A Simplified Approach to Analyzing Multi-regional Core-Periphery Models

Takashi Akamatsu and Yuki Takayama

18. August 2009

A Simplified Approach to Analyzing Multi-regional Core-Periphery Models

Takashi Akamatsu[*] *and Yuki Takayama*[*]

Abstract

This paper shows that the evolutionary process of spatial agglomeration in multi-regional core-periphery models can be explained analytically by a much simpler method than the continuous space approach of Krugman (1996). The proposed method overcomes the limitations of Turing's approach which has been applied to continuous space models. In particular, it allows us not only to examine whether or not agglomeration of mobile factors emerges from a uniform distribution, but also to trace the *evolution* of spatial agglomeration patterns (*i.e.*, bifurcations from various *polycentric patterns* as well as from a uniform pattern) with decreases in transportation cost.

Keywords: agglomeration, core-periphery model, multi-regional, stability, bifurcation
JEL classification: R12, R13, F15, F22, C62

1. Introduction

More than a decade has passed since the new economic geography (NEG) emerged with now well-known modeling techniques such as "Dixit-Stiglitz, Icebergs, Evolution, and the Computer", by Fujita, Krugman and Venables (1999). These new modeling techniques, first introduced in the core-periphery (CP) model developed by Krugman (1991), provided a full-fledged general equilibrium approach and led to numerous studies on extending the original framework. Furthermore, in recent years, there has been a proliferation of theoretical and empirical work applying the NEG framework to deal with various policy issues (Baldwin et al., (2003), Behrens and Thisse (2007), Combes et al., (2009)).

Despite the remarkable growth of the NEG theory, there remain some fundamental issues that need to be addressed before the theory provides a sound foundation for empirical work and practical applications. One of the most relevant issues is to reintroduce spatial aspects into the theory. Even though this direction was pursued in the early development stages of NEG (*e.g.*, Krugman (1993, 1996), Fujita et al., (1999)), recent theoretical studies have been almost exclusively limited to the two-region CP/NEG models in which many essential aspects of "*space/geography*" have almost vanished. As a result, little is known about the rich properties of the multi-regional CP model. In view of the fact that two-region analysis has

* Graduate School of Information Sciences, Tohoku University, Sendai, Japan.

some serious limitations[2] due to the "degeneration of space", it seems reasonable to argue that "*a theoretical analysis of economic geography must make an effort to get beyond the two-location case*" since "*real-world geographical issues cannot be easily mapped into two-regional analysis*" (Fujita et al., (1999, Chap.6)). In other words, advancing our understanding of the multi-regional NEG/CP models is a prerequisite for systematic empirical work as well as for systematic evaluation of policy proposals.

Why, despite the obvious needs, have there been very few theoretical studies on multi-regional CP models in the last decade? This seems to be a direct result of technical difficulties that inevitably arise in examining the properties of the multi-regional CP model. As is well known in the NEG theory, the two-regional CP model, depending on transportation costs, exhibits a "*bifurcation*" from a symmetric equilibrium to an asymmetric equilibrium. In dealing with the multi-regional CP model, we are likely to encounter more complex bifurcation phenomena and hence we need to devise better methods to analyze them. In contrast to the large number of works on the CP model that have flourished during the last decade, there has been very little progress in developing effective approaches to this bifurcation problem since the work of Krugman (1996) and Fujita et al., (1999).

The only method that has been used to analyze bifurcation in the multi-regional CP model is the Turing (1952) approach, in which one focuses on the onset of instability of a uniform equilibrium distribution ("flat earth equilibrium") of mobile agents. That is, assuming a certain class of adjustment process (*e.g.*, "*replicator dynamics*"), one examines a trend of the economy away from, rather than toward, the flat earth equilibrium whose instability implies the *emergence* of *some* agglomeration.[3] Krugman (1996) and Fujita et al., (1999, Chap.6) applied this approach to the CP model with a continuum of locations on the circumference and succeeded in showing that the steady decreases in transportation costs lead to the instability of the flat earth equilibrium state. Recently, a few studies have also applied this approach, and re-examined the robustness of the Krugman's findings in the CP model with continuous space racetrack economy. Mossay (2003) theoretically qualifies the Krugman's results in the case of workers' heterogeneous preferences for location. More recently, Picard and Tabuchi (2009) examined the impact of the shape of transport costs on the structure of spatial equilibria[4].

---

[2] For more elaborated discussions on the limitations of the two-regional analysis, see, for example, Fujita and Thisse (2009), Akamatsu et al., (2009), Behrens and Thisse (2007), Fujita and Krugman (2004).

[3] The first notable application of this approach to analyzing agglomeration in a spatial economy was made by Papageorgiou and Smith (1983).

[4] Tabuchi et al. (2005) also study the impact of falling transport costs on the size and number of cities in a multi-regional model that extends a two-regional CP model by Ottaviano et al., (2002). Oyama (2009) showed that the multi-regional CP model admits a potential function, which allowed to identify a stationary state that is uniquely absorbing and globally accessible under the perfect foresight dynamics. However, these analyses are restricted to a very special class of transport geometry in which regions are pairwise equidistant.

While this approach offers a remarkable way of thinking about a seemingly complex issue, it has two important limitations. First, it deals with only the first stage of agglomeration when the value of a parameter (*e.g.*, transportation cost) steadily changes; it cannot give a good description of what happens thereafter. Indeed, Krugman (1996) and Fujita et al., (1999, Chap.17) resort to rather ad hoc numerical simulations for analyzing the possible bifurcations in the later stages; recent studies of Mossay (2003) and Picard and Tabuchi (2009) are silent on the bifurcations in the later stages. Second, the eigenvalue analysis required in the approach becomes complicated, and it is, in general, almost impossible to analytically obtain the eigenvalues for an arbitrary configuration of mobile workers. This is one of the most difficult obstacles that prevent us from understanding the general properties of the multi-regional CP model.

In this paper, we show that the evolutionary process of spatial agglomeration in the multi-regional CP models can be readily explained by a much simpler method than the continuous space approach of Krugman (1996) and Fujita et al., (1999). The main features of the proposed method are as follows:

1) it is applicable to the CP model with an arbitrary *discrete* number of regions, in contrast to Krugman's approach that is restricted to a special limiting case (*i.e.*, *continuous* space).

2) it exploits the concept of a "*spatial discounting matrix* (SDM)" in a circular city/region system ("*racetrack economy* (RE)"). This together with the *discrete Fourier transformation* (DFT) provides an analytically tractable method of elucidating the agglomeration properties of the multi-regional CP model, without resorting to numerical techniques.

3) it allows us not only to examine whether or not agglomeration of mobile factors emerges from a uniform distribution, but also to trace the *evolution* of spatial agglomeration patterns (*i.e.*, bifurcations from various *polycentric patterns* as well as from a uniform pattern) with the decreases in transportation cost. That is, it overcomes the limitations of Turing's approach that Krugman (1996), Fujita et al., (1999), Mossay (2003), and Picard and Tabuchi (2009) encountered in their continuous space models.

To demonstrate the proposed method, we employed a pair of multi-regional CP models, which are "solvable" variants of Krugman's original CP model. By the term "solvable", we mean that an explicit form of the indirect utility function of a consumer for a short-run equilibrium (in which a location pattern of workers is fixed) can be obtained (**Proposition 1**). More specifically, each of the CP models presented here is a multi-regional version of the two-region CP models recently developed by Forslid and Ottaviano (2003) and Pflüger (2004).

In the analysis of these CP models, we intentionally restricted ourselves to the case of four regions for clarity of exposition, although the approach presented in this paper can deal with a model with an arbitrary number of regions. Interested readers can consult Akamatsu et al., (2009) for more general cases. The four-region setting allowed us to illustrate the essential feature of our approach without going into too much technical detail. Indeed, even in this simple setting, we observed a number of interesting properties of the multi-regional CP model that are not reported in the literature.

In order to understand the bifurcation mechanism of the CP model, we need to know how the eigenvalues of the Jacobian matrix of the adjustment process depend on bifurcation parameters (*e.g.*, the transportation cost coefficient τ). A combination of the RE (with discrete locations) and the resultant circulant properties of the SDM greatly facilitate this analysis. Indeed, it is shown (in **Proposition 2**) that the eigenvalue g_k of the Jacobian matrix of the adjustment process can be expressed as a quadratic function of the eigenvalue f_k of the SDM. The former eigenvalue g_k thus obtained has a natural economic interpretation as the strength of "*net agglomeration force*", and offers the key to understanding the agglomeration properties of the CP economy.

To investigate the evolutionary process of the spatial agglomeration in the multi-regional CP models, we considered the process in which the value of the spatial discounting factor (SDF) r steadily increased (which means transportation cost decreases) over time. Starting from $r = 0$ at which a uniform distribution of skilled labor is a stable equilibrium state, we investigated when and what spatial patterns of agglomeration *emerged* (*i.e.* a bifurcation occuring) with the increases in the SDF. The analytical expression of the eigenvalues allowed us to identify the "break point" and the associated patterns of agglomeration that emerged at the bifurcation (**Proposition 4**). Unlike the conventional two-region models that exhibit only a single time of bifurcation, this is not the end of the story in the four-region model. Indeed, it is shown (in **Proposition 5**) that the agglomeration pattern after the first bifurcation *evolves* over time with the steady increases in the SDF: it first grows to a duocentric pattern, which continues to be stable for a while; further increases in the SDF, however, trigger the occurrence of a second bifurcation, which in turn leads to the formation of a monocentric agglomeration. This result was derived by using a simple analytical technique based on a similarity transformation. Furthermore, it was theoretically deduced (in **Proposition 6**) that the *collapse* of agglomeration (that corresponds to "re-dispersion" in the two-region CP model) can occur for a high-SDF range.

The remainder of the paper is organized as follows. Section 2 presents the equilibrium conditions of the multi-regional CP models as well as definitions of the stability and bifurcation of the equilibrium state. Section 3 defines the SDM in a racetrack economy, whose eigenvalues are provided by a DFT. Section 4 analyzes the evolutionary process of spatial patterns observed in our models. Section 5 concludes the paper.

2. The Model

2.1. Basic Assumptions

We present a pair of multi-regional CP models whose frameworks follow Forslid and

Ottaviano(2003) and Pflüger(2004) (defined as FO and Pf). The basic assumptions of the multi-regional CP model are the same as those of the FO and Pf models except for the number of regions, but we provide them here for completeness. The economy is composed of K regions indexed by $i = 0,1,...,K-1$, two factors of production and two sectors. The two factors of production are skilled and unskilled labor. Each worker supplies one unit of his type of labor inelastically. The skilled worker is mobile across regions and h_i denotes the number of these factors located in region i. The total endowment of skilled workers is H. The unskilled worker is immobile and equally distributed across all regions. The unit of unskilled worker is chosen such that the world endowment $L = K$ (*i.e.*, the number of unskilled workers in each region is one). The two sectors are agriculture (abbreviated by A) and manufacturing (abbreviated by M). The A-sector output is homogeneous and produced using a unit input requirement of unskilled labor under constant returns to scale and perfect competition. This output is the numéraire and assumed to be produced in all regions. The M-sector output is a horizontally differentiated product and produced using both skilled and unskilled labor under increasing returns to scale and Dixit-Stiglitz monopolistic competition. The goods of both sectors are transported, but transportation of the A-sector goods is frictionless while transportation of the M-sector goods is inhibited by iceberg transportation costs. That is, for each unit of the M-sector goods transported from region i to j, only a fraction $1/\phi_{ij} < 1$ arrives.

All workers have identical preferences U over the M and A-sector goods. The utility of each consumer in region i is given by:

[FO model[5]] $$U(C_i^M, C_i^A) = \mu \ln C_i^M + (1-\mu) \ln C_i^A \qquad (0 < \mu < 1) \qquad (2.1)$$

[Pf model] $$U(C_i^M, C_i^A) = \mu \ln C_i^M + C_i^A \qquad (\mu > 0) \qquad (2.2)$$

$$C_i^M \equiv \sum_j \left(\int_{k \in n_j} q_{ji}(k)^{(\sigma-1)/\sigma} dk \right)^{\sigma/(\sigma-1)} \qquad (\sigma > 1)$$

where C_i^A is the consumption of the A-sector goods in region i; C_i^M represents the manufacturing aggregate in region i; $q_{ji}(k)$ is the consumption of variety $k \in [0, n_j]$ produced in region j and n_j is the number of varieties produced in region j; μ is the constant expenditure share on industrial varieties and σ is the constant elasticity of substitution between any two varieties. The budget constraint is given by:

$$C_i^A + \sum_j \int_{k \in n_j} p_{ji}(k) q_{ji}(k) dk = Y_i$$

where $p_{ji}(k)$ denotes the price in region i of the M-sector goods produced in region j, and Y_i denotes the income of a consumer in region i.

