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Theory and Methodology

Preference programming and inconsistent interval judgments

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Abstract

The problem of derivation of the weights of alternatives from pairwise comparison matrices is long standing. In this paper, Lexicographic Goal Programming (LGP) has been used to find out weights from pairwise inconsistent interval judgment matrices. A number of properties and advantages of LGP as a weight determination technique have been explored. An algorithm for identification and modification of inconsistent bounds is also provided. The proposed technique has been illustrated by means of numerical examples.

Keywords: Analytic hierarchy process; Interval judgment; Preference programming

1. Introduction

Since the introduction of the Analytic Hierarchy Process (AHP) as a decision making tool by Saaty [12] in the late 70s it has been successfully applied in solving a variety of real world problems, ranging from Sports, Medicine, Banking, Sociology (e.g., projection of average family size), Political Science (e.g., Presidential election), Computer Science (e.g., software selection, evaluation of data base management systems, etc) to Management Science [6,18]. The major reasons behind AHP's popularity are its conceptual simplicity and its capability of handling subjective criteria and inconsistencies in the decision making process.

AHP solves a discrete multi-criteria decision making(MCDM) problem in four steps:

(i) Break down the problem into a hierarchy consisting of a finite number of levels, each level consisting of a finite number of elements,

(ii) construct the pairwise comparison matrices for all the criteria and for all the alternatives with respect to one criteria at a time,

(iii) derive the underlying weights from comparison matrices by using a suitable weight determination technique, and

(iv) synthesize the foregoing local weights to obtain global weights of the alternatives.

The present paper is concerned with steps (ii) and (iii). Saaty [12] proposed to use single points as the elements of the comparison matrices. Each element represents the degree of preference of one factor over another and is taken from the (1/9-9) ratio scale. But at the time of collecting data to investigate the effect of splitting objectives in the AHP, some respondents proposed us to use range of numbers for certain comparisons [9]. More specifically, respondents feel ease in articulating their "strength of preference" by means of range of numbers(may be called interval judgment) in some particular situation. In fact, strength of preference with respect to fuzzy criteria, e.g., taste, attrac-

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tiveness, quality, comfort, etc, cannot be adequately expressed by a single number. The other reasons of adopting interval judgments are probabilistic uncertainty in the decision making environment [13], group decision [7], incomplete information [3], at the initial phase of the elicitation process in the interactive mode [14], etc.

As in the case of single judgments, the crucial problem arises regarding how to derive weights of the factors (meaning criteria or alternatives) from interval comparison matrices, especially when they are inconsistent. In the following section, we summarize the previous works and propose our new technique.

2. Previous works on interval judgments

To deal with subjective factors, Van Laarhoven and Pedrycz [17] considered the entries in the comparison matrices as fuzzy numbers having triangular membership functions. For the same purpose, Buckley [5] used fuzzy intervals (i.e., fuzzy numbers with trapezoidal membership functions). Boender et al. [4] pointed out the fallacy in the normalization procedure for the computed fuzzy weights in the works of Van Laarhoven and Pedrycz [17] and subsequently they modified it.

Saaty and Vargas [13] introduced simulation approach to find out weight intervals from matrices of interval judgments. They also pointed out the difficulties in this approach.

Arbel [1] applied a Linear Programming (LP) approach to find out weights w_i's from an interval comparison matrix $A = ([l_{ij}, u_{ij}])_{n\times n}$, where l_{ij} and u_{ij} respectively denote the lower and upper bounds of a certain interval judgment. Salo and Hämäläinen [15] extended Arbel's LP approach considerably. They synthesized the set of local priority weights to obtain global weight intervals for the alternatives. In order to find out local set of weights, Arbel [1] as well as Salo and Hämäläinen [15] searched the set

$$\boldsymbol{W} = \left\{ w = (w_1, w_2, \ldots, w_n) \,\middle|\, l_{ij}w_j \leqslant w_i \leqslant u_{ij}w_j \quad \forall i,j \text{ and } \sum_{i=1}^{n} w_i = 1 \right\}. \tag{1}$$

The inequalities in (1) constitute a set of linear constraints, some of which may contradict with each other. This contradiction arises from the inconsistency in the DM's judgments. Evidently, for an inconsistent interval judgment matrix, the set $\boldsymbol{W}$ is empty, i.e., no set of weights exists which can satisfy all the judgments simultaneously. Here we can recall Arbel's [1, p. 323] question, "What happens when one finds that the elicitation process has yielded a non-solvable system of inequalities?". Kress [11] pointed out the ineffectiveness of Arbel's own suggestion of solving $n(n-1)$ number of linear programs to deal with inconsistent interval judgments. While searching for a more effective technique, Arbel and Vargas [2] proposed nonlinear programming technique. But in this technique, local optimum may not be global optimum because of the nonconvexity of the feasible region.

