Nyquist Frequency in Sequentially Sampled Data

Nezameddin Faghih and Ali Faghih

Shiraz University, University of Maryland

2008

Nyquist Frequency in Sequentially Sampled Data

Nezameddin Faghih* and Ali Faghih**

ABSTRACT

This paper studies the sequential sampling scheme as a solution to the problem of aliasing, where the sampling interval is restricted to a minimum allowable value ΔT. In sequential sampling, the signal is sampled at intervals of ΔT, $\Delta T+\Delta\tau$, $\Delta T+2\Delta\tau$, $\Delta T+3\Delta\tau$, ...; where $\Delta\tau < \Delta T$ and $\Delta\tau$ may be selected as desirable. Sequential sampling is, however, analyzed and it is proved that when the ratio $\Delta T/\Delta\tau$ is an integral number, the associated spectral estimates give a Nyquist frequency $\frac{1}{2\Delta\tau}$. This sampling scheme can, therefore, be employed to yield a required cut- off frequency.

---

* Shiraz University (faghihnezam@ut.ac.ir)

** University of Maryland, College Park

INTRODUCTION

Some data acquisition systems have a minimum allowable sampling interval and do not provide a desired sampling period less than a minimum allowable value. This may be due to some restrictions set by the measuring instrument that has to be used [1-7].

Let the minimum allowable sampling time be ΔT; if the uniform sampling scheme is employed then, the Nyquist or cut-off frequency is known [8] to be given as:

$$f_c = \frac{1}{2\Delta T} \quad (1)$$

This would mean that if frequencies higher than f_c are present, aliasing will occur. Otherwise, the signal would have to be filtered so that only frequencies below f_c are passed and, therefore, the spectral analysis will be restricted [8].

The sequential sampling scheme can, however, be employed to obtain an autocorrelation function with estimates $\Delta\tau$ apart, where $\Delta\tau < \Delta T$, with the exception of coefficients lying inside the range $R(0) \rightarrow R(\Delta T)$. In this sampling scheme, the signal would be sampled at intervals of:

$\Delta T,\ \Delta T+\Delta\tau,\ \Delta T+2\Delta\tau,\ \Delta T+3\Delta\tau,\ \ldots$

From the sampled signal, an autocorrelation function can be obtained with the coefficients:

$R(0),\ R(\Delta T),\ R(\Delta T+\Delta\tau),\ R(\Delta T+2\Delta\tau),\ \ldots$

While ΔT is restricted, the value of $\Delta\tau$ may be chosen as desirable. It will be proved, in this paper, that the sequential sampling can give an increased cut-off frequency as:

$$f_{cs} = \frac{1}{2\Delta\tau} \quad (2)$$

The sequential sampling can, therefore, be employed to overcome aliasing and the restrictions of spectral analysis, by selecting a sufficiently small value for $\Delta\tau$.

THE CUT OFF FREQUENCY IN THE SEQUENTIAL SAMPLING

The cut- off frequency provided by the sequential sampling scheme is considered in this section. The analysis employs the impulse representation of a continuous signal as an approach to discretization [9-14].

In the sequential sampling, the signal is sampled at intervals of:

$\Delta T,\ \Delta T+\Delta\tau,\ \Delta T+2\Delta\tau,\ \Delta T+3\Delta\tau,\ \ldots$

The sampling instants are, therefore, given by:

$$t_i = 0,\quad \Delta T,\ 2\Delta T+\Delta\tau,\ 3\Delta T+3\Delta\tau,\ 4\Delta T+6\Delta\tau,\ \ldots \quad (3)$$

This can be written as:

$$t_i = i\Delta T + [\sum_{r=0}^{i} r - i]\Delta\tau , \quad i = 0,1,2,3,\ldots \tag{4}$$

Since, $\sum_{r=0}^{i} r = \frac{i}{2}(i + 1)$, then equation (4) gives:

$$t_i = i\Delta T + \frac{i}{2}(i - 1)\Delta\tau \tag{5}$$

When a continuous signal x(t) is sampled, the sample values $x(t_i)$ are acquired. A discrete autocorrelation function, with coefficients $R(\tau_j)$, may be obtained from the discrete signal, by contributions of the products $x(t_i)x(t_{i+j})$. Equation (5) can be used to give the time delay τ_j as:

$$\tau_j = t_{i+j} - t_i = \Delta T + [\frac{j}{2}(j - 1) + ij + (j - 1)\mu]\Delta\tau \tag{6}$$

i= o, 1, 2, ...

j= o, 1, 2, ...