The utility maximization of (2.1) or (2.2) yields the following demand $q_{ij}(k)$ of a

---

[5] We take logarithms of the Forslid and Ottaviano(2003) type (*i.e.*, Cobb-Douglas-type) utility function to facilitate the analysis. Note that this transformation has no influence on the properties of the model.

consumer in region i for a variety of the M-sector goods k produced in location j:

[FO model] $$q_{ji}(k)=\frac{\mu\{p_{ji}(k)\}^{-\sigma}}{\rho_i^{1-\sigma}}Y_i$$

[Pf model] $$q_{ji}(k)=\frac{\mu\{p_{ji}(k)\}^{-\sigma}}{\rho_i^{1-\sigma}}$$

where

$$\rho_i=\sum_j\left(\int_{k\in n_j}p_{ji}(k)^{1-\sigma}dk\right)^{1/(1-\sigma)} \tag{2.3}$$

denotes the price index of the differentiated product in region i. Since the total income and population in region i are $w_i h_i+1$ and h_i+1, respectively, we have the total demand $Q_{ji}(k)$:

[FO model] $$Q_{ji}(k)=\frac{\mu\{p_{ji}(k)\}^{-\sigma}}{\rho_i^{1-\sigma}}(w_i h_i+1) \tag{2.4a}$$

[Pf model] $$Q_{ji}(k)=\frac{\mu\{p_{ji}(k)\}^{-\sigma}}{\rho_i^{1-\sigma}}(h_i+1) \tag{2.4b}$$

The A-sector technology requires one unit of unskilled labor in order to produce one unit of output. With free trade in the A-sector, the choice of this goods as the numéraire implies that in equilibrium the wage of the unskilled worker w_i^L is equal to one in all regions, that is, $w_i^L=1\ \forall i$. In the M-sector, product differentiation ensures a one-to-one relation between firms and varieties. Specifically, in order to produce $x_i(k)$ unit of product k, a firm incurs a fixed input requirement of α unit of skilled labor and a marginal input requirement of $\beta x_i(k)$ unit of unskilled labor. With $w_i^L=1$, the total cost of production of a firm in region i is thus given by $\alpha w_i+\beta x_i(k)$, where w_i is the wage of the skilled worker. Given the fixed input requirement α, the skilled labor market clearing implies that in equilibrium the number of firms is determined by $n_i=h_i/\alpha$ so that the number of active firms in a region is proportional to the number of its skilled workers.

Due to the iceberg transportation costs, the total supply of the M-sector firm located in region i (*i.e.* $x_i(k)$) is given by:

$$x_i(k)=\sum_j\phi_{ij}Q_{ij}(k) \tag{2.5}$$

Therefore, a typical M-sector firm located in region i maximizes profit as given by:

$$\Pi_i(k)=\sum_j p_{ij}(k)Q_{ij}(k)-\left(\alpha w_i+\beta\sum_j\phi_{ij}Q_{ij}(k)\right).$$

Since we have a continuum of firms, each one is negligible in the sense that its action has no impact on the market (*i.e.*, the price indices). Hence, the first order condition for the profit maximization gives:

$$p_{ij}(k) = \frac{\sigma\beta}{\sigma - 1}\phi_{ij} \tag{2.6}$$

This expression implies that the price of the *M*-sector goods does not depend on variety *k*, so that $Q_{ij}(k)$ and $x_i(k)$ also do not depend on *k*. Thus we describe these variables without argument *k*. Substituting (2.6) into (2.3), the price index becomes

$$\rho_i = \frac{\sigma\beta}{\sigma - 1}\left(\sum_j h_j d_{ji}\right)^{1/(1-\sigma)} \tag{2.7}$$

where $d_{ji} \equiv \phi_{ji}^{1-\sigma}$ is a "*spatial discounting factor*" between region *i* and *j*: from (2.4), (2.6) and (2.7), d_{ji} is represented as $(p_{ji}Q_{ji})/(p_{ii}Q_{ii})$, which means that d_{ji} is the ratio of total expenditure in region *i* for each *M*-sector product produced in region *j* to their expenditure for a domestic product.

2.2. Short-Run Equilibrium

In the short run, the skilled workers are immobile between regions, that is, their spatial distribution ($\mathbf{h} \equiv [h_0, h_1, ..., h_{K-1}]^{\mathrm{T}}$) is taken as given. The short-run equilibrium conditions consist of the *M*-sector goods market clearing condition and the zero profit condition due to the free entry and exit of firms. The former condition can be written as (2.5). The latter condition requires that the operating profit of a firm is entirely absorbed by the wage bill of its skilled workers:

$$w_i(\mathbf{h}) = \frac{1}{\alpha}\left(\sum_j p_{ij}Q_{ij}(\mathbf{h}) - \beta x_i(\mathbf{h})\right) \tag{2.8}$$

Substituting (2.4), (2.5), (2.6) and (2.7) into (2.8), we have the short-run equilibrium wage equations:

[FO model] $$w_i(\mathbf{h}) = \frac{\mu}{\sigma}\sum_j\left(\frac{d_{ij}}{\Delta_j(\mathbf{h})}\right)(w_j(\mathbf{h})h_j + 1) \tag{2.9}$$

[Pf model] $$w_i(\mathbf{h}) = \frac{\mu}{\sigma}\sum_j\left(\frac{d_{ij}}{\Delta_j(\mathbf{h})}\right)(h_j + 1) \tag{2.10}$$

where $\Delta_j(\mathbf{h}) \equiv \sum_k d_{kj}h_k$ denotes the market size of the *M*-sector in region *j*. Thus, $d_{ij} / \Delta_j(\mathbf{h})$ defines the market share in region *j* of each *M*-sector product produced in region *i*.

To obtain the indirect utility function $v_i(\mathbf{h})$, we express the equilibrium wage $w_i(\mathbf{h})$ as an explicit function of **h**. For this, we rewrite (2.9) and (2.10) in matrix form by using the "spatial discounting matrix" **D** whose (i, j) entry is d_{ij}. Then, the equilibrium wage $\mathbf{w}(\mathbf{h}) \equiv [w_0(\mathbf{h}), w_1(\mathbf{h}), ..., w_{K-1}(\mathbf{h})]^{\mathrm{T}}$, is given by:

[FO model] $$\mathbf{w}(\mathbf{h}) = \frac{\mu}{\sigma}\left[\mathbf{I} - \frac{\mu}{\sigma}\mathbf{MH}\right]^{-1}\mathbf{w}^{(L)}(\mathbf{h}) \tag{2.11}$$

[Pf model] $$\mathbf{w}(\mathbf{h}) = \frac{\mu}{\sigma}\{\mathbf{w}^{(H)}(\mathbf{h}) + \mathbf{w}^{(L)}(\mathbf{h})\}$$

$$\mathbf{w}^{(H)}(\mathbf{h}) \equiv \mathbf{M}\mathbf{h}, \quad \mathbf{w}^{(L)}(\mathbf{h}) \equiv \mathbf{M}\mathbf{1} \tag{2.12}$$

where $\mathbf{1} \equiv [1,1,...,1]^{\mathrm{T}}$ and **I** is a unit matrix. **M** and **H** are defined as

$$\mathbf{M} \equiv \mathbf{D}\boldsymbol{\Delta}^{-1}, \quad \boldsymbol{\Delta} \equiv \mathrm{diag}[\mathbf{D}\mathbf{h}], \quad \mathbf{H} \equiv \mathrm{diag}[\mathbf{h}] \tag{2.13}$$

This leads to the following proposition.

Proposition 1: *The indirect utility* $\mathbf{v}(\mathbf{h}) \equiv [v_0(\mathbf{h}), v_1(\mathbf{h}),..., v_{K-1}(\mathbf{h})]^{\mathrm{T}}$ *of the multi-regional FO and Pf models can be expressed as an explicit function of* $\mathbf{h}$:

[FO model] $$\mathbf{v}(\mathbf{h}) = \mu\mathbf{S}(\mathbf{h}) + \ln[\mathbf{w}(\mathbf{h})] \tag{2.14}$$

[Pf model] $$\mathbf{v}(\mathbf{h}) = \mathbf{S}(\mathbf{h}) + \sigma^{-1}\{\mathbf{w}^{(H)}(\mathbf{h}) + \mathbf{w}^{(L)}(\mathbf{h})\} \tag{2.15}$$

where $\ln[\mathbf{w}] \equiv [\ln w_0, \ln w_1, ..., \ln w_{K-1}]^{\mathrm{T}}$. $\mathbf{w}(\mathbf{h})$, $\mathbf{w}^{(H)}(\mathbf{h})$ *and* $\mathbf{w}^{(L)}(\mathbf{h})$ *are defined in (2.11), (2.12), and* $\mathbf{S}(\mathbf{h}) \equiv (\sigma-1)^{-1}\ln[\mathbf{D}\mathbf{h}]$.

2.3. Long-Run Equilibrium and Adjustment Dynamics

In the long run, the skilled workers are inter-regionally mobile and will move to the region where their indirect utility is higher. We assume that they are heterogeneous in their preferences for location choice. That is, the indirect utility for an individual *s* located in region *i* is expressed as:

$$v_i^{(s)}(\mathbf{h}) = v_i(\mathbf{h}) + \varepsilon_i^{(s)}$$

where $\varepsilon_i^{(s)}$ denotes the utility representing the idiosyncratic taste for a residential location. The distribution of $\{\varepsilon_i^{(s)} \mid \forall s\}$ is assumed to be the Weibull distribution and to be identical and independent across regions. Under this assumption, the fraction $P_i(\mathbf{h})$ of the skilled workers choosing region *i* is given by:

$$P_i(\mathbf{h}) \equiv \frac{\exp[\theta\, v_i(\mathbf{h})]}{\sum_j \exp[\theta\, v_j(\mathbf{h})]} \tag{2.16}$$

where $\theta \in (0,\infty)$ is the parameter expressing the inverse of the variance of individual tastes. When $\theta \to \infty$, (2.16) means that the workers decide their location only by $v_i(\mathbf{h})$, which corresponds to the case without heterogeneity (*i.e.*, the skilled workers are homogeneous).

The long-run equilibrium is defined as the spatial distribution of the mobile workers **h** that satisfies the following condition:

$$h_i = HP_i(\mathbf{h}) \quad \forall i \tag{2.17}$$

or equivalently written as $\mathbf{F}(\mathbf{h}) \equiv H\,\mathbf{P}(\mathbf{h}) - \mathbf{h} = \mathbf{0}$, where H is the total endowment of the skilled worker, and $\mathbf{P}(\mathbf{h}) \equiv [P_0(\mathbf{h}), P_1(\mathbf{h}), ..., P_{K-1}(\mathbf{h})]^{\mathrm{T}}$. This condition means that the actual number of individuals h_i in each region is equal to the number $HP_i(\mathbf{h})$ of individuals who choose that region under the current distribution **h** of skilled workers.

For this equilibrium condition, it is natural to assume the following adjustment process:

$$\dot{\mathbf{h}} = \mathbf{F}(\mathbf{h}) \tag{2.18}$$

This is the well-known logit dynamics, which were developed in evolutionary game theory (Fudenberg and Levine (1998) and Sandholm (2009)).

The adjustment process of (2.18) allows us to define stability of long-run equilibrium $\mathbf{h}^*$ in the sense of local stability: the stability of the linearized system of (2.18) at $\mathbf{h}^*$. It is well known in dynamic system theory that the local stability of the equilibrium $\mathbf{h}^*$ is determined by examining the eigenvalues of the Jacobian matrix of the adjustment process[6]:

$$\nabla\mathbf{F}(\mathbf{h}) = H\,\mathbf{J}(\mathbf{h})\nabla\mathbf{v}(\mathbf{h}) - \mathbf{I} \tag{2.19}$$

where each of $\mathbf{J}(\mathbf{h})$ and $\nabla\mathbf{v}(\mathbf{h})$ is a K-by-K matrix whose (i, j) entry is $\partial P_i(\mathbf{v}(\mathbf{h}))/\partial v_j$ and $\partial v_i(\mathbf{h})/\partial h_j$, respectively.

3. Net agglomeration forces in a racetrack economy

3.1. Racetrack economy and spatial discounting matrix

Consider a "*racetrack economy*" in which 4 regions $\{0, 1, 2, 3\}$ are equidistantly located on a circumference with radius 1. Let $t(i, j)$ denote the distance between two regions i and j. We define the distance between two regions as that measured by the minimum path length:

$$t(i, j) = (2\pi/4)\cdot m(i, j)$$

where $m(i, j) \equiv \min.\{|i-j|,\ 4-|i-j|\}$. The set $\{t(i, j), (i, j = 0,1,2,3)\}$ of the distances determines the *spatial discounting matrix* **D** whose (i, j) entry, d_{ij}, is given by:

$$d_{ij} \equiv \exp[-(\sigma-1)\cdot\tau\cdot t(i, j)] \tag{3.1a}$$

Defining the *spatial discount factor* (SDF) by

$$r \equiv \exp[-(\sigma-1)\cdot\tau\,(2\pi/4)] \tag{3.1b}$$

we can represent d_{ij} as $r^{m(i,j)}$. It follows from the definition that the SDF r is a monotonically decreasing function of the transportation cost (technology) parameter τ, and hence the feasible range of the SDF (corresponding to $0 \le \tau < +\infty$) is given by $(0,1]$: $\tau = 0 \Leftrightarrow r = 1$, and $\tau \to +\infty \Leftrightarrow r \to 0$. Note here that the SDF yields the following expression for the SDM in the racetrack economy:

$$\mathbf{D} = \begin{bmatrix} 1 & r & r^2 & r \\ r & 1 & r & r^2 \\ r^2 & r & 1 & r \\ r & r^2 & r & 1 \end{bmatrix}$$

[6] See, for example, Hirsch and Smale (1974).

As easily seen from this expression, the matrix $\mathbf{D}$ is a *circulant* which is constructed from the vector $\mathbf{d}_0 \equiv [1, r, r^2, r]^{\mathrm{T}}$ (see Appendix 3 for the definition and properties of circulant matrices). This circulant property of the SDM plays a key role in the following analysis.

3.2. Stability, eigenvalues and Jacobi matrices

Stability of equilibrium solutions for the CP model can be determined by examining the eigenvalues $\mathbf{g} \equiv [g_0, g_1, g_2, g_3]^{\mathrm{T}}$ of the Jacobian matrix of the adjustment process (2.18). Specifically, the equilibrium solution $\mathbf{h}$ satisfying (2.17) is asymptotically stable if all the eigenvalues of $\nabla\mathbf{F}(\mathbf{h})$ have negative real parts; otherwise the solution is unstable (*i.e.*, at least one eigenvalue of $\nabla\mathbf{F}(\mathbf{h})$ has a positive real part), and the solution moves in the direction of the corresponding eigenvector. The eigenvalues, if they are represented as functions of the key parameters of the CP model (*e.g.* transport technology parameter τ), further enable us to predict whether or not a particular agglomeration pattern (bifurcation) will occur with changes in the parameter values.

