The Genuine Saving Criterion and the Value of Population in an Economy with Endogenous Population Changes

Kenneth J. Arrow and Alain Bensoussan and Qi Feng and Suresh P. Sethi

University of Texas at Dallas

18. January 2008

The Genuine Saving Criterion and the Value of Population in an Economy with Endogenous Population Changes

Kenneth J. Arrow

Department of Economics, Stanford University
Stanford, CA 94305.

Alain Bensoussan

International Center for Decision and Risk Analysis
School of Management, The University of Texas at Dallas
Richardson, TX 75083

Qi Feng

McCombs School of Business, The University of Texas at Austin
Austin, TX 78712

Suresh P. Sethi

Center for Intelligent Supply Networks
School of Management, The University of Texas at Dallas
Richardson, TX 75083

January 18, 2008

Abstract

We study an economy in which the rate of change of population depends on population policy decisions. This requires population as well as capital as state variables. By showing the algebraic relationship between the shadow price of the population and the shadow price of the per capita capital stock, we are still able to depict the optimal path and its convergence to the long-run equilibrium on a two-dimensional phase diagram. Moreover, we derive explicitly the expression of genuine savings in our model to evaluate the sustainability of the system.

Key Words: Savings, population policy, value of the population, economic growth, optimal control, phase diagram, dynamic programming.

1. Introduction

This paper extends the analysis of Arrow, Dasgupta, and Mäler (2003), who study a one-sector model of an economy with exogenous non-exponential population growth. Arrow et al. (2003) provide an analysis of the role of varying population in the measurement of savings. This is accomplished by recognizing population as another form of capital and formulated as a state variable of the system in its optimal control formulation.

Subsequently, Arrow, Bensoussan, Feng, and Sethi (2007) has provided a thorough analysis of the problem formulated in Arrow et al. (2003). By showing that the co-state of the population is only algebraically related to the co-state of the capital stock, they develop a two-dimensional phase diagram of the problem. Monotone properties of the optimal trajectories and a computation algorithm are also discussed in that paper.

Our objective in this study is to extend the analysis in Arrow et.al (2007) to an economy where population change is endogenous. We do this by introducing population policy measures as decisions in addition to consumption/investment decisions over time. While our main purpose is to develop a methodology of analysis, we do this in the case of a country with naturally declining population. We should mention here that population is already declining in Japan and Germany, and other countries such as Italy and Spain are expected to soon follow suit. Now imagine that the country under consideration is interested in encouraging indigenous population growth by such measures as education and baby bonuses. We will term such expenditures as population policy expenditures. Thus, in our model, the output at each instant needs to be optimally allocated between consumption, population policy measures, and investment.

Even though the population change is endogenous in our model, we continue to follow the tradition of "total utilitarianism" articulated by Henry Sidgwick and Francis Edgeworth in the 1870s. Thus, we maximize the integral of the total societal consumption utility over time. We are well aware of the ethical issues it raises. In particular, it leads to the view that, if the costs of encouraging population growth is low, then the ideal can be a very large population with very low per capita consumption. A contemporary philosopher, Derek Parfit (1984) has termed this the *repugnant conclusion*. But Parfit also raises a similar argument against average utilitarian standards. According to him, it may also lead to absurd results. In this paper, we try to avoid the repugnant conclusion of Parfit by putting an upper bound on the population growth rate. It is even possible to choose a zero growth rate as the upper bound.

We use dynamic programming approach to solve the problem. A steady state analysis is conducted, which represents a nontrivial extension of the analysis in the classical case of the exponential growth of the population (see Arrow and Kurz 1970). The analysis involves a study of a system of differential equations in capital and its co-state as a function of the population. We show that it is not the population itself, but its rate of growth that reaches a steady state. Of course, this rate may be negative, positive, or zero depending on the parameters of the problem. Using the algebraic relation between the co-states of the population and the capital stock, we are able to analyze the optimal trajectory in a two-dimensional phase diagram involving only the capital stock and its co-state. Our phase diagram reveals a similar structure to that in the classical model of Arrow of Kurz (1970). Furthermore, we show that both the optimal expenditure on population policy and the optimal consumption increase with the capital stock. The co-state of the population also increases with the capital stock.

The plan of this paper is as follows. In Section 2, we develop the notation and the model. Here the state variables are aggregate capital and population. The control variables are consumption and population policy expenditures. The objective is to maximize the present value of the society's utility of consumption over time. The model is transformed to per capita variables in Section 2.1. In Section 3, we use dynamic programming to study the problem. The steady state analysis is carried out in Section 4. We perform a phase diagram analysis in Section 5. In Section 6, we relate our analysis to the maximum principle formulation of the problem. We also obtain the expressions for genuine savings and conditions for sustainability. Section 7 concludes the paper.

2. Model Description

We introduce the following notation used in the paper:

$K(t)$: total stock of capital: a state variable;

$N(t)$:	population: a state variable;
$k(t)$:	per capita stock of capital: a state variable;
$c(t)$:	consumption per capita: a control variable;
$m(t)$:	population policy expenditure per capita: a control variable;
M :	upper bound on the rate of population policy expenditure, $M \geqslant 0$;
$F(K, N)$:	production function, concave with constant returns to scale;
$u(c)$:	utility of consumption, $u(0) = 0,\ u'(0) = \infty,\ u'(c) > 0$ and $u''(c) < 0$ for $c > 0$;
δ :	population decay rate;
r :	discount rate
$g(m)$:	population change rate, $g'(m) > 0,\ g''(m) < 0,\ g(M) > \delta$;
$f(k)$:	per capita production function, $f(0) = 0,\ f(k) > 0,\ f'(k) > 0$, and $f''(k) < 0$ for $k > 0$; $f'(0) > r$ and $f'(\infty) < r$.

We consider a one-sector economy in which the stock of capital $K(t)$ and population $N(t)$ are two state variables. We do not distinguish between population and labor force for convenience in exposition. The output rate $F(K, N)$ of the economy depends on the capital stock K and the population, or labor force, N. Let $c(t)$ be the rate of individual consumption, assumed to be same for all. We will refer to it simply as the per capita consumption rate. We also use $m(t)$ to denote the per capita expenditure on population policy measures. Then the capital stock dynamics is

$$\dot{K} = F(K, N) - Nc - Nm, \quad K(0) = K_0, \tag{1}$$

It is important to note that the population N enters the dynamics in a nontrivial way.

As for the evolution of population over time, we assume that it is affected by population policy expenditure m and that it is independent of consumption c. Specifically, the population N is assumed to grow at the rate of $g(m) - \delta$. Then $g(0) - \delta$ is the natural change rate of the population without population policy expenditure. Thus the population change equation is

$$\dot{N} = [g(m) - \delta]N, \quad N(0) = N_0. \tag{2}$$

In this paper, we assume $g(0) < \delta$ so that the natural rate of population change is negative. We should mention, however, that this is not a mathematical requirement, and the results derived in the paper go through without this assumption.

For each individual in the society, the rate of utility for consuming c units per unit time is $u(c)$. In the tradition of total utilitarianism, which argues for treating people more or less equally, the objective becomes one of maximizing the total utility of the society given by

$$J(c(\cdot), m(\cdot)) = \int_0^{+\infty} e^{-rt} N u(c) dt. \tag{3}$$

Note that in (3), we have weighted people by their futurity (discounting) but not according to number of their contemporaries.

