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FIAT MONEY AND THE VALUE OF BINDING PORTFOLIO CONSTRAINTS

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Abstract. We establish necessary and sufficient conditions for the individual optimality of a consumption-portfolio plan in an infinite horizon economy where agents are uniformly impatient and fiat money is the only asset available for inter-temporal transfers of wealth. Next, we show that fiat money has a positive equilibrium price if and only if for some agent the zero short sale constraint is binding and has a positive shadow price (now or in the future). As there is always an agent that is long, it follows that marginal rates of inter-temporal substitution never coincide across agents. That is, monetary equilibria are never full Pareto efficient. We also give a counter-example illustrating the occurrence of monetary bubbles under incomplete markets in the absence of uniform impatience.

Keywords: Binding credit constraints, Fundamental value of money, Asset pricing bubbles.

JEL classification: D50, D52.

1. Introduction

The uniform impatience assumption (see Hernández and Santos (1996, Assumption C.3) or Magill and Quinzii (1996, Assumptions B2, B4)), together with borrowing constraints, is a usual requirement for existence of equilibrium in economies with infinite lived agents. This condition is satisfied whenever preferences are separable over time and across states so long as (i) the intertemporal discounted factor is constant, (ii) individual endowments are uniformly bounded away from zero, and (iii) aggregate endowment is uniformly bounded from above.

The assumption of uniform impatience has important implications for asset pricing as it rules out speculation in assets in positive net supply for deflator processes in the non-arbitrage pricing kernel, yielding finite present values of aggregate wealth, as Santos and Woodford (1997) showed. The well-known example of a positive price of fiat money by Bewley (1980) highlighted the importance of the finiteness of the present value of aggregate wealth.

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What happens if we use as deflators the agents' inter-nodes marginal rates of substitution? These deflators may fail to be in the asset pricing kernel when some portfolio constraints are binding.[1] For these Kuhn-Tucker deflators, assets in positive net supply may be priced above the series of deflated dividends and the difference may be due to the presence of shadow prices rather than to a bubble. Giménez (2007) already made this comment and Araujo, Páscoa and Torres-Martínez (2008) worked along these lines when addressing the pricing of long-lived collateralized assets.

To consider a simple and provocative case, we look, as in Bewley (1980), at economies with a single asset, paying no dividends and in positive net supply. As usual, we call this asset fiat money (or simply money), although we are quite aware that we are just looking at its role as a store of value, i.e. as an instrument to transfer wealth across time and states of nature. In this context and under uniform impatience, we show that money can and will only be positively valued as a result of agents' desire to take short positions that they cannot. That is, under uniform impatience, the positive price of money must be due to the presence of shadow prices of binding constraints.

This result does not collide with the example by Bewley (1980) or the results by Santos and Woodford (1997). It complements these results. Under uniform impatience, a positive price of money implies that the present value of the aggregate wealth must be infinite for any deflator in the asset pricing kernel. Without uniform impatience, it would only imply an infinite supremum for the present value of aggregate wealth, when all deflators in that kernel are considered. However, for any Kuhn-Tucker deflator process of a certain agent, the present value of the endowments of this agent is finite, but this deflator may fail to be in the asset pricing kernel (when this agent has binding portfolio constraints) or the present value of aggregate wealth may fail to be finite (when the deflator is not the same for all agents and uniform impatience does not hold).

In Bewley's (1980) example, the two uniformly impatient agents were not allowed to take short positions and the economy had no uncertainty. The positive price of money was a bubble for the unique deflator process in the asset pricing kernel and for this deflator the present value of aggregate wealth was infinite. However, the zero short-sales constraint was binding infinitely often. Thus, for the Kuhn-Tucker deflator process of each agent, the fundamental value of money was positive, consisting of the shadow prices of debt constraints.

Hence, we obtain a result that may seem surprising: credit frictions create room for welfare improvements through transfers of wealth that become possible when money has a positive price. However, monetary equilibria are always Pareto inefficient. Otherwise, by definition, agents' rates

[1]Notice that under inequality constraints on portfolios, non-arbitrage (from one node to its immediate successors) is equivalent to the existence of a positive vector of state prices solving a linear system of inequalities relating asset prices and returns (see Jouini and Kallal (1995) or Araujo, Fajardo and Páscoa (2005)). The state prices that make all inequalities hold as equalities constitute the kernel of the linear operator that defines the system, but there may be other solutions (for example, those given by the Kuhn-Tucker multiplers).

of intertemporal substitution would coincide. However, as money is in positive net supply, at least one agent must go long, having a zero shadow price. Thus, the shadow prices of all agents should be zero and, therefore, the price of money could not be positive.

To clarify our results, we prove that when money has a positive value, there exists a deflator, but not one of the Kuhn-Tucker deflators, under which the discounted value of aggregated wealth is *infinite* and a pure bubble appears. That is, in our framework (that includes Bewley (1980) model) it is always possible to interpret monetary equilibrium as a bubble. However, when we focus on Kuhn-Tucker multipliers—deflators that make financial Euler conditions compatible with physical Euler conditions—the positive price of money is always a consequence of a positive fundamental value.

We close the paper with an example of a stochastic economy that does not satisfy the uniform impatience assumption. Money is positive valued in equilibrium, although shadow prices of zero short-sale constraints are equal to zero. For the Kuhn-Tucker deflator processes of both agents, aggregate wealth has a finite present value.

Our main mathematical tool is a duality approach to dynamic programming problems that was already used in the context of long-lived collateralized assets by Araujo, Páscoa and Torres-Martínez (2008). This approach allows us to characterize non-interior solutions and the respective Kuhn-Tucker multiplier processes. A recent related paper by Rincón-Zapatero and Santos (2008) addresses the uniqueness of this multiplier process and the differentiability of the value function, without imposing the usual interiority assumptions.

The paper is organized as follows. Section 2 characterizes uniform impatience. Section 3 presents the basic model. In Section 4, we develop the necessary mathematical tools: a duality theory of individual optimization. In Section 5 we define the concepts of fundamental value of money and asset pricing bubbles. Finally, Section 6 presents the results on monetary equilibria and Section 7 an example of monetary equilibrium in an economy without uniform impatience. Some proofs are left to the Appendix.

2. Characterizing uniform impatience when utilities are separable

In this section, we recall the assumption of uniform impatience and characterize it for separable utilities in terms of intertemporal discount factors. As a consequence, we show that the uniform impatience assumption does not hold for agents with hyperbolic intertemporal discounting (see Laibson (1998)).

Consider an infinite horizon discrete time economy where the set of dates is $\{0, 1, \ldots\}$ and there is no uncertainty at $t = 0$. Given a history of realizations of the states of nature for the first $t - 1$

dates, with $t \geq 1$, $\overline{s}_t = (s_0, \ldots, s_{t-1})$, there is a finite set $S(\overline{s}_t)$ of states that may occur at date t. A vector $\xi = (t, \overline{s}_t, s)$, where $t \geq 1$ and $s \in S(\overline{s}_t)$, is called a *node*. The only node at $t = 0$ is denoted by ξ_0. Let D be the *event-tree*, i.e., the set of all nodes.