The eigenvalues $\mathbf{g}$ of the Jacobian matrix $\nabla\mathbf{F}(\mathbf{h})$ at an *arbitrary distribution* $\mathbf{h}$ of the skilled labor cannot be obtained without resorting to numerical techniques. It is, however, possible in some *symmetric distributions* $\bar{\mathbf{h}}$ to obtain analytical expressions for the eigenvalues $\mathbf{g}$ of the Jacobian. The key tool for making this possible is a *circulant matrix*, which has several useful properties for the eigenvalue analysis. To take "Property 1" of a circulant in Appendix 3 for example, it implies that if $\nabla\mathbf{F}(\bar{\mathbf{h}})$ is a circulant then the eigenvalues $\mathbf{g}$ can be obtained by discrete Fourier transformation (DFT) of the first row vector $\mathbf{x}_0$ of $\nabla\mathbf{F}(\bar{\mathbf{h}}) : \mathbf{g} = \mathbf{Z}\mathbf{x}_0$, where $\mathbf{Z}$ is a 4-by-4 DFT matrix. Furthermore, "Property 2" assures us that $\nabla\mathbf{F}(\bar{\mathbf{h}})$ is indeed a circulant if $\mathbf{J}(\bar{\mathbf{h}})$ and $\nabla\mathbf{v}(\bar{\mathbf{h}})$ in the right-hand side of (3.2) are circulants (note here that a unit matrix $\mathbf{I}$ is obviously a circulant).

A uniform distribution of skilled workers, $\bar{\mathbf{h}} \equiv [H/4, H/4, H/4, H/4]$, which has intrinsic significance in examining the emergence of agglomeration, gives us a simple example for illustrating the use of the above properties of circulants. We first show below that the Jacobian matrix $\nabla\mathbf{F}(\bar{\mathbf{h}})$ at the uniform distribution $\bar{\mathbf{h}}$ is a circulant. This in turn allows us to obtain analytical expressions for the eigenvalues of $\nabla\mathbf{F}(\bar{\mathbf{h}})$ as will be shown in **3.3**. For clarity of exposition, we restrict the analysis below to the case for the Pf model while the same conclusion holds for the FO model (for the details, see Appendix 2).

In order to show that $\nabla\mathbf{F}(\bar{\mathbf{h}})$ is a circulant, we examine each of $\nabla\mathbf{v}(\bar{\mathbf{h}})$ and $\mathbf{J}(\bar{\mathbf{h}})$ in turn. For the configuration $\bar{\mathbf{h}}$ in which $h \equiv H/4$ skilled workers are equally distributed in each region (*i.e.*, $\bar{\mathbf{h}} \equiv [h, h, h, h]$), the definition of $\mathbf{M}$ in (2.13) yield $\mathbf{M}(\bar{\mathbf{h}}) = (hd)^{-1}\mathbf{D}$, where $d \equiv \mathbf{d}_0 \cdot \mathbf{1}$, and hence the Jacobian matrix of indirect utility functions at $\bar{\mathbf{h}}$ reduces to:

$$\nabla\mathbf{v}(\bar{\mathbf{h}}) = h^{-1}\{b[\mathbf{D}/d] - a[\mathbf{D}/d]^2\} \tag{3.2}$$

where a and b are constant parameters defined as:

$$a \equiv \sigma^{-1}(1+h^{-1}), \quad b \equiv (\sigma-1)^{-1}+\sigma^{-1} \tag{3.3}$$

Note that the right-hand side of (3.2) consists only of additions and multiplications of the circulant matrix **D**. It follows from this that $\nabla\mathbf{v}(\bar{\mathbf{h}})$ is a circulant. We now show that $\mathbf{J}(\bar{\mathbf{h}})$ is a circulant. From the definition (2.16) of the location choice probability functions **P**(**h**), we have the Jacobian matrix **J**(**h**) at $\bar{\mathbf{h}}$ as:

$$\mathbf{J}(\bar{\mathbf{h}}) = (\theta/4)\big(\mathbf{I}-(1/4)\mathbf{E}\big) \tag{3.4}$$

where **E** is a 4 by 4 matrix whose entries are all equal to 1. This clearly shows that $\mathbf{J}(\bar{\mathbf{h}})$ is a circulant because **I** and **E** are obviously circulants. Thus, both $\nabla\mathbf{v}(\bar{\mathbf{h}})$ and $\mathbf{J}(\bar{\mathbf{h}})$ are circulants, and this leads to the conclusion that the Jacobian matrix of the adjustment process at the configuration $\bar{\mathbf{h}}$:

$$\nabla\mathbf{F}(\bar{\mathbf{h}}) = H\,\mathbf{J}(\bar{\mathbf{h}})\nabla\mathbf{v}(\bar{\mathbf{h}}) - \mathbf{I} \tag{3.5}$$

is a circulant.

3.3. Net agglomeration forces

The fact that the matrices $\mathbf{J}(\bar{\mathbf{h}})$ and $\nabla\mathbf{v}(\bar{\mathbf{h}})$ as well as $\nabla\mathbf{F}(\bar{\mathbf{h}})$ are all circulants allows us to obtain the eigenvalues **g** of $\nabla\mathbf{F}(\bar{\mathbf{h}})$ by applying a similarity transformation based on the DFT matrix **Z**. Specifically, the similarity transformation of both sides of (3.5) yields:

$$\mathrm{diag}[\mathbf{g}] = H\ \mathrm{diag}[\boldsymbol{\delta}]\ \mathrm{diag}[\mathbf{e}] - \mathrm{diag}[\mathbf{1}] \tag{3.6a}$$

where $\boldsymbol{\delta}$ and $\mathbf{e}$ are the eigenvalues of $\mathbf{J}(\bar{\mathbf{h}})$ and $\nabla\mathbf{v}(\bar{\mathbf{h}})$, respectively. In a more concise form this can be written as:

$$\mathbf{g} = H[\boldsymbol{\delta}]\cdot[\mathbf{e}] - \mathbf{1} \tag{3.6b}$$

where $[\mathbf{x}]\cdot[\mathbf{y}]$ denote the component-wise products of vectors **x** and **y**. The two eigenvalues, $\boldsymbol{\delta}$ and $\mathbf{e}$, in the right-hand side of (3.6) can easily be obtained as follows. The former eigenvalues $\boldsymbol{\delta}$ are readily given by the DFT of the first row vector of $\mathbf{J}(\bar{\mathbf{h}})$ in (3.4):

$$\boldsymbol{\delta} = (\theta/4)[0,1,1,1]^{\mathrm{T}} \tag{3.7}$$

As for the latter eigenvalues $\mathbf{e}$, notice that $\nabla\mathbf{v}(\bar{\mathbf{h}})$ in (3.2) consists of additions and multiplications of the circulant **D**/*d*. This implies that the eigenvalues **e** can be represented as functions of the eigenvalues **f** of the spatial discounting matrix **D**/*d*:

$$\mathbf{e} = h^{-1}\{b[\mathbf{f}] - a[\mathbf{f}]^2\} \tag{3.8}$$

where $[\mathbf{x}]^2$ denotes the component-wise square of a vector $\mathbf{x}$ (*i.e.*, $[\mathbf{x}]^2 \equiv [\mathbf{x}]\cdot[\mathbf{x}]$). The eigenvalues $\mathbf{f} \equiv [f_0, f_1, f_2, f_3]^{\mathrm{T}}$, in turn, are obtained by the DFT of the first row vector $\mathbf{d}_0/d$ of the matrix $\mathbf{D}/d$:

$$\mathbf{f} = \mathbf{Z}\,\mathbf{d}_0 \,/\, d = \frac{1}{d}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \mathrm{i} & -1 & -\mathrm{i} \\ 1 & -1 & 1 & -1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \end{bmatrix}\begin{bmatrix} 1 \\ r \\ r^2 \\ r \end{bmatrix} = \begin{bmatrix} 1 \\ c(r) \\ c(r)^2 \\ c(r) \end{bmatrix} \tag{3.9}$$

where i denotes the imaginary unit, $c(r) \equiv (1-r)/(1+r)$. Thus, we have the following proposition characterizing the eigenvalues and eigenvectors of the Jacobian matrix $\nabla\mathbf{F}(\bar{\mathbf{h}})$:

Proposition 2: *Consider a uniform distribution* $\bar{\mathbf{h}} = [h,h,h,h]$ *of skilled workers in a racetrack economy with 4 regions. The Jacobian matrix* $\nabla\mathbf{F}(\bar{\mathbf{h}})$ *of the adjustment process (2.18) of the CP model at* $\bar{\mathbf{h}}$ *has the following eigenvector and the associated eigenvalues:*

1) *the* k^{th} *eigenvector* $(k = 0,1,2,3)$ *is given by the* k^{th} *row vector,* $\mathbf{z}_k$, *of the discrete Fourier transformation (DFT) matrix* **Z**.

2) *the* k^{th} *eigenvalue* g_k $(k = 1,2,3)$ *is given by a quadratic function of the* k^{th} *eigenvalue* f_k *of the spatial discounting matrix* $\mathbf{D}/d$:

$$g_k = \theta G(f_k(r)) \quad (k = 1,2,3), \qquad g_0 = -1 \tag{3.10a}$$

$$G(x) \equiv bx - ax^2 - \theta^{-1} \tag{3.10b}$$

$$f_1(r) = f_3(r) = c(r) \equiv (1-r)/(1+r), \qquad f_2(r) = c(r)^2 \tag{3.10c}$$

where *a* and *b* are constant parameters defined in (3.3) for the Pf model (For the FO model, see Appendix 2).

The eigenvectors $\{\mathbf{z}_k, (k = 0,1,2,3)\}$ in the first part of **Proposition 2** represent agglomeration patterns of skilled workers by the configuration pattern of the entries. For example, all entries of $\mathbf{z}_0$ are equal to one, and the entry pattern of $\mathbf{z}_0 = [1,1,1,1]$ corresponds to the state (configuration of skilled workers among four regions) in which skilled workers are uniformly distributed among four regions; $\mathbf{z}_2 = [1,-1,1,-1]$ has the alternate sequence of 1 and −1 representing a duocentric pattern in which skilled workers reside in two regions alternately; similarly, $\mathbf{z}_1$ and $\mathbf{z}_3$ correspond to a monocentric pattern.

The eigenvalue g_k in the second part of the proposition can be interpreted as the strength of "net agglomeration force" that leads the uniform distribution in the direction of the k^{th} agglomeration pattern (*i.e.*, the k^{th} eigenvector). By the term "net agglomeration force", we mean the net effect of the "agglomeration force" minus "dispersion force". Specifically, each of these two forces corresponds to bx and ax^2 in (3.10b), respectively. As is clear from the derivation of the eigenvalues **g**, the former term (bx) stems from $\nabla\mathbf{S}$ and the first term $\mathbf{M}^{\mathrm{T}}$ of $\nabla\mathbf{w}^{(H)}$ (see (A1.2) and (A1.3) in Appendix 1), each of which means the so-called "forward linkage" (or "price-index effect") and "backward linkage" (or "demand effect"), respectively. This implies that this term represents the centripetal force induced by the increase in variety of products that would be realized when the uniform distribution $\bar{\mathbf{h}}$

deviates to the agglomeration pattern $\mathbf{z}_k$. The latter term ($a\,x^2$), which stems from $\nabla\mathbf{w}^{(L)}$ and the second term of $\nabla\mathbf{w}^{(H)}$, represent the centrifugal force due to the increased market competition ("market crowding effect") in the agglomerated pattern $\mathbf{z}_k$.

4. Theoretical prediction of agglomeration patterns

4.1. Emergence of agglomeration

In order for the bifurcation from the uniform equilibrium distribution $\bar{\mathbf{h}} = [h,h,h,h]$ to occur with the changes in the SDF r (or the transportation cost τ), either of the eigenvalues $g_1\,(=g_3)$ and g_2 must change sign. Since the eigenvalues $g_k\ (k=1,2,3)$ are given by $G(f_k)$, the changes in sign mean that there should exist real solutions for the quadratic equation (4.1) with respect to f_k:

$$G(f_k) \equiv b\,f_k - a\,f_k^{\ 2} - \theta^{-1} = 0 \tag{4.1}$$

Moreover, the solutions must lie in the interval $[0,1)$ that is the possible range of the eigenvalue f_k (see (3.10)). These conditions lead us to the following proposition:

Proposition 3: *Suppose that we continuously increase the value of the SDF r (i.e., we decrease the transport cost parameter τ) of the CP model in a racetrack economy with 4 regions, starting from $r=0$. In order for a bifurcation from a uniform equilibrium distribution $\bar{\mathbf{h}} = [h,h,h,h]$ to some agglomeration to occur, the parameters of the CP model should satisfy:*

$$\Theta \equiv b^2 - 4a\,\theta^{-1} \geq 0\text{, and } b + \sqrt{\Theta} \leq 2a \tag{4.2}$$

The first inequality of (4.2) is the condition for the existence of real solutions of the (4.1) with respect to f_k. This condition is not necessarily satisfied for the cases when heterogeneity of consumers is very large (*i.e.*, θ is very small), which implies that no agglomeration occurs in such cases. The second inequality of (4.2), which stems from the requirement that the solutions of (4.1) are less than 1, corresponds to the "no black-hole" condition that is well known in literature dealing with the two-region CP model. For the cases when parameters of the CP model do not satisfy this condition, the eigenvalues $g_k\ (k=1,2,3)$ are positive even when the SDF is zero (*i.e.*, transportation cost τ is very high), which means that the uniform distribution $\bar{\mathbf{h}}$ cannot be a stable equilibrium.