The problem is to select $c(t) \geqslant 0$ and $0 \leqslant m(t) \leqslant M,\ t \geqslant 0$, so as to maximize $J(c(\cdot), m(\cdot))$, subject to the condition that $K(t) \geqslant 0,\ t \geqslant 0$.

2.1 Per Capita Model

Let k denote the per capita capital stock K/N. Since we have assumed that the production function $F(K, N)$ is concave with constant returns to scale, we have

$$F(K, N) = NF\left(\frac{K}{N}, 1\right) = NF(k, 1) \triangleq Nf(k).$$

Notice that

$$\dot{k} = \frac{\dot{K}}{N} - K\frac{\dot{N}}{N^2} = f(k) - c - m - k[g(m) - \delta].$$

Then the state equations (1) can be rewritten as follows:

$$\dot{k} = f(k) - k[g(m) - \delta] - c - m, \quad k(0) = k_0 = K_0/N_0, \tag{4}$$

We use dynamic programming for our analysis. As is standard, we shall let $k(0) = k$ and $N(0) = N$. We shall also assume $M = \infty$ for ease of exposition. Then we can write the value function as

$$v(k, N) = \max_{\substack{c(\cdot)\geqslant 0 \\ m(\cdot)\geqslant 0}} \int_0^{\infty} e^{-rt} N(t) u(c(t)) dt, \tag{5}$$

subject to

$$\dot{k} = f(k) - k[g(m) - \delta] - c - m, \quad k(0) = k, \tag{6}$$

$$\dot{N} = N[g(m) - \delta], \quad N(0) = N. \tag{7}$$

The initial conditions k and N are of course positive. We expect $k(t) \geqslant 0, \forall t$. We need to impose the restriction that $c(t) = 0$ and $m(t) = 0$ when $k(t) = 0$. But it is known that since $u'(0) = \infty$, we must have $c(t) > 0$. This implies that $k(t) > 0$. Note that

$$v(k, N) = N \max_{\substack{c(\cdot)\geqslant 0 \\ m(\cdot)\geqslant 0}} \int_0^\infty e^{-[r+\delta-g(m(t))]} u(c(t))dt. \tag{8}$$

In the classical exponential growth case $g(m) - \delta = \nu$, a constant, the condition $r > g(0) - \delta$ is required for the value function to be finite. In the absence of this condition, the discount rate is less than or equal to the rate of the population growth, and the value function $v(k, N)$ becomes infinite for $k > 0$, $N > 0$. The generalization of the condition $r > \nu$ in our case is the condition that

$$\int_0^\infty e^{-rt} N(t)dt < \infty, \tag{9}$$

where $N(t)$ is the solution of (7).

3. Bellman Equation

The dynamic programming (DP) or the Bellman equation corresponding to the optimal control problem (5), (6) and (7) is

$$rv = v_k[f(k) + k\delta] - v_N N\delta + \max_{c,m}\{v_k[-kg(m) - c - m] + Nu(c) + v_N N g(m)\}. \tag{10}$$

From the expression (8), we look for a solution of the form

$$v(k, N) = NW(k), \tag{11}$$

where $W(k)$ is called the per capita value function (independent of N). Then we have

$$v_k = NW'(k), \ v_N = W(k). \tag{12}$$

Substituting (11) and (12) into (10) and dividing by N gives

$$\begin{aligned}(r + \delta)W(k) \quad &= \quad W'(k)(f(k) + k\delta) + \max_c\{u(c) - cW'(k)\} \\ &\quad + \max_{m\geqslant 0}\{g(m)[W(k) - kW'(k)] - mW'(k)\}.\end{aligned} \tag{13}$$

It is easy to see that the optimal control $\hat{c}(k)$ and $\hat{m}(k)$satisfy

$$u'(\hat{c}) = W'(k), \tag{14}$$

$$g'(\hat{m})[W(k) - kW'(k)] - W'(k) = 0. \tag{15}$$

Since we expect W_k to be finite, we have $\hat{c} > 0$ on account of our assumption that $u'(0) = \infty$; see, e.g., Karatzas et al. (1986) for explanations. In turn, we expect $k(t) > 0$. Note, however, that if $k(0) = 0$, then the optimal consumption is $\hat{c}(t) = 0$ for $t \geqslant 0$.

Dividing (15) by $W'(k)$ and rearranging terms, we obtain

$$k + \frac{1}{g'(m)} = \frac{W(k)}{W'(k)}. \tag{16}$$

Relations (14) and (16) suggest the definitions of the adjoint variables

$$p(k) := W'(k) = u'(c), \tag{17}$$

$$\psi(k) := \frac{W(k)}{W'(k)} = k + \frac{1}{g'(m)}. \tag{18}$$

In view of the Envelope Theorem[1], we can differentiate the Bellman equation (13) with respect to k, and obtain the adjoint equation

$$p'(k)[f(k) + k\delta] + p(k)[f'(k) + \delta] - (\delta + r)p(k) - cp'(k) - g(m)kp'(k) - mp'(k) = 0, \tag{19}$$

which can be written as

$$p'(k) = \frac{p(k)[r - f'(k)]}{f(k) - k[g(m) - \delta] - c - m}. \tag{20}$$

Furthermore, differentiating (18) with respect to k and using (17), (18) and (20) gives

$$\begin{aligned} \psi'(k) &= \frac{W'(k)^2 - W(k)W''(k)}{W'(k)^2} = 1 - \frac{\psi(k)}{p(k)}p'(k) \\ &= 1 - \frac{\psi(k)[r - f'(k)]}{f(k) - k[g(m) - \delta] - c - m} \\ &= \frac{\psi(k)[f'(k) - r] + f(k) - k[g(m) - \delta] - c - m}{f(k) - k[g(m) - \delta] - c - m}. \end{aligned} \tag{21}$$

Next we show that $p(k)$ and $\psi(k)$ are linked by an algebraic relation. To see this, we substitute (17) and (18) into (13) to obtain the relation

$$p(k) = \frac{\psi(k)p(k)[r + \delta - g(m)] - u(c)}{f(k) - k[g(m) - \delta] - c - m}. \tag{22}$$

[1]See Derzko, Sethi and Thompson (1984) for a proof of the theorem. This theorem is often used on economics, see, e.g., Arrow and Kurz (1970).

Dividing both sides of (22) by $p(k)$, we obtain

$$f(k) - k(g(m) - \delta) - c - m - \psi(k)(\delta + r - g(m)) + L(c) = 0, \tag{23}$$

where

$$L(c) = \frac{u(c)}{u'(c)} = \frac{u(c)}{p(k)}. \tag{24}$$

The quantity $L(c)$ is interpreted as *value of life*; see Section 6 for explainations.