Given $\xi = (t, \overline{s}_t, s)$ and $\mu = (t', \overline{s}_{t'}, s')$, we say that μ is a *successor* of ξ, and we write $\mu \geq \xi$, if $t' \geq t$ and $\overline{s}_{t'} = (\overline{s}_t, s, \ldots)$. We write $\mu > \xi$ to say that $\mu \geq \xi$ but $\mu \neq \xi$ and we denote by $t(\xi)$ the date associated with a node ξ. Let $\xi^+ = \{\mu \in D : (\mu \geq \xi) \wedge (t(\mu) = t(\xi) + 1)\}$ be the set of immediate successors of ξ. The (unique) predecessor of $\xi > \xi_0$ is denoted by ξ^- and $D(\xi) := \{\mu \in D : \mu \geq \xi\}$ is the sub-tree with root ξ. The set of nodes with date T in $D(\xi)$ is denoted by $D_T(\xi)$, and $D^T(\xi) = \bigcup_{k=t(\xi)}^{T} D_k(\xi)$ denotes the set of successors of ξ with date less than or equal to T. When $\xi = \xi_0$ notations above will be shorten to D_T and D^T.

At any node $\xi \in D$, a finite set of perishable commodities is available for trade, L. There is a finite set of infinite-lived agents, H. Each agent $h \in H$ has at any $\xi \in D$ a physical endowment $w^h(\xi) \in \mathbb{R}^L_+$ and has preferences over consumption plans, $(x(\xi); \xi \in D) \in \mathbb{R}^{L \times D}_+$, which are represented by a function $U^h : \mathbb{R}^{L \times D}_+ \to \mathbb{R}_+ \cup \{+\infty\}$. Aggregated physical endowments at a node ξ are given by $W(\xi) \in \mathbb{R}^L_{++}$.

Assumption 1 (Separability of preferences). For any agent $h \in H$, the utility function $U^h((x(\xi); \xi \in D)) = \sum_{\xi \in D} u^h(\xi, x(\xi))$, where for any $\xi \in D$, $u^h(\xi, \cdot) : \mathbb{R}^L_+ \to \mathbb{R}_+$ is a continuous, concave and strictly increasing function. Also, $\sum_{\xi \in D} u^h(\xi, W(\xi))$ is finite.

Assumption 2 (Uniform impatience). *There are $\pi \in [0, 1)$ and $(\Delta(\mu); \mu \in D) \in \mathbb{R}^{L \times D}_+$ such that, given a consumption plan $(x(\mu); \mu \in D)$, with $0 \leq x(\mu) \leq W(\mu)$, for any $h \in H$, we have*

$$u^h(\xi, x(\xi) + \Delta(\xi)) + \sum_{\mu > \xi} u^h(\mu, \pi' x(\mu)) > \sum_{\mu \geq \xi} u^h(\mu, x(\mu)), \qquad \forall \xi \in D, \ \forall \pi' \geq \pi.$$

Moreover, there is $\kappa > 0$ such that, $w^h(\xi) \geq \kappa \Delta(\xi) > 0$, $\forall \xi \in D$.

The requirements of impatience above depend on both preferences and physical endowments. As particular cases we obtain the assumptions imposed by Hernández and Santos (1996)—for any $\mu \in D$, $\Delta(\mu) = W(\mu)$—and Magill and Quinzii (1994, 1996)—initial endowments are uniformly bounded away from zero by a bundle $\underline{w} \in \mathbb{R}^L_{++}$, and $\Delta(\mu) = (1, 0, \ldots, 0)$, $\forall \mu \in D$.

Our characterization of uniform impatience is,

Proposition 1. *Suppose that Assumption 1 holds, that $(W(\xi); \xi \in D)$ is a bounded consumption plan and that there is $\underline{w} \in \mathbb{R}^L_+ \setminus \{0\}$ such that, $w^h(\xi) \geq \underline{w}$, $\forall \xi \in D$. Moreover, assume that there* *exists a function* $u^h : \mathbb{R}^L_+ \to \mathbb{R}_+$ *such that, for any* $\xi \in D$, $u^h(\xi, \cdot) \equiv \beta^h_{t(\xi)} \rho^h(\xi) u^h(\cdot)$, *where* $\beta^h_{t(\xi)}$ *is a strictly positive discount factor and* $\rho^h(\xi)$ *denotes the probability to reach node* ξ, *which satisfies* $\rho^h(\xi) = \sum_{\mu \in \xi^+} \rho^h(\mu)$, *with* $\rho^h(\xi_0) = 1$.

For each $t \geq 0$, *let* $a^h_t = \frac{1}{\beta^h_t} \sum_{r=t+1}^{+\infty} \beta^h_r$. *Then, the function* U^h *satisfies uniform impatience (Assumption 2) if and only if the sequence* $(a^h_t)_{t \geq 0}$ *is bounded.*

Proof. Assume that $(W(\xi); \xi \in D)$ is a bounded consumption plan. That is, there is $\overline{W} \in \mathbb{R}^L_+$ such that, $W(\xi) \leq \overline{W}$, $\forall \xi \in D$. If $(a^h_t)_{t\geq 0}$ is bounded, then there exists $\overline{a}^h > 0$ such that, $a^h_t \leq \overline{a}^h$, for each $t \geq 0$. Also, since $\mathbb{F} := \{x \in \mathbb{R}^L_+ : x \leq \overline{W}\}$ is compact, the continuity of u^h assures that there is $\pi \in (0,1)$ such that $u^h(x) - u^h(\pi' x) \leq \frac{u^h(\overline{W} + \underline{w}) - u^h(\overline{W})}{2\overline{a}^h}$, $\forall x \in \mathbb{F}$, $\forall \pi' \geq \pi$. Thus, uniform impatience follows by choosing $\kappa = 1$ and $\Delta(\xi) = \underline{w}$, $\forall \xi \in D$. Indeed, given a plan $(x(\mu); \mu \in D) \in \mathbb{R}^{L \times D}_+$ such that, $x(\mu) \leq W(\mu)$ $\forall \mu \in D$, the concavity of u^h assures that, for any $\xi \in D$ and $\pi' \geq \pi$,

$$\begin{aligned}\sum_{\mu > \xi} \beta^h_{t(\mu)} \rho^h(\mu) u^h(x(\mu)) - \sum_{\mu > \xi} \beta^h_{t(\mu)} \rho^h(\mu) u^h(\pi' x(\mu)) \quad &\leq \quad \frac{\beta^h_{t(\xi)} a^h_t}{2\overline{a}^h} \rho^h(\xi) \left(u^h(\overline{W} + \underline{w}) - u^h(\overline{W}) \right) \\ &< \quad \beta^h_{t(\xi)} \rho^h(\xi) u^h(x(\xi) + \Delta(\xi)) - \beta^h_{t(\xi)} \rho^h(\xi) u^h(x(\xi)).\end{aligned}$$

Reciprocally, suppose that uniform impatience property holds. Then, there are $(\pi, \kappa) \in [0,1) \times \mathbb{R}_{++}$ and $(\Delta(\mu); \mu \in D) \in \mathbb{R}^{L\times D}_+$ satisfying, for any $\xi \in D$, $w^h(\xi) \geq \kappa \Delta(\xi)$, such that, given $(x(\mu); \mu \in D) \in \mathbb{R}^{L \times D}_+$ with $x(\mu) \leq W(\mu)$, for all $\mu \in D$, we have that, for any node $\xi \in D$,

$$\frac{1}{\beta^h_{t(\xi)} \rho^h(\xi)} \left[\sum_{\mu > \xi} \beta^h_{t(\mu)} \rho^h(\mu) u^h(x(\mu)) - \sum_{\mu > \xi} \beta^h_{t(\mu)} \rho^h(\mu) u^h(\pi x(\mu)) \right] \quad < \quad u^h(x(\xi) + \Delta(\xi)) - u^h(x(\xi)).$$

It follows that, for any node ξ,

$$\frac{1}{\beta^h_{t(\xi)} \rho^h(\xi)} \left[\sum_{\mu > \xi} \beta^h_{t(\mu)} \rho^h(\mu) u^h(\underline{w}) - \sum_{\mu > \xi} \beta^h_{t(\mu)} \rho^h(\mu) u^h(\pi \underline{w}) \right] \quad < \quad u^h\left(\left(1 + \frac{1}{\kappa}\right) \overline{W} \right).$$

