Dynamic Markets with Randomly Arriving Agents

Maher Said

Yale University

5. August 2008

DYNAMIC MARKETS WITH RANDOMLY ARRIVING AGENTS

MAHER SAID†
AUGUST 5, 2008

ABSTRACT. We develop a model of a dynamic market with randomly arriving participants. Both buyers and sellers arrive probabilistically over time. The valuation of each buyer for each object is independently distributed and private information to each buyer. Equilibrium prices are determined by a sequence of second-price auctions. We examine the manner in which equilibrium behavior and payoffs are influenced by both current market conditions and anticipated future dynamics.

KEYWORDS: Dynamic markets, Random arrivals, Endogenous option value, Sequential auctions, Stochastic equivalence.

JEL CLASSIFICATION: C73, D44, D83.

†DEPARTMENT OF ECONOMICS, YALE UNIVERSITY, 28 HILLHOUSE AVENUE, NEW HAVEN, CT 06511
E-mail address: maher.said@yale.edu.
This paper is based on work previously circulated as "Sequential Auctions with Random Arrivals." I am grateful to my advisor, Dirk Bergemann, for his advice and guidance, as well as to Laure Athias, Luigi Balletta, Alessandro Bonatti, Chris Conlon, Rahul Deb, Eduardo Faingold, Ben Polak, and Colin Stewart for their many helpful conversations and suggestions, as well as to participants in the Yale Micro Theory Lunch and the 2008 World Congress of the Game Theory Society. This work was generously supported by a National Science Foundation Graduate Research Fellowship and a Cowles Foundation Anderson Fellowship.

1. Introduction

In this paper, we consider a model of dynamic markets in which buyers and sellers arrive randomly to the market. We examine the equilibrium behavior of market participants in response to these arrivals, as well as to changes in market conditions. In particular, we investigate the role of current and anticipated future conditions in determining endogenous outside options in this setting. We characterize the influence of these outside options on equilibrium price determination, and study the manner in which equilibrium payoffs and behavior are affected by both current conditions and future dynamics.

Consider the case of an individual who wants to purchase a new home. After surveying the houses available on the market, a potential buyer will determine which house best matches her individual preferences as well as her budgetary constraints, and will make an offer on that home. Obviously, the amount of her offer will depend on the physical characteristics of the home: the size of the house, the neighborhood or school district in which it lies, the potential for resale in the future, and so on. Note that this heterogeneity is evaluated differently by different buyers. In addition, however, market conditions will play a large role. In a "seller's market" in which demand for housing is large relative to supply, prices will be higher. Abstracting away from immediate housing needs, expectations about future market conditions and the dynamics governing them will also play an important role. If high demand is expected to be only a short-lived phenomenon, prices will be attenuated somewhat, especially if buyers are patient. Similarly, if a large number of homes are expected to come onto the market in the near future, prices will be further depressed. Thus, the offer of a potential homebuyer will depend on both the current competitiveness of the market as well as the anticipated characteristics of the market in the near future.

As an alternative example, consider online auction markets. eBay, for instance, has developed into a multi-billion dollar marketplace for the sale and resale of both new and used goods. Buyers on eBay for any particular good have available to them a wide array of different sellers from which they may purchase the object, with different participants in each auction and, often, slightly differing characteristics of the object in question. For instance, there is frequently some exogenous variation in the condition of a used object, and different sellers can, and do, ship their items from various locations. Therefore, as individuals place their bids, they must take into account the fact that they are not participating in a one-shot auction—they are free to costlessly participate in a later auction if they are outbid and lose.

The present work abstracts away from many of the details of these various markets, focusing on what we view as the essential features. In particular, we examine the situation in which buyers are confronted with an infinite sequence of auctions for stochastically equivalent objects that arrive at random times. Moreover, new buyers probabilistically arrive on the market in each period. Thus, in any given auction, buyers are presented with the outside option of participating in a future auction for an "equivalent" object, but with a potentially different number of competitors. We provide a precise characterization of this option value, and explore the manner in which it varies with the number of buyers currently present on the market, as well as the expectations of future market conditions.