In the following analyses, we assume that the parameters (h,σ,θ) of the CP model satisfy (4.2). We then have two real solutions for the quadratic equation (4.1) with respect to f_k:

$$x_+^* \equiv (b+\sqrt{\Theta})/(2a) \quad\text{and}\quad x_-^* \equiv (b-\sqrt{\Theta})/(2a) \tag{4.3}$$

Each of the solutions $x_\pm^*$ means a critical value ("*break point*") at which a bifurcation from the flat earth equilibrium $\bar{\mathbf{h}}$ occurs when we regard the eigenvalue f_k as a bifurcation parameter. Since we are interested in the process of increasing the SDF r, it is more meaningful to express the break point in terms of the SDF (rather than in terms of f_k). Note here that each of the eigenvalues $\{f_k(r)\}$ is a monotonically decreasing function of the SDF (see (3.10c) in **Proposition 2**). This implies that, in the process of increasing the SDF, the eigenvalue $f_k(r)$ first crosses the critical value x_+^* before reaching x_-^*. We can also see from (3.10c) that the eigenvalue $f_2(r)$ first reaches the critical value x_+^* before $f_1(r)$ does, since $f_2(r)$ is always smaller than $f_1(r)$:

$$f_1(r) \equiv c(r) > f_2(r) \equiv c(r)^2 \quad \forall r \in (0,1]$$

Thus, we can conclude that, in the course of the steady increases in the SDF, the first bifurcation occurs when the SDF first reaches the critical value r_+^* that satisfies:

$$x_+^* = f_2(r_+^*) \equiv c(r_+^*)^2$$

To be more specific, the critical value r_+^* of the SDF is given by:

$$r_+^* = (1-\sqrt{x_+^*})/(1+\sqrt{x_+^*}) \tag{4.4}$$

It is worth noting that (4.4) implicitly provides information on the changes in r_+^* with the changes in values of the CP model parameters (h, σ, θ) since x_+^* is explicitly represented as a function of the CP model parameters in (4.3). To take but one example, consider the effect of an increase in the skilled workers h of the Pf model with homogeneous consumers (*i.e.*, $\theta \to +\infty$), for which the solution x_+^* in (4.3) reduces to:

$$\lim_{\theta \to +\infty} x_+^* = b/a = b\sigma/(1+h^{-1})$$

It can easily be seen that the increase in the number of skilled workers h increases x_+^*, and this in turn decreases the critical value r_+^*.

The fact that the eigenvalue $f_2(r)$ first crosses the critical value x_+^* also enables us to identify the associated agglomeration pattern that emerges at the first bifurcation. Recall here that the equilibrium solution moves in the direction of the eigenvector whose associated eigenvalue first hits the critical value. As stated in **Proposition 2**, this direction is the second eigenvector $\mathbf{z}_2 = [1, -1, 1, -1]$. Therefore, the pattern of agglomeration that first emerges is:

$$\mathbf{h} = \bar{\mathbf{h}} + \delta \mathbf{z}_2 = [h+\delta, h-\delta, h+\delta, h-\delta] \qquad (0 \le \delta \le h)$$

in which skilled workers agglomerate in alternate two regions. Thus, we can characterize the bifurcation from the uniform distribution as follows.

Proposition 4: *Suppose that the conditions of (4.2) in* ***Proposition 3*** *are satisfied for the CP model, and that the uniform distribution* $\bar{\mathbf{h}} = [h, h, h, h]$ *of skilled workers is a stable equilibrium at some value of the SDF r. Starting from this state, we consider the process where the value of the SDF continuously increases (i.e., the transportation cost* τ *decreases).*

1) *The net agglomeration force (i.e., the eigenvalue)* g_k *for each agglomeration pattern (i.e., the eigenvector)* $\mathbf{z}_k$ *increases as the SDF increases, and the uniform distribution* *becomes unstable (i.e., agglomeration emerges) at the break point* $r = r_+^*$ *given by (4.3) and (4.4).*

2) *The critical value* r_+^* *for the bifurcation decreases, as a) the heterogeneity of skilled workers (in location choice) is smaller (i.e., θ is large), b) the number of skilled workers relative to that of the unskilled workers are larger (i.e., h is large), c) the elasticity of substitution between two varieties is smaller (i.e., σ is small).*

3) *The pattern of agglomeration that first emerges is* $\mathbf{h} = [h+\delta, h-\delta, h+\delta, h-\delta]$ $(0 \le \delta \le h)$, *in which skilled workers agglomerate alternately in two regions.*

4.2. Evolution of agglomeration

In conventional CP models with two regions, increases in the SDF (or decreases in transportation cost τ) lead to the occurrence of a bifurcation from the uniform distribution $\bar{\mathbf{h}}$ to a monocentric agglomeration. In the CP model with four (or more) regions, the first bifurcation shown in **4.1** does not directly branch to the monocentric pattern; instead, further bifurcations (leading either to a more concentrated pattern or a more dispersed pattern) can repeatedly occur. In what follows, we will examine such evolution of agglomeration after the first bifurcation, restricting ourselves to *the homogeneous consumer case* (*i.e.*, $\theta \to +\infty$).

4.2.1. Evolution to a duocentric pattern $\mathbf{h}^*$***–Sustain point for*** $\mathbf{h}^*$

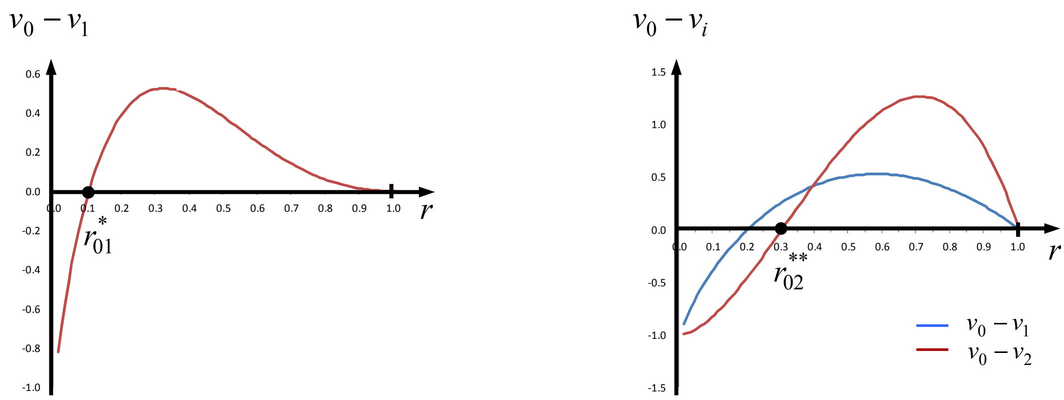
For the Pf model with homogeneous consumers, the deviation δ from the uniform distribution $\bar{\mathbf{h}}$ monotonically increases with the increases in the SDF after the first bifurcation. Shortly after the increases in the SDF from the break point $r = r_+^*$, this leads to a duocentric pattern, $\mathbf{h}^* = [2h, 0, 2h, 0]$, where skilled workers equally exist only in the alternate two regions. The fact that the duocentric pattern $\mathbf{h}^*$ may exist as an equilibrium solution of the Pf model can be confirmed by examining the "*sustain point*" for $\mathbf{h}^*$. The sustain point is the value of the SDF above which the equilibrium condition for $\mathbf{h}^*$,

$$v_0(\mathbf{h}^*) = v_2(\mathbf{h}^*) = \max_k \{v_k(\mathbf{h}^*)\} \tag{4.5}$$

is satisfied. As shown in Appendix 4, the condition of (4.5) indeed holds for any r larger than r_{01}^*, which is the sustain point for $\mathbf{h}^*$. This is illustrated in Figure 1(a), where the horizontal axis denotes the SDF r, and the curve represents the utility difference $v_0(\mathbf{h}^*) - v_1(\mathbf{h}^*)$ as a function of r. As can be seen from this figure, $v_0(\mathbf{h}^*) - v_1(\mathbf{h}^*)$, is positive for any r larger than $r = 0.1$ (the sustain point), which means that the duocentric pattern $\mathbf{h}^*$ continues to be an equilibrium for the range of the SDF above the sustain point.

4.2.2. Bifurcation from the duocentric pattern–Break point at $\mathbf{h}^*$

After the emergence of the duocentric pattern $\mathbf{h}^* = [2h, 0, 2h, 0]$, further increases in the SDF (above the sustain point $r = r_{01}^*$) can lead to further bifurcations (*i.e.*, $\mathbf{h}^*$ become unstable). In order to investigate such a possibility, we need to obtain the eigenvectors and the associated eigenvalues for the Jacobian matrix of the adjustment process at $\mathbf{h}^*$:

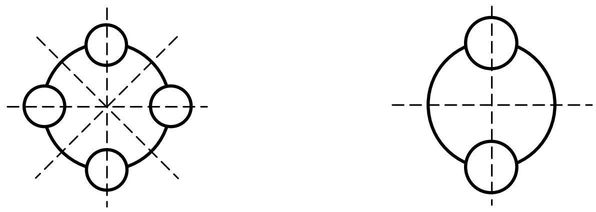


(a) The sustain point for $\mathbf{h}^*$ (b) The sustain point for $\mathbf{h}^{**}$

Figure 1. The sustainable regions for $\mathbf{h}^* = [2h, 0, 2h, 0]$ and $\mathbf{h}^{**} = [4h, 0, 0, 0]$

$$\nabla\mathbf{F}(\mathbf{h}^*) = H\,\mathbf{J}(\mathbf{h}^*)\,\nabla\mathbf{v}(\mathbf{h}^*) - \mathbf{I}$$

A seeming difficulty we encounter in obtaining the eigenvalues is that the Jacobian $\nabla\mathbf{F}(\mathbf{h}^*)$ at $\mathbf{h}^*$ is no longer a circulant matrix, unlike the Jacobian $\nabla\mathbf{F}(\bar{\mathbf{h}})$ at $\bar{\mathbf{h}}$. This is because of the loss of symmetry in the configuration of skilled workers: D_4 symmetry of $\bar{\mathbf{h}}$ is reduced to D_2 symmetry of $\mathbf{h}^*$ (see Figure 2), which leads to the fact that $\mathbf{J}(\mathbf{h}^*)$ and $\nabla\mathbf{v}(\mathbf{h}^*)$ are not circulants. However, as it turns out, it is still possible to find a closed form expression for the eigenvalues of $\nabla\mathbf{F}(\mathbf{h}^*)$ by using the fact that the duocentric pattern $\mathbf{h}^*$ has partial symmetry and the submatrices of $\mathbf{J}(\mathbf{h}^*)$ and $\nabla\mathbf{v}(\mathbf{h}^*)$ are circulants (see Lemma A.1).



D_4 symmetry D_2 symmetry

Figure 2. Symmetry of the two configurations $\bar{\mathbf{h}} = (h, h, h, h)$ and $\mathbf{h}^* = (2h, 0, 2h, 0)$

In order to exploit the symmetry remaining in the duocentric pattern $\mathbf{h}^*$, we begin by dividing the set $C = \{0,1,2,3\}$ of regions into two subsets: the subset $C_0 = \{0,2\}$ of regions with skilled workers, and the subset $C_1 = \{1,3\}$ of regions without skilled workers. Corresponding to this division of the set of regions, we consider the following permutation σ of the set $C = \{0,1,2,3\}$:

$$\sigma(0) = 0,\ \sigma(1) = 2,\ \sigma(2) = 1,\ \sigma(3) = 3$$

so that the first half elements $\{\sigma(0), \sigma(1)\}$ and the second half elements $\{\sigma(2), \sigma(3)\}$ of the set $C^P = \{\sigma(0), \sigma(1), \sigma(2), \sigma(3)\}$ correspond to the subsets C_0 and C_1, respectively. For this permutation, we then define the associated permutation matrix $\mathbf{P}$:

$$\mathbf{P}=\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$$

It can be readily verified that, for a 4 by 4 matrix **A** whose (i,j) element is denoted as a_{ij}, $\mathbf{PAP}^{\mathrm{T}}$ yields a matrix whose (i,j) element is $a_{\sigma(i)\sigma(j)}$, and that $\mathbf{P}\mathbf{P}^{\mathrm{T}}=\mathbf{P}^{\mathrm{T}}\mathbf{P}=\mathbf{I}$; that is, the similarity transformation $\mathbf{PAP}^{\mathrm{T}}$ gives a consistent renumbering of the rows and columns of **A** by the permutation σ.

The permutation σ (or the similarity transformation based on the permutation matrix **P**) constitutes a "new coordinate system" for analyzing the Jacobian matrix of the adjustment process. Under the new coordinate system, the SDM **D** can be represented as:

$$\mathbf{D}^{\times}\equiv\mathbf{P}\mathbf{D}\mathbf{P}^{\mathrm{T}}=\left[\begin{array}{c:c}\mathbf{D}^{(0)}&\mathbf{D}^{(1)}\\\hdashline\mathbf{D}^{(1)}&\mathbf{D}^{(0)}\end{array}\right]\tag{4.6}$$

where each of the submatrices $\mathbf{D}^{(0)}$ and $\mathbf{D}^{(1)}$ is a 2-by-2 circulant generated from a vector $\mathbf{d}_0^{(0)}\equiv[1,r^2]^{\mathrm{T}}$ and $\mathbf{d}_0^{(1)}\equiv[r,r]^{\mathrm{T}}$, respectively. Similarly, the Jacobian matrix $\nabla\mathbf{F}(\mathbf{h}^*)$ of the adjustment process is transformed into:

$$\nabla^{\times}\mathbf{F}(\mathbf{h}^*)\equiv\mathbf{P}\nabla\mathbf{F}(\mathbf{h}^*)\mathbf{P}^{\mathrm{T}}=H\,\mathbf{J}^{\times}(\mathbf{h}^*)\,\nabla^{\times}\mathbf{v}(\mathbf{h}^*)-\mathbf{I}\tag{4.7a}$$

where the Jacobian matrices in the right-hand side are respectively defined by:

$$\nabla^{\times}\mathbf{v}(\mathbf{h}^*)\equiv\mathbf{P}\nabla\mathbf{v}(\mathbf{h}^*)\mathbf{P}^{\mathrm{T}}=\frac{1}{2h}\left[\begin{array}{c:c}\mathbf{V}^{(00)}&\mathbf{V}^{(01)}\\\hdashline\mathbf{V}^{(10)}&\mathbf{V}^{(11)}\end{array}\right]\tag{4.7b}$$

$$\mathbf{J}^{\times}(\mathbf{h}^*)\equiv\mathbf{P}\mathbf{J}(\mathbf{h}^*)\mathbf{P}^{\mathrm{T}}=\left[\begin{array}{c:c}\mathbf{J}^{(00)}&\mathbf{J}^{(01)}\\\hdashline\mathbf{J}^{(10)}&\mathbf{J}^{(11)}\end{array}\right]\tag{4.7c}$$

and the submatrices $\mathbf{V}^{(ij)}$ and $\mathbf{J}^{(ij)}$ $(i,j=0,1)$ are 2-by-2 matrices.

