Conditional heteroskedasticity in nonlinear simultaneous equations

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Abstract

We show in this paper that the treatment of conditional heteroskedasticity inside nonlinear systems of simultaneous equations is a sufficiently manageable matter for some types of multivariate ARCH error structures. Reparameterization makes it possible to estimate the model by means of the (nearly) standard algorithms developed in the past and widely used for estimating nonlinear simultaneous equations where the error structure is of the i.i.d. type with unrestricted contemporaneous covariance matrix. The method is discussed in this paper and empirical applications exemplify the efficiency gains.

Keywords: nonlinear simultaneous equations, conditional heteroskedasticity, instrumental variables, nonlinear FIML, demand–supply model, long term Treasury bonds.

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1 Introduction

Various and more or less sophisticated versions of ARCH models are now routinely applied, with controversial results, to long financial time series in the attempt of modelling the time varying variance (and possibly covariance) of the error process. It is sometimes observed that even when a moderate amount of conditional heteroskedasticity is present, a simple ARCH structure can successfully be employed.

Monthly monetary models based on simultaneous equations are employed by several institutions (e.g. Banca d'Italia, see Angeloni et al., 1992, or Giovannini et al., 1994). In the functional form specification of this type of models, it is customary that endogenous variables are subject to some nonlinear transformation. For example the log of a price index may be involved as explanatory variable in a log-linear equation, while the price index level can be used to deflate variables of interest in some other equation and in turn it can be an endogenous variable being explained by another equation elsewhere in the system (or in another system when monetary and real models are kept separated).

When prices or monetary variables such as money supply and interest rates are involved in a model, even if the series are observed quarterly (e.g. Engle, 1982) or monthly (e.g. Bianchi, Calzolari and Sterbenz, 1991, Fiorentini and Maravall, 1994), it is common to find evidence of possible heteroskedasticity that can successfully be removed with ARCH models.

The purpose of this paper is to introduce a simple and yet usefull way of managing conditional heteroskedasticity in nonlinear systems of simultaneous equations. The method proposed is a type of multivariate ARCH that, after an appropriate reparameterization, can suitably be handled with the (nearly) standard techniques developed in the past and used for estimating nonlinear simultaneous equations with i.i.d. error terms.

While the hypothesis of conditionally multivariate normal disturbances leads to a sort of Nonlinear FIML (Amemiya, 1977), if no assumption is made on the conditional density function resort to techniques like Nonlinear 2SLS, Nonlinear 3SLS, or Best Nonlinear 3SLS could lead to a kind of semiparametric multivariate ARCH for simultaneous equations.

The few results available in the literature for estimation of simultaneous equations with ARCH-GARCH errors are confined to linear systems (and to the best of our knowledge no example of application is available, either with simulated or with real data). They are briefly summarized at the beginning of section

2. We then propose, for ARCH errors, a reparameterization that is illustrated first in the context of a single equation (univariate case), then for linear simultaneous equations, finally for nonlinear simultaneous equations, with a unified treatment.

Two applications are presented, the first of which is an application to a simple nonlinear model, with simulated data. Its purpose is to show the benefits involved by the appropriate joint treatment of endogeneity and ARCH error structure in a nonlinear system of simultaneous equations (section 3). In section 4 we present the results of an application to a demand-supply model for the long term Treasury bonds in Italy based on monthly data.

2 Model Specification and Estimator Properties

Multiple equation models with multivariate ARCH structure of the errors introduced by Kraft and Engle (1983) and Bollerslev, Engle, and Wooldridge (1988) are exemplified in the literature. On the contrary, simultaneous equation models with time varying covariance matrix has received only little attention. Baba, Engle, Kraft, and Kroner (1991), and Harmon (1988) introduced the general theoretical framework for the SEM-GARCH model.

In a multivariate context the general GARCH error specification is given by

$$\epsilon_t = H^{1/2} e_t \quad e_t \ \text{ i.i.d.} \quad E(e_t) = 0 \quad Var(e_t) = I \tag{1}$$

where ϵ_t denotes the $n \times 1$ error vector at time t (n being the number of equations) and H_t is the $n \times n$ time varying conditional covariance matrix that is parameterized as

$$h_t = vech(H_t) = c + \sum_{i=1}^{q} A_i \, vech(\epsilon_{t-i}\epsilon'_{t-i}) \; + \; \sum_{i=1}^{p} B_i \, vech(H_{t-i}) \tag{2}$$