Essentially, losing in an auction today yields the opportunity to participate in another auction in the future; however, the potential for entry by additional buyers and the random arrival times of auctions implies, in contrast to much of the literature on sequential auctions, that the competitive environment in the future may differ significantly from the present one. Thus, the expected payoff of a buyer currently present on the market is directly linked to her expected payoffs with a different number of competitors present in future periods. This leads to a difference equation characterizing the "outside option" available to each buyer. We show that this outside option is, in fact, an appropriately discounted sum of expected payoffs from participating in each of the infinite sequence of auctions with differing numbers of participants, where the weight on each auction is a combination of pure time discounting and the likelihood of market dynamics leading to the corresponding state. Moreover, optimal bidding behavior accounts for the option value in a straightforward and separable manner. Finally, this result is robust to various assumptions regarding the evolution of valuations, as well as to changes in the trading institution employed.

The present work is related to, and complements, several strands of the literature on dynamic markets and bargaining. In an insightful early work, Taylor (1995) examines bargaining power and price formation as it relates to the number of traders on each side of a market. His model assumes that agents are homogeneous—all buyers have the same commonly known value for purchasing an object—and that trade is conducted via posted prices. Our model enriches his setting by allowing for heterogeneous buyers and objects, employing an auction mechanism for price determination. The theme and questions of this paper are also connected to those of Satterthwaite and Shneyerov (2007). These authors consider a world in which a continuum of both buyers and sellers enter in every period, following time-invariant strategies in a steady-state equilibrium. The present work, on the other hand, is concerned with the behavior of agents in a dynamic environment with constantly changing conditions, and so a steady-state analysis runs counter to its goals.

Inderst (2008) considers a bargaining model in which a seller is randomly visited by heterogeneous buyers. If the seller is currently engaged in bargaining with one agent when another arrives, she may choose to switch from bargaining with one buyer to bargaining with the other. However, this switch is permanent, implying that the arrival of a new buyer either "restarts" the game or is completely irrelevant. Fuchs and Skrzypacz (forthcoming) take a different approach: they consider an incomplete information bargaining problem between a buyer and a seller, and allow for the possibility of the arrival of various "events." These events end the game and yield an exogenously determined expected payoff to each agent. The suggested interpretation is that these events may be viewed as triggers for some sort of multi-lateral mechanism involving new entrants (a second-price auction, for example) for which the expected payoffs are a reduced-form representation. Thus, while both works are primarily concerned with characterizing the endogenous option value that results from the potential arrival of additional participants to the market, they do this in a framework of bilateral bargaining which fails to capture the dynamic nature of direct competition between several current and future potential market participants.

The present work is also closely linked to certain elements of the sequential auctions literature. Engelbrecht-Wiggans (1994) studies sequential auctions in which a fixed number of perfectly patient buyers with single-unit demand compete in a finite sequence of second-price auctions for stochastically equivalent objects. His model, however, does not allow for several features of the present work; in particular, it does not allow for the entry of new buyers or consider the role of market dynamics in price determination. Said (2008) looks at a model with entry dynamics similar to the present work, but makes the complementary assumption of private values that remain constant over time.

Sailer (2006) and Zeithammer (2006) both conduct empirical examinations of eBay auctions while taking into consideration the sequential nature of that market. Although the latter is differentiated by the introduction of an assumption that buyers are able to observe their valuation for some upcoming objects, both authors assume a fixed number of competing buyers in each period, and therefore are unable to account for fluctuations in market conditions and competitiveness. Essentially, they assume away the existence of variation in market conditions. Thus, they are closely related to the special case of our model in which an auction occurs in every period and the winning bidder is always replaced by exactly one new buyer, ensuring that market conditions remain stationary.[1]

The paper is organized as follows. Section 2 presents our model, and Section 3 solves for the equilibrium. Section 4 discusses some comparative statics results. Section 5 demonstrates the robustness of the model in a setting where market characteristics may vary from one period to the next. Finally, Section 6 concludes and suggests some avenues for further research. All proofs are found in the Appendix.