Therefore, we conclude that, for any $\xi \in D$,

$$\frac{1}{\beta^h_{t(\xi)}} \left(u^h(\underline{w}) - u^h(\pi \underline{w}) \right) \sum_{t = t(\xi)+1}^{+\infty} \beta^h_t \quad < \quad u^h\left(\left(1 + \frac{1}{\kappa}\right) \overline{W} \right),$$

which implies that the sequence $(a^h_t)_{t\geq 0}$ is bounded. □

Under the conditions of Proposition 1, if intertemporal discount factors are constant, i.e. $\exists \lambda^h \in \mathbb{R}_{++} : \frac{\beta^h_{t(\xi)+1}}{\beta^h_{t(\xi)}} = \lambda^h$, $\forall \xi \in D$, then both $\lambda^h < 1$ and $a^h_t = \frac{\lambda^h}{1 - \lambda^h}$, for each $t \geq 0$. In this case, the utility function U^h satisfies the uniform impatience condition.

However, even with bounded plans of endowments, uniform impatience is a restrictive condition when intertemporal discount factors are time varying. For instance, if we consider *hyperbolic intertemporal discount factors*, that is, $\beta_t^h = (1+\theta t)^{-\frac{\tau}{\theta}}$, where $(\tau,\theta) \gg 0$, then the function U^h, as defined in the statement of Proposition 1, satisfies Assumption 1 and the sequence $(a_t^h)_{t\geq 0}$ goes to infinity as t increases. Therefore, in this case, uniform impatience does not hold.

3. A monetary model with uniform impatience agents

We assume that there is only one asset, *money*, that can be traded along the event-tree. Although this security does not deliver any physical payment, it can be used to make intertemporal transfers. Let $q = (q(\xi); \xi \in D)$ be the plan of monetary prices. We assume that money is in positive net supply that does not disappear from the economy neither deteriorates. Denote money endowments at a node $\xi \in D$ by $e^h(\xi) \in \mathbb{R}_+$. We denote by $z^h(\xi) \in \mathbb{R}_+$ the quantity of money that h negotiates at ξ. Let $p(\xi) := (p(\xi,l); l \in L)$ be the commodity price at $\xi \in D$ and $p = (p(\xi); \xi \in D)$.

Given prices (p,q), let $B^h(p,q)$ be the choice set of agent $h \in H$, that is, the set of plans $(x,z) := ((x(\xi),z(\xi)); \xi \in D) \in \mathbb{R}_+^{L\times D} \times \mathbb{R}_+^D$, such that, at any $\xi \in D$, the following budget constraint holds,

$$g_\xi^h(y^h(\xi), y^h(\xi^-); p,q) := p(\xi)\left(x^h(\xi) - w^h(\xi)\right) + q(\xi)\left(z^h(\xi) - e^h(\xi) - z^h(\xi^-)\right) \quad \leq \quad 0,$$

where $y^h(\xi) = (x^h(\xi), z^h(\xi))$, $y^h(\xi_0^-) := (x^h(\xi_0^-), z^h(\xi_0^-)) = 0$.

Agent's h individual problem is to choose a plan $y^h = (x^h, z^h)$ in $B^h(p,q)$ in order to maximize her utility function $U^h : \mathbb{R}_+^{L\times D} \to \mathbb{R}_+ \cup \{+\infty\}$.

Definition 1.

An equilibrium for our economy is given by a vector of prices (p,q) jointly with individual plans $\big((x^h,z^h); h \in H\big)$, such that,

(a) *For each $h \in H$, the plan $(x^h,z^h) \in B^h(p,q)$ is optimal at prices (p,q).*
(b) *Physical and asset markets clear,*

$$\sum_{h\in H}\left(x^h(\xi); z^h(\xi)\right) = \left(W(\xi), \sum_{h\in H}\left(e^h(\xi) + z^h(\xi^-)\right)\right).$$

Note that, a *pure spot market equilibrium*, i.e. an equilibrium with zero monetary price, always exists provided that preferences satisfy Assumption 1 above.

4. Duality theory for individual optimization

In this section, we determine necessary and sufficient conditions for individual optimality.

Some previous definitions and notations are necessary. By normalization, we assume that prices (p,q) belong to $\mathbb{P} := \{(p,q) \in \mathbb{R}_+^{L\times D} \times \mathbb{R}_+^D : \sum_{l\in L} p(\xi,l) + q(\xi) = 1, \;\; \forall \xi \in D\}$. Given a concave function $f : X \subset \mathbb{R}^L \to \mathbb{R} \cup \{-\infty\}$ the super-differential at $x \in X$, denoted by $\partial f(x)$, is defined as the set of vectors $p \in \mathbb{R}^L$ such that, for all $x' \in X$, $f(\xi, x') - f(\xi, x) \leq p(x' - x)$.

Definition 2.

Given $(p, q) \in \mathbb{P}$ and $y^h = (x^h, z^h) \in B^h(p, q)$, we say that $(\gamma^h(\xi); \xi \in D) \in \mathbb{R}^D_+$ constitutes a family of Kuhn-Tucker multipliers (associated to y^h) if there exist, for each $\xi \in D$, super-gradients $u'(\xi) \in \partial u^h(\xi, x^h(\xi))$ such that,

(a) For every $\xi \in D$, $\gamma^h(\xi)\, g^h_\xi(y^h(\xi), y^h(\xi^-); p, q) = 0$.

(b) The following Euler conditions hold,

$$\gamma^h(\xi)p(\xi) - u'(\xi) \geq 0, \qquad \left(\gamma^h(\xi)p(\xi) - u'(\xi)\right)\, x^h(\xi) = 0,$$

$$\gamma^h(\xi)q(\xi) - \sum_{\mu \in \xi^+} \gamma^h(\mu)q(\mu) \geq 0, \qquad \left(\gamma^h(\xi)q(\xi) - \sum_{\mu \in \xi^+} \gamma^h(\mu)q(\mu)\right) z^h(\xi) = 0.$$

(c) The following transversality condition holds: $\lim_{T \to +\infty} \sum_{\xi \in D_T} \gamma^h(\xi)q(\xi)z^h(\xi) = 0.$

Since we only know that, for any plan $(p, q) \in \mathbb{P}$, the choice set $B^h(p, q)$ belongs to $\mathbb{R}^{D\times L}_+ \times \mathbb{R}^D_+$, it is not obvious that a plan of Kuhn-Tucker multipliers will exist. Thus, we need to develop a duality theory. As individual admissible plans are determined by countably many inequalities, we will construct Kuhn-Tucker multipliers using the Kuhn-Tucker Theorem for Euclidean spaces.

First of all, we want to note that, when Kuhn-Tucker multipliers exist and are used as intertemporal deflators, the discounted value of individual endowments is finite.

Proposition 2. (Finite discounted value of individual endowments)

Fix a plan $(p, q) \in \mathbb{P}$ and $y^h = (x^h, z^h) \in B^h(p, q)$ such that $U^h(x^h) < +\infty$. If Assumption 1 holds then for any family of Kuhn-Tucker multipliers associated to y^h, $(\gamma^h(\xi);\, \xi \in D)$, we have $\sum_{\xi \in D} \gamma^h(\xi)\, \left(p(\xi)\, w^h(\xi) + q(\xi)e^h(\xi)\right) < +\infty$.