In the simplest case of $n = 2$ and GARCH(1,1) error process, the variance equation becomes

$$h_t = \begin{bmatrix} h_{1,1,t} \\ h_{1,2,t} \\ h_{2,2,t} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} + \begin{bmatrix} a_{1,1} a_{1,2} a_{1,3} \\ a_{2,1} a_{2,2} a_{2,3} \\ a_{3,1} a_{3,2} a_{3,3} \end{bmatrix} \begin{bmatrix} \epsilon^2_{1,t-1} \\ \epsilon_{1,t-1}\epsilon_{2,t-1} \\ \epsilon^2_{2,t-1} \end{bmatrix} + \begin{bmatrix} g_{1,1} g_{1,2} g_{1,3} \\ g_{2,1} g_{2,2} g_{2,3} \\ g_{3,1} g_{3,2} g_{3,3} \end{bmatrix} \begin{bmatrix} h_{1,1,t-1} \\ h_{1,2,t-1} \\ h_{2,2,t-1} \end{bmatrix} \tag{3}$$

where there are 21 parameters, and many complicated restrictions need be imposed to guarantee positive definiteness of H_t for any t.

When the general multivariate conditional heteroskedastic covariance matrix is imposed on the error process, the number of ARCH (or GARCH) parameters increases dramatically with the number of equations (for 5 equations still with GARCH(1,1) there would be 465 parameters, as observed by Harmon, 1988).

To guarantee a more efficient inference some restrictions are usually assumed on the conditional variance parameters. Bollerslev, Engle, and Wooldridge (1988) proposed the so-called diagonal representation, that still suffers from the need of imposing restrictions to guarantee positive definiteness of H_t.

The set of constraints that, in an unrestricted model, guarantees the variance covariance matrix to be positive-definite is in general very hard to derive analytically. Baba et al. (1991) proposed a representation (BEEK) where the number of parameters is slightly larger than for the diagonal representation, but a positive definite covariance matrix is implied by the peculiar type of parameterization adopted.

Bollerslev (1990) reduced even more the number of variation free parameters introducing the fixed correlation GARCH, where the covariance matrix is time varying but the correlation structure remains constant over time.

In this paper we propose another approach. We start from a system of simultaneous equations and we generalize it to allow the tractability of some multivariate ARCH error structure. We show that a linear SEM with ARCH errors can be transformed in a suitable way in a nonlinear SEM with constant covariance matrix; while if we start with a system of nonlinear simultaneous equation with conditional heteroskedasticity the reparameterization we adopt leads to another system of nonlinear equations, which has a more complicated structural form, but the error term are no more heteroskedastic. The important point to be stressed is that the introduction of ARCH errors in a nonlinear SEM does not affect very much the analytical tractability of these type of models and some of the usual estimation techniques remain essentially the same

2.1 The univariate case

In the simple univariate linear case, with ARCH(1) errors, we have

$$y_t = b_1 + b_2 z_t + \epsilon_t \qquad \epsilon_t \mid \mathcal{I}_{t-1} \sim N(0, h_t) \qquad h_t = \alpha_0 + \alpha_1 \epsilon^2_{t-1} \tag{4}$$

so that ϵ_t can be written as

$$\epsilon_t = \sqrt{\alpha_0 + \alpha_1 \epsilon_{t-1}^2}\, e_t \qquad\qquad e_t \;\; \text{i.i.d.} \;\; N(0,1) \tag{5}$$

Inserting the above expression in equation (4), and rearranging terms, we get

$$\frac{y_t - b_1 - b_2 z_t}{\sqrt{1 + (\alpha_1/\alpha_0)\epsilon_{t-1}^2}} = \sqrt{\alpha_0}\, e_t \tag{6}$$

Let us call $\alpha_0 = \sigma^2$, $\alpha_1/\alpha_0 = b_3$, and $\sigma e_t = u_t$ i.i.d. $N(0, \sigma^2)$. Equation (6) is reparameterized as

$$\frac{y_t - b_1 - b_2 z_t}{\sqrt{1 + b_3(y_{t-1} - b_1 - b_2 z_{t-1})^2}} = u_t \quad \longrightarrow \quad \mathrm{f}(y_t, x_t, a) = u_t \tag{7}$$

where x_t is the vector of all exogenous and lagged dependent variables ($x_t = [1, z_t, y_{t-1}]'$), and $a = [b_1, b_2, b_3]'$ is the vector of coefficients, two of which (b_1 and b_2) come from the mean equation and one (b_3) comes from the variance equation. The error terms u_t are i.i.d. $N(0, \sigma^2)$, with σ^2 unknown.

For an ARCH(2) specification of the error structure, a similar reparameterization with $\alpha_2/\alpha_0 = b_4$ would lead to

$$\frac{y_t - b_1 - b_2 z_t}{\sqrt{1 + b_3(y_{t-1} - b_1 - b_2 z_{t-1})^2 + b_4(y_{t-2} - b_1 - b_2 z_{t-2})^2}} = u_t \longrightarrow \mathrm{f}(y_t, x_t, a) = u_t \tag{8}$$

where $x_t = [1, z_t, y_{t-1}, y_{t-2}]'$, $a = [b_1, b_2, b_3, b_4]'$ and u_t i.i.d. $N(0, \sigma^2)$. Analogous result follows from reparameterization of a general ARCH(q).