Salo's [14] "extended region" approach is based on reconstruction of the inconsistent comparison matrix by means of

$$\tilde{u}_{ij} = \max_k\{u_{ik}u_{kj}\}, \qquad \tilde{l}_{ij} = 1/\tilde{u}_{ji}, \quad \text{for all } i \text{ and } j.$$

However, the difference $(\tilde{u}_{ij} - \tilde{l}_{ij})$ may be much greater than $(u_{ij} - l_{ij})$, which may be unrealistic to the DM. Apart from this, $\tilde{u}_{ij}$ may violate the bounds of the fundamental ratio scale drastically. The larger the bounds, the slower is the rate of convergence. Although by applying Salo's [14] approach the weights can be determined, absolute dominance or pairwise dominance bounds for the weights of alternatives are to be obtained upon interaction with the DM. But at the time of analysis, the DM may not be available, or even if he is available, he may not be persuaded to change his old judgments. Above all one should not force the DM to be artificially consistent. Saaty [12, p. 237] writes: "...improving consistency does not mean getting an answer closer to real life situation. It only means that the ratio estimates in the matrix as a sample collection, are closer to being logically related than to being randomly chosen."

In this paper, we investigate the problem which is stated as: "If the DM is no longer available after filling up a comparison matrix with ranges of numbers and if the matrix is inconsistent, then what method should be adopted to determine the local weights?".

We have already mentioned that no set of weights exists to satisfy all the interval judgments, if the matrix turns out to be inconsistent. If this is the case, then our main objective is to satisfy as many judgments (i.e., cells in the comparison matrices) as possible. Evidently this task can be done by lexicographic goal programming(LGP).

3. Formulation of LGP as a weight determination technique

As already stated in the previous section, each judgment $[l_{ij}, u_{ij}]$ of the matrix $A = ([l_{ij}, u_{ij}])_{n\times n}$ gives rise to the following two inequalities

$$l_{ij} \leqslant w_i/w_j \leqslant u_{ij}, \tag{2}$$

i.e.

$$-w_i + l_{ij}w_j \leqslant 0, \qquad w_i - u_{ij}w_j \leqslant 0.$$

Upon introduction of the deviational variables p_{ij}, n_{ij}, p'_{ij} and n'_{ij}, the foregoing inequalities can be transformed into equalities as:

$$-w_i + l_{ij}w_j + n_{ij} - p_{ij} = 0, \tag{3}$$

$$w_i - u_{ij}w_j + n'_{ij} - p'_{ij} = 0. \tag{4}$$

Proceeding in this manner for all the entries of the concerned interval matrix, we have the following requisite formulation of LGP as a weight determination technique:

Lex. minimize

$$\boldsymbol{a} = \left(p_{nn} + n_{nn}, \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(p_{ij} + p'_{ij})\right)$$

subject to

$$\begin{aligned} &-w_i + l_{ij}w_j + n_{ij} - p_{ij} = 0,\\ &w_i - u_{ij}w_j + n'_{ij} - p'_{ij} = 0,\\ &\quad i = 1,2,\dots,n-1, \quad j = i+1,\dots,n,\\ &\sum_{i=1}^{n} w_i + n_{nn} - p_{nn} = 1,\\ &w_i, n_{ij}, p_{ij}, n'_{ij}, p'_{ij} \geqslant 0, \quad \text{for all } i \text{ and } j. \end{aligned} \tag{5}$$

Note that satisfaction of the normalization constraint has been kept at the first priority level followed by the priority of satisfaction of all the constraints. Although the term $p_{nn} + n_{nn}$ is always zero in the lexicographical minimization by LGP, it has been kept to show the number of priority levels. If the DM is more confident on some judgments, then the concerned deviational variables can be taken in the second and the rest in the subsequent priority level(s).

4. Some properties of LGP

After calculating the weights w_i by LGP from inconsistent interval judgment matrix, the ratios w_i/w_j for all i and j may or may not belong to the interval $[l_{ij}, u_{ij}]$, i.e., because of inconsistency some of the ratios w_i/w_j may violate the concerned interval $[l_{ij}, u_{ij}]$. Accordingly, we have the following lemma:

Lemma 1. *The deviational variables n_{ij}, p_{ij}, n'_{ij} and p'_{ij}, for some particular judgment $I_{ij} = [l_{ij}, u_{ij}]$, satisfy the following relations*:

$$n_{ij}p_{ij} = 0, \qquad n'_{ij}p'_{ij} = 0, \qquad p_{ij}p'_{ij} = 0. \tag{6}$$

Proof. The first two equalities directly follow from the theory of goal programming.