Proposition 3. (Necessary and sufficient conditions for individual optimality)

Under Assumption 1, fix a plan $(p, q) \in \mathbb{P}$ and $y^h = (x^h, z^h) \in B^h(p, q)$. If $U^h(x^h) < +\infty$ and y^h is an optimal plan for agent $h \in H$ at prices (p, q), then there exists a family of Kuhn-Tucker multipliers associated to y^h. Reciprocally, if there exists a family of Kuhn-Tucker multipliers associated to y^h, then y^h is an optimal plan for agent h at prices (p, q).

5. Frictions induced by debt constraints, fundamental values and bubbles

In a frictionless world, that is, where financial debt constraints are non-saturated, there are two (equivalent) definitions of the fundamental value of an asset. The fundamental value is either (1) equal to the discounted value of future deliveries that an agent will receive for one unit of the asset that she buys and keeps forever; or, (2) equal to the discounted value of rental services, which coincides with the value of deliveries, given the absence of any friction associated to debt constraint.

These concepts do not coincide when frictions are allowed. Thus, we adopt the second definition, that includes the role that money has: it allows for intertemporal transfers, although its deliveries are zero.

Proposition 4.

Under Assumption 1, given an equilibrium $\left[(p,q);((x^h,z^h);h\in H)\right]$*, at each node* $\xi\in D$ *and for any* $h\in H$*,* $q(\xi)\geq F(\xi,q,\gamma^h)$*, where* $\gamma^h:=(\gamma^h(\xi);\xi\in D)$ *denotes the agent's h plan of Kuhn-Tucker multipliers and*

$$F(\xi,q,\gamma^h):=\frac{1}{\gamma^h(\xi)}\sum_{\mu\in D(\xi)}\left(\gamma^h(\mu)q(\mu)-\sum_{\nu\in\mu^+}\gamma^h(\nu)q(\nu)\right),$$

is the fundamental value of money at $\xi\in D$*.*

Note that the rental services that one unit of money gives at $\mu\in D$ are equal to the difference between the amount of resources that an agent pays at μ to have one unit of money, less the amount of resources (discounted to node μ) that the agent receives at nodes $\nu\in\mu^+$ when he sells this unit of money. That is, $q(\mu)-\sum_{\nu\in\mu^+}\frac{\gamma^h(\nu)}{\gamma^h(\mu)}q(\nu)$. Thus, the *fundamental value* of money at a node ξ, as was defined above, coincides with the discounted value of (unitary) future rental services.

On the other hand, under Assumption 1, it follows from Propositions 3 and 4 that, given an equilibrium $\left[(p,q);((x^h,z^h);h\in H)\right]$, there are, for each agent $h\in H$, Kuhn-Tucker multipliers $(\gamma^h(\xi);\xi\in D)$, such that,

$$q(\xi)=F(\xi,q,\gamma^h)+\lim_{T\to+\infty}\sum_{\{\mu\geq\xi:t(\mu)=T\}}\frac{\gamma^h(\mu)}{\gamma^h(\xi)}q(\mu),$$

where the second term in the right hand side is called the *bubble* component of $q(\xi)$. When $q(\xi)>F(\xi,q,\gamma^h)$ we say that fiat money has a bubble at ξ under γ^h.

Finally, we say that *debt constraints induce frictions over agent h in $\tilde{D} \subset D$* if the plan of shadow prices $(\eta^h(\mu);\ \mu \in \tilde{D})$ that is defined implicitly, at each $\mu \in \tilde{D}$, by the conditions:

$$\begin{aligned} 0 &= \eta^h(\mu) z^h(\mu), \\ \gamma^h(\mu) q(\mu) &= \sum_{\nu \in \mu^+} \gamma^h(\nu) q(\nu) + \eta^h(\mu), \end{aligned}$$

is different from zero.

Note that the fundamental value of money, $F(\xi, q, \gamma^h)$, can be expressed in terms of the shadow prices $(\eta^h(\mu);\ \mu \in D)$ as $F(\xi, q, \gamma^h) = \frac{1}{\gamma^h(\xi)} \sum_{\mu \in D(\xi)} \eta^h(\mu)$. In other words, the rental services that one unit of money gives are measured by the shadow prices of zero short-sales constraints.

6. Characterizing monetary equilibria

Let us see under what conditions can we have equilibria with positive price of money, also called *monetary equilibria*.

Theorem 1.

Suppose that Assumptions 1 holds and fix an equilibrium $\left[(p,q);((x^h,z^h);h \in H)\right]$. If Assumption 2 holds and at some node $\xi \in D$ the monetary price $q(\xi)$ is strictly positive, then debt constraints induce frictions over each agent in $D(\xi)$. Reciprocally, if debt constraints induce frictions over some agent in the sub-tree $D(\xi)$, then $q(\xi) > 0$.

Observation. This theorem is related to the result in Santos and Woodford (1997), Theorem 3.3, that asserted that, under uniform impatience, assets in positive net supply are free of price bubbles for deflators, in the asset pricing kernel, that yield finite present values of aggregate wealth. However, we may have a positive price of money due to the presence of shadow prices in the Kuhn-Tucker deflator process (and, in this case, for any kernel deflator, the present value of aggregate wealth will be infinite).

Proof of the Theorem 1.

By definition, if for some $h \in H$, $(\eta^h(\mu); \mu \geq \xi) = 0$ then $F(\xi, q, \gamma^h(\xi)) = 0$. Thus, a monetary equilibrium is a *pure bubble*. However, under uniform impatience (Assumption 2) bubbles are ruled out in equilibrium for the deflators given by the Kuhn-Tucker multipliers. Indeed, at each $\xi \in D$, and for any agent $h \in H$, $q(\xi) z^h(\xi) \geq 0$. Therefore, by the impatience property, $0 \leq (1-\pi) q(\xi) z^h(\xi) \leq p(\xi)\Delta(\xi)$. Moreover, as money is in positive net supply, it follows that $\left(\frac{q(\xi)}{p(\xi)\Delta(\xi)}; \xi \in D\right)$ is uniformly bounded. Since by Proposition 2, for any $h \in H$, $\sum_{\xi \in D} \gamma^h(\xi) p(\xi) w^h(\xi) < +\infty$, it follows from

Assumption 2 that bubbles do not arise in equilibrium. Therefore, we conclude that, if $q(\xi) > 0$ then $(\eta^h(\mu); \mu \geq \xi) \neq 0$, for all $h \in H$.

Reciprocally, by definition, if debt constraints induce frictions over some agent in the sub-tree $D(\xi)$, then the fundamental value of money at ξ is strictly positive, which implies that $q(\xi) > 0$. □

Corollary 1.

Under Assumption 1, given a monetary equilibrium $\left[(p,q); ((x^h, z^h); h \in H)\right]$, there always exists a plan of non-arbitrage deflators in the asset-pricing kernel, $(\nu(\xi); \xi \in D)$, for which the price of money is a pure bubble.

Proof. Fix an agent $h \in H$. It follows from Proposition 3 that there is a family of Kuhn-Tucker multipliers associated with the plan (x^h, z^h), namely $(\gamma^h(\xi); \xi \in D)$. Also, by Euler conditions, if $q(\xi) = 0$ at some node $\xi \in D$, then $q(\mu) = 0$, $\forall \mu > \xi$.