For these Jacobian matrices under the new coordinate system, we can show (Lemma A.1) that all the submatrices $\mathbf{V}^{(ij)}$ and $\mathbf{J}^{(ij)}$ $(i,j=0,1)$ are circulants. This fact allows us to conclude (Lemma A.2) that knowing only the eigenvalues $\mathbf{e}^{(00)}$ of the submatrix $\mathbf{V}^{(00)}$ $=\{dv_i/dh_j\ (i,j\in C_0)\}$ is sufficient to obtain the eigenvalues $\mathbf{g}^*$ of the Jacobian $\nabla\mathbf{F}(\mathbf{h}^*)$. Furthermore, it can be readily shown that the submatrix $\mathbf{V}^{(00)}$ is a circulant consisting only of submatrices $\mathbf{D}^{(0)}$ and $\mathbf{D}^{(1)}$ of the SDM **D** (these are also circulants), which means that we can obtain analytical expressions for the eigenvalues $\mathbf{e}^{(00)}$. These considerations lead to the following lemma:

Lemma 1: *The Jacobian matrix* $\nabla\mathbf{F}(\mathbf{h}^*)$ *of the adjustment process (2.18) of the CP model at* $\mathbf{h}^*$ *has the following eigenvector and the associated eigenvalues:*

1) *the* k^{th} *eigenvector* $(k=0,1,2,3)$ *is given by the* k^{th} *row vector,* $\mathbf{z}_k^*$*, of the discrete Fourier transformation (DFT) matrix* $\mathbf{Z}^*\equiv\mathbf{P}^{\mathrm{T}}\mathrm{diag}[\mathbf{Z}_{[2]},\mathbf{Z}_{[2]}]\,\mathbf{P}$, where $\mathbf{Z}_{[2]}\equiv\begin{bmatrix}1&1\\1&-1\end{bmatrix}$.

2) the eigenvalues $\{g_k^* \ (k=0,1,2,3)\}$ *are given by* $g_k^* = -1 \ (k=0,1,3)$, and

$$g_2^* = \theta G^*(c(r^2)), \tag{4.9a}$$

$$G^*(x) \equiv b\,x - a^* x^2 - \theta^{-1}, \tag{4.9b}$$

where $c(r^2) \equiv (1-r^2)/(1+r^2)$, and the parameters a^* and b are defined as

$$a^* \equiv \sigma^{-1}(1+(2h)^{-1}), \qquad b \equiv (\sigma-1)^{-1} + \sigma^{-1}.$$

Proof: see Appendix 5.

The eigenvalues $\mathbf{g}^*$ obtained in Lemma 1 allow us to determine the critical value ("break point") at which a bifurcation from the duocentric pattern $\mathbf{h}^* = [2h, 0, 2h, 0]$ to a more concentrated pattern occurs. In a similar manner to the discussion for the first bifurcation from the uniform distribution $\bar{\mathbf{h}} = [h,h,h,h]$, we see that the second bifurcation from the duocentric pattern $\mathbf{h}^*$ occurs when the eigenvalue $g_2^*(x)$ changes sign. Since $g_2^*(x)$ is given by (4.9), the critical values of $x \equiv c(r^2)$ at which the eigenvalue changes sign are the solutions of the quadratic equation $G^*(x) = 0$. For the homogeneous consumer case (*i.e.*, $\theta \to +\infty$), the quadratic equation reduces to:

$$\lim_{\theta \to +\infty} G^*(x) = b\,x - a^* x^2 = 0 \tag{4.10}$$

and hence, the critical values (the two solutions of (4.10)) are given by:

$$x_+^{**} = b/a^* = b\sigma/(1+(2h)^{-1}), \text{ and } \ x_-^{**} = 0 \tag{4.11}$$

Note here that $x \equiv c(r^2)$ is a monotonically decreasing function of the SDF. This implies that, in the course of increasing the SDF, the x first crosses x_+^{**}. Therefore, the second bifurcation occurs when the SDF first reaches the critical value r_+^{**} that satisfies $x_+^{**} = c(r_+^{**\,2})$. That is, the critical value of the second bifurcation in terms of the SDF is given by:

$$r_+^{**} = [(1-x_+^{**})/(1+x_+^{**})]^{1/2} \tag{4.12}$$

We can also identify the associated agglomeration pattern that emerges at this bifurcation. The moving direction away from $\mathbf{h}^*$ at this bifurcation is the second eigenvector $\mathbf{z}_2^* = (1, 0, \text{-}1, 0)$. Accordingly, the emerging spatial configuration is given by:

$$\mathbf{h} = \mathbf{h}^* + \delta \mathbf{z}_2^* = (2h+\delta, 0, 2h-\delta, 0) \qquad (0 \le \delta \le 2h)$$

Thus, the properties of the second bifurcation can be summarized as follows:

Proposition 5: *Suppose that the SDF r is larger than the sustain point* r_{01}^* *of the duocentric pattern* $\mathbf{h}^* = [2h, 0, 2h, 0]$ *and* $\mathbf{h}^*$ *is a stable equilibrium for the CP model with homogeneous consumers. With the increases in the SDF, the duocentric pattern* $\mathbf{h}^*$ *become unstable at the second break point* $r = r_+^{**}$ *given by (4.11) and (4.12), and then a more concentrated pattern* $\mathbf{h} = [2h+\delta, 0, 2h-\delta, 0]$ $(0 \le \delta \le 2h)$ *emerges.*

4.2.3. Evolution to a monocentric pattern $\mathbf{h}^{**}$ *–Sustain point for* $\mathbf{h}^{**}$

After the second bifurcation, the deviation δ from the duocentric pattern $\mathbf{h}^* = [2h, 0, 2h, 0]$ monotonically increases with the increase in the SDF, which leads to a monocentric pattern, $\mathbf{h}^{**} = [4h, 0, 0, 0]$. This fact can be confirmed by examining a sustain point for the monocentric pattern. As shown in Appendix 4, the equilibrium condition for $\mathbf{h}^{**}$:

$$v_0(\mathbf{h}^{**}) = \max_k \{v_k(\mathbf{h}^{**})\} \tag{4.13}$$

is satisfied for any *r* larger than some critical value r_{02}^{**}, which is the sustain point for $\mathbf{h}^{**}$. This is illustrated in Figure 1(b), where the horizontal axis denotes the SDF, and each of the red and blue curves is the utility difference $v_0(\mathbf{h}^{**}) - v_2(\mathbf{h}^{**})$ and $v_0(\mathbf{h}^{**}) - v_1(\mathbf{h}^{**})$ as a function of *r*, respectively. As can be seen from this figure, $v_0(\mathbf{h}^{**})$ gives the largest utility among $\{v_i(\mathbf{h}^{**})\ (i = 0,1,2,3)\}$ for any *r* larger than 0.3 (the sustain point), which means that the monocentric pattern $\mathbf{h}^{**}$ continues to be an equilibrium for the range $1 > r > r_{02}^{**}$.

The results obtained so far can be summarized as a schematic representation in Figure 3.

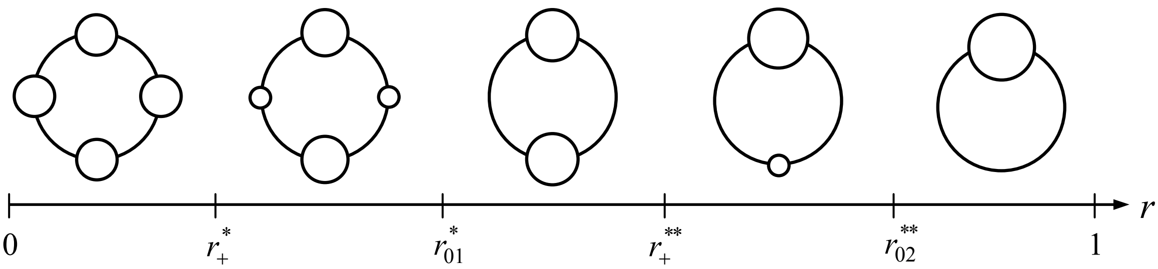


Figure 3. A series of agglomeration patterns that emerge in the course of increasing the SDF.

4.3. Collapse of agglomeration

In **4.1**, we derived two critical values (of the eigenvalue f_k of the matrix **D**), x_+^* and x_-^*, at each of which a bifurcation from the uniform distribution $\bar{\mathbf{h}} = [h, h, h, h]$ occurs, and only the properties of the bifurcation at the former critical value (x_+^*) have been shown. We now examine the bifurcation at the latter critical value (x_-^*). Before starting a detailed discussion on the bifurcation at x_-^*, we should recall that the eigenvalue g_k for each agglomeration pattern *k* is a quadratic function of f_k as seen in **Proposition 2**. It follows that each of the $\{g_k\}$ is a unimodal function of the SDF because f_k is a decreasing function of the SDF. In other words, all the net agglomeration forces monotonically decrease[7] after a monotonic increasing process in the course of increasing the SDF. This fact implies that agglomeration

[7] As seen from (3.12b), the absolute values of both the agglomeration force (the first term, bf_k, of g_k) and dispersion force (the second term, af_k^2, of g_k) decline as f_k decreases. It should be noted that the rate of decay (with the decrease in f_k) of the agglomeration force, *b*, is constant, while that of the dispersion force, $2af_k$, is proportional to f_k. This means that the latter gets smaller than the former for $f_k < b/(2a)$. In other words, for the range $[r_k^{-1}(b/(2a)), 1]$ of the SDF, the agglomeration force declines faster than the dispersion force (*i.e.*, the net agglomeration force decreases) as the SDF increases.

patterns observed in **4.1** and **4.2** might revert to the dispersed distribution $\bar{\mathbf{h}}$ if every net agglomeration force at $\bar{\mathbf{h}}$ could decrease to negative values for a (relatively) high SDF range. Whether or not this can happen decisively depends on the value of x_-^*, which in turn depends on the heterogeneity of the consumers. Thus, we divide the following discussion into two cases: the first is when consumers are homogeneous with respect to location choice, and the second is when consumers are heterogeneous. As we shall see below, these two cases exhibit significantly different properties for the bifurcation at x_-^*.

For the homogeneous consumers case (*i.e.*, $\theta \to +\infty$), $\Theta = b^2$ holds and hence the critical value given by (4.3) reduces to the following simpler expression:

$$\lim_{\theta \to +\infty} . \ x_-^*(\theta) = 0$$

Substituting this into the inverse function, $r_k(\cdot)$, of the eigenvalue $f_k(r)$, we see that

$$r_k(x_-^*) = r_k(0) = 1 \ \forall k$$

This means that the net agglomeration forces **g** are always positive for the interval $[r_+^*, 1)$ of the SDF. This fact is illustrated in Figure 4(a), where the horizontal axis denotes the SDF r, and the red curve denotes the eigenvalue g_2 as a function of r, while the blue curve denotes the eigenvalue g_1; both of the two curves are unimodal functions whose values reach zero at $r = 1$. Therefore, the critical value of the SDF at which the bifurcation from agglomeration equilibrium to the flat earth equilibrium occurs is:

$$\lim_{\theta \to +\infty} . \ r_-^* = 1$$

That is, no matter how large the SDF is, agglomeration never breaks down, except for the maximum limit at $r = 1$.

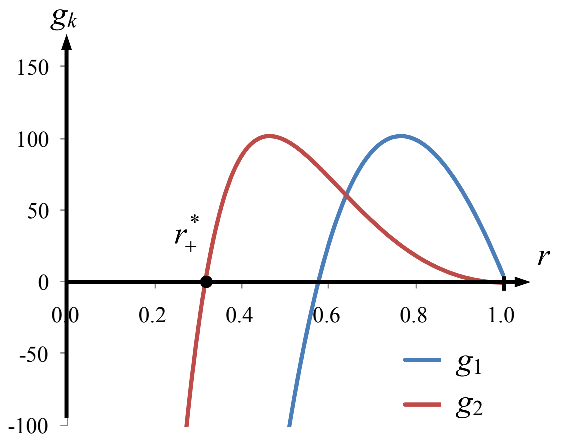
For the case in which consumers are heterogeneous (*i.e.*, θ is finite), we see from (4.3) that the critical value x_-^* is always positive regardless of the values of the CP model parameters, and hence:

$$r_k(x_-^*) < 1 \quad \forall k$$

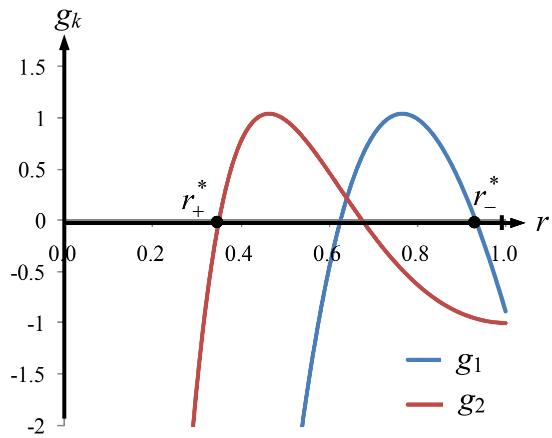
This means that the net agglomeration force g_k can be negative on the interval $[r_k(x_-^*), 1]$ of the SDF. This fact is illustrated in Figure 4(b), where the logit choice parameter θ is set equal to 20; the horizontal axis, the red and blue curves denote the SDF, the eigenvalues g_2 and g_1, respectively. Unlike the case of the homogeneous consumers, both of the two eigenvalues reach zero at $r < 1$. Accordingly, in the course of increasing the SDF, a bifurcation from some agglomeration pattern to the uniform distribution ("re-dispersion") occurs at the critical value:

$$r_-^* = r_1(x_-^*) = (1 - x_-^*)/(1 + x_-^*) < 1 \qquad (4.14)$$

above which all the net agglomeration forces are negative. That is, the economy with heterogeneous consumers necessarily moves from agglomeration to dispersion when the SDF increases (the transportation cost τ decreases) to a relatively high range. This is a generalized result of the "*bell-shaped*" bifurcation diagram obtained in Tabuchi and Thisse (2002) and Murata (2003), in which consumers (skilled workers) in the two-region CP model are supposed to exhibit idiosyncratic taste differences in residential locations.