From (21) and (23), we get

$$\psi'(k) = \frac{\psi(k)[f'(k) + \delta - g(m)] - L(c)}{f(k) - k[g(m) - \delta] - c - m}. \tag{25}$$

In (17), we express c in terms of p. Since $\psi(k)$ is related to $p(k)$ algebraically, we can also express m in terms of p. In fact, we can use (17) and (18) in (23) to obtain

$$\begin{aligned} m + \frac{r + \delta - g(m)}{g'(m)} &= f(k) - rk - c + L(c) \\ &= f(k) - rk - u'^{-1}(p) + u(u'^{-1}(p))/p. \end{aligned} \tag{26}$$

4. Steady State Analysis

The initial condition is obtained at the steady state for which the numerator and the denominator of the right-hand side of the differential equation (20) vanish. In doing so, we must also observe the maximization conditions (17) and (26). These provide us with four equations in k, p, c and m. By bringing in the condition (18), we can rewrite the steady state relations as follows:

$$f'(k) = r, \tag{27}$$

$$f(k) - (g(m) - \delta)k - c - m = 0, \tag{28}$$

$$u'(c) = p, \tag{29}$$

$$p\psi(g(m) - \delta - r) + u(c) = 0, \tag{30}$$

$$\psi = k + \frac{1}{g'(m)}. \tag{31}$$

(32)

If there exists a solution of these equations, then this solution, denoted as $\bar{k}$, $\bar{c}$, $\bar{m}$, $\bar{\psi}$, and $\bar{p}$, represents the steady state values of the per capita capital, the consumption rate, the population policy expenditure rate, and the marginal valuations p and ψ. Note that the population $N(t)$ does not have a constant value in the steady state. Rather, it grows at the constant rate of $g(\bar{m}) - \delta$.

We now attend to the question of the existence of a solution to the system of steady state relations (27)-(31). This analysis will also provide us with the conditions required for existence. We shall treat two cases: $M = 0$ and $M > 0$.

4.1 The Classical Model of Exponential Population Growth and No Population Policy: The Case $M = 0$

With $M = 0$, equation (7) reduces to

$$\dot{N} = [g(0) - \delta]N, \ N(0) = N_0.$$

The usual assumption is that $r > g(0) - \delta$, so that the objective function J remains bounded.

In this case, only (27), (28) and (29) are relevant with $m = 0$. These relations reduce to

$$f'(k_\infty) = r, \ f(k_\infty) - [g(0) - \delta]k_\infty - c_\infty = 0, \ u'(c_\infty) = p_\infty,$$

where k_∞, c_∞ and p_∞ are the equilibrium values of per capita capital stock, per capita consumption, and the costate variable associated with the capital.

The analysis of this classical economic growth model is well known (see, e.g., Arrow and Kurz (1970)), and will not be repeated here.

Next we study a model with population policy measures. We introduce assumptions that simplify the exposition. More general cases can also be analyzed; their analysis is similar but tedious.

4.2 The Model with Population Policy: The Case $M > 0$

This is a case with $M > 0$. From (27) and the conditions on $f(k)$, we obtain

$$\bar{k} = f'^{-1}(r) > 0. \tag{33}$$

From (29) and (31), we can obtain $\bar{p}$ and $\bar{\psi}$ in terms of $\bar{k}$, $\bar{c}$ and $\bar{m}$, if $\bar{c}$ and $\bar{m}$ exist. Thus we need to study only the existence of $\bar{c}$ and $\bar{m}$.

Let us define the function

$$c(m) = f(\bar{k}) + \bar{k}\delta - \bar{k}g(m) - m. \tag{34}$$

corresponding to (28). Using (24) and (31) in (30), we obtain

$$L(c(m)) = [\delta + r - g(m)] \left[\bar{k} + \frac{1}{g'(m)}\right], \tag{35}$$

which is an algebraic equation for m. From (8), it is desirable to have $\delta + r - g(m) > 0$ at the equilibrium so that the value function is bounded. Thus, we look for solutions such that the right-hand side of (35) is positive, which implies $L(c(m)) > 0$. In view of $c > 0$ and therefore $u'(c) > 0$, this condition on L implies $u(c(m)) > 0$. Thus define $\underline{c} \geqslant 0$ such that $u(\underline{c}) = 0$, so that $c(m) > \underline{c}$.

Remark 4.1 For $u(c) = c^\gamma,\ 0 < \gamma < 1$, we have $\underline{c} = 0$. For $u(c) = \ln c,\ \underline{c} = 1$.

Substituting from (34) into (35), we get

$$L(f(\bar{k}) + \bar{k}\delta - \bar{k}g(m) - m) = [\delta + r - g(m)] \left[\bar{k} + \frac{1}{g'(m)}\right]. \tag{36}$$

This is an algebraic equation that yields $\bar{m}$ and $\bar{c} = c(\bar{m})$. To solve (36), define

$$\phi(m) = L(f(\bar{k}) + \bar{k}\delta - \bar{k}g(m) - m) - [\delta + r - g(m)] \left[\bar{k} + \frac{1}{g'(m)}\right]. \tag{37}$$

Then,

$$\phi'(m) = -[\bar{k}g'(m) + 1][L'(f(\bar{k}) + \bar{k}\delta - \bar{k}g(m) - m) - 1] + [\delta + r - g(m)]\frac{g''(m)}{g'^2(m)}. \tag{38}$$

In view of (8) and (9), if $r + \delta - g(m) \leqslant 0$ in the steady state, the optimal value function in (8) becomes infinite. Thus, we focus on the case when $r + \delta - g(M) > 0$ so that $r + \delta - g(m) > 0$ for $0 \leqslant m \leqslant M$.

We can now prove the following result.

Theorem 4.1 *Assume*

$$r > g(0) - \delta, \quad c(M) < \underline{c}, \tag{39}$$

$$\phi(0) = L(f(\bar{k}) + \bar{k}\delta - \bar{k}g(0)) - [\delta + r - g(0)] \left[\bar{k} + \frac{1}{g'(0)}\right] > 0. \tag{40}$$

Then there exists a unique $\hat{m} < M$*, such that*

$$c(\hat{m}) = f(\bar{k}) + \bar{k}\delta - \bar{k}g(\hat{m}) - \hat{m} = \underline{c}. \tag{41}$$

Assume $r+\delta-g(\hat{m}) > 0$*. Then there exists a solution* $(\bar{k}, \bar{c}, \bar{m}, \bar{\psi}, \bar{p})$ *of* (27)-(31)*. Moreover,* $\bar{c} > \underline{c} > 0,\ 0 < \bar{m} < \hat{m},\ \bar{p} > 0$ *and* $\bar{\psi} > 0$*. Furthermore, in the interval* $[0,\hat{m}]$*,* $\hat{m}$ *is uniquely defined and the other steady state values are also unique.*

Proof: From (39) and (40), we can easily conclude that $u(c(0)) > 0$. Hence, $c(0) > \underline{c}$. But $c(M) < \underline{c}$ and $c'(m) = -\bar{k}g'(m) - 1 < 0$. Therefore, $\hat{m} > 0$ is uniquely defined.

Using the definition (24) of $L(c)$ we have

$$L'(c) = 1 - \frac{u(c)u''(c)}{u'^2(c)}. \tag{42}$$

Substituting (42) into (38), we obtain

$$\phi'(m) = -[\bar{k}g'(m)+1]\left[-\frac{u(c(m))u''(c(m))}{u'^2(c(m))}\right] + [\delta + r - g(m)]\frac{g''(m)}{g'^2(m)}.$$

Since $\delta + r - g(m) > 0$ for $0 \leqslant m < \hat{m}$ by assumption, and $g''(m) < 0$, we have $\phi'(m) < 0$ for $0 \leqslant m < \hat{m}$.