Define $\nu := (\nu(\xi) : \xi \in D)$ by $\nu(\xi_0) = 1$, and

$$\begin{aligned} \nu(\mu) &= 1, & \forall \xi \in D, \forall \mu > \xi : q(\xi) = 0, \\ \frac{\nu(\mu)}{\nu(\xi)} &= \frac{\gamma^h(\mu)}{\gamma^h(\xi) - \frac{\eta^h(\xi)}{q(\xi)}}, & \forall \xi \in D, \forall \mu \in \xi^+ : q(\xi) > 0. \end{aligned}$$

Euler conditions on $(\gamma^h(\xi); \xi \in D)$ imply that, for each $\xi \in D$, $\nu(\xi)q(\xi) = \sum_{\mu \in \xi^+} \nu(\mu) q(\mu)$. Therefore, using the plan of deflators ν, *financial Euler conditions* hold and the positive price of money is a bubble. □

Under Assumption 2, the plan $\left(\frac{q(\xi)}{p(\xi)\Delta(\xi)}; \xi \in D\right)$ is uniformly bounded along the event-tree and, therefore, the existence of a bubble for a plan of deflators $(\nu(\xi); \xi \in D)$ implies that the deflated value of future individual endowments, $\sum_{\xi \in D} \nu(\xi) p(\xi) w^h(\xi)$, has to be infinite for any agent $h \in H$. This plan of deflators, which is incompatible with physical Euler conditions, is compatible with zero shadow prices and our observation conforms to the results by Santos and Woodford (1997): a monetary bubble may only occur, for a plan of deflators in the asset pricing kernel, if the associated present value of aggregate wealth is infinite.

Some remarks:

∘ Adding an (unrestricted) asset that pays returns would not necessarily make fiat money useless, since the degree of market incompleteness may still leave room for a spanning role of money. On the other hand, if we allow for an increasing number of non-redundant securities in order to assure that aggregated wealth can be replicated by the deliveries of a portfolio trading plan, money will have zero price. Indeed, in this context, independently of the non-arbitrage deflator, the discounted value of future wealth must be finite (see Santos and Woodford (1997)). Therefore, if money has a positive value, we obtain a contradiction, since as we say above, we may always construct a deflator in the asset pricing kernel under which the discounted value of aggregated wealth is *infinite*. However, the issue of new assets in order to achieve that property of the financial markets can be too costly.

∘ In models addressing the role of money as a medium of exchange, starting with Clower (1967), it is instead liquidity frictions that become crucial. Grandmont and Younès (1972) consider a temporary equilibrium model where fiat money is the only store of value and prove that equilibrium exists as a consequence of some "viscosity" in the exchange process. In a recent work along those lines, but in the general equilibrium context, Santos (2006) showed that monetary equilibrium only arises when cash-in-advance constraints are binding infinitely often for all agents. Also, in a cashless economy with zero short-sales restrictions, Giménez (2007) provided examples of monetary bubbles that can be reinterpreted as positive fundamental values.

Next, we show that given a monetary equilibrium allocation, there is always another feasible allocation that makes some agent better off without hurting others, that is, monetary equilibria are inefficient in the Pareto sense. This claim is shown by noticing that, in our context, Pareto efficiency requires marginal rates of inter-temporal substitution to be equal across agents.[2]

Proposition 5. *Under Assumption 1, if for each* $\xi \in D$, $u^h(\xi, \cdot)$ *is differentiable in* $\mathbb{R}^L_{++}$ *and* $\lim_{\|x\|_{min}\to 0^+} \nabla u^h(\xi, x) = +\infty$, *then any monetary equilibrium is Pareto inefficient.*

Proof. Suppose that there exists an efficient monetary equilibrium. Since $\lim_{\|x\|_{min}\to 0^+} \nabla u^h(\xi, x) = +\infty$, $\forall (h, \xi) \in H \times D$, all agents have interior consumption along the event-tree. Positive net supply of money implies that there exists, at each $\xi \in D$, at least one lender. Therefore, by the efficiency property, it follows that *all* individuals have zero shadow prices. Therefore, it follows from the transversality condition of Definition 2, jointly with Proposition 4, that $q(\xi) = 0$ for any node $\xi \in D$. A contradiction. □

[2] More formally, an allocation $((\overline{x}^i(\xi))_{\xi\in D}, i \in H)$ is Pareto efficient if for each agent $i \in H$ it maximize the utility of the agent, $U^i((x^i(\xi))_{\xi\in D})$ among the allocations $((x^j(\xi))_{\xi\in D}, j \in H)$ that satisfies both $U^j((x^j(\xi))_{\xi\in D}) \geq U^i((\overline{x}^j(\xi))_{\xi\in D})$, $\forall j \neq i$; and $\sum_{j\in H} x^j(\xi) \leq W(\xi)$, $\forall \xi \in D$. Under the conditions of Proposition 5, the necessary Kuhn-Tucker conditions for this problems imply that the marginal rates of intertemporal substitution must be equal across agents.

The inefficiency of monetary equilibrium was previously addressed in the context of temporary equilibrium models with cash-in-advance constraints by Grandmont and Younès (1973). Also, as was analyzed by Hahn (1973) (see also Starrett (1973)), the existence of transactions costs may lead to inefficient allocations.

7. Monetary equilibrium in the absence of uniform impatience

To highlight the role of uniform impatience we adapt Example 1 in Araujo, Páscoa and Torres-Martínez (2008) in order to prove that without uniform impatience on preferences money may have a bubble for deflators that give a finite present value of aggregate wealth, even for Kuhn-Tucker multipliers. Essentially this happens because individuals will believe that, as time goes on, the probability that the economy may fall in a path in which endowments increase without an upper bound converges to zero fast enough. Notice that it must be the case that the supremum over all asset pricing kernel deflators of the present value of aggregate wealth is infinite (see Santos and Woodford (1997), Theorem 3.1 and Corollary 3.2).

Example. Assume that each $\xi \in D$ has two successors: $\xi^+ = \{\xi^u, \xi^d\}$. There are 'two agents $H = \{1,2\}$ and only one commodity. Each $h \in H$ has physical endowments $(w^h(\xi))_{\xi \in D}$, receives financial endowments $e^h \geq 0$ only at the first node, and has preferences represented by the utility function $U^h(x) = \sum_{\xi \in D} \beta^{t(\xi)} \rho^h(\xi)\, x(\xi)$, where $\beta \in (0,1)$ and the plan $(\rho^h(\xi))_{\xi \in D} \in (0,1)^D$ satisfies $\rho(\xi_0) = 1$, $\rho^h(\xi) = \rho^h(\xi^d) + \rho^h(\xi^u)$ and

$$\rho^1(\xi^u) = \frac{1}{2^{t(\xi)+1}} \rho^1(\xi), \qquad \rho^2(\xi^u) = \left(1 - \tfrac{1}{2^{t(\xi)+1}}\right) \rho^2(\xi).$$

Suppose that agent $h = 1$ is the only one endowed with the asset, i.e. $(e^1, e^2) = (1, 0)$ and that, for each $\xi \in D$,

$$w^1(\xi) = \begin{cases} 1 + \beta^{-t(\xi)} & \text{if } \xi \in D^{du}, \\ 1 & \text{otherwise}; \end{cases} \qquad w^2(\xi) = \begin{cases} 1 + \beta^{-t(\xi)} & \text{if } \xi \in \{\xi_0^d\} \cup D^{ud}, \\ 1 & \text{otherwise}; \end{cases}$$

where D^{du} is the set of nodes attained after going down followed by up, that is, $D^{du} = \{\eta \in D : \exists \xi,\ \eta = (\xi^d)^u\ \}$ and D^{ud} denotes the set of nodes reached by going up and then down, that is, $D^{ud} = \{\eta \in D : \exists \xi,\ \eta = (\xi^u)^d\ \}$.