To prove the last equality, let us consider the following cases:

Case I: $w_i/w_j \in [l_{ij}, u_{ij}]$. In this case, we have

$$-w_i + l_{ij}w_j \leqslant 0, \qquad w_i - u_{ij}w_j \leqslant 0.$$

So, from (3) and (4), we get $n_{ij}, n'_{ij} \geqslant 0$. Hence, from the first two equalities, it follows that

$$p_{ij} = p'_{ij} = 0 \quad \Rightarrow \quad p_{ij}p'_{ij} = 0.$$

Case II: $w_i/w_j \leqslant l_{ij}$. In this case, we have

$$-w_i + l_{ij}w_j \geqslant 0, \qquad w_i - u_{ij}w_j \leqslant 0.$$

So, again from (3) and (4), we can infer that

$$p_{ij} > 0 \text{ and } p'_{ij} = 0 \quad \Rightarrow \quad p_{ij}p'_{ij} = 0.$$

A similar argument follows for the case $w_i/w_j \geqslant u_{ij}$. □

Theorem 1. *If the interval ${}^{(1)}I_{ij} = [l_{ij}, u_{ij}]$ of any pairwise comparison matrix $([l_{ij}, u_{ij}])_{n\times n}$ changes to ${}^{(2)}I_{ij} = [l_{ij} - \alpha_{ij}, u_{ij} + \beta_{ij}]$ for $\alpha_{ij}, \beta_{ij} \geqslant 0$, $i \neq j$, and weight vectors in the two cases are denoted by ${}^{(1)}w$ and ${}^{(2)}w$, respectively, then the deviational variables satisfy the following relation*:

$$(u_{ij}-l_{ij})({}^{(2)}w_j-{}^{(1)}w_j)+(\alpha_{ij}+\beta_{ij}){}^{(2)}w_j + ({}^{(1)}n_{ij}+{}^{(1)}n'_{ij})({}^{(2)}n_{ij}+{}^{(2)}n'_{ij}) - ({}^{(1)}p_{ij}+{}^{(1)}p'_{ij})+({}^{(2)}p_{ij}+{}^{(2)}p'_{ij})=0. \quad (7)$$

Proof. In the first case, we have the equations:

$$-{}^{(1)}w_i+l_{ij}{}^{(1)}w_j+{}^{(1)}n_{ij}-{}^{(1)}p_{ij}=0, \quad (8)$$

$${}^{(1)}w_i-u_{ij}{}^{(1)}w_j+{}^{(1)}n'_{ij}-{}^{(1)}p'_{ij}=0. \quad (9)$$

Adding (8) and (9), we have

$$(l_{ij}-u_{ij}){}^{(1)}w_j+({}^{(1)}n_{ij}+{}^{(1)}n'_{ij}) - ({}^{(1)}p_{ij}+{}^{(1)}p'_{ij})=0, \quad \text{or}$$

$$(u_{ij}-l_{ij}){}^{(1)}w_j-({}^{(1)}n_{ij}+{}^{(1)}n'_{ij}) + ({}^{(1)}p_{ij}+{}^{(1)}p'_{ij})=0. \quad (10)$$

Similarly, in the modified case we have the equation

$$(u_{ij}-l_{ij}){}^{(2)}w_j+(\alpha_{ij}+\beta_{ij}){}^{(2)}w_j + ({}^{(2)}p_{ij}+{}^{(2)}p'_{ij})-({}^{(2)}n_{ij}+{}^{(2)}n'_{ij})=0. \quad (11)$$

Subtraction of (10) from (11) gives the required relation. □

Theorem 2. *Let ${}^{(1)}w_i$ and ${}^{(2)}w_i$, $i=1,2,\dots,n$, be the two sets of weights obtained from the inconsistent interval judgment matrices $([l_{ij},u_{ij}])_{n\times n}$ and $([l_{ij}-\alpha_{ij},u_{ij}+\beta_{ij}])_{n\times n}$, $\alpha_{ij},\beta_{ij}\geqslant 0$, $i\neq j$, respectively. Then*

$$\sum_{i,j}{}^{(1)}p_{ij}\leqslant\sum_{i,j}\alpha_{ij}{}^{(2)}w_j+\sum_{(i,j)\in D}{}^{(2)}p_{ij} - \sum_{(i,j)\in D'}{}^{(2)}n_{ij}, \quad (12)$$

