K]$ is the action space, where $K > 0$; it denotes the units ordered by the manager,
- $u(x, a) := p \min\{x, 1\} - l(a)$ is the immediate return function, where $l(a) = c_0 + c_1 a$ for $a > 0$, and $l(0) = 0$.

Clearly, the manager will place an order, if

$$\delta(py) > l(y), \quad y > 0.$$

From Theorem 2, there exists a bounded continuous function v^* such that it holds

$$v^*(x) = \max_{0 \leq a \leq K} [u(x, a) + \delta(v^*(\max\{0, x - 1\} + a))]$$

for all $x \in X$. Moreover, there exists an optimal ordering plan $\mathbf{a}^* = \{a_t\}_0^\infty \in \Pi(x_0)$ $(x_0 = x)$ that satisfies the equations

$$v^*(x_t) = u(x_t, a_t^*) + \delta(v^*(\max\{0, x_t - 1\} + a_t^*)), \quad \text{for } t \geq 0.$$

Example 6 The one-sector growth model (compare to Ramsey (1928)). Let c_t denote consumption in time period t and let k_t denote the capital stock accumulated during period t and used for production in period $t+1$. The agent starts with an initial capital k_0. Consider the sequences of consumption level $\mathbf{c} = \{c_t\}_1^\infty$, and capital stocks $\mathbf{k} = \{k_t\}_1^\infty$. Let f denote a production function. Income $f(k_{t-1})$ is freely divided between consumption c_t and capital stock k_t. A consumption sequence $\mathbf{c}$ is called feasible from k_0, if

$$c_t \geq 0, \; k_t \geq 0, \quad k_t = f(k_{t-1}) - c_t, \quad t = 1, 2, \ldots.$$

The agent is equipped with a one-period utility function $u : [0, \infty) \mapsto \underline{R}$ and a discount function δ satisfying (A1).

Then, such model can be viewed as a system described in Section 3, where

- the state space is the set of non-negative real numbers with a generic element k,
- the action space is $A := [0, \infty)$ with a generic element c,
- the available action set is $\Psi(k) := [0, f(k)]$.

Proposition 2 *Let u be continuous, increasing, and such that $u(0) = -\infty$. Assume that f is continuous, non-decreasing, $f(0) = 0$, and there exists $\bar{k} > 0$ such that*

$$f(k) > k \quad for \quad k \in (0, \bar{k}) \quad and \quad f(k) \leq k \quad for \quad k \geq \bar{k}. \tag{24}$$

Then there exists an upper semicontinuous optimal return function V^* such that it holds

$$V^*(k) = \max_{0 \leq c \leq f(k)} [u(c) + \delta(V^*(f(k) - c))], \quad k \in X. \tag{25}$$

Moreover, there exist an optimal consumption plan $\boldsymbol{c}^* = \{c_t^*\}_1^\infty \in \Pi(k_0)$ ($k_0 = k$) and a corresponding sequence of capital stocks $\boldsymbol{k} = \{k_t\}_1^\infty$ such that, for any $t \geq 1$, we have

$$V^*(k_{t-1}) = u(c_t^*) + \delta(V^*(f(k_{t-1}) - c_t^*))].$$

The proof is based on truncated models and Theorem 3.

Remark 6 Assumption (24) is common and has been used by Bhattacharya and Majumdar (2007); Stokey et al. (1989). It plays a crucial role in our proof, because it allows to consider submodels on a truncated space X_n. This approach, to a limited extent, resembles the ideas proposed by Rincón-Zapatero and Rodriguez-Palmero (2003, 2009). However, their results and related ones in Alvarez and Stokey (1998); Le Van and Morhaim (2002) concern standard discounting function $\delta(z) = \beta z$ with $\beta \in (0, 1)$. Therefore, equation (25) is an extension of the Bellman equation obtained for such models with a linear discounting function (see, for instance, Becker and Boyd III (1997), Chapter 1 in Bhattacharya and Majumdar (2007), or Chapters 4 and 5 in Stokey et al. (1989)).

Remark 7 The choice of a suitable discount function in a particular model seems to be an interesting and open issue. In a number of sequential decision-making problems related to operation research (like Example 5), we may consider a subbaditive discount function δ. This property reflects a tacit assumption that the requested rate of return of the decision maker is an increasing function of capital (as in Example 3). In other models, related more to the theory of resource extraction or optimal growth, the discount function δ can be drawn from a social or political background. Then, the behaviour of a decision maker may be consistent with a majority opinion on time preferences. In that case, the variable discount rate might be increasing as in empirical studies of Benzion et al. (1989); Green et al. (1997); Kirby (1997); Thaler (1981) partially reflected in Example 2.

5 Conclusions

We consider dynamic programming problems with the variable discounting represented by a discount function δ satisfying pretty general condition (A1). The recursive utilities induced by the aggegator W, introduced in (9), exceed a class of utilities studied in the literature. This happens, if the derivative of δ equals 1 at at least one point. On one hand, the possibility of an application of various discount functions allows to take into account different aspects of discounting discussed in the area of finance, economics, psychology (see Section 2). On the other hand, the variable discounting leads to many interesting open problems in dynamic programming, such as stability of optimal paths, differentiability of a value function, etc. We believe that extensions to stochastic dynamic programming are possible in different directions. Some of them may lead to non-stationary dynamic programming as in Hinderer (1970) or "non-expected utilities", see Kreps and Porteus (1978), Marinacci and Montrucchio (2010) and the references therein. These issues are our objectives in future reseach.

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