From (41) and the definition of $\underline{c}$, we have

$$u(f(\bar{k}) + \bar{k}\delta - \bar{k}g(\hat{m}) - \hat{m}) = 0. \tag{43}$$

Since $c(\hat{m}) = \underline{c}$ and $c'(m) < 0$, we can conclude that for $0 < m < \hat{m}$, we have $c(m) > \underline{c} > 0$ and, therefore,

$$u(f(\bar{k}) + \bar{k}\delta - \bar{k}g(m) - m) > 0, \quad 0 < m < \hat{m}. \tag{44}$$

From (24), (37) and (43), we have

$$\phi(\hat{m}) = -[\delta + r - g(\hat{m})]\left[\bar{k} + \frac{1}{g'(\hat{m})}\right] < 0. \tag{45}$$

From (40), (45) and $\phi'(m) < 0$, there exists a unique $\bar{m},\ 0 < \bar{m} < \hat{m}$, such that $\phi(\bar{m}) = 0$. Furthermore, $\bar{c} = c(\bar{m}) > \underline{c}$, and $\bar{p}$ and $\bar{\psi}$ obtained uniquely from (29) and (31) satisfy $\bar{p} > 0$ and $\bar{\psi} > 0$. $\square$

The condition $r + \delta - g(m) > 0$ for $0 \leqslant m < \hat{m}$ means that the population under these population policy measures grows slower than the discount rate, so that the value function remains bounded. Condition (40) is a bit harder to interpret. However, we can provide the following insight into this condition. One can see that with $c = \underline{c}$, $\phi(0) > 0 \Rightarrow f(\bar{k}) - (g(0) - \delta)\bar{k} - \underline{c} > 0$, which in turn implies that $\dot{k} > 0$ at $\bar{k}$ with $m = 0$ and $c = \underline{c}$. Thus, per capita capital increases with the reduced consumption $\underline{c}$ and no population policy.

Finally, the condition $r + \delta - g(\hat{m}) > 0$ allows us to have a steady state which is not explosive. This does not preclude a better solution, where $\bar{m} > \hat{m}$ and the objective function is infinite.

Finally, we note that it is quite possible that $\bar{m}$ is such that $g(\bar{m}) - \delta < 0$, in which case the population decreases to zero as $t \to \infty$.

Thus, we see that the conditions imposed in Theorem 4.1 argue for the steady state population policy expenditure to be between 0 and $\hat{m}$.

4.3 An Example

Consider a model of Section 4.2 with $g(m) = \sqrt{m}$ and $u(c) = \ln c$. Then $\underline{c} = 1$. The condition (40) becomes

$$\phi(0) = [f(\bar{k}) + \bar{k}\delta] \ln(f(\bar{k}) + \bar{k}\delta) - [\delta + r]\bar{k} > 0. \tag{46}$$

This needs $f(\bar{k}) + \bar{k}\delta > 1$. Since we can rewrite (46) as

$$\phi(0) = (f(\bar{k}) + \bar{k}\delta)[\ln(f(\bar{k}) + \bar{k}\delta) - 1] + f(\bar{k}) - r\bar{k},$$

and since $f(\bar{k}) - r\bar{k} > 0$ from (27) and the concavity of $f(k)$, we obtain $\phi(0) > 0$, if for instance $f(\bar{k}) + \bar{k}\delta > e$.

In this case, using (41), we can define $\hat{m}$ to be a solution of

$$f(\bar{k}) + \bar{k}\delta - \bar{k}\sqrt{\hat{m}} - \hat{m} = 1. \tag{47}$$

Since (47) is a quadratic equation in $\sqrt{\hat{m}}$ and since we are assuming $\bar{k}$ to be large enough for (46) to hold, the equation has two real roots. However, we want $g(\hat{m}) = \sqrt{\hat{m}} \geqslant 0$, so we choose the positive root for $\sqrt{\hat{m}}$. This gives

$$\sqrt{\hat{m}} = \frac{\sqrt{\bar{k}^2 + 4(f(\bar{k}) + \bar{k}\delta - 1)} - \bar{k}}{2} > 0. \tag{48}$$

Let us now assume that $\bar{k}$ is such that $\sqrt{\hat{m}} < r + \delta$. Then from Theorem 4.1, we have a steady state population policy expenditure $\bar{m}$ such that $0 \leqslant \bar{m} < \hat{m}$.

5. Phase Diagram Analysis

In this section we analyze the phase diagram of the problem. We first derive the range of the solutions in Section 5.1. Then we determine the optimal trajectory of $p(k)$ by discussing the corresponding differential equation in Section 5.2. In Section 5.3, we define the curves $\dot{k}(t) = 0$ and $\dot{p}(t) = 0$. A numerical example is presented in Section 5.4 to illustrate the results.

5.1 Range of Solutions

From (26) and (17),

$$\Psi(m) := m + \frac{r + \delta - g(m)}{g'(m)} = f(k) - rk - {u'}^{-1}(p) + \frac{u({u'}^{-1}(p))}{p}. \tag{49}$$

Note that

$$\Psi'(m)m_p = -\frac{u({u'}^{-1}(p)))}{p^2}, \tag{50}$$

$$\Psi'(m)m_k = f'(k) - r, \tag{51}$$

$$\Psi'(m) = -\frac{[r + \delta - g(m)]g''(m)}{[g'(m)]^2}. \tag{52}$$

We are interested in a solution m satisfying $r + \delta - g(m) > 0$. In this case, we have the following properties.

Lemma 5.1 *If there is an $m(k, p) \in [0, g^{-1}(r + \delta)]$ that solves (49), then*

$$\Psi'(m) \geqslant 0,$$

$$m_p(k, p) \leqslant 0,$$

$$\begin{cases} m_k(k, p) \geqslant 0 & \text{for } k \leqslant \bar{k}, \\ m_k(k, p) < 0 & \text{for } k > \bar{k}. \end{cases}$$

Since $\Psi(m)$ increases on $0 \leqslant m \leqslant \tilde{m} = g^{-1}(r + \delta)$, the corresponding (k, p) must satisfy

$$\Psi(0) \leqslant f(k) - rk - u'^{-1}(p) + \frac{u({u'}^{-1}(p))}{p} \leqslant \Psi(\tilde{m}). \tag{53}$$

Under the condition (53), there exist a unique solution $m(k,p)$ to (22).

Now define

$$H(p) = \frac{u(u^{'-1}(p))}{p} - u'^{-1}(p).$$

Since $H(p)$ is decreasing in p, the inverse of H exist. Thus, the requirement of (53) is equivalent to

$$H^{-1}(\Psi(\tilde{m}) - f(k) + rk) \leqslant p \leqslant H^{-1}(\Psi(0) - f(k) + rk). \tag{54}$$

5.2 Optimal Trajectory $p(k)$

Substituting the solution $m(k,p)$ of (22) into (20) together with the steady state values $(\bar{k},\bar{p})$ define the optimal trajectory $p(k)$. Note that both the denominator and the numerator in the right-hand side of (20) vanish at the steady state. In order to determine the curve $p(k)$, we also need to derive the value $p'(\bar{k})$.

Let

$$\chi(k,p) := f(k) + k\delta - u^{'-1}(p) - m - kg(m).$$

Then,

$$\begin{aligned}
\chi_k &= f'(k) + \delta - g(m) - (1 + kg'(m))m_k, \\
\chi_p &= -\frac{1}{u''(u'^{-1}(p))} - (1 + kg'(m))m_p.
\end{aligned}$$

For (k,p) in the neighborhood of $(\bar{k},\bar{p})$, we have

$$\begin{aligned}
\chi(k,p) &\approx \chi_k(\bar{k},\bar{p})(k-\bar{k}) + \chi_p(\bar{k},\bar{p})p'(\bar{k})(k-\bar{k}) \\
&= \Big[(r + \delta - g(\bar{m})) - p'(\bar{k})[-\frac{1}{u''(u'^{-1}(\bar{p}))} + (1 + \bar{k}g'(\bar{m}))m_{\bar{p}}]\Big](k-\bar{k}).
\end{aligned}$$

In deriving the last equation, we have used the fact that $m_{\bar{k}} = 0$.