$$\sum_{i,j}{}^{(1)}p'_{ij}\leqslant\sum_{i,j}\beta_{ij}{}^{(2)}w_j+\sum_{(i,j)\in D}{}^{(2)}p'_{ij} - \sum_{(i,j)\in D'}{}^{(2)}n'_{ij}, \quad (13)$$

where D denotes the set of ordered pairs (i,j) for which w_i/w_j violates lower (or upper) bounds for both the matrices $([l_{ij},u_{ij}])_{n\times n}$ and $([l_{ij}-\alpha_{ij},u_{ij}+\beta_{ij}])_{n\times n}$ and D' denotes the set of ordered pairs (i,j) where w_i/w_j violates the lower (or upper) bound of $[l_{ij},u_{ij}]$ but not that of $[l_{ij}-\alpha_{ij},u_{ij}+\beta_{ij}]$.

Proof. For an inconsistent interval judgment matrix, only one of the lower and upper bounds is violated. Accordingly, either p_{ij} or p'_{ij} is positive. Without any loss of generality, let us assume that the lower bound is violated. Then by using Lemma 1, from the following equations

$$-{}^{(1)}w_i+l_{ij}{}^{(1)}w_j+{}^{(1)}n_{ij}-{}^{(1)}p_{ij}=0,$$

$$-{}^{(2)}w_i+l_{ij}{}^{(2)}w_j-\alpha_{ij}{}^{(2)}w_j+{}^{(2)}n_{ij}-{}^{(2)}p_{ij}=0,$$

we have

$${}^{(1)}p_{ij}=-{}^{(1)}w_i+l_{ij}{}^{(1)}w_j,$$

$$\alpha_{ij}{}^{(2)}w_j+{}^{(2)}p_{ij}=-{}^{(2)}w_i+l_{ij}{}^{(2)}w_j$$

(assuming lower bounds have got violated in both the cases), or

$${}^{(1)}p_{ij}-\alpha_{ij}{}^{(2)}w_j-{}^{(2)}p_{ij} = ({}^{(2)}w_i-{}^{(1)}w_i)+l_{ij}({}^{(1)}w_j-{}^{(2)}w_j)\leqslant 0$$

(since lower bound has been decreased) or

$${}^{(1)}p_{ij}\leqslant\alpha_{ij}{}^{(2)}w_j+{}^{(2)}p_{ij}. \quad (14)$$

If ${}^{(1)}n_{ij}=0$ and ${}^{(2)}n_{ij}>0$, then the computed ratio ${}^{(2)}w_i/{}^{(2)}w_j$ must be greater than $l_{ij}-\alpha_{ij}$. In this case, the inequality (14) will be replaced by

$${}^{(1)}p_{ij}\leqslant\alpha_{ij}{}^{(2)}w_j-{}^{(2)}n_{ij}. \quad (15)$$

On summing over all i and j in both the inequalities (14) and (15) we have

$$\sum_{i,j}{}^{(1)}p_{ij}\leqslant\sum_{i,j}\alpha_{ij}{}^{(2)}w_j+\sum_{(i,j)\in D}{}^{(2)}p_{ij} - \sum_{(i,j)\in D'}{}^{(2)}n_{ij},$$

where D and D' are as defined previously. The case of violation of the upper bounds can be treated similarly. □

Theorem 3. *Let w_i, ${}^{(1)}w_i$, ${}^{(2)}w_i$, $i=1,2,\dots,n$, denote the three sets of weights obtained respectively from the matrices $(\frac{1}{2}(l_{ij}+u_{ij}))_{n\times n}$, $([l_{ij},u_{ij}])_{n\times n}$, and $([l_{ij}-\alpha_{ij},u_{ij}+\beta_{ij}])_{n\times n}$, $\alpha_{ij},\beta_{ij}\geqslant 0$, and $i\neq j$. Then*

$$\sum_i|w_i-{}^{(1)}w_i|\leqslant\sum_i|w_i-{}^{(2)}w_i|. \quad (16)$$

Proof. Trivial. □

Theorem 4. *If there are m alternate solutions* $(w_1^k, w_2^k, \ldots, w_n^k)$, $k = 1, 2, \ldots, m$, *of* (5) *for a consistent matrix* $([l_{ij}, u_{ij}])_{n\times n}$, *i.e.,*

$$l_{ij} \leqslant w_i^k / w_j^k \leqslant u_{ij},$$

for $i = 1, 2, \ldots, n-1$, $j = i+1, \ldots, n$ *and* $k = 1, 2, \ldots, m$, *then*