Agents will use positive endowment shocks in low probability states to buy money and sell it later in states with higher probabilities. Let prices be $(p(\xi), q(\xi))_{\xi \in D} = (\beta^{t(\xi)}, 1)_{\xi \in D}$ and suppose that consumption of agent h is given by $x^h(\xi) = w^{h'}(\xi)$, where $h \neq h'$. It follows from budget constraints that, at each ξ, the portfolio of agent h must satisfy $z^h(\xi) = \beta^{t(\xi)}(w^h(\xi) - w^{h'}(\xi)) + z^h(\xi^-)$, where $z^h(\xi_0^-) := e^h$ and $h \neq h'$.

Thus, the consumption allocations above jointly with the portfolios $(z^1(\xi_0),\, z^1(\xi^u),\, z^1(\xi^d)) = (1,\, 1,\, 0)$ and $(z^2(\xi))_{\xi\in D} = (1 - z^1(\xi))_{\xi\in D}$ are budget and market feasible. Finally, given $(h,\xi) \in H\times D$, let $\gamma^h(\xi) = \rho^h(\xi)$ be the candidate for Kuhn-Tucker multiplier of agent h at node ξ. It follows that conditions below hold and they assure individual optimality (see Proposition 3 in the Appendix),

$$\begin{aligned}
(\gamma^h(\xi)p_\xi, \gamma^h(\xi)q(\xi)) &= (\beta^{t(\xi)}\rho^h(\xi), \gamma^h(\xi^u)q(\xi^u) + \gamma^h(\xi^d)q(\xi^d)),\\
\sum_{\{\eta\in D:\, t(\eta)=T\}} \gamma^h(\eta)q(\eta)z^h(\eta) \longrightarrow &\ 0, \qquad \text{as } T\to+\infty.
\end{aligned}$$

By construction, the plan of shadow prices associated to zero short-sales constraints is zero. Therefore, money has a zero fundamental value and a bubble under Kuhn-Tucker multipliers.

Intuitively, the price of the consumption good is falling at a rate that compensates the intertemporal impatience of the agents. Thus, since along the event-tree the purchase power of fiat money is rising, it is not worthy for agents to have short positions on money and, therefore, binding zero short-sale constraints do not induce frictions.

Also, the diversity of individuals' beliefs about the uncertainty (probabilities $\rho^h(\xi)$) implies that both agents perceive a finite present value of aggregate wealth.[3] Finally, Assumption 2 is not satisfied, because aggregated physical endowments were unbounded along the event-tree.[4] □

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[3]Using agent' h Kuhn-Tucker multipliers as deflators, the present value of aggregated wealth at $\xi\in D$, denoted by $PV^h(\xi)$, satisfies,

$$\begin{aligned}
PV^h(\xi) &= \sum_{\mu\geq\xi}\frac{\gamma^h(\mu)}{\gamma^h(\xi)}\,p(\mu)\,W(\mu) = \frac{2}{\rho^h(\xi)}\sum_{\mu\geq\xi}\rho^h(\mu)\beta^{t(\mu)} + \frac{1}{\rho^h(\xi)}\sum_{\{\mu\geq\xi:\mu\in D^{ud}\cup D^{du}\cup\{\xi_0^d\}\}}\rho^h(\mu)\\
&= 2\frac{\beta^{t(\xi)}}{1-\beta} + \sum_{\{\mu\geq\xi:\mu\in D^{ud}\cup D^{du}\cup\{\xi_0^d\},\, t(\mu)\leq t(\xi)+1\}}\frac{\rho^h(\mu)}{\rho^h(\xi)} + \sum_{s=t(\xi)+1}^{+\infty}\left[\frac{1}{2^{s+1}}\left(1-\frac{1}{2^s}\right)+\left(1-\frac{1}{2^{s+1}}\right)\frac{1}{2^s}\right]\\
&= 2\frac{\beta^{t(\xi)}}{1-\beta} + \frac{3}{2}\frac{1}{2^{t(\xi)}} - \frac{1}{3}\frac{1}{4^{t(\xi)}} + \frac{1}{\rho^h(\xi)}\sum_{\{\mu\geq\xi:\mu\in D^{ud}\cup D^{du},\, t(\mu)\leq t(\xi)+1\}}\rho^h(\mu) < +\infty.
\end{aligned}$$

[4]If Assumption B holds, there are $(\kappa,\pi)\in\mathbb{R}_{++}\times(0,1)$ such that, for any $\xi\in D^{uu} := \{\mu\in D: \exists\eta\in D; \mu = (\eta^u)^u\}$,

$$\frac{1}{\kappa} = \frac{w^h(\xi)}{\kappa} > \frac{1-\pi}{\beta^{t(\xi)}\rho^h(\xi)}\sum_{\mu>\xi}\rho^h(\mu)\beta^{t(\mu)}W(\mu), \quad \forall h\in H.$$

Thus, for all $(\xi,h)\in D^{uu}\times H$, $\beta^{t(\xi)}\left(\frac{1}{\kappa(1-\pi)} + W(\xi)\right) > PV^h(\xi)$. On the other hand, given $\xi\in D^{uu}$,

$$PV^1(\xi) \geq \frac{1}{\rho^1(\xi)}\sum_{\{\mu\geq\xi:\mu\in D^{ud}\cup D^{du},\, t(\mu)\leq t(\xi)+1\}}\rho^1(\mu) = 1 - \frac{1}{2^{t(\xi)+1}}.$$

Therefore, as for any $T\in\mathbb{N}$ there exists $\xi\in D^{uu}$ with $t(\xi)=T$, we conclude that, $\beta^T\left(\frac{1}{\kappa(1-\pi)}+2\right) > 0.5$, for all $T>0$. A contradiction.

Conclusion

It is well known that, under uniform impatience, positive net supply assets are free of bubbles for non-arbitrage kernel deflators that yield finite present values of wealth. However, this does not mean that prices cannot be above the series of deflated dividends for the deflators given by the agents' marginal rates of substitution, which also yield finite present values of individual wealth.

In this paper we showed that binding zero short-sales constraints lead to positive prices of fiat money. These monetary equilibria are Pareto improvements, but they always induce an inefficient distribution of physical resources.

Appendix

Proof of Proposition 2. Let $\mathcal{L}^h_\xi : \mathbb{R}^{L+1} \times \mathbb{R}^{L+1} \to \mathbb{R} \cup \{-\infty\}$ be the function defined by $\mathcal{L}^h_\xi(y(\xi), y(\xi^-)) = v^h(\xi, y(\xi)) - \gamma^h(\xi)\, g^h_\xi(y(\xi), y(\xi^-); p, q)$, where $y(\xi) = (x(\xi), z(\xi))$ and $v^h(\xi, \cdot) : \mathbb{R}^L \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is given by

$$v^h(\xi, y(\xi)) = \begin{cases} u^h(\xi, x(\xi)) & \text{if } x(\xi) \geq 0; \\ -\infty & \text{otherwise.} \end{cases}$$

It follows from Assumption 1 and Euler conditions that, for each $T \geq 0$,

$$\sum_{\xi \in D^T} \mathcal{L}^h_\xi(0,0) - \sum_{\xi \in D^T} \mathcal{L}^h_\xi(y^h(\xi), y^h(\xi^-)) \leq - \sum_{\xi \in D_T} \gamma^h(\xi) q(\xi)(0 - z^h(\xi)).$$