2. THE MODEL

We consider the continuous-time limit of a discrete-time market model; periods of length Δ are indexed by $t \in \mathbb{N}$. There is a finite number n_t of risk-neutral buyers with single-unit demand on the market in any given period t. Each buyer $i \in \{1, \ldots, n_t\}$ has a valuation v_i^t, where v_i^t has distribution F on $\mathbb{R}_+$, where we assume that F has finite variance and a continuous density function f. Valuations are private information, and are independently and identically distributed, both across buyers and across objects over time; that is, the objects are, as in Engelbrecht-Wiggans (1994), stochastically equivalent. Finally, we assume that buyers discount the future exponentially with discount factor $\delta = e^{-r\Delta}$, where the discount rate is $r > 0$.

In each period, there is at most one seller present. The arrival of sellers is stochastic; in particular, there is some exogenously fixed probability $p = \lambda\Delta, \lambda > 0$ that a new seller arrives on the market in each period. Similarly, additional buyers may arrive on the market in each period. For simplicity, we will assume that at most one buyer arrives at a time, and that this arrival occurs with some exogenously given probability $q = \rho\Delta, \rho > 0$. In the event that a seller has arrived on the market, each buyer i observes both v_i^t and the number of competing buyers present. The seller then conducts a second-price auction to allocate their object.

Note that we assume that sellers are not strategic—they are unable to set a reserve price and are not permitted to remain on the market for more than one period. Conversely, buyers must participate in each auction that takes place when they are present on the market, but may submit

[1] Nekipelov (2007) also models eBay auctions, taking into account the random nature of entry in that market. However, his model is concerned with the role of new entrants *within* a *single* auction, and hence is best viewed as complementary to the present work.

negative bids. The only exception is the case in which a buyer would be the sole participant in an auction; in such situations, the buyer is given the option of receiving the object at a price of zero or waiting until the next period.[2]

We will write $Y_k^{(n)}$ to denote the k-th highest of n independent draws from F, and $G_k^{(n)}$ and $g_k^{(n)}$ will denote the corresponding distribution and density functions, respectively, of this random variable. In addition, we will define, for all $n \in \mathbb{N}$,

$$\widehat{Y}(n) := \mathbb{E}\left[Y_1^{(n)}\right] - \mathbb{E}\left[Y_1^{(n-1)}\right].$$

This is the expected difference between the highest of n and $n-1$ independent draws from the distribution F, where, by convention, we let $\mathbb{E}\left[Y_1^{(0)}\right] = 0$.

3. Equilibrium

Let $V(v_i^t, n)$ denote the expected payoff to a bidder when her valuation is v_i^t and she is one of n bidders present on the market. Recall that a seller must be currently present on the market for buyers to be aware of their valuations. Furthermore, let $W(n)$ denote the expected value to a buyer when she is one of n buyers present on the market at the beginning of a period, *before* the realization of the buyer and seller arrival processes. At the beginning of a period when there are n buyers present, there are four possible outcomes: with probability pq, both a buyer and a seller may arrive, leading to an auction with $n+1$ participants; with probability $p(1-q)$ only a seller arrives, yielding an auction with n participants; with probability $(1-p)q$ only a buyer may arrive, leading to the next period starting with $n+1$ participants; or, with the remaining probability $(1-p)(1-q)$, neither a buyer nor a seller may arrive, leading to the next period being identical to the current one. Thus,

$$\begin{aligned} W(n) := & \, pq\mathbb{E}[V(v_i^t, n+1)] + p(1-q)\mathbb{E}[V(v_i^t, n)] \\ & + (1-p)q\delta W(n+1) + (1-p)(1-q)\delta W(n). \end{aligned} \tag{1}$$

Let us now consider the problem facing buyer i when there are $n > 1$ buyers on the market and an object is currently available (and, hence, an auction is "about" to occur). This buyer, with valuation v_i^t must choose her bid b_i^t. If she wins the auction, she receives a payoff of v_i^t less the second-highest bid. On the other hand, if she loses, she remains on the market as one of $n-1$ buyers tomorrow, yielding her a payoff of $\delta W(n-1)$. Therefore,

$$V(v_i^t, n) = \max_{b_i^t} \left\{ \begin{array}{c} \Pr\left(b_i^t > \max_{j \neq i}\{b_j^t\}\right) \mathbb{E}\left[v_i^t - \max_{j \neq i}\{b_j^t\} | b_i^t > \max_{j \neq i}\{b_j^t\}\right] \\ + \Pr\left(b_i^t < \max_{j \neq i}\{b_j^t\}\right) \delta W(n-1) \end{array} \right\}.$$

