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Martingale Model

Rossano Giandomenico

Abstract: The model determines a stochastic continuous process as continuous limit of a stochastic discrete process so to show that the stochastic continuous process converges to the stochastic discrete process such that we can integrate it. Furthermore, the model determines the expected volatility and the expected mean so to show that the volatility and the mean are increasing function of the time.

Introduction

In the 1827 Robert Brown was the first to observe and to describe the motion of a small particle suspended in a liquid as result of the successive and casual impacts of the near particle so to note that its variance is an increasing function of the time, from this the term Brownian motion.
In the 1905 Albert Einstein proposed on this a mathematics theory that was developed and presented in a more rigorous way by Norbert Wiener in the 1923.
The first stochastic model of the market price that used a Brownian motion was developed in the 1900 by Louis Bachelier in its extraordinary thesis at Sorbone of Paris. He has modelled the market price in the continuous time such that the uncertainty on the immediately future was preserved. This process usually called Martingale has the problem to permit to the market price to assume a negative value but as we will see this problem is easy to solve. In fact, if we consider that it models the rate of return, instead the arithmetic change of the market price, we get a stochastic continuous process that follows a lognormal distribution in each finite interval known as geometric Brown process. Many authors have given formal mathematics assumptions to a generalized stochastic equation but we show that it can be get as continuous limit of a discrete casual variable trying to explicit the economics assumptions implicit in the mathematics assumptions. The first implication is the continuous negotiations that can seem an abstraction from the reality but if the interval of the time is very short or indeterminate small the continuous solution is a good approximation of the discrete solution.

Martingale

The influences that can determine the fluctuations of the market are many, past events, present events and future events, which are incorporated in the market price.
Thus, the current fluctuations are function not only of the precedent fluctuations but either of the market expectation that depends from an infinite number of factors such that it is impossible a mathematics prediction.
Furthermore, we can note that speculators of the market use two kind of probability:

1) Mathematics probability determinable at priori (objective)

2) Probability of the future events impossible to predict in a mathematics way (subjective)

The last one is the probability that speculators try to predict. In every instant the sellers believe in a decrease of the market price, instead, the buyers believe in an increase of it. Thus, the market believes neither in an increasing neither in a decreasing of the market price because the number of sellers are exactly equal to the number of buyers.

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But if the market believes neither in an increasing neither in a decreasing of the market price we can assume a fluctuation of a given width with a given probability at priori.
Formally, in the discrete time we have:

$$S(\tau) = S(t) + \varepsilon(t,\tau)$$

Where $\varepsilon(t,\tau)$ is a casual variable with normal distribution, mean zero and variance $\sigma^2(t,\tau)$ such that $Cov[\varepsilon(t,\tau), \varepsilon(\tau,s)] = 0 \quad \forall \quad \tau < s$, the process is independent from the others on each finite interval.
We can note that the conditional expected value is:

$$E[S(\tau) | S(t)] = S(t) \qquad \forall \qquad \tau > t$$

The process looks just one step back. This is what is called Martingale and its interpretation is easy enough, the conditional expected value is influenced neither from the past value neither from any current information, but depends only from the current price.
This is equal to assume that the market is efficient such that any available information on the past, present and future, is incorporated quickly in the current price. As such, the past prices don't have provisional value. In fact, if this is not the case the speculators can beat the market by using the technical analysis of historical series.
We have a Supermartingale when:

$$E[S(\tau) | S(t)] \leq S(t)$$

We have Submartingale when:

$$E[S(\tau) | S(t)] \geq S(t)$$

We call this either Martingale with drift because $E[\varepsilon(t,\tau)] = \mu(t,\tau)$ such that $\mu(t,\tau)$ is the drift of the stochastic process.
At this point, we have to construct a continuous process as limit of the discrete process with the same characteristics such that it is independent from the width of the time interval Δt .
If we take an interval of time $[t, \tau]$ and we share it in n interval of the same width we have:

$$\Delta t = [(\tau - t) / n] .$$

From this we can write the discrete time model as follows:

$$S(\tau) - S(t) = \sum_{k=0}^{n-1} S(t_k + \Delta t) - S(t_k) \qquad 1.1$$

Where:

$$S(t_k + \Delta t) - S(t_k) = S(t_{k+1}) - S(t_k) - E t_k [S(t_{k+1}) - S(t_k)] = \varepsilon(k)$$

$$E t_k [S(t_k + \Delta t) - S(t_k)] = 0$$

$$Cov[\varepsilon(k), \varepsilon(k+1)] = 0$$

Now we put:

$$S(t_k + \Delta t) - S(t_k) = \frac{\sigma_k \, \varepsilon(k) \sqrt{\Delta t}}{\sqrt{\sigma_k^2 \Delta t}} \qquad 1.2$$