Therefore, as for each $\xi \in D$, $\gamma^h(\xi)\, g^h_\xi(y^h(\xi), y^h(\xi^-); p, q) = 0$, we have that, for any $S \in \mathbb{N}$,

$$\begin{aligned} 0 \leq \sum_{\xi \in D^S} \gamma^h(\xi) \left(p(\xi)\, w^h(\xi) + q(\xi) e^h(\xi)\right) &\leq \limsup_{T \to +\infty} \sum_{\xi \in D^T} \gamma^h(\xi) \left(p(\xi)\, w^h(\xi) + q(\xi) e^h(\xi)\right) \\ &\leq U^h(x^h) + \limsup_{T} \sum_{\xi \in D_T} \gamma^h(\xi) q(\xi) z^h(\xi) \\ &\leq U^h(x^h) < +\infty, \end{aligned}$$

which concludes the proof. □

Proof of Proposition 3. Suppose that $(y^h(\xi))_{\xi \in D}$ is optimal for agent $h \in H$ at prices (p, q). For each $T \in \mathbb{N}$, consider the truncated optimization problem,

$$(P^{h,T}) \qquad \begin{aligned} &\max \sum_{\xi \in D^T} u^h(\xi, x(\xi)) \\ &\text{s.t.} \begin{cases} g^h_\xi\,(y(\xi), y(\xi^-);\, p, q) \leq 0, & \forall \xi \in D^T, \text{ where } y(\xi) = (x(\xi), z(\xi)), \\ (x(\xi), z(\xi)) \geq 0, & \forall \xi \in D^T. \end{cases} \end{aligned}$$

It follows that, under Assumption 1, each truncated problem $P^{h,T}$ has a solution $(y^{h,T}(\xi))_{\xi\in D^T}$.[5] Moreover, the optimality of $(y^h(\xi))_{\xi\in D}$ in the original problem implies that $U^h(x^h)$ is greater than or equal to $\sum_{\xi\in D^T} u^h(\xi, x^{h,T}(\xi))$. In fact, the plan $(\tilde{y}_\xi)_{\xi\in D}$ that equals to $\tilde{y}_\xi = y^{h,T}_\xi$, if $\xi \in D^T$, and equals to $\tilde{y}_\xi = 0$, if $\xi \in D \setminus D_T$, is budget feasible in the original economy and, therefore, the allocation $(y^{h,T}(\xi))_{\xi\in D^T}$ cannot improve the utility level of agent h.

Define $v^h(\xi,\cdot) : \mathbb{R}^L \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ by

$$v^h(\xi, y(\xi)) = \begin{cases} u^h(\xi, x(\xi)) & \text{if } x(\xi) \geq 0; \\ -\infty & \text{in other case.} \end{cases}$$

where $y(\xi) = (x(\xi), z(\xi))$. Given a multiplier $\gamma \in \mathbb{R}$, let $\mathcal{L}^h_\xi(\cdot, \gamma; p, q) : \mathbb{R}^{L+1} \times \mathbb{R}^{L+1} \to \mathbb{R} \cup \{-\infty\}$ be the Lagrangian at node ξ, i.e.,

$$\mathcal{L}^h_\xi(y(\xi), y(\xi^-), \gamma;\, p, q) = v^h(\xi, y(\xi)) - \gamma\, g^h_\xi(y(\xi), y(\xi^-); p, q).$$

It follows from Rockafellar (1997, Theorem 28.3) that there exist non-negative multipliers $(\gamma^{h,T}(\xi))_{\xi\in D^T}$ such that the following saddle point property

$$\sum_{\xi\in D^T} \mathcal{L}^h_\xi(y(\xi), y(\xi^-), \gamma^{h,T}(\xi);\, p, q) \leq \sum_{\xi\in D^T} \mathcal{L}^h_\xi(y^{h,T}(\xi), y^{h,T}(\xi^-), \gamma^{h,T}(\xi);\, p, q), \tag{1}$$

is satisfied, for each plan $(y(\xi))_{\xi\in D^T} = (x(\xi), z(\xi))_{\xi\in D^T} \in \mathbb{R}^{L\times D^T}_+ \times \mathbb{R}^{D^T}_+$. Moreover, at each node $\xi \in D^T$, multipliers satisfy $\gamma^{h,T}(\xi)\, g^h_\xi(y^{h,T}(\xi), y^{h,T}(\xi^-); p, q) = 0$.

Analogous arguments to those made in Claims A1-A3 in Araujo, Páscoa and Torres-Martínez (2008) implies that,

[5] In fact, as $(y^h(\xi))_{\xi\in D}$ is optimal and $U^h(x^h) < +\infty$, it follows that there exists a solution for $P^{h,T}$ if and only if there exists a solution for the problem,

$$(\tilde{P}^{h,T}) \qquad \begin{array}{ll} \max & \sum_{\xi\in D^T} u^h(\xi, x(\xi)) \\ \text{s.t.} & \begin{cases} g^h_\xi\,(y(\xi), y(\xi^-);\, p, q) \leq 0, & \forall \xi \in D^T, \text{ where } y(\xi) = (x(\xi), z(\xi)), \\ z(\xi) \geq 0, & \forall \xi \in D^{T-1} \text{ such that } q(\xi) > 0 \\ z(\xi) = 0, & \text{if } \left[\xi \in D^{T-1} \text{ and } q(\xi) = 0\right] \text{ or } \xi \in D_T, \\ x(\xi) \geq 0, & \forall \xi \in D^T. \end{cases} \end{array}$$

Indeed, it follows from the existence of an optimal plan which gives finite utility that if $q(\xi) = 0$ for some $\xi \in D$, then $q(\mu) = 0$ for each successor $\mu > \xi$. Now, budget feasibility assures that,

$$z(\xi) \leq \frac{p(\xi)w^h(\xi)}{q(\xi)} + z(\xi^-), \ \forall \xi \in D^{T-1} \text{ such that } q(\xi) > 0.$$

As $z(\xi_0^-) = 0$, the set of feasible financial positions is bounded in the problem $(\tilde{P}^{h,T})$. Thus, budget feasible consumption allocations are also bounded and, therefore, the set of admissible strategies is compact. As the objective function is continuous, there is a solution for $(\tilde{P}^{h,T})$.

Claim. *Under Assumption 1, the following conditions hold:*

(i) For each $t < T$,

$$0 \leq \sum_{\xi \in D^t} \gamma^{h,T}(\xi) \left(p(\xi) w^h(\xi) + q(\xi) e^h(\xi) \right) \leq U^h(x^h).$$

(ii) For each $0 < t < T$,

$$\sum_{\xi \in D_t} \gamma^{h,T}(\xi) q(\xi) z^h(\xi^-) \leq \sum_{\xi \in D \setminus D^{t-1}} u^h(\xi, x^h(\xi)).$$

(iii) For each $\xi \in D^{T-1}$ and for any $y(\xi) = (x(\xi), z(\xi)) \geq 0$,

$$u^h(\xi, x(\xi)) - u^h(\xi, x^h(\xi)) \leq \left(\gamma^{h,T}(\xi) p(\xi) \, ; \; \gamma^{h,T}(\xi) q(\xi) - \sum_{\mu \in \xi^+} \gamma^{h,T}(\mu) q(\mu) \right) \cdot (y(\xi) - y^h(\xi)) + \sum_{\eta \in D \setminus D^T} u^h(\eta, x^h(\eta)).$$

Now, at each $\xi \in D$, $\underline{w}^h(\xi) := \min_{l \in L} w^h(\xi, l) > 0$. Also, as a consequence of monotonicity of $u^h(\xi)$, $||p(\xi)||_\Sigma > 0$. Thus, item (i) above guarantees that, for each $\xi \in D$,

$$0 \leq \gamma^{h,T}(\xi) \leq \frac{U^h(x^h)}{\underline{w}^h(\xi) \, ||p(\xi)||_\Sigma}, \quad \forall T > t(\xi).$$

Therefore, the sequence $(\gamma^{h,T}(\xi))_{T \geq t(\xi)}$ is bounded, node by node. As the event-tree is countable, there is a common subsequence $(T_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ and non-negative multipliers $(\gamma^h(\xi))_{\xi \in D}$ such that, for each $\xi \in D$, $\gamma^{h,T_k}(\xi) \to_{k \to +\infty} \gamma^h(\xi)$, and

$$\gamma^h(\xi) g^h_\xi(p, q, y^h(\xi), y^h(\xi^-)) = 0 \, ; \tag{2}$$

$$\lim_{t \to +\infty} \sum_{\xi \in D_t} \gamma^h(\xi) q(\xi) z^h(\xi^-) = 0 \, , \tag{3}$$

where equation (2) follows from the strictly monotonicity of $u^h(\xi)$, and equation (3) is a consequence of item (ii) (taking the limit as T goes to infinity and, afterwards, the limit in t).