$(a)\quad h = 0.1,\ \sigma = 2.0,\ \theta = 1000$



$(b)\quad h = 0.1,\ \sigma = 2.0,\ \theta = 20$

Figure 4. Eigenvalues g_1 and g_2 as functions of the spatial discounting factor.

Some remarks regarding the "re-dispersion" are in order. First, for the "re-dispersion" to occur in the CP model, it is sufficient that the model parameter satisfies $x_-^* > 0$. This requirement, even if the consumers are homogeneous, can be satisfied by adding an extra negative (constant) term to the function G in (3.10) determining the critical values x_-^*. Therefore, we can deduce that "re-dispersion" occurs whenever some dispersion forces, such as land rent or congestion externality, that increase with agglomeration (but do not depend on the SDF r) are introduced into the CP model. This deduction is, of course, applicable to the conventional two-region model as well as the current four-city model. This gives a consistent theoretical explanation of several "re-dispersion" results reported in studies on the two-region CP model: Tabuchi (1998), Helpman (1998)[8], Alonso-Villar (2007), in each of which residential land rent (that is an increasing function of the number of skilled workers in each city) is introduced into the two-region CP model with homogeneous consumers.

Second, somewhat surprisingly, the "re-dispersion" can be observed not only at the critical value r_-^* but also in the course of evolving agglomeration; the re-dispersion at r_-^* is the last among multiple re-dispersions in the course of increasing the SDF. The mechanism by which

[8] Strictly speaking, the Helpman's model and Murata-Thisse's model do not exhibi "re-dispersion"; in their models, a uniform distribution is unstable (*i.e.*, an agglomerated pattern is a stable equilibrium) when the SDF is very low (*i.e.*, the transportation cost is very high). This is because their models do not satisfy the "no black-hole" condition (note that their models are not endowed with immobile laborers that function as a dispersion force in the Krugman's CP model), which causes the degeneration of the first bifurcation from the uniform distribution. As a result, their model exhibits only a single time of bifurcation from the agglomeration to dispersion, and the one and only bifurcation corresponds to the "re-dispersion" in the current context.

such repetitions of agglomeration and dispersion may occur can best be explained by the example in Figure 5. Each of the red and the blue curves in this figure respectively represent the eigenvalue g_2 and g_1 at the uniform distribution $\bar{\mathbf{h}}$ as a function of the SDF r (the horizontal axis). It follows that a bifurcation from agglomeration to dispersion occurs at the time if all the eigenvalues become negative. We can see from the figure that this actually occurs twice during the process of increasing r:

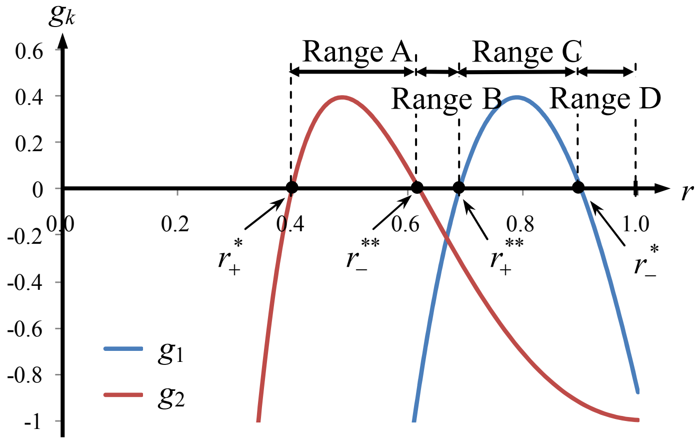


Figure 5. Repetition of agglomeration and dispersion ($h = 0.1$, $\sigma = 2.4$, $\theta = 20$)

Range A: the uniform distribution $\bar{\mathbf{h}}$ is unstable because the eigenvalue g_2 associated with the eigenvector $\mathbf{z}_2 = [1, -1, 1, -1]$ at $\bar{\mathbf{h}}$ is positive.

Range B: the uniform distribution $\bar{\mathbf{h}}$ is stable because all the eigenvalues $\{g_k\}$ at $\bar{\mathbf{h}}$ are negative. The bifurcation from agglomeration $(\bar{\mathbf{h}} + \delta \mathbf{z}_2)$ to dispersion occurs at the boundary r_-^{**} between A and B.

Range C: the uniform distribution $\bar{\mathbf{h}}$ is unstable because the eigenvalues g_1 and g_3 associated with the eigenvectors $\mathbf{z}_1 = [1, \mathrm{i}, -1, -\mathrm{i}]$ and $\mathbf{z}_3 = [1, -\mathrm{i}, -1, \mathrm{i}]$ at $\bar{\mathbf{h}}$ are positive. The bifurcation from dispersion to agglomeration $(\bar{\mathbf{h}} + \delta(\mathbf{z}_1 + \mathbf{z}_3))$ occurs at the boundary r_+^{**} between B and C.

Range D: the uniform distribution $\bar{\mathbf{h}}$ is stable because all the eigenvalues $\{g_k\}$ at $\bar{\mathbf{h}}$ are negative. The bifurcation from agglomeration $(\bar{\mathbf{h}} + \delta(\mathbf{z}_1 + \mathbf{z}_3))$ to dispersion occurs at the boundary r_-^* between C and D.

The observations so far can be summarized as the following proposition:

Proposition 6: *Starting with an agglomeration state, we consider the process where the SDF continuously increases (i.e., the transportation cost τ decreases).*

1) *The net agglomeration force g_k for each $\mathbf{z}_k$ monotonically decreases with the increase in the SDF after a monotonic increase process (i.e., g_k is a unimodal function of the SDF).*

2) *For the CP model with homogeneous consumers, all the net agglomeration forces reach zero only at the limit of $r = 1$. That is, the agglomeration equilibrium never reverts to the uniform distribution equilibrium during the course of increasing the SDF.*

3) For the CP model with heterogeneous consumers, all the net agglomeration forces reaches zero at a strictly positive value of the SDF. That is, a bifurcation from some agglomeration to the uniform distribution equilibrium ("re-dispersion") occurs at $r = r_{-}^{*}$*, where* r_{-}^{*} *is given by (4.3) and (4.14). Furthermore, the "re-dispersion" can occur not only from the monocentric agglomeration but also from the duocentric agglomeration. That is, repetitions of agglomeration and dispersion may be observed for the CP model with heterogeneous consumers (see Figure 5)*

5. Concluding remarks

This paper provided a simple approach to analyzing the bifurcation phenomena in the multi-regional core-periphery (CP) model. The proposed method allows us not only to examine whether or not agglomeration of mobile factors emerges from a uniform distribution, but also to trace the evolution of spatial agglomeration patterns (*i.e.*, bifurcations from various *polycentric patterns* as well as a uniform pattern) with the steady decreases in transportation cost. Furthermore, it is theoretically deduced that the evolutionary process in the CP model may exhibit repetitions of agglomeration and dispersion, which gives a theoretical explanation for several "bell-shaped development" results reported in studies on the two-region CP models. Although we restricted ourselves to illustrating the evolutionary process in the four-region case, the approach can be extended to more general cases. The detailed discussion on the method available for models with an arbitrary number of regions can be found in Akamatsu et al., (2009).

Through the analysis of the CP model, we also demonstrated that the spatial discounting matrix (SDM) encapsulates the essential information required for analyzing the multi-regional CP model. Specifically, in order to know spatial concentration-dispersion patterns that may emerge in the CP model, it suffices *a*) to obtain the eigenvalues of the SDM, and *b*) to represent the Jacobian matrix of the indirect utility as a function of the SDM. It should be emphasized that this fact holds for a wide variety of models dealing with the formation of spatial patterns in economic activities. Indeed, the same procedure as that of this paper can be readily applied to other variants of the multi-regional CP model. Therefore, it would be valuable for future research to investigate the bifurcation behaviors in these models by exploiting the method presented in this paper. We believe that such systematic studies would substantially improve our understanding of the nature of agglomeration economies.

Acknowledgements

We are grateful for comments and suggestions made by Masahisa Fujita, Kiyohiro Ikeda, Tomoya Mori, Se-il Mun, Yasusada Murata, Toshimori Otazawa, Daisuke Oyama, Yasuhiro Sato, Takatoshi Tabuchi, Kazuhiro Yamamoto, Dao-Zhi Zeng, as well as participants at the

23th Applied Regional Science Conference Meeting in Yamagata. Funding from the Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research (B) 19360227/21360194 is greatly acknowledged.

References

Akamatsu, T., Takayama, Y. and Ikeda, K. (2009) Spatial Discounting, Fourier, and Racetrack Economy: A Recipe for the Analysis of Spatial Agglomeration Models. *TUTUP Working Paper*, *Tohoku University*.

Alonso-Villar, O. (2008) A Model of Economic Geography with Demand-Pull and Congestion Costs. *Papers in Regional Science*, 87: 261-276.

Baldwin, R., Forslid, R., Martin, P., Ottaviano, G. and Robert-Nicoud, F. (2003) *Economic Geography and Public Policy*. Princeton University Press.

Behrens, K. and Thisse, J.-F. (2007) Regional Economics: A New Economic Geography Perspective. *Regional Science and Urban Economics*, 37: 457-465.

Combes, P-P., Mayer, T. and Thisse, J-F. (2008) *Economic Geography*: *The Integration of Regions and Nations*. Princeton University Press.

Forslid, R. and Ottaviano, G. (2003) An Analytically Solvable Core -Periphery Model. *Journal of Economic Geography*, 3: 229-240.

Fudenberg, D. and Levine, D. K. (1998) *The Theory of Learning in Games*. MIT Press.

Fujita, M. Krugman, P., and Venables, A.J. (1999) *The Spatial Economy*. MIT Press.

Fujita, M. and Krugman, P. (2004) The new economic geography: Past, present and the future, *Papers in Regional Science*, 83: 139-164.

Fujita, M. and Thisse, J.-F. (2002) *Economics of Agglomeration*. Cambridge University Press.

Fujita, M. and Thisse, J.-F. (2009) New Economic Geography: An Appraisal on the Occasion of Paul Krugman's 2008 Nobel Prize in Economic Sciences. *Regional Science and Urban Economics*, 39: 109-119.

Gray, R. M. (2006) Toeplitz and Circulant Matrices: A Review, *Foundations and Trends in Communications and Information Theory*. 2: 155-239.

Helpman, E. (1998) The Size of Regions. in D. Pines, E. Sadka, and Y. Zilcha (eds.), *Topics in Public Economics*. *Theoretical and Applied Analysis*, Cambridge: Cambridge University Press, 33-54.

Hirsch, M.W. and Smale, S. (1974) *Differential Equations, Dynamical Systems, and Linear Algebra*. Academic Press.

Krugman, P. (1991) Increasing Returns and Economic Geography. *Journal of Political Economy*, 99: 483-499.

Krugman, P. (1993) On the Number and Location of Cities. *European Economic Review,* 37: 293-298.

Krugman, P. (1996) *The Self-Organizing Economy*. Blackwell Publishers.

Mossay, P. (2003) Increasing Returns and Heterogeneity in a Spatial Economy. *Regional Science and Urban Economics*, 33: 419-444.

Murata, Y. (2003) Product Diversity, Taste Heterogeneity, and Geographic Distribution of Economic Activities: Market vs. Non-market Interactions. *Journal of Urban Economics*, 53: 126-144.

Murata, Y. and Thisse, J.F. (2005) A Simple Model of Economic Geography à la Helpman-Tabuchi. *Journal of Urban Economics*, 58: 137-155.

Ottaviano, G., Tabuchi, T. and Thisse, J.-F. (2002) Agglomeration and Trade Revisited. *International Economic Review*, 43: 409-436.

Oyama, D. (2009) Agglomeration under Forward-looking Expectations: Potentials and Global Stability, *Regional Science and Urban Economics*, 39: 696-713.

Papageorgiou, Y.Y. and Smith, T.R. (1983) Agglomeration as Local Instability of Spatially Uniform Steady-States. *Econometrica*, 51: 1109-1120.

Picard, P.M. and Tabuchi, T. (2009) Self-Organized Agglomerations and Transport Costs. *Economic Theory*, 42: 565-589.

Pflüger, M. (2004) A Simple Analytically Solvable, Chamberlinian Agglomeration Model. *Regional Science and Urban Economics*, 34: 565-573.

Sandholm, W.H. (2009) *Population Games and Evolutionary Dynamics*. MIT Press.

Tabuchi, T. (1998) Urban Agglomeration and Dispersion: A Synthesis of Alonso and Krugman. *Journal of Urban Economics*, 44: 333-551.

Tabuchi, T. and Thisse, J.F. (2002) Taste Heterogeneity, Labor Mobility and Economic Geography. *Journal*

of Development Economics, 69: 155-177.
Tabuchi, T., Thisse, J.-F. and Zeng, D.Z. (2005) On the Number and Size of Cities. *Journal of Economic Geography*, 5: 423-448.
Turing, A.M. (1952) The Chemical Basis of Morphogenesis. *Philosophical Transactions of the Royal Society of London*, 237: 37-72.