In all cases we get a nonlinear implicit equation with additive i.i.d. error terms with unknown variance. The characteristics of this equation are such that it cannot be viewed as a nonlinear regression model. In fact the Jacobian $J_t = \partial u_t/\partial y_t = \partial f_t/\partial y_t$ is a time dependent function of the coefficients, while for the standard univariate nonlinear regression $y_t = \mathrm{q}(x_t, b) + u_t$ the Jacobian J_t would be equal to 1 for every t, and therefore M.L. would be equal to L.S..

As an alternative to maximum likelihood, Engle and González-Rivera (1991) consider a semiparametric extension of GARCH models, retaining Bollerslev's as a general form for heteroskedasticity, yet allowing the density of ϵ_t to be of an unknown form, and using nonparametric estimates of the score function of ϵ_t to estimate the parameters of a GARCH process. They report Monte Carlo simulation results showing that their method outperforms the Gaussian pseudo-maximum likelihood, but it does not seem to be adaptive, in the sense that it is not fully efficient with respect to the GARCH process parameters.

González-Rivera (1993) develops sufficient conditions under which the former estimator behaves better then QML and derives a family of distribution functions in which the variance equation parameters can be estimated adaptively. Linton (1993) examines this semiparametric model further, considering only the situation where the unknown error density is symmetric about zero and constructing an estimator asymptotically equivalent to MLE. Drost and Klaassen (1993) point out that a complete adaptive estimation of the conditional variance parameters is not feasible for the original GARCH formulation due to the presence of a scale parameter (the unconditional variance of the disturbance term); they present a reparameterization of the model, which resembles the GARCH-M model and allows for a semiparametric estimator which performs better then QML and whose difference from the MLE becomes negligible if the sample size is large.

2.2 The linear simultaneous equations case

Let us suppose to deal with a linear system of simultaneous equations and to impose some sort of multivariate ARCH structure on the error terms (without yet specifying the type of multivariate structure). For example, we may consider a standard textbook model for demand and supply (y_1 is price and y_2 is quantity)

$$\begin{aligned} y_{1,t} &= b_{1,1} + b_{1,2} y_{2,t} + b_{1,3} z_{1,t} + \epsilon_{1,t} \\ y_{2,t} &= b_{2,1} + b_{2,2} y_{1,t} + b_{2,3} z_{2,t} + \epsilon_{2,t} \end{aligned} \tag{9}$$

where both ϵ_1 and ϵ_2 are ARCH(1)

$$\begin{aligned} \epsilon_{1,t} &= \sqrt{\alpha_{1,0} + \alpha_{1,1}\epsilon_{1,t-1}^2}\, e_{1,t} = \sqrt{1 + b_{1,4}\epsilon_{1,t-1}^2}\, u_{1,t} \qquad u_{1,t} \text{ i.i.d. } N(0, \sigma_{1,1}) \\ \epsilon_{2,t} &= \sqrt{\alpha_{2,0} + \alpha_{2,1}\epsilon_{2,t-1}^2}\, e_{2,t} = \sqrt{1 + b_{2,4}\epsilon_{2,t-1}^2}\, u_{2,t} \qquad u_{2,t} \text{ i.i.d. } N(0, \sigma_{2,2}) \end{aligned} \tag{10}$$

having posed $\sigma_{1,1} = \alpha_{1,0}$, $\sigma_{2,2} = \alpha_{2,0}$, $b_{1,4} = \alpha_{1,1}/\alpha_{1,0}$, and $b_{2,4} = \alpha_{2,1}/\alpha_{2,0}$.

If we do not assume $u_{1,t}$, and $u_{2,t}$ to be contemporaneously uncorrelated, substitution of (10) into (9) gives

$$\begin{aligned} \frac{y_{1,t}-b_{1,1}-b_{1,2}y_{2,t}-b_{1,3}z_{1,t}}{\sqrt{1+b_{1,4}(y_{1,t-1}-b_{1,1}-b_{1,2}y_{2,t-1}-b_{1,3}z_{1,t-1})^2}} &= u_{1,t} \\ \frac{y_{2,t}-b_{2,1}-b_{2,2}y_{1,t}-b_{2,3}z_{2,t}}{\sqrt{1+b_{2,4}(y_{2,t-1}-b_{2,1}-b_{2,2}y_{1,t-1}-b_{2,3}z_{2,t-1})^2}} &= u_{2,t} \end{aligned} \tag{11}$$

$$\begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix} \text{ i.i.d. } \mathrm{N}\left(0\,, \begin{bmatrix} \sigma_{1,1} & \sigma_{1,2} \\ \sigma_{2,1} & \sigma_{2,2} \end{bmatrix}\right) \tag{12}$$

We have thus obtained two equations still simultaneous, but nonlinear, each with 4 coefficients instead of the 3 it had at the beginning, with error terms that are additive i.i.d. multivariate normal with unrestricted contemporaneous covariance matrix. This turns out to be a particular type of multivariate ARCH error structure introduced into the original model.