We may use this expression in order to determine equilibrium bid functions, as demonstrated in the following

[2]This may be viewed as the reduced-form version of a bargaining game between the buyer and the seller: since the seller is impatient, she is willing to accept any offer made by the buyer. The buyer, however, may not wish to exercise her own outside option and wait for a new draw from the distribution F.

LEMMA 1 (Equilibrium bids)**.**
In equilibrium, a buyer with value v_i^t who is one of $n > 1$ buyers on the market bids $b_i^t = b^*(v_i^t, n)$, where

$$b^*(v_i^t, n) := v_i^t - \delta W(n-1). \tag{2}$$

Given this bidding strategy, the probability of i winning the auction in period t is simply $\Pr(v_i^t > \max_{j\neq i}\{v_j^t\})$, and the surplus gained in this case becomes $v_i^t - \max_{j\neq i}\{v_j^t\}$. Therefore, we may rewrite V as

$$\begin{aligned} V(v_i^t, n) &= \Pr(v_i^t > \max_{j\neq i}\{v_j^t\})\mathbb{E}\left[v_i^t - \max_{j\neq i}\{v_j^t\} | v_i^t > \max_{j\neq i}\{v_j^t\}\right] + \delta W(n-1) \\ &= \Pr\left(Y_1^{(n-1)} < v_i^t\right)\mathbb{E}\left[v_i^t - Y_1^{(n-1)} | Y_1^{(n-1)} < v_i^t\right] + \delta W(n-1) \\ &= G_1^{(n-1)}(v_i^t)\left(v_i^t - \mathbb{E}\left[Y_2^{(n)} | Y_1^{(n)} = v_i^t\right]\right) + \delta W(n-1), \end{aligned} \tag{3}$$

where the equivalence between the second and third lines relies on the properties of the highest and second-highest order statistics.[3]

Note that, *ex ante*, any one of the $n > 1$ buyers present on the market in any period is equally likely to have the highest value amongst her competitors. Thus, we may use the above to show that the ex ante expected utility of a buyer when there is an object available for sale is simply the sum of her probability of winning the object multiplied by the expected difference between the highest and second-highest values and the discounted option value of losing the object and remaining on the market in the next period. Formally, we have the following

LEMMA 2 (Expected auction payoffs)**.**
The expected payoff to a bidder from an auction with $n > 1$ participants is

$$\mathbb{E}[V(v_i^t, n)] = \widehat{Y}(n) + \delta W(n-1). \tag{4}$$

With this result in hand, we may rewrite Equation 1 for $n > 1$ in terms of W and $\widehat{Y}$ alone:

$$\begin{aligned} W(n+1) = &\frac{1 - \delta pq - \delta(1-p)(1-q)}{\delta(1-p)q} W(n) - \frac{p(1-q)}{(1-p)q} W(n-1) \\ &- \frac{p}{\delta(1-p)}\widehat{Y}(n+1) - \frac{p(1-q)}{\delta(1-p)q}\widehat{Y}(n). \end{aligned} \tag{5}$$

In the case in which $n = 1$, however, the expression differs due to a lone buyer's ability to "pass" on purchasing an object. In particular, we have

$$V(v_i^t, 1) = \max\left\{v_i^t, \delta W(1)\right\}.$$

Therefore,

$$\begin{aligned} W(1) = &pq\big(\widehat{Y}(2) + \delta W(1)\big) + p(1-q)\left(F\left(\delta W(1)\right)\delta W(1) + \left(1 - F\left(\delta W(1)\right)\right)\mathbb{E}\left[v | v > \delta W(1)\right]\right) \\ &+ (1-p)q\delta W(2) + (1-p)(1-q)\delta W(1). \end{aligned} \tag{6}$$