Appendix 1 Jacobian Matrices for the Pf model

The Jacobian matrix of the adjustment process for the Pf model is

$$\nabla \mathbf{F}(\mathbf{h}) = H\,\mathbf{J}(\mathbf{h})\nabla \mathbf{v}(\mathbf{h}) - \mathbf{I}.$$

The Jacobian matrices, $\mathbf{J}$ and $\nabla \mathbf{v}$, in the right-hand side of (2.19) is given by

$$\mathbf{J}(\mathbf{h}) \equiv \left[\frac{\partial P_i(\mathbf{V})}{\partial V_j}\right], \qquad \frac{\partial P_i}{\partial V_j} = \begin{cases} -\theta P_i P_j & \text{if} \quad i \neq j \\ \theta P_i(1-P_j) & \text{if} \quad i = j \end{cases}. \tag{A1.1}$$

$$\nabla \mathbf{v}(\mathbf{h}) = \nabla \mathbf{S}(\mathbf{h}) + \sigma^{-1}[\nabla \mathbf{w}^{(L)}(\mathbf{h}) + \nabla \mathbf{w}^{(H)}(\mathbf{h})] \tag{A1.2}$$

where the matrices $\nabla \mathbf{S}$, $\nabla \mathbf{w}^{(L)}$ and $\nabla \mathbf{w}^{(H)}$ in the right-hand of (A1.2) are given by

$$\nabla \mathbf{S} = (\sigma - 1)^{-1}\mathbf{M}^{\mathrm{T}}, \tag{A1.3a}$$

$$\nabla \mathbf{w}^{(L)} = -\mathbf{M}\mathbf{M}^{\mathrm{T}}, \qquad \nabla \mathbf{w}^{(H)} = \mathbf{M} - \mathbf{M}\mathbf{H}\,\mathbf{M}^{\mathrm{T}}. \tag{A1.3b}$$

Appendix 2 Jacobian Matrices and Its Eigenvalues for the FO model

We show the Jacobian matrix $\nabla \mathbf{v}(\overline{\mathbf{h}})$ and eigenvalues $\mathbf{g}$ of $\nabla \mathbf{F}(\overline{\mathbf{h}})$ for the FO model in turn.

(1) Differentiating the indirect utility function (2.14) with respect to $\mathbf{h}$ and substituting $\mathbf{h} = \overline{\mathbf{h}}$ into the result, $\nabla \mathbf{v}(\overline{\mathbf{h}})$ is given by:

$$\nabla \mathbf{v}(\overline{\mathbf{h}}) = h^{-1}\kappa_{-}(\mathbf{D}/d) + \overline{w}^{-1}\,\nabla \mathbf{w}(\overline{\mathbf{h}}) \tag{A2.1}$$

where $\kappa_{-} \equiv \mu/(\sigma-1)$ and $\overline{w} \equiv w_i(\overline{\mathbf{h}}) = \kappa/[(1-\kappa)h]$. The matrix $\nabla \mathbf{w}(\overline{\mathbf{h}})$ in the right-hand side is obtained by differentiating both sides of (2.9) with respect to $\mathbf{h}$ and substituting $\mathbf{h} = \overline{\mathbf{h}}$ into this equation:

$$\nabla \mathbf{w}(\overline{\mathbf{h}}) = h^{-1}\,\overline{w}\,[\mathbf{I} - \kappa(\mathbf{D}/d)]^{-1}(\mathbf{D}/d)[\kappa \mathbf{I} - (\mathbf{D}/d)] \tag{A2.2}$$

where $\kappa \equiv \mu/\sigma$. Substituting (A2.2) into (A2.1), we obtain the Jacobian matrix of the indirect utility:

$$\nabla \mathbf{v}(\overline{\mathbf{h}}) = h^{-1}\,\{\kappa_{-}(\mathbf{D}/d) + [\mathbf{I} - \kappa(\mathbf{D}/d)]^{-1}(\mathbf{D}/d)[\kappa \mathbf{I} - (\mathbf{D}/d)]\} \tag{A2.3}$$

(2) To obtain $\mathbf{g}$, it is sufficient to calculate the eigenvalues $\mathbf{e}$ of $\nabla \mathbf{v}(\overline{\mathbf{h}})$, since $\nabla \mathbf{F}(\overline{\mathbf{h}})$ and $\mathbf{J}(\overline{\mathbf{h}})$ are expressed as (3.5) and (3.4), respectively. From the fact that the right-hand side of (A2.3) consists only of $\mathbf{D}$ and $\mathbf{I}$ (*i.e.*, $\nabla \mathbf{v}(\overline{\mathbf{h}})$ is a circulant), we can obtain $\mathbf{e}$ by the DFT of the first row vector of this matrix.

$$e_k = h^{-1}\{\kappa_{-} f_k + [1 - \kappa f_k]^{-1} f_k [\kappa - f_k]\} \tag{A2.4}$$

Substituting (A2.4) and (3.7) into (3.6b), we have:

$$g_k = \begin{cases} -1 & \text{if} \quad k = 0 \\ \theta G(f_k)[1 - \kappa f_k]^{-1} & \text{if} \quad k \neq 0 \end{cases} \tag{A2.5}$$

where $G(f_k) \equiv b\,f_k - a\,f_k^{\,2} - \theta^{-1}$, $a \equiv \kappa\kappa_{-} + 1$, $b \equiv \kappa_{-} + \kappa(1+\theta^{-1})$.

Appendix 3 Properties of Circulant Matrices

A circulant $\mathbf{C}$ is defined as a square matrix of the form:

$$\mathbf{C} \equiv \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{K-2} & c_{K-1} \\ c_{K-1} & c_0 & c_1 & c_2 & \cdots & c_{K-2} \\ \vdots & \vdots & & & & \vdots \\ c_2 & c_3 & \cdots & c_{K-1} & c_0 & c_1 \\ c_1 & c_2 & \cdots & c_{K-2} & c_{K-1} & c_0 \end{bmatrix}$$

The elements of each row of **C** are identical to those of the previous row, but are moved one position to the right and wrapped around. The whole circulant is evidently determined by the first row vector $\mathbf{c}=[c_0, c_1, \ldots, c_{K-1}]$. Circulant matrices satisfy the following two well-known properties[9].

Property 1: Every circulant matrix **C** is diagonalized by the following similarity transformation:

$$\mathbf{Z}^*\mathbf{C}\mathbf{Z} = \mathrm{diag}(\boldsymbol{\lambda})$$

where **Z** is the DFT matrix whose (j,k) entry is given by $\omega^{jk} = \exp[\mathrm{i}(2\pi\, jk/K)]$, $\mathrm{i} \equiv \sqrt{-1}$; $\boldsymbol{\lambda} \equiv [\lambda_0, \lambda_1, \cdots, \lambda_{K-1}]^{\mathrm{T}}$, and $\mathbf{Z}^*$ denotes the conjugate transpose of **Z**. The k^{th} eigenvalues and the eigenvectors of **C** are therefore λ_k and the k^{th} row of the DFT matrix **Z**, respectively. Furthermore, $\boldsymbol{\lambda}$ is directly given by the DFT of the first row vector **c** of **C**: $\boldsymbol{\lambda} = \mathbf{Z}\mathbf{c}^{\mathrm{T}}$.

Property 2: If $\mathbf{C}_1$ and $\mathbf{C}_2$ are circulant matrices, the sum $\mathbf{C}_1 + \mathbf{C}_2$ and the product $\mathbf{C}_1\mathbf{C}_2$ are circulants. Also, if $\mathbf{C}_1$ is nonsingular, its inverse $\mathbf{C}_1^{-1}$ is a circulant.

Appendix 4 Sustain points for $\mathbf{h}^* = [2h, 0, 2h, 0]$ and $\mathbf{h}^{**} = [4h, 0, 0, 0]$

We will show the derivation of sustain points for $\mathbf{h}^* = [2h, 0, 2h, 0]$ and $\mathbf{h}^{**} = [4h, 0, 0, 0]$, in turn.

(1) For the duocentric pattern $\mathbf{h}^* = [2h, 0, 2h, 0]$, we can easily obtain the indirect utility for each region by substituting $\mathbf{h} = \mathbf{h}^*$ into (2.15):

$$v_i(\mathbf{h}^*) = \begin{cases} \sigma^{-1}[1 + h^{-1}] + (\sigma-1)^{-1}\ln(1+r^2) & (i = 0,2) \\ \sigma^{-1}[\,(2h)^{-1}(x(r)^{-1} + (1+2h)\,x(r))\,] + (\sigma-1)^{-1}\ln(2r) & (i = 1,3) \end{cases}$$

where $x(r) \equiv 2r/(1+r^2)$. To obtain the sustain point for $\mathbf{h}^*$, we represent the utility difference between the "core" regions and the "periphery" regions as a function of the SDF:

$$v_{01}(r) \equiv v_0(\mathbf{h}^*) - v_1(\mathbf{h}^*) = 2(1+h)x - 1 - (1+2h)\,x^2 - 2h\sigma(\sigma-1)^{-1}(x\ln x)$$

By inspecting the function $v_{01}(r)$, we see that it takes zero value at $r = 1$ and $r = r_{01}^* > 0$ (*i.e.*, the equation $v_{01}(r) = 0$ has two positive solutions 1 and r_{01}^*) and that:

$$v_{01}(r) \begin{cases} < 0 \ \text{ for } \ 0 < r < r_{01}^* \\ \geq 0 \ \text{ for } \ r_{01}^* \leq r < 1 \end{cases}$$

This means that the equilibrium condition for $\mathbf{h}^*$, $v_0(\mathbf{h}^*) = v_2(\mathbf{h}^*) = \max_k\{v_k(\mathbf{h}^*)\}$, is satisfied for any r larger than r_{01}^*; that is, $r = r_{01}^*$ is the sustain point for $\mathbf{h}^*$.

(2) The sustain point for the monocentric pattern $\mathbf{h}^{**} = [4h, 0, 0, 0]$ can be obtained in a similar manner. The indirect utility at $\mathbf{h}^{**}$ is given by:

$$v_i(\mathbf{h}^{**}) = \begin{cases} \sigma^{-1}[1 + h^{-1}] & (i = 0) \\ \sigma^{-1}[\,(2h)^{-1}(r^{-1} + r) + r\,] + (\sigma-1)^{-1}\ln r & (i = 1,3) \\ \sigma^{-1}[\,(4h)^{-1}(2 + r^{-2} + r^2) + r^2\,] + 2(\sigma-1)^{-1}\ln r & (i = 2) \end{cases}$$

Define the following utility difference functions at $\mathbf{h}^{**} = (4h, 0, 0, 0)$:

---

[9] For the proofs of these properties, see, for example, Gray (2006).

$$v_{01}(r) \equiv v_0(\mathbf{h}^{**}) - v_1(\mathbf{h}^{**}) = 2(1+h)r - 1 - (1+2h)r^2 - 2h\sigma(\sigma-1)^{-1}(r\ln r)$$
$$v_{02}(r) \equiv v_0(\mathbf{h}^{**}) - v_2(\mathbf{h}^{**}) = 2(1+2h)r^2 - 1 - (1+4h)r^4 - 4h\sigma(\sigma-1)^{-1}(r^2\ln r^2)$$

After a tedious calculation, we can show that:

$$v_{0i}(r)\begin{cases} < 0 \text{ for } 0 < r < r_{0i}^{**} \\ \geq 0 \text{ for } r_{0i}^{**} \leq r < 1 \end{cases} \quad (i=1,2), \qquad \text{and} \quad 0 < r_{01}^{**} < r_{02}^{**}$$

where r_{0i}^{*} is the solution of $v_{0i}(r) = 0 \quad (i=1,2)$. That is, the equilibrium condition for $\mathbf{h}^{**}$, $v_0(\mathbf{h}^{**})$ $= \max_k\{v_k(\mathbf{h}^{**})\}$, is satisfied for any r larger than r_{02}^{**}; that is, $r = r_{02}^{**}$ is the sustain point for $\mathbf{h}^{**}$.

Appendix 5 Proof of Lemma 1

The following two lemmas help us prove Lemma 1:

Lemma A.1: *All the submatrices* $\mathbf{V}^{(ij)}$ *and* $\mathbf{J}^{(ij)}$ $(i, j = 0,1)$ *defined in (4.7) are circulants.*

Lemma A.2: *The eigenvalues* g_k^* $(k = 0,1,2,3)$ *of the Jacobian matrix* $\nabla\mathbf{F}(\mathbf{h}^*)$ *at* $\mathbf{h}^* = [2h,\ 0,\ 2h,\ 0]$ *are represented as* $g_2^* = \theta e_1^{(00)} - 1$ *and* $g_k^* = -1$ $(k = 0,1,3)$, *where* $\mathbf{e}^{(00)} \equiv [e_0^{(00)}, e_1^{(00)}]^{\mathrm{T}}$ *denotes the eigenvalues of the Jacobian matrix* $\mathbf{V}^{(00)}$.

Proof of Lemma A.1: We prove that each of the submatrices $\mathbf{J}^{(ij)}$ and $\mathbf{V}^{(ij)}$ is a circulant in turn.