The advantage of this reparameterization is that the final notation is the one traditionally adopted for simultaneous systems of nonlinear implicit equations with i.i.d. error terms, that is a notation usually employed for macroeconometric models in structural form, and for which a huge literature is available. For example our two reparameterized equations (11) would perfectly fit the notation by Amemiya (1977)

$$\begin{aligned} f_{1,t} &= f_1(y_t, x_t, a_1) = \\ f_{2,t} &= f_2(y_t, x_t, a_2) = \end{aligned} \begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix} \text{ i.i.d. } \mathrm{N}(0\,,\Sigma\,) \tag{13}$$

where $a_1 = [b_{1,1}, b_{1,2}, b_{1,3}, b_{1,4}]'$, $a_2 = [b_{2,1}, b_{2,2}, b_{2,3}, b_{2,4}]'$, and Σ is the unrestricted contemporaneous covariance matrix, constant over time, that can be estimated in the usual way from the structural form residuals. We observe that the two equations do not share common coefficients, that is to say there are no cross-equation restrictions.

Assuming for a model like (13) a multivariate normal distribution of the error terms, Amemiya (1977) shows that a fully efficient estimate of the coefficients (nonlinear FIML) can be obtained by iterating to convergence an instrumental variables method

$$\hat{a}_{(m+1)} = \hat{a}_{(m)} - \left[\hat{G}'_{(m)}\left(\hat{\Sigma}^{-1}_{(m)} \otimes I\right) G_{(m)}\right]^{-1} \hat{G}'_{(m)}\left(\hat{\Sigma}^{-1}_{(m)} \otimes I\right) \mathrm{vec}\,\hat{U}_{(m)} \tag{14}$$

where $a = [a_1', a_2']'$ is the vector of coefficients (which in our case are 8), G is a block-diagonal matrix whose blocks G_i ($i = 1, 2$) have rows $g'_{i,t}$ (obtained as $g_{i,t} = \partial f_{i,t}/\partial a_i$); the block-diagonal matrix of instruments $\hat{G}$ has blocks

$$\hat{G}'_i = G'_i - \left(\frac{1}{T}\sum_{t=1}^{T}\frac{\partial g_{i,t}}{\partial u'_t}\right) U' \tag{15}$$

where $\partial g_{i,t}/\partial u_{i,t} = (\partial g_{i,t}/\partial y'_t)(\partial f_t/\partial y'_t)^{-i}$, all derivatives being evaluated at $\hat{a}_{(m)}$ and $\hat{U}_{(m)}$ (this choice for $\hat{G}$ is not unique, as shown in Calzolari and Sampoli, 1993).

When no assumption is made on the functional form of the density of the error vector, for general systems of nonlinear simultaneous equations one could build other instruments $\hat{G}$ that, if used in equation (14), would produce some kind of semiparametric estimator, such as Nonlinear 3SLS ($\hat{G}$ is essentially G evaluated at $U = 0$), or Amemiya's Best Nonlinear 3SLS ($\hat{G}$ is the conditional expectation of G, whose feasible forms were proposed by Newey, 1990 and Robinson, 1991). These forms of instruments for semiparametric estimation seem to be not suitable for our case. The problem of semiparametric estimation and of the semiparametric efficiency bound in this context deserves further investigation.

2.3 The nonlinear simultaneous equations case

In the previous section we started with a linear model with some form of conditional heteroskedasticity and, after reparameterization, we ended up with a nonlinear model without heteroskedasticity. When the model specification is nonlinear at the outset, the same ARCH structure of the disturbances does not imply any further complication, provided that our reparameterization is adopted.

Suppose that our model is made of two simultaneous nonlinear implicit equations, whose structural form will be indicated as

$$\begin{aligned} w_{1,t} &= w_1(y_t, x_t, b_1) = \epsilon_{1,t} \\ w_{2,t} &= w_2(y_t, x_t, b_2) = \epsilon_{2,t} \end{aligned} \tag{16}$$

where $y_t = [y_{1,t}, y_{2,t}]'$ is the vector of jointly dependent endogenous variables at time t, x_t is the vector of exogenous and lagged endogenous variables, b_1 and b_2

are the vectors of model coefficients, and ϵ_t is the vector of ARCH(1) disturbances as specified in the previous section.