Thus, the expected payoff to a buyer is given by a solution to the second-order non-homogenous linear difference equation in Equation 5 and boundary condition provided by Equation 6. While

[3] See, for example, David and Nagaraja (2003, Chapter 3).

it is possible to find a solution to this system, moving to the continuous-time limit leads to a much more tractable solution. Thus, recalling that $\delta = e^{-r\Delta}, p = \lambda\Delta,$ and $q = \rho\Delta$ and taking the limit as Δ goes to zero, we have

$$W(n+1) = \frac{r+\lambda+\rho}{\rho}W(n) - \frac{\lambda}{\rho}\left(\widehat{Y}(n) + W(n-1)\right) \text{ for all } n > 1, \text{ and} \tag{7}$$

$$W(1) = \frac{\lambda\,(1-F(W(1)))}{r+\lambda(1-F(W(1)))+\rho}\mathbb{E}\left[v|v > W(1)\right] + \frac{\rho}{r+\lambda(1-F(W(1)))+\rho}W(2). \tag{8}$$

We may then rewrite this second-order difference equation as a first-order system of difference equations. In particular, we have, for all $k > 0$,

$$\begin{pmatrix} W(k+2) \\ W(k+1) \end{pmatrix} = \begin{bmatrix} \frac{r+\lambda+\rho}{\rho} & -\frac{\lambda}{\rho} \\ 1 & 0 \end{bmatrix} \begin{pmatrix} W(k+1) \\ W(k) \end{pmatrix} + \begin{pmatrix} -\frac{\lambda}{\rho}\widehat{Y}(k+1) \\ 0 \end{pmatrix}. \tag{9}$$

Note that there is an infinite number of solutions (in general) to this difference equation; indeed, even accounting for the boundary condition in Equation 8, there is a continuum of solutions which satisfy the difference equation. However, we are able to rule out solutions in which expected utility does not diverge to infinity (positive or negative) as the number of buyers on the market grows—it is possible to show that there exists a unique bounded (and hence "sensible") solution to the difference equation. To characterize this solution, define

$$\zeta_1 := \frac{r+\lambda+\rho-\sqrt{(r+\lambda+\rho)^2-4\lambda\rho}}{2\rho},$$
$$\zeta_2 := \frac{r+\lambda+\rho+\sqrt{(r+\lambda+\rho)^2-4\lambda\rho}}{2\rho}.$$

These two constants are the eigenvalues of the "transition" matrix in Equation 9. It is straightforward to show that $\zeta_1\zeta_2 = \frac{\lambda}{\rho}$ and $0 < \zeta_1 < 1 < \zeta_2$ for all $r, \lambda, \rho > 0$.

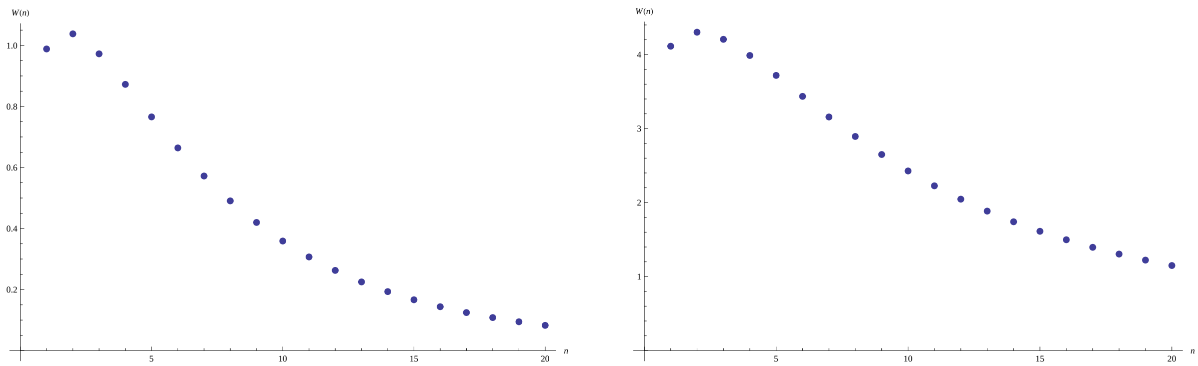
PROPOSITION 1 (Equilibrium payoffs with buyer arrivals)**.**
The unique symmetric equilibrium of this infinite-horizon sequential auction game is defined by the ex ante *expected payoff function given by*