$$l_{ij} \leqslant \bar{w}_i^k / \bar{w}_j^k \leqslant u_{ij},$$

where

$$\bar{w}_i^k = \frac{1}{m} \sum_{k=1}^{m} w_i^k, \quad i = 1, 2, \ldots, n.$$

Proof. Since $(w_1^k, w_2^k, \ldots, w_n^k)$, $k = 1, 2, \ldots, m$, are solutions of LGP in (5) for the consistent matrix $([l_{ij}, u_{ij}])_{n\times n}$, we have

$$l_{ij} \leqslant w_i^k / w_j^k \quad k = 1, 2, \ldots, m, \quad i = 1, 2, \ldots, n-1,\ j = i+1, \ldots, n,$$

$$\Rightarrow w_i^k \geqslant l_{ij} w_j^k,$$

$$\Rightarrow \frac{1}{m} \sum_{k=1}^{m} w_i^k \geqslant l_{ij} \frac{1}{m} \sum_{k=1}^{m} w_j^k,$$

$$\Rightarrow \bar{w}_i^k / \bar{w}_j^k \geqslant l_{ij},$$

Similarly, $\bar{w}_i^k / \bar{w}_j^k \leqslant u_{ij}$ for $k = 1, 2, \ldots, m$, $i = 1, 2, \ldots, n-1$ and $j = i+1, \ldots, n$. □

Theorems 2 and 3 are useful in comparing the sum of the values of the deviational variables and the difference of weights, respectively, obtained from the given as well as the modified interval pairwise comparison matrices. Theorem 4 has been used to determine the single weight vector where alternate weight vectors are available.

5. Advantages of LGP

There are some advantages of LGP, namely,

(i) Since linear programming is a special case of LGP, the problem which can be solved by LP can also be solved by LGP. In addition to this, since goal programming does not give infeasible solution, it is always capable of finding weights even in the presence of inconsistency.

(ii) In general, a single comparison matrix may consist of no judgment, single point judgment, multiple point judgments, interval judgment, multiple interval judgments, interval judgment having only one specified bound (e.g., $3 \leqslant w_i/w_j$ or $w_i/w_j \leqslant 5$) for some particular comparison. Moreover, the judgment(s) of this comparison may be inconsistent with other(s). Keeping all these points into account, LGP can be conveniently used to determine weights.

(iii) While forming comparison matrices, the DM may not be equally certain for all the pairs of comparisons, i.e., the degree of confidence for all the comparisons may not be equal. By degree of confidence, we mean how easily the DM has specified his judgments. Also, it can be easily checked that the marginal impact of all the cells on the resulting solution is not equal (Takeda et al. [16]). In addition to this, the DM may prefer to satisfy some particular judgments for some specific reason. If this is the case, then it can be easily accomplished by placing the concerned constraint at the higher priority level in the goal programming formulation. Even within same priority level, weights can be assigned to some particular deviational variables, if the DM wishes to do so. This advantage cannot be achieved by any of the existing methods.

(iv) If the DM's matrix turns out to be inconsistent, then there exists no set of weights which can satisfy all the judgments simultaneously. In this case, the DM may be satisfied for minimum number of violations of approximately articulated judgments. Typically this job can be done by LGP.

6. Identification and modification of inconsistent bounds

To determine weights by using LGP, if the value of

$$a_2 = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (p_{ij} + p'_{ij}),$$

the second component of the achievement vector, becomes positive, then this positivity indicates inconsistencies present in the articulated judgments. Presence of inconsistency can also be checked by the infeasibility of Arbel's LP model. To identify the bounds which contribute the largest amount of inconsistency, we propose the following algorithm:

Step 1. Calculate the weights of alternatives by LGP. If $a_2 > 0$, then go to Step 2, otherwise stop.

Step 2. Construct the matrices $\boldsymbol{L} = (l_{ij})_{n\times n}$ and $\boldsymbol{U} = (u_{ij})_{n\times n}$ taking the lower and upper bounds from the matrix $\boldsymbol{A} = ([l_{ij}, u_{ij}])_{n\times n}$, respectively. (Note that $\boldsymbol{L}$ and $\boldsymbol{U}$ are not reciprocal matrices.)