After reparameterization, defining the additional coefficient for the first equation $b_1^* = \alpha_{1,1}/\alpha_{1,0}$, and for the second equation $b_2^* = \alpha_{2,1}/\alpha_{2,0}$, we get the structural form equations

$$\begin{array}{l} \frac{w_1(y_t,x_t,b_1)}{\sqrt{1+b_1^*[w_1(y_{t-1},x_{t-1},b_1)]^2}} = \\ \frac{w_2(y_t,x_t,b_2)}{\sqrt{1+b_2^*[w_2(y_{t-1},x_{t-1},b_2)]^2}} = \end{array} \begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix} \text{ i.i.d. N} \left(0, \begin{bmatrix} \sigma_{1,1} & \sigma_{1,2} \\ \sigma_{2,1} & \sigma_{2,2} \end{bmatrix}\right) \tag{17}$$

that can be written as

$$\begin{aligned} f_{1,t} &= f_1(y_t,x_t,a_1) = u_{1,t} \\ f_{2,t} &= f_2(y_t,x_t,a_2) = u_{2,t} \end{aligned} \tag{18}$$

where $a_1 = [b_1', b_1^*]'$, and $a_2 = [b_2', b_2^*]'$, that is two nonlinear simultaneous implicit equations, whose additive errors are i.i.d. normal with unrestricted covariance matrix Σ.

An ARCH(2) error structure would simply introduce an additional coefficient in the two equations (18); ARCH(q) is analogous.

2.4 Fixed correlation multivariate ARCH

Proposition 1: Let $w_i(y_t,x_t,b_i) = \epsilon_{i,t}$ with $h_{i,t} = \alpha_{i,0} + \alpha_{i,1}\epsilon_{i,t-1}^2$ denote a generic equation in the original nonlinear system. After reparameterization the equation becomes $f_i(y_t,x_t,b_i) = u_{i,t}$, being the vector of all error terms in the system u_t i.i.d. $N(0,\Sigma)$. Apart from simultaneity, the error structure in the reparameterized model is equivalent to the Bollerslev (1990) fixed-correlation multivariate ARCH with $\rho_{i,j} = \sigma_{i,j}/\sqrt{\sigma_{i,i}\sigma_{j,j}}$. □

All we have to show is that

$$\frac{\sigma_{i,j}}{\sqrt{\sigma_{i,i}\sigma_{j,j}}} = \frac{h_{i,j,t}}{\sqrt{h_{i,t}h_{j,t}}} = \rho_{i,j} \qquad \forall t = 1,\ldots,T \tag{19}$$

that is, conditional correlation is constant over time.

We have in fact

$$\frac{\sigma_{i,j}}{\sqrt{\sigma_{i,i}\sigma_{j,j}}} = \frac{E_{t-1}(u_{i,t}\ u_{j,t})}{\sqrt{\alpha_{i,0}\alpha_{j,0}}} = \frac{E_{t-1}\left(\frac{\epsilon_{i,t}\sqrt{\alpha_{i,0}}}{\sqrt{h_{i,t}}} \frac{\epsilon_{j,t}\sqrt{\alpha_{j,0}}}{\sqrt{h_{j,t}}}\right)}{\sqrt{\alpha_{i,0}\alpha_{j,0}}} \tag{20}$$

and, since $E_{t-1}(h_{i,t}) = h_{i,t}$, the proof follows immediately.

A time varying conditional covariance with a constant correlation was found to be a quite plausible hypothesis in many empirical situations and it considerably simplifies estimation and inference. For a related application (in a multivariate but not simultaneous context) see Baillie and Bollerslev (1990), or Ng(1991).

3 Estimation Efficiency: An Example

A simulation experiment has been performed in order to appreciate the behaviour of the estimators previously discussed.

Let us suppose to have a nonlinear version of the demand-supply model already used (eq. 9)

$$\begin{aligned} \log p_t^d &= b_{1,1} + b_{1,2}\log q_t^d + b_{1,3}z_{1,t} && +\epsilon_{1,t} \\ q_t^s &= b_{2,1} + b_{2,2}p_t^s + b_{2,3}z_{2,t} && +\epsilon_{2,t} \\ p_t^d &= p_t^s - 10z_{3,t} \\ q_t^s &= q_t^d \end{aligned} \tag{21}$$

The demand function is log-linear, while the supply equation is linear. The last equation is the standard market-clearing condition, while the prices for demand and supply are allowed to differ for the presence of exogenous institutional intervention such as incentive or subsidies to production, taxes, tariffs, etc.