$$W(n) = \zeta_1^{n-1}W(1) + \frac{\zeta_1^n\zeta_2}{\zeta_2-\zeta_1}\sum_{k=1}^{n-1}\left(\zeta_1^{-k}-\zeta_2^{-k}\right)\widehat{Y}(k+1) + \frac{\zeta_1\zeta_2^n-\zeta_1^n\zeta_2}{\zeta_2-\zeta_1}\sum_{k=n}^{\infty}\zeta_2^{-k}\widehat{Y}(k+1), \tag{10}$$

where $W(1)$ is the solution to

$$\begin{aligned}\frac{r+\lambda(1-F(W(1)))+\rho}{\rho}W(1) - \frac{\lambda}{\rho}(1-F(W(1)))\mathbb{E}[v|v \geq W(1)]& \\ = \zeta_1 W(1) + \zeta_1\zeta_2\sum_{k=1}^{\infty}\zeta_2^{-k}\widehat{Y}(k+1).&\end{aligned} \tag{11}$$

We may now calculate the expected payoffs of buyers for each value of $n \in \mathbb{N}$. Figure 1 displays an example of these paths for two distributions of values: the uniform and the exponential. A consistent feature of the plot of W is the "hump" shape, in which buyer values are initially increasing in n and then decreasing asymptotically towards zero. The reason for this is the manner



(a) Uniformly distributed values   (b) Exponentially distributed values

Figure 1: $W(n)$ with $r = 0.05,\ \lambda = 1$, and $\rho = 1$.

in which buyers shade their bids downwards; in particular, when a buyer is alone on the market and chooses to purchase an object, she receives the object for free, while if she were to be one of several buyers present, the amount she would have to pay when winning an object may be negative. Thus, for low levels of n, a buyer's expected payoff may be increasing in n. However, as the number of buyers present on the market increases, the likelihood of winning an object decreases, as does the likelihood of the second-highest value being low enough for the price to be negative. It is important to note that this observation is not an artifact of the second-price auction, as revenue equivalence implies that any standard auction format will lead to identical expected payments. Rather, this is due to the assumption that sellers may not set reserve prices and are willing to accept negative prices.[4]

Note that this solution is easily generalized to trading institutions other than the sequential second-price auction. By revenue equivalence, $\widehat{Y}(n)$ is the *ex ante* expected payoff of a buyer in any standard one-shot auction mechanism with n buyers. Therefore, Equation 10 continues to hold, as is, for markets in which objects are sold via, for instance, sequential first-price auctions—while buyers' bids will differ, their expected payoffs will remain the same, and hence Equation 10 continues to characterize equilibrium payoffs.

On the other hand, if a different trading institution were to be employed, then replacing $\widehat{Y}(n)$ by the appropriate *ex ante* expected payoff of a buyer in that mechanism would yield the corresponding solution for that institution. For example, suppose that each seller employs a (fixed) multi-lateral bargaining game for allocating her object. Letting $\widetilde{Y}(n)$ denote the *ex ante* expected payoff to each of n buyers from participating in this one-shot bargaining game, equilibrium in the resulting dynamic market is then characterized by the analogue to Equation 10 where $\widehat{Y}$ is replaced by $\widetilde{Y}$.

[4]This assumption is ultimately motivated by technical factors—if reserve prices (even one at zero) were permitted, then the equation characterizing W would become a second-order non-homogeneous and *nonlinear* difference equation. Solving such a difference equation numerically, let alone analytically, appears to be an exercise in futility.

4. Comparative Statics

In order to better understand the effects of time and entry on the payoffs of agents in this market, we consider some comparative statics. Unfortunately, changes in the parameters often have countervailing effects, and clean comparative statics results are difficult, if not impossible to derive. In particular, the effect of changes in the parameters depends upon the relationship between them as embodied in ζ_1 and ζ_2, and such effects may go in either direction.