where μ is a constant given by:

$$\mu=\Delta T/\Delta\tau \tag{7}$$

It is seen from equation (6) that for j=0, the time delay is zero and for j=1, the time delay is $\Delta T+i\Delta\tau$ (where i=0, 1, 2,3, ...). An autocorrelation function is, therefore, obtainable at discrete values of the time delay given as:

$$\tau_n = o, \Delta T+n\Delta\tau n= 0, 1, 2, 3, \ldots \tag{8}$$

If the ratio μ is an integral number, then higher values of j would also provide more contributions to the autocorrelation estimates at the above time delays τ_n. This is because j(j-1)/2 is always even, and any value of j would hence add a multiple of $\Delta\tau$ to ΔT.

The discrete autocorrelation function may be represented as:

$$R^*(\tau) = \Delta\tau . R(\tau)\delta_b(\tau) \tag{9}$$

where $R(\tau)$ is the continuous autocorrelation function and $\delta_b(\tau)$ is the following form of the delta comb:

$$\delta_b(\tau) = \delta(\tau) + \delta(\tau - \Delta T) + \delta(\tau - \Delta T - \Delta\tau) + \delta(\tau - \Delta T - 2\Delta\tau) + L \tag{10}$$

It is established [9-10] that the Fourier transform of equation (10) can be written as:

$$\Delta_b(\omega) = 1 + e^{-j\omega\Delta T}\sum_{0}^{\infty} e^{-j\omega n\Delta\tau} \qquad (11)$$

which by manipulation [9-10] can be re-written as:

$$\Delta_b(\omega) = \frac{1 - e^{-j\omega\Delta\tau} + e^{-j\omega\mu\Delta\tau}}{1 - e^{-j\omega\Delta\tau}} \qquad (12)$$

where substitution has also been made for ΔT from equation (7).

The Fourier transformation of $R^*(\tau)$ gives the spectral density $S^*(\omega)$ corresponding to the sampled signal and that of $R(\tau)$ would yield the spectral density $S(\omega)$ of the original continuous signal. The approach adopted for the Fourier transformation of equation (9) is based on the convolution and residue theorems [9]. By evaluating the residue terms [9] and using the convolution property [9], for substitution into equation (9), the Fourier transform of this equation ca be obtained as:

$$S^*(\omega) = \sum_{-\infty}^{\infty} e^{j2\pi n\mu} S(\omega + 2n\omega_{cs}), \quad \omega_{cs} = \pi_{/\Delta\tau} \qquad (13)$$

However, if the ratio $\mu=(\Delta T/\Delta\tau)$ is an integral number then,

$$e^{j2\pi n\mu} = 1$$

noting that n is also an integer. Substituting this into equation (13) gives:

$$S^*(\omega) = \sum_{-\infty}^{\infty} S(\omega + 2n\omega_{cs}), \quad \omega_{cs} = \pi_{/\Delta\tau} \qquad (14)$$

Now, consider the periodicity of $S^*(\omega)$; this can also be examined by applying the corresponding methods [9-14]. Using equations (9), (10), (11) and the rules established for discrete Fourier transformation [9-10], it can be written:

$$\Delta\tau[R(0) + e^{-j\omega\Delta T}\sum_{0}^{\infty} R(\Delta T + n\Delta\tau)e^{-j\omega n\Delta\tau}] = \Delta\tau[R(0) + e^{-j\omega\mu\Delta\tau}\sum_{0}^{\infty} R(\Delta T + n\Delta\tau)e^{-j\omega n\Delta\tau}] \qquad (15)$$

and then for an integer m:

$$\Delta\tau[R(0) + e^{-j(\omega+2m\omega_{cs})\mu\Delta\tau}\sum_0^{\infty} R(\Delta T + n\Delta\tau)e^{-j(\omega+2m\omega_{cs})n\Delta\tau}] =$$

$$\Delta\tau[R(0) + e^{-j\omega\mu\Delta\tau}e^{-j2m\omega_{cs}\mu\Delta\tau}\sum_0^{\infty} R(\Delta T + n\Delta\tau)e^{-j\omega n\Delta\tau}.e^{-j2m\omega_{cs}n\Delta\tau}] =$$

$$\Delta\tau[R(0) + e^{-j\omega\mu\Delta\tau}.e^{-j2m\mu\pi}\sum_0^{\infty} R(\Delta T + n\Delta\tau)e^{-j\omega n\Delta\tau}.e^{-j2mn\pi}] =$$

$$\Delta\tau[R(0) + e^{-j\omega\mu\Delta\tau}.e^{-j2m\mu\pi}\sum_0^{\infty} R(\Delta T + n\Delta\tau)e^{-j\omega n\Delta\tau}] \qquad (16)$$