By a perturbation argument, we have

$$p'(\bar{k}) = \frac{-\bar{p}f''(\bar{k})}{r + \delta - g(\bar{m}) - p'(\bar{k})\left[-\frac{1}{u''(u'^{-1}(\bar{p}))} + (1 + \bar{k}g'(\bar{m}))m_{\bar{p}}\right]}. \tag{55}$$

That is,

$$-\left[\frac{1}{u''(u'^{-1}(p))} + (1 + \bar{k}g'(\bar{m}))m_{\bar{p}}\right](p'(\bar{k}))^2 + (r + \delta - g(\bar{m}))p'(\bar{k}) + \bar{p}f''(\bar{k}) = 0.$$

It is easy to see that the above equation has one positive root and one negative root. We take the negative solution of $p'(\bar{k})$ because of the following consideration. With the negative solution, we can prove that the ODE (20) has a smooth solution such that $p'(k) < 0$. This is discussed in Theorem 5.1.

Theorem 5.1 *The optimal trajectory $p(k)$ defined by (20) is decreasing in k within the range (54). Moreover,*

$$\begin{cases} f(k) - k[g(m(k,p(k))) - \delta] - u'^{-1}(p(k)) - m(k,p(k)) > 0 & \text{for } k < \bar{k}, \\ f(k) - k[g(m(k,p(k))) - \delta] - u'^{-1}(p(k)) - m(k,p(k)) < 0 & \text{for } k > \bar{k}. \end{cases}$$

Proof. We first prove the result for $k < \bar{k}$. Define

$$\pi(k) = f(k) - kg(m(k,p(k))) + k\delta - u'^{-1}(p) - m(k,p(k)).$$

Since $p'(\bar{k}) < 0$, we have $p(\bar{k} - \varepsilon) > p(\bar{k})$ for a small positive ε. Also since $\bar{p} > 0$ and $f''(k) < 0$, equation (55) implies

$$\pi'(\bar{k}) = f'(\bar{k}) + \delta - g(\bar{m}) - p'(\bar{k}) \left[-\frac{1}{u''(u'^{-1}(\bar{p}))} + (1 + \bar{k}g'(\bar{m}))m_{\bar{p}} \right] < 0.$$

Therefore the derivative $p'(k)$ at $k - \varepsilon$ is well defined and $p'(k - \varepsilon) < 0$. We can proceed as long as

$$\begin{aligned} \pi'(k) &= f'(k) + \delta - g(\bar{m}) - p'(\bar{k}) \left[\frac{1}{u''(u'^{-1}(p))} + (1 + kg'(m))m_p \right] \\ &\quad + (1 + kg'(m))m_k < 0. \end{aligned} \tag{56}$$

This implies $f(k) - k[g(m(k,p(k))) - \delta] - u'^{-1}(p(k)) - m(k,p(k)) > 0$ and $p'(k) < 0$.

Suppose there is a point $\tilde{k} < \bar{k}$ with $\pi(\tilde{k}) = 0$. Since $\pi(\tilde{k} + \varepsilon) > 0$, we have $\pi'(\tilde{k}) \geqslant 0$. On the other hand, $\pi(\tilde{k}) = 0$ in (20) implies $p'(\tilde{k}) = -\infty$, which, in turn, implies $\pi'(k) = -\infty$ in (56). This leads to a contradiction. Thus, we have proven the result for $k < \bar{k}$.

The result for $k > \bar{k}$ follows in a similar way. The details are omitted here. □

Corollary 5.1 *Along the optimal trajectory $p(k)$ defined by (20), the optimal population policy expenditure rate $m(k, p(k))$ satisfies*

$$\frac{dm(k,p(k))}{dk} \geqslant 0.$$

Moreover, the co-state of the population ψ defined in (18) is also increasing in k.

Proof. From (50), (51) and (20), we have

$$\begin{aligned}\frac{dm}{dk} &= m_k + m_p p'(k) \\ &= \frac{f'(k)-r}{\Psi'(m)} + \frac{u(u'^{-1}(p))/p^2}{\Psi'(m)} \cdot \frac{p(f'(k)-r)}{f(k)-k[g(m)-\delta]-u'^{-1}(p)-m} \\ &= \frac{f'(k)-r}{\Psi'(m)} \cdot \frac{f(k)-kg(m)+k\delta-m-u'^{-1}(p)+L(u'^{-1}(p))}{f(k)-k[g(m)-\delta]-u'^{-1}(p)-m} \\ &= -\frac{p'(k)}{p\Psi'(m)}[f(k)-kg(m)+k\delta-m-u'^{-1}(p)+L(u'^{-1}(p))].\end{aligned}$$

From Theorem 5.1, we have $p'(k) < 0$. We also have $\Psi'(m) \geqslant 0$ and $p = u'(c) > 0$. We need to show that the term inside the parenthesis is positive. By (49),

$$\begin{aligned}& f(k)-kg(m)+k\delta-m-u'^{-1}(p)+L(u'^{-1}(p)) \\ =\ & f(k)-rk-u'^{-1}(p)+L(u'^{-1}(p))+k(r+\delta-g(m) \\ =\ & m+\left[1+\frac{1}{g'(m)}\right](r+\delta-g(m)).\end{aligned}$$

Under the condition (54), we have $r+\delta \geqslant g(m)$. Hence, we conclude that $dm/dk \geqslant 0$.

Also from (18), we deduce that

$$\frac{d\psi}{dk} = 1 + \frac{-g''(m(k,p(k)))\frac{dm}{dk}}{(g'(m))^2} > 0.$$

This concludes the proof. □

5.3 Curves $\dot{k}(t) = 0$ and $\dot{p}(t) = 0$

Along the curve $\chi(k,p) = 0$, we have $\dot{k}(t) = 0$. Differentiating with respect to k, we obtain

$$\chi_k + \chi_p p'(k) = 0.$$

Thus, the curve $\dot{k}(t) = 0$ is defined by

$$p'(k) = \frac{f'(k)+\delta-g(m)-(1+kg'(m))m_k}{\frac{1}{u''(u'^{-1}(p))}+(1+kg'(m))m_p}$$

with the boundary $p(\bar{k}) = \bar{p}$. Note that $\chi(k,p)$ increases in p. Thus, $\dot{k}(t) > 0$ for any point above the curve, and $\dot{k}(t) < 0$ for any point below the curve.

The curve $\dot{p}(t) = 0$ is defined by $k = f'^{-1}(r) = \bar{k}$. We have $\dot{p}(t) < 0$ for $k < \bar{k}$ and $\dot{p}(t) > 0$ for $k > \bar{k}$.