A sample period of 2000 observations has been adopted, to ensure a behaviour of estimators close to asymptotic. Strictly exogenous variables have been generated as

$$\begin{bmatrix} z_{1,t} \\ z_{2,t} \\ z_{3,t} \end{bmatrix} \text{ i.i.d. N} \left(0, \begin{bmatrix} 1 & .5 & .5 \\ .5 & 1 & .5 \\ .5 & .5 & 1 \end{bmatrix}\right) \tag{22}$$

As usual for maximum likelihood estimation of simultaneous equation, the definitional equations must first be substituted out into the stochastic equations (at least in principle). Let y_1 and y_2 denote the price p_t^s and the quantity q_t respectively, we obtain a nonlinear structural system in the form of eq. (16)

$$\begin{aligned} w_{1,t} &= \log(y_{1,t} - 10z_{3,t}) - b_{1,1} - b_{1,2}\log y_{2,t} - b_{1,3}z_{1,t} = \epsilon_{1,t} \\ w_{2,t} &= y_{2,t} - b_{2,1} - b_{2,2}y_{1,t} - b_{2,3}z_{2,t} = \epsilon_{2,t} \end{aligned} \tag{23}$$

The coefficient values used in the simulation experiment are displayed in the first column of table 1. In particular the values $b_{1,2} = -1$ and $b_{2,2} = 1$ have been chosen to ensure the proper slopes of the demand and supply curves.

For the generation of ϵ_1 and ϵ_2 we have adopted a fixed-correlation multivariate ARCH(1) specification with $\alpha_{1,0} = 0.025$, $\alpha_{1,1} = 0.7$, $\alpha_{2,0} = 10.0$, $\alpha_{2,1} = 0.7$, and $\rho = 0.9$.

Hence, the reparameterized model becomes

$$\begin{aligned} f_{1,t} &= w_{1,t} \times \left(1 + b_{1,4}w_{1,t-1}^2\right)^{-1/2} &= u_{1,t} \\ f_{2,t} &= w_{2,t} \times \left(1 + b_{2,4}w_{2,t-1}^2\right)^{-1/2} &= u_{2,t} \end{aligned} \qquad u_t \ i.i.d.\ N(0, \Sigma) \tag{24}$$

(as there are 3 coefficients in each of the two equations, the additional coefficients b_1^* and b_2^* have been labelled $b_{1,4}$ and $b_{2,4}$, respectively).

We have obtained, in this way, two nonlinear simultaneous implicit equations, each of which has 4 coefficients, with the last one derived from the conditional variance parameters (with the α's adopted in the generation process, the "true" $b_{1,4}$ and $b_{2,4}$ are $b_{1,4} = 28.0$ and $b_{2,4} = 0.07$, see table 1, first column). The error terms are additive, i.i.d. and with unrestricted covariance matrix Σ.

Simultaneity ignored: we may estimate the two stochastic equations of the original model (eq. 21) ignoring both simultaneity (or regressors' endogeneity) and heteroskedasticity, and just apply OLS. As expected, the resulting estimates are completely misleading (for example, the estimated slope of the demand equation is equal to 0.417 instead of -1.0).

We may properly consider heteroskedasticity, but still ignore simultaneity (or regressors' endogeneity), and estimate, by maximum likelihood, a traditional univariate ARCH(1) for each of the two stochastic equations in the original model (eq. 21 or 23). The results change, but are still completely misleading (the slope of the demand equation is 0.321 instead of -1.0).

Simultaneity considered: let us now take into account simultaneity. Again, we may ignore heteroskedasticity by estimating the model in its original form (the

Table 1: Estimation of nonlinear demand-supply model

		Simultaneity Ignored		Simultaneity Considered					
				Ignoring ARCH			Considering ARCH		
Coeff.	*True*	OLS	ARCH(1)	IIV	NL3SLS	FIML	Semipar. methods		FIML
$b_{1,1}$	9.0	2.639	3.069	9.228	9.204	9.070	—	—	8.853
		(.064)	(.109)	(.430)	(.318)	(.557)	(—)	(—)	(.123)
$b_{1,2}$	-1.0	0.417	0.321	-1.045	-1.043	-1.013	—	—	-0.966
		(.014)	(.024)	(.096)	(.071)	(.124)	(—)	(—)	(.027)
$b_{1,3}$	0.10	-0.049	-0.038	0.100	0.099	0.087	—	—	.099
		(.003)	(.003)	(.012)	(.007)	(.008)	(—)	(—)	(.003)
$b_{1,4}$	28.0	—	13.12	—	—	—	—	—	32.34
		(—)	(4.49)	(—)	(—)	(—)	(—)	(—)	(2.54)
$b_{2,1}$	0.0	-17.28	-16.32	2.742	2.876	6.292	—	—	-0.12
		(.545)	(.698)	(1.76)	(1.27)	(2.92)	(—)	(—)	(.455)
$b_{2,2}$	1.0	1.189	1.175	0.972	0.970	0.933	—	—	1.00
		(.006)	(.008)	(.019)	(.014)	(.031)	(—)	(—)	(.005)
$b_{2,3}$	10.0	10.1	10.1	9.926	9.93	9.86	—	—	10.1
		(.099)	(.068)	(.130)	(.118)	(.231)	(—)	(—)	(.050)
$b_{2,4}$	0.07	—	0.103	—	—	—	—	—	0.072
		(—)	(.013)	(—)	(—)	(—)	(—)	(—)	(.006)

choice between equations 21 or 23 is determined only by computational simplicity), or we may consider heteroskedasticity estimating the reparameterized model (eq. 24). In both cases we may take into account the normal density function of the error terms (correctly or not), or ignore it. And, again, we may apply a limited information or a full information method.