Moreover, using item (iii), and taking the limit as T goes to infinity, we obtain that, for each $y(\xi) = (x(\xi), z(\xi)) \geq 0$,

$$u^h(\xi, x(\xi)) - u^h(\xi, x^h(\xi)) \leq \left(\gamma^h(\xi) p(\xi) \, ; \; \gamma^h(\xi) q(\xi) - \sum_{\mu \in \xi^+} \gamma^h(\mu) q(\mu) \right) \cdot (y(\xi) - y^h(\xi)).$$

It follows that $\left(\gamma^h(\xi) p(\xi) \, ; \gamma^h(\xi) q(\xi) - \sum_{\mu \in \xi^+} \gamma^h(\mu) q(\mu) \right)$ belongs to the super-differential set of the function $v^h(\xi, \cdot) + \delta(\cdot, \mathbb{R}^L_+ \times \mathbb{R}_+)$ at point $y^h(\xi)$, where $\delta(y, \mathbb{R}^L_+ \times \mathbb{R}_+) = 0$, when $y \in \mathbb{R}^L_+ \times \mathbb{R}_+$ and $\delta(y, \mathbb{R}^L_+ \times \mathbb{R}_+) = -\infty$, otherwise. Notice that, for each $y \in \mathbb{R}^L_+ \times \mathbb{R}_+$, $\varsigma \in \partial \delta(y, \mathbb{R}^L_+ \times \mathbb{R}_+) \Leftrightarrow 0 \leq \varsigma(y' - y), \; \forall y' \in \mathbb{R}^L_+ \times \mathbb{R}_+$.

Now, by Theorem 23.8 in Rockafellar (1997), for all $y \in \mathbb{R}^L_+ \times \mathbb{R}_+$, if $v'(\xi)$ belongs to $\partial\left[v^h(\xi,y)+\delta(y,\mathbb{R}^L_+ \times \mathbb{R}_+)\right]$ then there exists $\tilde v'(\xi) \in \partial v^h(\xi,y)$ such that both $v'(\xi) \geq \tilde v'(\xi)$ and $(v'(\xi)-\tilde v'(\xi))\cdot(x,z)=0$, where $y=(x,z)$. Therefore, it follows that there exists, for each $\xi \in D$, a super-gradient $\tilde v'(\xi) \in \partial v^h(\xi, y^h(\xi))$ such that,

$$\begin{aligned}\left(\gamma^h(\xi)p(\xi)\,;\ \gamma^h(\xi)q(\xi)-\sum_{\mu\in\xi^+}\gamma^h(\mu)q(\mu)\right)-\tilde v'(\xi) &\geq 0,\\ \left[\left(\gamma^h(\xi)p(\xi)\,;\ \gamma^h(\xi)q(\xi)-\sum_{\mu\in\xi^+}\gamma^h(\mu)q(\mu)\right)-\tilde v'(\xi)\right]\cdot(x^h(\xi),z^h(\xi)) &= 0.\end{aligned}$$

As $\tilde v'(\xi) \in \partial v^h(\xi, y^h(\xi))$ if and only if there is $u'(\xi) \in \partial u^h(\xi, x^h(\xi))$ such that $\tilde v'(\xi) = (u'(\xi),0)$, it follows from last inequalities that Euler conditions hold.

On the other side, item (i) in claim above guarantees that, $\sum_{\xi\in D}\gamma^h(\xi)(p(\xi)w^h(\xi)+q(\xi)e^h(\xi)) < +\infty$ and, therefore, equations (2) and (3) assure that,

$$\begin{aligned}\lim_{t\to+\infty}\sum_{\xi\in D_t}\gamma^h(\xi)q(\xi)z^h(\xi) &\leq \lim_{t\to+\infty}\sum_{\xi\in D_t}\gamma^h(\xi)\left(p(\xi)w^h(\xi)+q(\xi)e^h(\xi)+q(\xi)z^h(\xi^-)\right)\\ &\leq \lim_{t\to+\infty}\sum_{\xi\in D_t}\gamma^h(\xi)q(\xi)z^h(\xi^-) = 0,\end{aligned}$$

which implies that transversality condition holds.

Reciprocally, it follows from Euler conditions that, for each $T \geq 0$,

$$\sum_{\xi\in D^T}\mathcal{L}^h_\xi(y(\xi),y(\xi^-),\gamma^h(\xi);\,p,q)-\sum_{\xi\in D^T}\mathcal{L}^h_\xi(y^h(\xi),y^h(\xi^-),\gamma^h_\xi;\,p,q) \leq -\sum_{\xi\in D_T}\gamma^h(\xi)q(\xi)(z(\xi)-z^h(\xi)).$$

Moreover, as at each node $\xi \in D$ we have that $\gamma^h(\xi)g^h_\xi(y^h(\xi),y^h(\xi^-);p,q)=0$, each budget feasible plan $y=((x(\xi),z(\xi));\xi\in D)$ must satisfy

$$\sum_{\xi\in D^T}u^h(\xi,x(\xi))-\sum_{\xi\in D^T}u^h(\xi,x^h(\xi)) \leq -\sum_{\xi\in D_T}\gamma^h(\xi)q(\xi)(z(\xi)-z^h(\xi)).$$

Now, as the sequence $\left(\sum_{\xi\in D_T}\gamma^h(\xi)q(\xi)z^h(\xi)\right)_{T\in\mathbb{N}}$ converges, it is bounded. Thus,

$$\limsup_{T\to+\infty}\left(-\sum_{\xi\in D_T}\gamma^h(\xi)q(\xi)(z(\xi)-z^h(\xi))\right) \leq \limsup_{T\to+\infty}\left(-\sum_{\xi\in D_T}\gamma^h(\xi)q(\xi)z(\xi)\right) \leq 0$$

Therefore,

$$U^h(x)=\limsup_{T\to+\infty}\sum_{\xi\in D^T}u^h(\xi,x(\xi)) \leq U^h(x^h),$$

which guarantees that the plan $(x^h(\xi),z^h(\xi))_{\xi\in D}$ is optimal. □

Proof of Proposition 4. By Proposition 3, there are, for each agent $h \in H$, non-negative shadow prices $(\eta^h(\xi); \xi \in D)$, satisfying for each $\xi \in D$,

$$\begin{aligned} 0 &= \eta^h(\xi) z^h(\xi); \\ \gamma^h(\xi) q(\xi) &= \sum_{\mu \in \xi^+} \gamma^h(\mu) q(\mu) + \eta^h(\xi). \end{aligned}$$

Therefore,

$$\gamma^h(\xi) q(\xi) = \sum_{\mu \geq \xi} \eta^h(\mu) + \lim_{T \to +\infty} \sum_{\mu \in D_T(\xi)} \gamma^h(\mu) q(\mu).$$

As multipliers and monetary prices are non-negative, the infinite sum in the right hand side of equation above is well defined because its partial sums are increasing and bounded by $\gamma^h(\xi) q(\xi)$. This also implies that the limit of the (discounted) asset price exists. □

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