5.4 Examples

Consider the example in Section 4.3 with $g(m) = \sqrt{m}$, $f(k) = \sqrt{k}$, and $u(c) = \ln c$. The curves $\dot{k}(t) = 0$ and $\dot{p}(t) = 0$ are defined as

$$p'(k) = -p^2 \frac{1/(2\sqrt{k}) - r + (r + \delta - \sqrt{m})(\sqrt{k}/(4m) - rk/(2m) + 1)}{1 + (1/\sqrt{m} + k/(2m))(\delta + r - \sqrt{m}) \ln p},$$
$$k = 1/4r^2.$$

The optimal trajectory is given by

$$p'(k) = \frac{p[r - 1/(2\sqrt{k})]}{\sqrt{k} - k[\sqrt{m} - \delta] - 1/p - m(p(k), k)},$$

where $m(p, k)$ is the solution of

$$m + 2\sqrt{m}[r + \delta - g(m)] = \sqrt{k} - rk - 1/p - \frac{\ln p}{p}.$$

In Figure 1, we plot the phase diagram for the example with $r = 0.25$ and $\delta = 1$. In this particular example, the steady state values are $(\bar{k}, \bar{c}, \bar{m}, \bar{\psi}, \bar{p}) = (4, 2.925, 0.435, 1.819, 0.342)$. Since $g(\bar{m}) - \delta = -0.34 < 0$, the population keeps decreasing exponentially at the steady stage. In Figure 1, we also plot the classical curve for which the population grows at a constant rate -0.34 with no population policy expenditure. The steady state of the classical curve is at $(k_\infty, p_\infty) = (4, 0.298)$ with a consumption rate of $c_\infty = 3.356 > \bar{c}$.

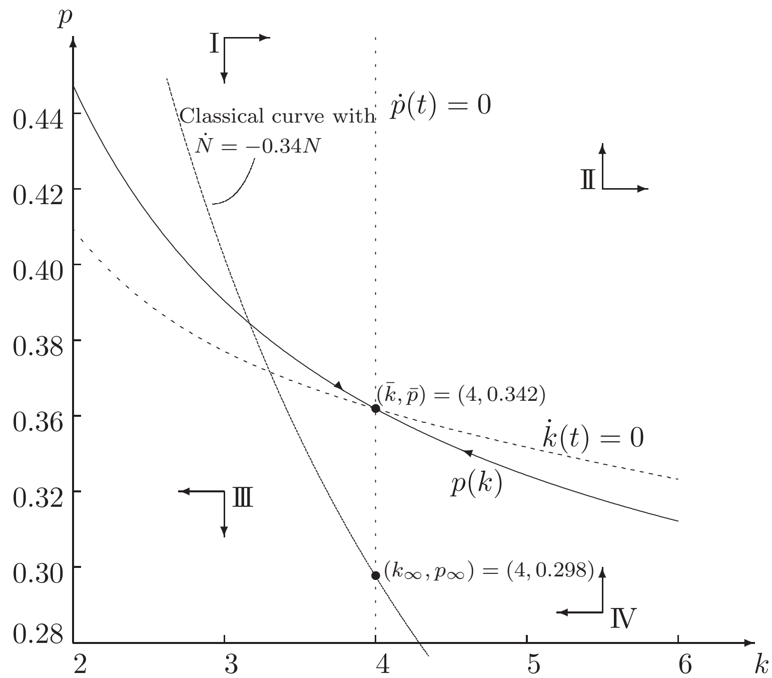
In this case, $\tilde{m} = (\delta + r)^2$. The range defined in (53) is

$$-\sqrt{k} + rk \leqslant -\ln p/p - 1/p \leqslant (\delta + r)^2 - \sqrt{k} + rk$$

Note that $-\infty < -\ln p/p - 1/p \leqslant 0$ for $0 < p \leqslant 1 = u'(\underline{c})$. In Figure 2, we plot the bounds of the optimal path and the optimal population policy expenditure of the previous example.

In Figures 3 and 4, we provide an example with a positive population growth rate of 0.118 at the equilibrium. In this example, the steady state values are $(\bar{k}, \bar{c}, \bar{m}, \bar{\psi}, \bar{p}) = (11.111, 1.349, 0.700, 9.450, 0.741)$. We note that the curve $\dot{k} = 0$ is not monotone in k. The optimal trajectory $p(k)$ is at its lower bound and $m = \tilde{m} = 0.723$ for $k \geqslant 11.66$. We also observe that the classical curve with no population policy is well below the optimal trajectory $p(k)$.

Figure 1: The phase diagram when $r = 0.25$ and $\delta = 1$.



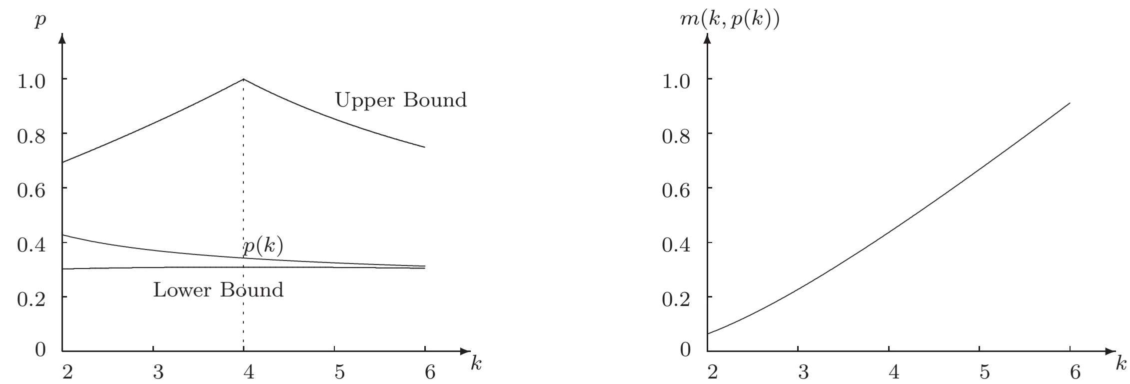
In Figures 5 and 6, we provide an example with the population level saturated at the equilibrium. In this example, the steady state values are $(\bar{k}, \bar{c}, \bar{m}, \bar{\psi}, \bar{p}) = (4.725, 1.951, 0.223, 2.906, 0.513)$. Since $\sqrt{\bar{m}} - \delta = 0$, the population level stays constant once the equilibrium is reached. We note that the curve $\dot{k} = 0$ is not convex for $k < \bar{k}$ and concave for $k > \bar{k}$. The optimal trajectory $p(k)$ is at its lower bound and $m = \tilde{m} = 0.493$ for $k \geqslant 6.78$.

6. Genuine Savings and Relationship to the Maximum Principle

Economists often use the maximum principle for studying optimal control problems like (5)-(7). It would therefore be useful to relate our analysis to multipliers that arise with the application of the maximum principle to the problems (5)-(7). We formulate the Hamiltonian (see, Sethi and Thompson 2000)

$$H = Nu(c) + \lambda[f(k) - k(g(m) - \delta) - c - m] + \mu N[g(m) - \delta], \tag{57}$$

Figure 2: The example with $r = 0.25$ and $\delta = 1$.



where the adjoint equations satisfy

$$\dot{\lambda} = (r - f'(k))\lambda + \lambda(g(m) - \delta), \tag{58}$$

$$\dot{\mu} = r\mu - u(c) - \mu(g(m) - \delta). \tag{59}$$

It is known that λ and μ provide marginal valuations of k and N, respectively, i.e., $\lambda = \theta_k$ and $\mu = v_N$. From (12) and the definitions of p and ψ in (17) and (18), we can relate λ and μ to p and ψ as follows:

$$\lambda(t) = v_k(k(t), N(t)) = N(t)p(k(t)) \text{ and } \mu(t) = v_N(k(t), N(t)) = \psi(k(t))p(k(t)). \tag{60}$$