Where:

$$\sigma_k^2 = E t_k [\varepsilon^2(k)]/\Delta t$$

We can note that as $n \to \infty$ and Δt becomes infinitesimal, we have for the theorem of the central limit:

$$dS(t) = \sigma(t)\ N[0,1]\sqrt{dt}$$

Where:

$$\sigma(t) = \sigma(t,\tau) / \sqrt{(\tau - t)} \qquad \text{denotes the instantaneous volatility}$$

This shows that the continuous negotiations bring to have a normal distribution for the dynamic of the price. We can write it as follows:

$$dS(t) = \sigma(t)\ dW(t)$$

Where:

$$dW(t) = N[0,1]\sqrt{dt} \qquad \text{denotes a Wiener process}$$

We can note that the Wiener process is not derivable with respect to the time. In fact, the derivative becomes infinite as dt tends to zero:

$$dW / dt = N[0,1](1/2)(dt^{-1/2})$$

If a Wiener process is not derivable with respect to the time we can't integrate it. In fact, either if the Wiener process is continuous, it is a function of an infinite variation. Thus, we can't compute the integral for each single trajectory, but given our assumption that the stochastic continuous process is the limit toward the discrete time process converges as the interval of time becomes infinitesimal, we can assume that the Wiener process is given and that the diffusion process $\sigma(t)$ is deterministic, because the stochastic continuous process as the time increase converges to the stochastic discrete process. Thus we have:

$$\Delta W(t) = N[0,1]\sqrt{\Delta t}$$

That is equal to:

$$\int_t^{\tau} dW(t) = W(\tau) - W(t) = N[0,1]\sqrt{\tau - t}$$

We remind that an integral is a sum. Thus, we have:

$$E\left[\int_t^{\tau} dW(t)^2\right] = (\tau - t)$$

$$E\left[\int_t^{\tau} \sigma(t)\ dW(t)\right] = 0$$

$$E\left\{\left[\int_t^{\tau} \sigma(t)\ dW(t)\right]^2\right\} = \int_t^{\tau} E[\sigma(t)^2]\ dt$$

We can note that 1.2 becomes at limit the stochastic differential equation:

$$dS(t) = \sigma(t)\, dW(t)$$

$$S(t) = S(o)$$

Instead, the 1.1 becomes:

$$S(\tau) = S(t) + \int_t^\tau \sigma(t)\, dW(t)$$

We have the following characteristics for the stochastic differential equation:

$$E[\, dS\,] = 0$$

$$Var[\, dS\,] = \sigma^2(t)\, dt$$

We can note that the variance is an increasing function of the time and the expected value is zero. Moreover, we can note that we kept the same characteristics of the discrete time model.

$$E[\, S(\tau)\,] = S(t)$$

$$Var[\, S(\tau)\,] = \sigma^2(t)\,(\tau - t) = \sigma^2(t,\tau)$$

Along the same line, we have for a Martingale with drift the following stochastic differential equation:

$$dS(t) = \mu(t)\, dt + \sigma(t)\, dW(t)$$

$$S(t) = S(o)$$

Where:

$\mu(t) = \mu(t,\tau) / (\tau - t)$ denotes the instantaneous mean

Again we have:

$$S(\tau) = S(t) + \int_t^\tau \mu(t)\, dt + \int_t^\tau \sigma(t)\, dW(t)$$

We can note that :

$$E[\, dS\,] = \mu(t)\, dt$$

$$Var[\, dS\,] = \sigma^2(t)\, dt$$

Thus:

$$E[\, S(\tau)\,] = S(t) + \mu(t)\,(\tau - t) = S(t) + \mu(t,\tau)$$

$$Var[\, S(\tau)\,] = \sigma^2(t)\,(\tau - t) = \sigma^2(t,\tau)$$

At this point, we have to note that the instantaneous volatility and the drift of the stochastic continuous process are forward processes, in the sense that they are based on the effective volatility and mean of the discrete time process. As such, they are deterministic, but in the reality we can’t observe the future volatility and the future mean. As result, the volatility and the drift of the process are stochastic; in this case we can approximate them by using the expected

value. We have to note that the drift can be stochastic either with respect to a deterministic value of the expected mean of the discrete time process so that it will revert to it. This is what we usually call mean-reverting process such that the stochastic continuous time process goes from a Supermartingale to a Submartingale in continuous time.
At this point, we have to note that the price can assume negative value, but as we said we can solve this problem by assuming that the Martingale models the rate of return, instead, the arithmetic change of the market price. Thus, we get a stochastic continuous process that follows a lognormal distribution in each finite interval known as geometric Brown process. In fact, we can see that if we put the following process:

$$dS(t) / S(t) = \mu(t)\, dt$$

We have the following solution:

$$S(\tau) = S(t)\, e^{\mu(t)\,(\tau - t)}$$

As result, the geometric Brown process follows a lognormal distribution in each finite interval. Thus, we have the following:

$$dS(t) / S(t) = \mu(t)\, dt + \sigma(t)\, dW(t)$$

We can note that:

$$E[\, S(\tau)\,] = S(t)\, e^{\mu(t)\,(\tau - t)}$$

Expected Volatility and Expected Mean

Now we have to note that we showed that the volatility is an increasing function of the time with respect to a finite interval of time. In fact, the stochastic continuous process converges to the stochastic discrete process. This doesn't mean that the volatility is an increasing function of time. In fact, we can have a greater finite interval with the same volatility. Thus, we need a model that permits us to get the expected volatility and its behaviour.
We can construct a stochastic continuous process with the instantaneous historical variance like drift. This permits us to get the expected volatility and to observe the behaviour of the volatility from one step to another. However, we can find a probability solution of a stochastic continuous process just if we use a geometric Brown process.
Thus, we have:

$$\sigma(o,t)^2 = \sigma^2(o)(t - o)$$ denotes the total historical variance

$$\sigma^2(o) = \sigma(o,t)^2 / (t - o)$$ denotes the instantaneous historical variance

We can put the following geometric Brown process:

$$d\sigma(t,t)^2 / \sigma(t,t)^2 = \mu(t)\, dt + \delta(t)\, dW(t)$$

$$\sigma(t,t)^2 = \sigma(o,t)^2$$

Where:

$$\mu(t) \;=\; \sigma^2(o)/\,\sigma(o,t)^2 \;=\; 1\,/\,(t-o)$$

$\delta(t)$ denotes the instantaneous volatility of the variance

Where we have assumed that the initial time is in $t > o$.
We can note that the expected value of the process is an increasing function of time.
Thus, we have:

$$\mathrm{E}\,[\,\sigma(t,\tau)^2\,] \;=\; \sigma(t,t)^2\, e^{\{[1\,/\,(t-o)]\,(\tau-t)\}}$$

Therefore, the expected instantaneous volatility is:

$$\mathrm{E}\,[\,\sigma(t)\,] = \sqrt{\{\mathrm{E}\,[\,\sigma(t,\tau)^2\,] \;-\; \sigma(o,t)^2\}\,/\,(\tau-t)} \;=\; \sqrt{\sigma^2(o)(e-1)}$$

Where:

$$(\tau - t) \;=\; (t - \mathrm{o})$$

We got the expected instantaneous volatility in the interval $(\tau - t)$ that is an increasing function of the number of steps, this shows that the expected volatility is an increasing function of the time. The limit of the model is that the expected instantaneous volatility can be negative, but for rational value of the parameter $\delta(t)$ the probability that this can happen is very low.
We conclude by noting that this is an expected value and that the effective instantaneous volatility can be different from it. At this point, we can construct the same process for the mean so to show that the expected mean is an increasing function of time.

Conclusion

The model has showed that a stochastic continuous process is the limit of a discrete time process such that we can integrate it either if it is a function of an infinite variation. Furthermore, we have determined the expected volatility and the expected mean on the base of the historical volatility and the historical mean so to show that they are an increasing function of the time.

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