Step 3. Calculate the supplementary matrices $\boldsymbol{C} = (c_{ij})_{n\times n}$ and $\boldsymbol{D} = (d_{ij})_{n\times n}$ where

$$c_{ij} = \left(\prod_{k=1}^{n} l_{ik} l_{kj}\right)^{1/n} \quad \text{for } i = 1,2,\ldots,n-1,\; j = i+1,\ldots,n,$$

$$c_{ji} = 1/c_{ij}$$

and

$$d_{ij} = \left(\prod_{k=1}^{n} u_{ik} u_{kj}\right)^{1/n} \quad \text{for } i = 1,2,\ldots,n-1,\; j = i+1,\ldots,n,$$

$$d_{ji} = 1/d_{ij}.$$

Here, four cases may arise, namely,

(i) $l_{ij}, u_{ij} \geqslant 1 \quad c_{ij}, d_{ij} \geqslant 1$,
(ii) $l_{ij}, u_{ij} \geqslant 1 \quad c_{ij}, d_{ij} < 1$,
(iii) $l_{ij}, u_{ij} < 1 \quad c_{ij}, d_{ij} \geqslant 1$,
(iv) $l_{ij}, u_{ij} < 1 \quad c_{ij}, d_{ij} < 1$,

for $i = 1,2,\ldots,n-1$, $j = i+1,\ldots,n$.

Step 4. Let

$$l'_{ij} = \begin{cases} l_{ij} & \text{if } l_{ij} \geqslant 1, \\ 1/l_{ij} & \text{if } l_{ij} < 1, \end{cases}$$

$$c'_{ij} = \begin{cases} c_{ij} & \text{if } c_{ij} \geqslant 1, \\ 1/c_{ij} & \text{if } c_{ij} < 1. \end{cases}$$

Similar notations are used for the upper bound case.

Step 5. Calculate the absolute deviations

$$\rho_{ij} = |l'_{ij} * c'_{ij} - \gamma_{ij}| \quad \text{and} \quad \eta_{ij} = |u'_{ij} * d'_{ij} - \delta_{ij}|,$$

for all $i = 1,2,\ldots,n-1$, $j = i+1,\ldots,n$, where '$*$' denotes the '$-$' sign in cases (i) and (iv) and the '$+$' sign in cases (ii) and (iii) in Step 3. The values of γ_{ij} and δ_{ij} are 0 in cases (i) and (iv) and 2 in cases (ii) and (iii).

Step 6. Calculate the maximum of all the deviations ρ_{ij} and η_{ij}, $i = 1,2,\ldots,n-1$, $j = i+1,\ldots,n$. If ρ_{pq} is maximum, then modify l_{pq}, otherwise if η_{pq} gives the maximum value, change u_{pq}.

If $l_{pq} - c_{pq} < 0$, then l_{pq} should be increased, otherwise the modified value should be less than l_{pq}. Similar treatment follows for the matrix of upper bounds.

Go to Step 1.

The rationality of the above algorithm can be stated as folows:

When the matrix is inconsistent, we have $a_{ij} \neq a_{ik}a_{kj}$ for some k. In the presence of inconsistency, we take the geometric average of all possible a_{ij} values as

$$a_{ij} = \left(\prod_{k=1}^{n} a_{ik} a_{kj}\right)^{1/n}.$$

Next, we find the deviation of this average value from the actual value of a_{ij}. The process is repeated for all i and j. In this regard, we have considered the assertion, "the larger the deviation of the average value from the articulated value, the more is the amount of inconsistency captured by the element itself". The element corresponding to largest deviation is termed as the 'most inconsistent bound'.

7. Numerical examples

Example 1. Let us take the following interval judgment matrix [11].

	A	B	C	D
A	1	[1,2]	[1,2]	[2,3]
B		1	[3,5]	[4,5]
C			1	[6,8]
D				1

Kress [11] has shown that the matrix is inconsistent and hence cannot be solved by linear programming and also by the proposed modified method of Arbel [1]. The problem of weight determination from the foregoing matrix is solved by lexicographic goal programming and the calculated weights are 0.3030, 0.4545, 0.1515, and 0.0910 for A, B, C, and D, respectively, with $a_2 = 0.5758$.

Since the value of a_2 is positive, we apply the algorithm given in Section 6 for identification of the inconsistent bounds. It is observed that ρ_{34} (= 3.6834) and η_{34} (= 3.7705) are the maximum values among all ρ_{ij} and η_{ij}, respectively. Perhaps the judgment for the ordered pair (C, D), i.e., [6, 8], has been given carelessly. Changing this judgment to [2, 3] we get the modified set of weights as: 0.3000, 0.4500, 0.1500, and 0.1000, respectively, with $a_2 = 0.20$. Since a_2 is still positive, we calculate the absolute deviations ρ_{ij} and η_{ij} and it is observed that ρ_{23} (= 1.4349) and η_{14} (= 1.5663) are the maximum among the concerned deviations. Let us take the new bounds $u_{14} = 5$ and $l_{23} = 2$. The modified set of weights are obtained as (0.3846, 0.3846, 0.1538, 0.0769) or (0.3704, 0.3704, 0.1852, 0.0741) with $a_2 = 0.0$. Furthermore, instead of eliminating total inconsistency, a certain amount thereof can be eliminated by using the proposed algorithm. At this point, expectedly, there should be a cut-off value of a_2 (like the AHP consistency index) below which solution may be well acceptable. This point is under further exploration. On the other hand, if the DM does not wish to modify any of the inconsistent judgment(s), the the set of weights (0.3030, 0.4545, 0.1515, 0.0900) calculated by LGP should be used to calculate the overall weights.