since m and n are integers. If μ is also an integral number, this would reduce to:

$$\Delta\tau[R(0) + e^{-j\omega\mu\Delta\tau}\sum_0^{\infty} R(\Delta T + n\Delta\tau)e^{-j\omega n\Delta\tau}]$$

from which it follows that:

$$S^*(\omega + 2m\omega_{cs}) = S^*(\omega) \qquad (17)$$

This is the mathematical statement for $S^*(\omega)$ to be periodic with period $2\omega_{cs}$. Otherwise, if μ is not an integral number,

$$S^*(\omega + 2m\omega_{cs}) \neq S^*(\omega) \qquad (18)$$

and the requirement for periodicity is not met.

It is, therefore, seen that when the ratio $\mu(=\Delta T/\Delta\tau)$ is an integral number, the periodic pattern, conforming with the Nyquist theorem,[9-14] is obtained. That is, the sequential sampling gives a cut-off frequency $\omega_{cs} = \pi/_{\Delta\tau}$ or $f_{cs} = \frac{1}{2\Delta\tau}$. On the contrary, when μ is not a whole number, $S^*(\omega)$ is related to the true spectral density by equation (13); it includes a complex term and is not periodic.

CONCLUSIONS

This paper has considered the sequential sampling scheme, as a solution to the problem of aliasing, where the sampling interval is restricted to a minimum allowable value ΔT. In the sequential sampling, the signal is sampled at intervals of $\Delta T, \Delta T + \Delta\tau, \Delta T + 2\Delta\tau, \Delta T + 3\Delta\tau, \ldots;$ where $\Delta\tau$ p ΔT and may be selected as desirable.

The sequential sampling was considered analytically and it was proved that, when the ratio $\Delta T / \Delta\tau$ is an integral number, the corresponding spectral estimates give a cut- off frequency of $\frac{1}{2\Delta\tau}$. On the contrary, when the ratio is not a whole number, the associated spectrum of the sequentially sampled data was found to comprise a complex term in its relation to the true spectrum and would not be periodic in terms of the cut-off frequency.

.

REFERENCES

1 - Wirth, W.D. (1995), "Energy Saving by Coherent Sequential Detection of Radar Signals with Unknown Doppler Shift", IEE Proceedings on Radar, Sonar and Navigation, 142, 145-52.

2 - Willis, N.P. and Bresler, Y.(1992), "A New Approach to the Time-Sequential Sampling Problem", Proceedings of the IEEE International Conference on Acoustics, Speech and Signal Processing, 3, 277-80.

3 - Allebach, J.P. (1984), "Design of Antialiasing Patterns for Time-Sequential Sampling of Spatiotemporal Signals", IEEE Transactions on Acoustics, Speech and Signal Processing, 32, 1, 137- 44.

4 - Allebach, J.P. (1981), "Design of Sampling Patterns for Time Sequential Sampling of Spatio- Temporal Signals", Proceedings of Micro-Delcon Delaware Bay Computer Conference, 9-13.

5 - Aizawa, A.N. and Wah, B.W. (1994), "A Sequential Sampling Procedure for Genetic Algorithms", Computer and Mathematics with Applications, 27, 9-10, 77-82.

6 - Gaster, M. and Bradbury, L.J.S. (1976), "The Measurement of the Spectra of Highly Turbulent Flows by a Randomly Triggered Pulsed-Wire Anemometer", J. Fluid Mech, 77, 499-509.

7 - Bradbury, L.J.S. (1978), "Examples of the Use of the Pulsed Wire Anemometer in Highly Turbulent Flow", Proceedings of the Dynamic Flow Conference, Marseille, 489-509.

8 - Bendat, J.S. and Piersol, A.G. (1986), "Random Data: Analysis and Measurement Procedures", Wiley-Interscience.

9 - Saucedo, R. and Schiring, E. (1968), "Introduction to Continuous and Digital Control Systems", Macmillan.

10 - Papoulis, A. (1962), "The Fourier Integral and its Applications", McGraw- Hill.

11 - Oppenheim, A.V. et al (1983), “Signals and Systems”, Prentice- Hall.

12 - Carlson, A.B. (1986), “Communication Systems”, McGraw-Hill.

13 - Jones, R.H. and Steele, N.C. (1989), “Mathematics in Communication Theory”, Ellis Horwood Publishers.

14 - Steward, E.G. (1989), “Fourier Optics: an Introduction”, Ellis Horwood Publishers.