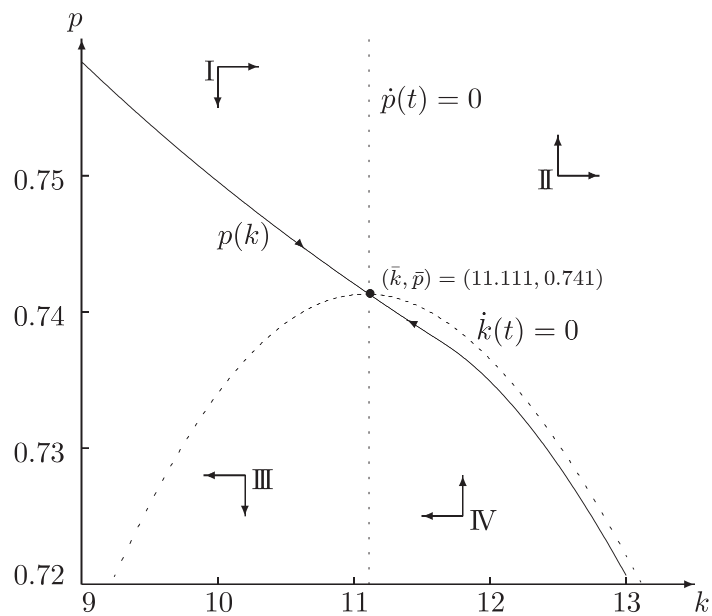
Furthermore, the necessary optimality conditions $H_c = 0$ and $H_m = 0$ give rise to the same conditions as (14) and (15), respectively.

We should mention that we did not use the maximum principle formulation for our analysis of the steady state, since in our problem the population does not settle down to a stationary value. Rather, it is the rate of change of the population, $g(m) - \delta$, that reaches a steady state. Another thing we should mention is that the optimal consumption must satisfy $u'(c) = \lambda/N = p$, and it is λ/N that must remain bounded. Likewise, it is $\mu/p = \psi$ that must remain bounded. One can then see that the steady state equations (27)-(31) obtained in Section 4 correspond to $\dot{k} = 0$, $d(\lambda/N)/dt = 0$, $d(\mu/N)/dt$, $H_c = 0$ and $H_m = 0$ in the maximum principle framework.

Genuine savings (see, Arrow et al. 2003) in our model can be defined as

$$\dot{v} = \frac{dv}{dt} = \frac{\partial v}{\partial k}\dot{k} + \frac{\partial v}{\partial N}\dot{N}$$

Figure 3: The phase diagram when $r = 0.15$ and $\delta = 0.7$.



$$= \quad \lambda\dot{k} + \mu\dot{N} = Np\dot{k} + \psi pN(g(m) - \delta). \tag{61}$$

Here we have used the relations

$$\frac{\partial v}{\partial k} = Np \text{ and} \frac{\partial v}{\partial N} = \psi p, \tag{62}$$

obtained from (7), (12), (16), (17), and (18).

If we divide (61) by p, we get the genuine savings expressed in commodity terms. Dividing further by N, we get the expression for *per capita genuine savings in commodity terms* as

$$\frac{1}{Np}\frac{dv}{dt} = \dot{k} + \psi(g(m) - \delta). \tag{63}$$

Since we can write the value function in terms of K and N as

$$V(K, N) = v(\frac{K}{N}, N) = v(k, N),$$

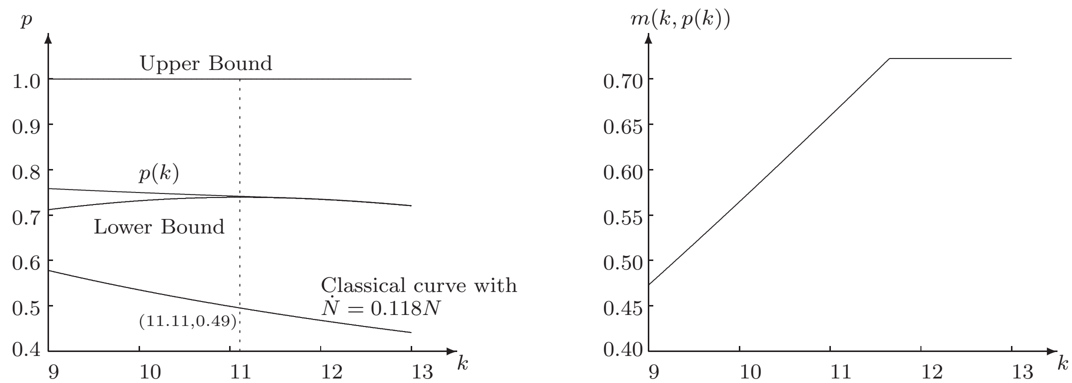
it is easy to see that

$$\partial V/\partial K = v_k/N = p$$

and

$$\frac{\partial V}{\partial N} \quad = \quad -\frac{v}{k} \cdot \frac{K}{N^2} + v_N,$$

Figure 4: The example with $r = 0.15$ and $\delta = 0.7$.



$$= \quad -\frac{\lambda}{N} \cdot k + \mu,$$
$$= \quad -pk + p\psi,$$

which gives

$$\psi = (\partial V / \partial N)/p + k = q + k, \tag{64}$$

We can now see that per capita genuine savings in commodity terms given by (63) is the same as the expression (14) in Arrow et al. (2003).

Furthermore, an optimal path is *sustainable* at time t in the sense of Pezzey (1992), if and only if,

$$\dot{k}(t) + \psi(k(t))(g(m(t)) - \delta) > 0. \tag{65}$$

Let us define

$$\rho(t) = e^{-\int_0^t [f'(k)+\delta-g(m)]ds}. \tag{66}$$

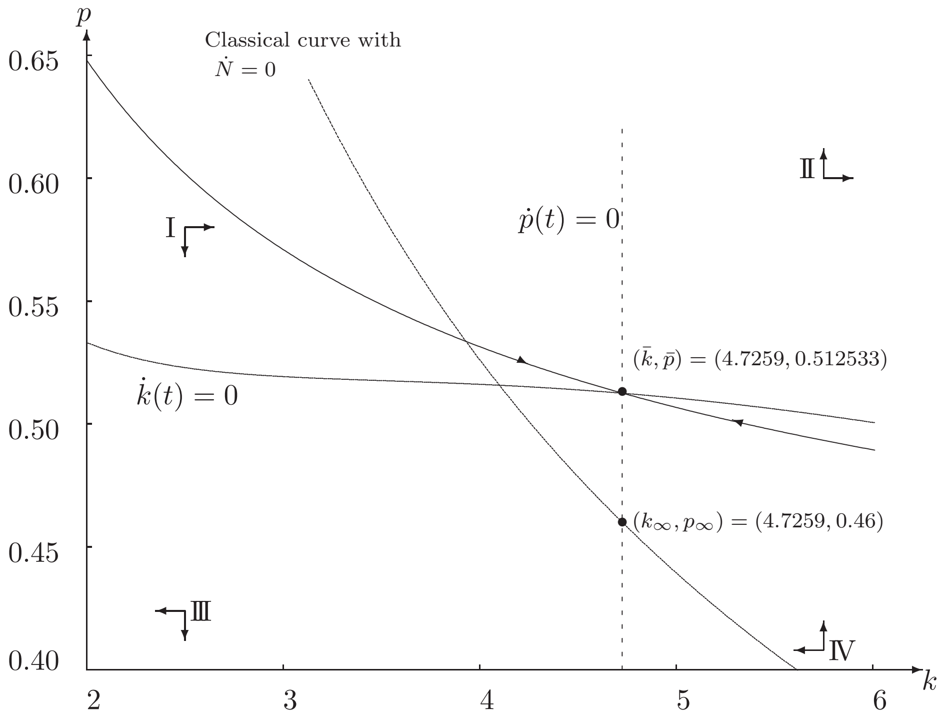
Then,

$$\dot{\rho} = -\rho[f'(k) + \delta - g(m)], \ \rho(0) = 1. \tag{67}$$

Furthermore, let $\varphi(t) = \psi(k(t))$ and consider

$$\frac{d(\varphi\rho)}{dt} = \rho\dot{\varphi} + \varphi\dot{\rho}. \tag{68}$$

Figure 5: The phase diagram when $r = 0.23$ and $\delta = 0.472036$.