Thus we have a variety of alternative estimates, some of which are exemplified in table 1.

Heteroskedasticity and normality ignored: we may apply Iterative Instrumental Variables (IIV) to the model (21 or 23), instruments being obtained from the simultaneous solution of the model without error terms, ignoring correlation between equations (thus in a limited information framework, see Dutta and Lyttkens, 1974, for linear systems, or Angeloni et al., 1992, for an application to a large scale nonlinear model). The results are in the corresponding column of the table, showing that the estimation method properly considers regressors' endogeneity and provides consistent estimates of the coefficients (the slopes of the two curves are properly estimated with values -1.045 and 0.972; the constant term of the second equation is not very satisfactory, 2.742, but its large standard error shows that it is not significantly different from zero, that would be the "true" value).

Still ignoring heteroskedasticity and normality, we may now take advantage of the equations' cross-correlation. Thus we estimate the model still in the form 21 or 23, still calculating instruments from the simultaneous solution of the system without error terms, as above, but then apply a Full Information formula, thus obtaining an estimate of the NL3SLS class (Amemiya, 1977). The estimator may also be viewed as an iterated version of Brundy and Jorgenson's (1971) FIVE method, recalling that (Amemiya, 1977) the nonlinearity of the model would prevent iterations to converge to FIML even if the error terms were normally distributed (Durbin, 1963 and 1988, Hausman, 1974).

The numerical results are consistent with expectation, being coefficients quite close to those of the limited information method (the previous column), but standard errors considerably smaller, as the estimation method takes advantage of the large correlation between the two equations.

Heteroskedasticity ignored, normality erroneously considered: we may apply Full Information Maximum Likelihood to the model 21 or 23, as if the error terms of the two equations were i.i.d. normal. This would be an inappropriate application of FIML (Amemiya, 1977) because the error terms had been generated with a conditionally heteroskedastic structure and are therefore not normal (high kurtosis). We still get good estimates of the coefficients, but the standard errors are larger than for the previous case (and most of them also larger than for IIV). We must remark that we are operating in a QML framework, so the standard errors must properly be calculated as in White (1982, 1983) using jointly Hessian and outer products of first derivatives of the log-likelihoods. It would be interesting to observe that if standard errors were incorrectly calculated only from the Hessian or only from the matrix of outer products, they would be smaller than those of NL3SLS, coherently with the theoretical result that, when the model is nonlinear, FIML is more efficient than any estimator of the NL3SLS class (these smaller but incorrect standard errors have been calculated, but are not displayed in the table).

Heteroskedasticity considered, normality ignored: we may estimate the model after reparameterization, that is in the form 24. Instrumental variables of the limited information or full information type could be applied, thus providing some sort of semiparametric estimates that do not exploit information from the distribution of the error terms. This part deserves further investigation; results are not displayed in the table.

Heteroskedasticity and normality correctly considered: we may estimate the reparameterized model 24 with Nonlinear FIML, as the error terms are i.i.d normal with unrestricted covariance matrix. Estimation is performed by iterating the instrumental variables method summarized in section (2.2), and taking advantage of the computational tools discussed in Calzolari, Panattoni and Weihs (1987). Again the results are consistent with the theory: the estimated coefficients are good and the standard errors are the smallest among all consistent estimators (these are in fact expected to be the most efficient estimates). We note in particular that only this method provides accurate estimates of the parameters derived from the variance equations ($b_{1,4}$ and $b_{2,4}$).

4 An Example Based on Real Data

A monthly econometric model of the financial sector has been recently proposed for Italy in Reale and Tirelli (1994), (see also Giovannini, Reale and Tirelli, 1994). The block modelling the Treasury security market consists of two simultaneous linear equations where the dependent variables are net demand and supply of long term Treasury bonds measured in billions of Italian lire. The bid mechanism for the bonds in the primary market, based on a competitive system of auction, is such that demand is an explanatory variable for supply and viceversa.

The two equations in the original model exhibit significant conditional heteroskedasticity. Several 0-1 dummy variables have been inserted in the original equations. One of the effects is the reduction of outliers and the reduction of the heteroskedasticity effect.