Remark 1. In the usage of goal programming, two different forms of the 'achievement vector', namely, the lexicographic order of all the deviational variables (here deviational variables are uniformly weighted) and the linear weighting form (i.e., Archimedian form) are the most relevant to determine local weights from an interval pairwise comparison matrix. In the formulation (5) of goal programming, we have considered only the first form. The difficulty in using the second form is to provide the priority weights for the intervals. Nevertheless, if the DM is able to provide the same, this form can be used to estimate the weights. For instance, in the Archimedian form, we get the weight vector (0.3871, 0.3871, 0.1935, 0.0323) and $a = 0.7079$, assuming the objective function

$$\begin{aligned}\text{min: } a = (&1.5p_{12} + 1.5p'_{12} + p_{13} + p'_{13} + p_{14} + p'_{14} \\ &+ p_{23} + p'_{23} + p_{24} + p'_{24} + 1.5p_{34} \\ &+ 1.5p'_{34} + 2p_{44} + 2n_{44}).\end{aligned}$$

Remark 2. While deriving weights, we have considered only the upper triangular part of an interval matrix as it was considered by Arbel [1]. Actually, no new information is embodied in the lower triangular part, as the elements in this part are reciprocals of the corresponding elements in the upper triangular part. For the same reason, the most inconsistent bound(s) has been identified only from the elements in the upper triangular part.

Example 2. We adopt this example from Islam et al. [10]. Suppose, a person is interested to invest his money to any one of the four portfolios: bank deposit (BD), debentures (DB), government bonds (GB), and shares (SH). Out of these portfolios he has to choose only one based upon four criteria: return (Re), risk (Ri), tax benefits (Tb), and liquidity (Li).

The pairwise comparison matrices for all the criteria as well as for all the alternatives are as follows:

	Re	Ri	Tb	Li
Re	1	[3,4]	[5,6]	[6,7]
Ri		1	[4,5]	[5,6]
Tb			1	[3,4]
Li				1

Re	BD	DB	GB	SH
BD	1	[1/4, 1/3]	[3,4]	[1/6, 1/5]
DB		1	[6,7]	[1/5, 1/4]
GB			1	[1/7, 1/6]
SH				1

Ri	BD	DB	GB	SH
BD	1	[3,4]	[4,5]	[6,7]
DB		1	[3,4]	[5,6]
GB			1	[4,5]
SH				1

Tb	BD	DB	GB	SH
BD	1	1	[1/6,1/5]	[1/4,1/3]
DB		1	[1/6,1/5]	[1/4,1/3]
GB			1	[4,5]
SH				1

Li	BD	DB	GB	SH
BD	1	[3,4]	6	[6,7]
DB		1	[3,4]	[3,4]
GB			1	[3,4]
SH				1

Traditional LP does not give a solution to any one of the above matrices, implying the presence of some contradictory inequalities in all the cases. Keeping in view the satisfaction of as many cells as possible, LGP has been applied to calculate the weights. Initially, the length of each of the intervals $I_{ij} = [l_{ij}, u_{ij}]$ is assumed to be unity. In order to see variation in the weights, subsequently the intervals have been changed to $[l_{ij} - 0.5, u_{ij} + 0.5]$, $[l_{ij} - 1.0, u_{ij} + 1.0]$, and $[l_{ij} - 2.0, u_{ij} + 2.0]$, for all $i = 1, 2, \ldots, n-1$, $j = i+1, i+2, \ldots, n$ (for $l_{ij}, u_{ij} < 1$, changes are made in the denominators), where the respective lengths of changed intervals are 2, 3, and 5 units. The weights of the criteria and local weights of all the four portfolios are shown in Table 1 and Table 2, respectively. From these tables it is clear that the amount of inconsistency has been gradually decreased at the widening of the intervals. It is important to note that the lengths of the intervals have been increased to observe the nature of the varied weights, not to improve the consistency of the matrix. Consistency of the matrix has to be improved by using the algorithm given in the preceding section.