But from (25),

$$\dot{\varphi} = \psi'(k)\dot{k} = \psi[f'(k) + \delta - g(m)] - L(c).$$

Therefore, (68) reduces to

$$\frac{d(\varphi\rho)}{dt} = -\rho L(c). \tag{69}$$

As $t \to \infty$,

$$f'(k) + \delta - g(m) \to r + \delta - g(\bar{m}) > r + \delta - g(\hat{m}) > 0.$$

Therefore, $\rho(\infty) \to 0$. So we can solve (69) as

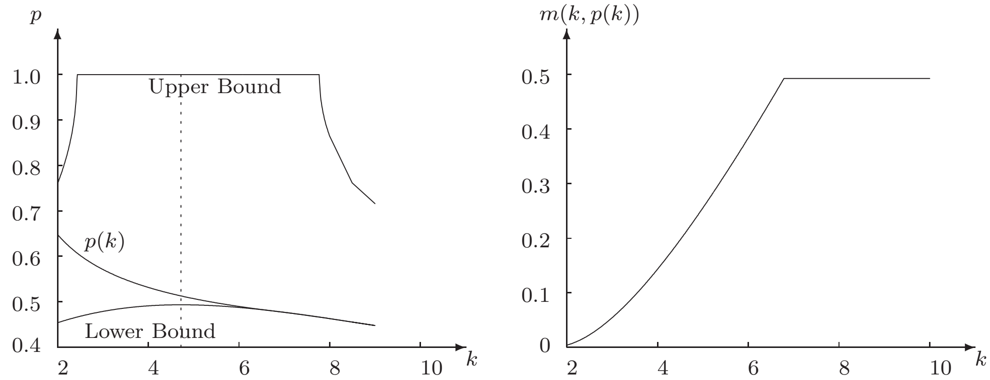
$$\varphi(t)\rho(t) = \int_t^\infty \rho(s)L(c(s))ds. \tag{70}$$

From (66), $\rho(t) > 0$. So from (64) and (70), we have

$$k(t) + q(t) = \varphi(t) = \int_t^\infty \frac{\rho(s)}{\rho(t)}L(c(s))ds > 0, \tag{71}$$

provided $u(c) > 0$, which we could assume, as is natural, to be positive in the relevant range. Many standard utility functions even satisfy $u(c) > 0$ for $c > 0$. Note from Section 4.1 that

Figure 6: The example with $r = 0.23$ and $\delta = 0.472036$.



$u(\bar{c}) > 0$ in equilibrium. Also if the net population growth is positive at least in equilibrium, then from (65) we see that genuine savings exceed increases in per capita capital.

Arrow et al. (2003) interpret $L(c)$ as *value of life*. To see this, let σ be the probability of survival. At any moment of time, the individual enjoys satisfaction $u(c)$ in case of survival and 0 otherwise, so that expected satisfaction is $\sigma u(c)$. A constant value of this expression defines an indifference curve between probability of survival and consumption. Then the marginal willingness to pay in consumption for an increase in survival probability is given by the negative of the slope of the indifference curve, that is,

$$-dc/d\sigma = u(c)/\sigma u'(c). \tag{72}$$

If we start from a situation of certain survival, i.e., $\sigma = 1$, then in view of (24) and (72), $dc = -L(c)d\sigma$ is the increased consumption ($dc > 0$) that would compensate for a decrease of $d\sigma$ ($d\sigma < 0$) in survival probability. The quantity $\varphi(t) = k(t) + q(t)$ is the value of life discounted at the marginal productivity of the capital adjusted by the population growth rate.

In comparing (71) to the relation (15) obtained in Arrow et al., we note that if we define

$$\begin{aligned} R(t) &= e^{-\int_0^t F_K(K(s),N(s))ds}, \\ Q(t) &= e^{\int_0^t (g(m(s))-\delta)} ds, \end{aligned}$$

then $q(t) = R(t)Q(t)$. Hence,

$$\varphi(t) = q(t) + k(t) = \int_t^\infty \frac{R(s)}{R(t)} \cdot \frac{Q(s)}{Q(t)} L(c(s)) ds.$$

Note that $\phi = \psi(t) > 0$ does not mean that the population itself is good. That depends on the sign of q, which may be negative. From (60), we know that the shadow price of the population $\mu = \psi p$. Population is certainly good if $\mu > 0$, i.e., if $p > 0$, since we know $\varphi = \psi > 0$ from (71).

7. Concluding Remarks

We have studied a one-sector model of an economy with the population changing at an exponential rate affected by the population policy in effect. This rate can be positive, negative or zero. We use dynamic programming for our analysis. We also show briefly how our analysis is related to the maximum principle.

By showing that the co-state of the population is only algebraically related to the co-state of the capital stock, we are able to develop a two-dimensional phase diagram of the problem. The phase diagram analysis is very similar to the classical model (Arrow and Kurz 1970) with an exponentially growing population with a positive constant growth rate, even though the controlled growth rate may be negative in our model.

As a topic for future research, it is interesting to consider a more general model in which the population growth rate depends on both the current population and the population policy expenditures, i.e., $\dot{N} = N(g(N, m) - \delta)$.

References

[1] K. J. Arrow, A. Bensoussan, Q. Feng, and S. P. Sethi (2007). Optimal savings and the value of population, *Proceedings of the National Academy of Sciences (PNAS)*, **104**, 47, 18421-18426.

[2] K. J. Arrow, P. Dasgupta, and K.-G. Mäler (2003). The genuine savings criterion and the value of population, *Economic Theory*, **21**, 217-225.

[3] K. J. Arrow and M. Kurz (1970). *Public investment, the rate of return, and optimal fiscal policy*, Johns Hopkins Press, Baltimore, MD.

[4] N. A. Derzko, S. P. Sethi and G. L. Thompson (1984). Necessary and sufficient conditions for optimal control of quasilinear partial differential systems. *J. of Optimization Theory and Applications*, **43**, 1989-1001.

[5] D. Parfit (1984). *Reasons and persons*, Oxford University Press, UK.

[6] J. Pezzey (1992), Sustainable development concepts: an economic analysis, World Environment Paper No. 2, World Bank.

[7] S. P. Sethi and G. L. Thompson (2000). *Optimal control theory: applications to management science and economics,* Second Edition, Kluwer Academic Publishers, Boston, MA.