(1) Let each of $p_{(0)}$ and $p_{(1)}$ denote the location choice probability of the subset of regions C_0 $(i = 0)$ and C_1 $(i = 1)$ at the agglomerating pattern $\mathbf{h}^*$, respectively:

$$p_{(i)} \equiv \frac{\exp[-\theta v_i(\mathbf{h}^*)]}{2\{\exp[-\theta v_0(\mathbf{h}^*)] + \exp[-\theta v_1(\mathbf{h}^*)]\}} = \begin{cases} (1-\varepsilon)/2 & (i=0) \\ \varepsilon/2 & (i=1) \end{cases} \tag{A5.1}$$

A straightforward calculation of the definition (4.7) of $\mathbf{J}^{\times}(\mathbf{h}^*) \equiv \mathbf{P}\mathbf{J}(\mathbf{h}^*)\mathbf{P}^{\mathrm{T}}$ yields:

$$\mathbf{J}^{(ij)} = \theta\, p_{(i)}\mathbf{p}^{(ij)} \quad \text{where} \quad \mathbf{p}^{(ij)} = \begin{bmatrix} \delta_{ij} - p_{(j)} & -p_{(j)} \\ -p_{(j)} & \delta_{ij} - p_{(j)} \end{bmatrix} \quad (i, j = 0,1)$$

This explicitly shows that the submatrices $\mathbf{J}^{(ij)}$ $(i, j = 0,1)$ are circulants generated from:

$$\mathbf{J}_0^{(ij)} = \begin{cases} \theta\, p_{(i)}[1 - p_{(j)}, -p_{(j)}] & (i = j) \\ \theta\, p_{(i)}[-p_{(j)}, -p_{(j)}] & (i \neq j) \end{cases} \tag{A5.2}$$

(2) As is shown in (A1.2), the Jacobian matrix $\nabla\mathbf{v}(\mathbf{h}^*)$ consists of additions and multiplications of $\mathbf{M}(\mathbf{h}^*) \equiv \mathbf{D}\{\boldsymbol{\Delta}(\mathbf{h}^*)\}^{-1}$. It follows from this that the Jacobian matrix $\nabla^{\times}\mathbf{v}(\mathbf{h}^*) \equiv \mathbf{P}\nabla\mathbf{v}(\mathbf{h}^*)\mathbf{P}^{\mathrm{T}}$ in the new coordinate system consists of those of $\mathbf{P}\mathbf{M}(\mathbf{h}^*)\mathbf{P}^{\mathrm{T}}$, which in turn is composed of submatrices $\mathbf{M}^{(ij)}$ $(i, j = 0,1)$:

$$\mathbf{P}\mathbf{M}(\mathbf{h}^*)\mathbf{P}^{\mathrm{T}} \equiv (2h)^{-1}\begin{bmatrix} \mathbf{M}^{(00)} & \mathbf{M}^{(01)} \\ \mathbf{M}^{(10)} & \mathbf{M}^{(11)} \end{bmatrix}$$

Therefore, in order to prove that $\mathbf{V}^{(ij)}$ $(i, j = 0,1)$ are circulants, it suffices to show that the submatrices $\mathbf{M}^{(ij)}$ $(i, j = 0,1)$ are circulants. Note here that $\mathbf{P}\mathbf{M}(\mathbf{h}^*)\mathbf{P}^{\mathrm{T}}$ can be represented as:

$$\mathbf{P}\mathbf{M}(\mathbf{h}^*)\mathbf{P}^{\mathrm{T}} = [\mathbf{P}\mathbf{D}\mathbf{P}^{\mathrm{T}}][\mathbf{P}\{\boldsymbol{\Delta}(\mathbf{h}^*)\}^{-1}\mathbf{P}^{\mathrm{T}}] \tag{A5.3}$$

The first bracket of the right-hand side of (A5.3) is given in (4.6), and a simple calculation of the second bracket yields:

$$\mathbf{P}\{\boldsymbol{\Delta}(\mathbf{h}^*)\}^{-1}\mathbf{P}^{\mathrm{T}} = (2h)^{-1}\{\mathrm{diag}[d_{(0)}, d_{(0)}, d_{(1)}, d_{(1)}]\}^{-1}$$

where $d_{(0)} \equiv 1+r^2$ and $d_{(1)} \equiv 2r$. Thus, we have:

$$\mathbf{P}\mathbf{M}(\mathbf{h}^*)\mathbf{P}^{\mathrm{T}} = (2h)^{-1}\left[\begin{array}{c:c} d_{(0)}^{-1}\mathbf{D}_{(0)} & d_{(0)}^{-1}\mathbf{D}_{(1)} \\ \hdashline d_{(1)}^{-1}\mathbf{D}_{(1)} & d_{(1)}^{-1}\mathbf{D}_{(0)} \end{array}\right] \tag{A5.4}$$

which shows that the submatrices $\mathbf{M}^{(ij)}$ $(i,j=0,1)$ are circulants.

<u>Proof of Lemma A.2</u>: We prove that the eigenvalues of the Jacobian $\nabla\mathbf{F}(\mathbf{h}^*)$ are given by $[0,0,\theta e_1^{(00)},0]^{\mathrm{T}} - \mathbf{1}$ when $p_{(0)} \to 1/2$. Consider the Jacobian $\nabla^{\times}\mathbf{F}(\mathbf{h}^*)$ in the new coordinate system:

$$\nabla^{\times}\mathbf{F}(\mathbf{h}^*) \equiv \mathbf{P}\nabla\mathbf{F}(\mathbf{h}^*)\mathbf{P}^{\mathrm{T}} = H\,\mathbf{J}^{\times}(\mathbf{h}^*)\nabla^{\times}\mathbf{v}(\mathbf{h}^*) - \mathbf{I} \equiv 2\left[\begin{array}{c:c} \mathbf{F}^{(00)} & \mathbf{F}^{(01)} \\ \hdashline \mathbf{F}^{(10)} & \mathbf{F}^{(11)} \end{array}\right] - \mathbf{I} \tag{A5.5a}$$

$$\mathbf{F}^{(ij)} \equiv \sum\nolimits_{k=0,1} \mathbf{J}^{(ik)}\mathbf{V}^{(kj)} \qquad (i,j=0,1) \tag{A5.5b}$$

Note here that the submatrices $\mathbf{F}^{(ij)}$ of the Jacobian matrix $\nabla^{\times}\mathbf{F}(\mathbf{h}^*)$ are circulants because all submatrices $\mathbf{J}^{(ij)}$ and $\mathbf{V}^{(ij)}$ $(i,j=0,1)$ are circulants. This enables us to diagonalize each of the submatrices $\mathbf{F}^{(ij)}$ by using a 2-by-2 DFT matrix $\mathbf{Z}_{[2]}$:

$$(\mathbf{Z}^{\times})^{-1}\nabla^{\times}\mathbf{F}\,\mathbf{Z}^{\times} = 2\left[\begin{array}{c:c} \mathrm{diag}(\mathbf{f}^{(00)}) & \mathrm{diag}(\mathbf{f}^{(01)}) \\ \hdashline \mathrm{diag}(\mathbf{f}^{(10)}) & \mathrm{diag}(\mathbf{f}^{(11)}) \end{array}\right] - \mathbf{I} \tag{A5.6}$$

where $\mathbf{f}^{(ij)}$ is the eigenvalues of $\mathbf{F}^{(ij)}$, and $\mathbf{Z}^{\times} \equiv \mathrm{diag}[\mathbf{Z}_{[2]}, \mathbf{Z}_{[2]}]$. It also follows that applying the similarity transformation based on $\mathbf{Z}_{[2]}$ to both sides of (A5.5b) yields:

$$\mathbf{f}^{(ij)} = \sum\nolimits_{k=0,1} [\boldsymbol{\delta}^{(ik)}]\cdot[\mathbf{e}^{(kj)}] \qquad (i,j=0,1) \tag{A5.7}$$

where $\boldsymbol{\delta}^{(ij)} \equiv [\delta_0^{(ij)}, \delta_1^{(ij)}]^{\mathrm{T}}$ and $\mathbf{e}^{(ij)} \equiv [e_0^{(ij)}, e_1^{(ij)}]^{\mathrm{T}}$ denote the eigenvalues of the Jacobian matrices $\mathbf{J}^{(ij)}$ and $\mathbf{V}^{(ij)}$ $(i,j=0,1)$, respectively. The former eigenvalues $\boldsymbol{\delta}^{(ij)}$ can be given analytically by the DFT of the first row vector $\mathbf{J}_0^{(ij)}$ of the submatrix $\mathbf{J}^{(ij)}$:

$$\boldsymbol{\delta}^{(ij)} = \mathbf{Z}_{[2]}\mathbf{J}_0^{(ij)} \qquad (i,j=0,1) \tag{A5.8}$$

Substituting (A5.1) and (A5.2) into (A5.8), we have:

$$\boldsymbol{\delta}^{(00)} = (\theta/2)(1-\varepsilon)[\varepsilon \quad 1]^{\mathrm{T}}, \qquad \boldsymbol{\delta}^{(11)} = (\theta/2)\varepsilon[1-\varepsilon \quad 1]^{\mathrm{T}}$$

$$\boldsymbol{\delta}^{(01)} = (\theta/2)(1-\varepsilon)[-\varepsilon \quad 0]^{\mathrm{T}}, \quad \boldsymbol{\delta}^{(10)} = (\theta/2)\varepsilon[\varepsilon-1 \quad 0]^{\mathrm{T}}$$

It follows from this that:

$$\lim_{p_{(0)}\to 1/2} \boldsymbol{\delta}^{(ij)} = \lim_{\varepsilon\to 1/2} \boldsymbol{\delta}^{(ij)} = \begin{cases} (\theta/2)[0 \quad 1]^{\mathrm{T}} & \text{if } i=j=0 \\ \mathbf{0} & \text{otherwise} \end{cases}$$

Substituting these $\boldsymbol{\delta}^{(ij)}$ $(i,j=0,1)$ into (A5.7) yields:

$$\lim_{p_{(0)}\to 1/2} \mathbf{f}^{(ij)} = \begin{cases} (\theta/2)[0 \quad e_1^{(0j)}]^{\mathrm{T}} & \text{if } i=0 \\ \mathbf{0} & \text{if } i=1 \end{cases} \qquad (j=0,1)$$

Thus, when $p_{(0)} \to 1/2$, (A5.6) reduces to:

$$(\mathbf{Z}^{\times})^{-1}\nabla^{\times}\mathbf{F}\,\mathbf{Z}^{\times} = \theta\left[\begin{array}{cc:cc} 0 & 0 & 0 & 0 \\ 0 & e_1^{(00)} & 0 & e_1^{(01)} \\ \hdashline \multicolumn{2}{c:}{\mathbf{0}} & \multicolumn{2}{c}{\mathbf{0}} \end{array}\right] - \mathbf{I}$$

Converting this into the original coordinate system, we obtain:

$$\mathbf{P}^{\mathrm{T}}[(\mathbf{Z}^{\times})^{-1}\nabla^{\times}\mathbf{F}\,\mathbf{Z}^{\times}]\,\mathbf{P} = (\mathbf{Z}^{*})^{-1}\nabla\mathbf{F}\,\mathbf{Z}^{*} = \theta\left[\begin{array}{c:cc} \mathbf{0} & \multicolumn{2}{c}{\mathbf{0}} \\ \hdashline \mathbf{0} & e_1^{(00)} & e_1^{(01)} \\ & 0 & 0 \end{array}\right] - \mathbf{I}$$

where $\mathbf{Z}^{*} \equiv \mathbf{P}^{\mathrm{T}}\mathbf{Z}^{\times}\mathbf{P}$. Since eigenvalues of an upper-triangular matrix are given by the diagonal entries, we can conclude that the eigenvalues of the Jacobian $\nabla\mathbf{F}(\mathbf{h}^{*})$ are given by $[0,0,\theta e_1^{(00)},0]^{\mathrm{T}} - \mathbf{1}$.

<u>Proof of Lemma 1</u>: Substituting (A1.2) and (A5.4) into the definition of $\nabla^{\times}\mathbf{v}(\mathbf{h}^{*})$ in (4.7), we see that the Jacobian matrix $\mathbf{V}^{(00)}$ consists of additions and multiplications of $\mathbf{D}^{(0)}$ and $\mathbf{D}^{(1)}$:

$$\mathbf{V}^{(00)} \equiv b[\mathbf{D}^{(0)}/d_{(0)}] - \{a^{*}[\mathbf{D}^{(0)}/d_{(0)}]^{2} + a_{(1)}^{*}[\mathbf{D}^{(1)}/d_{(1)}]^{2}\}$$

where $d_{(0)} \equiv 1+r^{2}$, $d_{(1)} \equiv 2r$, $a^{*} \equiv \sigma^{-1}(1+(2h)^{-1})$, $a_{(1)}^{*} \equiv \sigma^{-1}(2h)^{-1}$, and $b \equiv (\sigma-1)^{-1} + \sigma^{-1}$. Since $\mathbf{D}^{(0)}$ and $\mathbf{D}^{(1)}$ are circulants, we have the following expressions for the eigenvalues $\mathbf{e}^{(00)}$ of the Jacobian $\mathbf{V}^{(00)}$:

$$\mathbf{e}^{(00)} = b\,\mathbf{f}_{(0)} - \{a^{*}(\mathbf{f}_{(0)})^{2} + a_{(1)}^{*}(\mathbf{f}_{(1)})^{2}\} \equiv \begin{bmatrix} e_0^{(00)} \\ e_1^{(00)} \end{bmatrix} \tag{A5.9}$$

where each vector $\mathbf{f}_{(i)}$ $(i = 0,1)$ is the eigenvalues of $\mathbf{D}^{(i)}/d_{(i)}$, each of which is obtained by DFT of vectors $\mathbf{d}_0^{(i)}$:

$$\mathbf{f}_{(0)} = \frac{1}{d_{(0)}}\mathbf{Z}_{[2]}\mathbf{d}_0^{(0)} = \frac{1}{1+r^{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ r^{2} \end{bmatrix} = \begin{bmatrix} 1 \\ c(r^{2}) \end{bmatrix} \tag{A5.10a}$$

$$\mathbf{f}_{(1)} = \frac{1}{d_{(1)}}\mathbf{Z}_{[2]}\mathbf{d}_0^{(1)} = \frac{1}{2r}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} r \\ r \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{A5.10b}$$

Substituting (A5.10) into (A5.9) yields:

$$e_0^{(00)} = 0, \quad e_1^{(00)} = b\,x - a^{*}x^{2}, \text{ where } x \equiv c(r^{2}) \tag{A5.11}$$

Combining (A5.11) and Lemma A.2, we obtain Lemma 1.