Table 1
Criteria weights for various lengths of intervals

Interval length	Re	Ri	Tb	Li	a_2
1	0.6383	0.2128	0.1064	0.0426	0.5745
2	0.5927	0.2371	0.0912	0.0790	0.3070
3	0.5581	0.2791	0.0930	0.0698	0.0465
5	0.6154	0.2308	0.0769	0.0769	0.0000

Table 2
Local weights of alternatives for various lengths of intervals

Criterion	Interval length	BD	DB	GB	SH	a_2
Re	1	0.1132	0.1698	0.0377	0.6792	0.6415
	2	0.1025	0.1903	0.0410	0.6662	0.4597
	3	0.1026	0.2051	0.0769	0.6154	0.2308
	5	0.1751	0.2355	0.0589	0.5305	0.0000
Ri	1	0.6250	0.2083	0.1250	0.0417	0.5417
	2	0.5985	0.2394	0.1088	0.0532	0.3096
	3	0.5333	0.2667	0.1333	0.0667	0.0667
	5	0.6000	0.2000	0.1333	0.0667	0.0000
Tb	1	0.1054	0.1054	0.6313	0.1578	0.3158
	2	0.0965	0.0965	0.6276	0.1793	0.1241
	3	0.0910	0.0910	0.6361	0.1819	0.0000
	5	0.0897	0.0897	0.7179	0.1026	0.0000
Li	1	0.6087	0.2029	0.1014	0.0870	0.3188
	2	0.5882	0.2353	0.0980	0.0784	0.1078
	3	0.5581	0.2791	0.0930	0.0698	0.0465
	5	0.6923	0.1154	0.1154	0.0769	0.0000

There are a number of alternate solutions for all the comparison matrices where the interval lengths of the entries are equal to five. In order to remove ambiguity in the choice of the preferred one from amongst the alternative solutions, following Salo and Hämäläinen [15] the global weights of the portfolios are obtained as ranges of numbers:

$$\mathrm{BD} = [0.1663, 0.3958], \quad \mathrm{DB} = [0.1873, 0.3430],$$
$$\mathrm{GB} = [0.1016, 0.2163], \quad \mathrm{SH} = [0.2340, 0.4199].$$

From these, we notice that the weight intervals of all the four portfolios overlap with each other, except the pair GB and SH. This overlapping creates the problem of choosing the best alternative. To overcome this difficulty, there have been two dominance relations, viz, absolute dominance and pairwise dominance [15], by which the most preferred alternative can be identified. For the present case, it is verified that no alternative emerges as most preferred by any of the two relations. This urges the DM to revise his earlier judgments to

Table 3
Global weights of alternatives obtained by LGP for various interval lengths of entries in the comparison matrices

Interval length	BD	DB	GB	SH
1	0.2431	0.1736	0.1163	0.4671
2	0.2583	0.1977	0.1114	0.4326
3	0.2558	0.2212	0.1249	0.3981
5	0.2576	0.2709	0.1639	0.3074

get the most preferred one. In case the DM does not wish to do so, then several alternatives are left before him. In one of them, he can take the average of the corresponding components of the local alternative solutions. Theorem 4 assures that the average weights satisfy the concerned set of constraints. For all the interval lengths, the global weights of the portfolios are shown in Table 3.

Theorem 3 has been verified for all the matrices of this example. In the course of verification, we considered the interval lengths as 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, and 5 units respectively, for each of the entries of the comparison matrices. The detailed numerical results are hereby omitted.

8. Summary and concluding remarks

Approximate articulation of preference ratios by means of range of numbers is a flexible way to deal with probabilistic or fuzzy uncertainties to choose the most preferred alternative in some decision making problem. In course of articulating these preference ratios, quite often, the DM becomes inconsistent in his own judgments. In fact, it is almost impossible to be consistent, especially when one uses the bounded (1/9-9) ratio scale. It may be said that consistency in AHP is exception rather than a rule. There has been arguments over the choice of the most suitable method to extract weights from inconsistent pairwise comparison matrices. This argument has not yet been settled.

While there is a number of methods to find out weights from inconsistent single judgment matrices, virtually there is not a single one for the inconsistent interval judgment matrices. In this paper, we have shown how LGP can be used to find out weights from such inconsistent interval judgment matrices. A number of advantages and properties of the proposed method are discussed. In addition to the proposed method, we have also provided one algorithm by which the most inconsistent bound can be identified and revised. There is greater possibility that a pairwise comparison matrix to be inconsistent and involve intervals for some of the entries to solve practical problems. In view of this the proposed method promises a substantial real-world applications.

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