The model used in our experiments has been derived from the block of equations modelling the Treasury security market, with the following changes. Net demand and supply have been substituted by gross demand and supply; both endogenous variables became strictly positive in the sample period. The linear demand equation has been substituted by a log-linear equation. Dummy variables have been suppressed. The model is the following:

$$\mathrm{BTS}_t = b_{1,1} + b_{1,2}\mathrm{BTS}_{t-1} + b_{1,3}\mathrm{BTD}_t + b_{1,4}\mathrm{BRQ}_t \quad + u_{1,t} \qquad (25)$$

$$\log \mathrm{BTD}_t = b_{2,1} + b_{2,2}\log \mathrm{BTD}_{t-1} + b_{2,3}\log \mathrm{BTS}_t + b_{2,4}\log \mathrm{DFA}_{t-1} + b_{2,5}\mathrm{DIR}_{t-1} + b_{2,6}\mathrm{ITDR}_{t-2} \quad + u_{2,t} \qquad (26)$$

BTS = Supply of long term Treasury bonds
BTD = Demand of long term Treasury bonds
BRQ = Treasury borrowing requirment
DFA = first differences of the stock of financial assets held by the private sector
DIR = first differences of the inflation rate
IT = Net interest rate on Treasury bonds in the secondary market
DR = Discount rate
ITDR = IT - DR

Estimation results are displayed in Table 2, for the sample period 1986.2-1993.7 (90 observations).

From the values estimated for $b_{1,5}$, $b_{2,7}$ and for the matrix Σ, values of α_1 for the two equations and values of the cross equation correlation ($\rho_{1,2}$) are displayed in Table 3 for the various estimation methods.

5 Conclusion

Treatment of conditional heteroskedasticity inside nonlinear systems of simultaneous equations is a sufficiently manageable matter for multivariate ARCH

Table 2: Nonlinear demand-supply model of Italian T-Bills market

	Simultaneity Ignored		Simultaneity Considered					
			Ignoring ARCH			Considering ARCH		
Coeff.	OLS	ARCH(1)	LIV	NL3SLS	FIML	Semipar. methods		FIML
$b_{1,1}$	1025.0	932.2	1254.	1261.	1575.9	—	—	862.5
	(423.)	(484.)	(482.)	(480.)	(784.)	(—)	(—)	(779.)
$b_{1,2}$	0.083	0.057	0.264	0.260	0.482	—	—	0.627
	(.060)	(.065)	(.085)	(.086)	(.113)	(—)	(—)	(.223)
$b_{1,3}$	0.466	0.461	0.283	0.285	0.063	—	—	0.016
	(.035)	(.038)	(.059)	(.061)	(.094)	(—)	(—)	(.144)
$b_{1,4}$	-0.038	-0.050	-0.060	-0.060	-0.085	—	—	-0.108
	(.022)	(.019)	(.026)	(.026)	(.045)	(—)	(—)	(.042)
$b_{1,5}$	—	1.47E-7	—	—	—	—	—	0.48E-7
	(—)	(.7E-7)	(—)	(—)	(—)	(—)	(—)	(.5E-7)
$b_{2,1}$	-1.622	-0.801	-1.470	-1.430	-1.76	—	—	-2.16
	(.663)	(.429)	(1.03)	(1.03)	(1.17)	(—)	(—)	(.603)
$b_{2,2}$	0.085	0.096	0.108	0.124	0.074	—	—	0.119
	(.038)	(.038)	(.057)	(.057)	(.066)	(—)	(—)	(.045)
$b_{2,3}$	1.037	1.065	0.989	0.966	1.063	—	—	1.008
	(.043)	(.043)	(.148)	(.150)	(.125)	(—)	(—)	(.074)
$b_{2,4}$	0.115	-.004	0.119	0.118	0.117	—	—	0.170
	(.068)	(.037)	(.069)	(.068)	(.094)	(—)	(—)	(.053)
$b_{2,5}$	-0.156	-.070	-0.172	-0.186	-0.152	—	—	-0.104
	(.121)	(.132)	(.122)	(.121)	(.125)	(—)	(—)	(.157)
$b_{2,6}$	0.275	0.273	0.269	0.255	0.282	—	—	0.318
	(.057)	(.053)	(.061)	(.060)	(.086)	(—)	(—)	(.068)
$b_{2,7}$	—	13.51	—	—	—	—	—	9.57
	(—)	(9.89)	(—)	(—)	(—)	(—)	(—)	(5.01)

Table 3: Conditional covariance coefficients

Coeff.	ARCH(1)	NL3SLS	FIML	FIML-ARCH
$\alpha_{1,1}$	.32	—	—	.31
$\alpha_{2,1}$	.58	—	—	.47
$\rho_{1,2}$	—	-.18	-.13	.05

error structures, with time varying conditional variances and covariances, but a constant conditional correlation. Reparameterization makes it possible to estimate the model by means of (nearly) standard algorithms widely used in the past for estimating nonlinear simultaneous equations where the error structure was of the i.i.d. type with unrestricted contemporaneous covariance matrix. The method has been discussed in this paper and empirical applications have exemplified the efficiency gains.

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