Moreover, the changes in the parameters have, in addition to scale effects on $W(n)$, *shape* effects: the size, or even existence, of the "hump" described above is affected by changes in these parameters. This is most clearly seen by considering Figure 2, which plots W for various values of ρ. It is immediately apparent that the relationship between ρ and W is non-monotonic and dependent upon n and the other parameters.

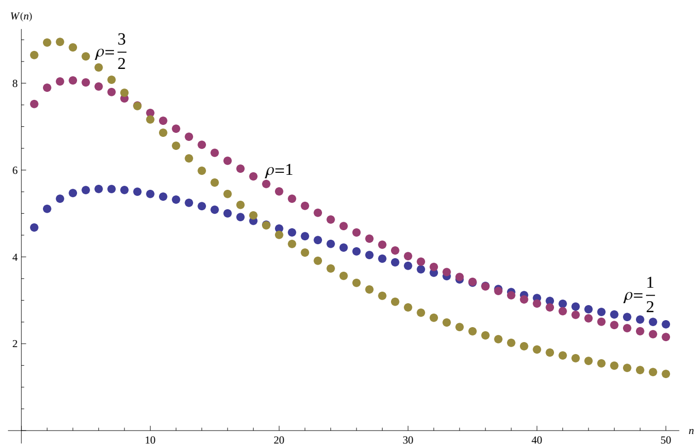


Figure 2: $W(n)$ with exponentially distributed values, $r = 0.05$, $\lambda = 2$, and $\rho \in \{\frac{1}{2}, 1, \frac{3}{2}\}$.

More positive results are possible for understanding the effects of changes in the distribution of values on payoffs. Notice that buyer welfare is an increasing function of $\widehat{Y}(k)$ for all $k \in \mathbb{N}$, and recall that

$$\widehat{Y}(k) = \mathbb{E}\left[Y_1^{(k)}\right] - \mathbb{E}\left[Y_1^{(k-1)}\right] = \frac{1}{k}\left(\mathbb{E}\left[Y_1^{(k)}\right] - \mathbb{E}\left[Y_2^{(k)}\right]\right).$$

Thus, a distributional change that systematically affects this difference will effectively lead to a change in buyer welfare. Notice, however, that replacing F by a distribution G such that stochastically dominates F is not sufficient for increasing $\widehat{Y}$. Although such a change increases both $\mathbb{E}\left[Y_1^{(k)}\right]$ and $\mathbb{E}\left[Y_2^{(k)}\right]$, it may decrease the difference between the two.[5] Thus, standard stochastic dominance is not sufficient for our purposes, as the ordering of the distributions is reversed when considering the difference between order statistics.[6]

The statistics literature, however, has identified two different conditions that are sufficient for our purposes. First, Kochar (1999) shows that if G dominates F in terms of the hazard rate order

[5] This difference is referred to by the statistics literature as a "sample spacing." The curious reader is referred to Boland, Shaked, and Shanthikumar (1998) for an overview of stochastic ordering of order statistics, and to Xu and Li (2006) for additional results on the stochastic ordering of sample spacings.

[6] We should point out that the bounded support of these two distributions is *not* essential to this argument.

and either F or G display a decreasing hazard rate, then $\widehat{Y}(k|G) \geq \widehat{Y}(k|F)$ for all $k \in \mathbb{N}$. Recall that the hazard rate of a distribution H with density h is given by

$$\lambda_H(t) = \frac{h(t)}{1 - H(t)}.$$

Therefore, G dominates F in terms of the hazard rate if

$$\lambda_G(t) \leq \lambda_F(t) \text{ for all } t.$$

Note that hazard rate dominance implies first-order stochastic dominance. Second, Bartoszewicz (1986) demonstrates that, if G dominates F in terms of the dispersive order, then we again have $\widehat{Y}(k|G) \geq \widehat{Y}(k|F)$ for all $k \in \mathbb{N}$. A distribution G dominates the distribution F in terms of the dispersive order if, for all $0 \leq x < y \leq 1$,

$$G^{-1}(y) - G^{-1}(x) \geq F^{-1}(y) - F^{-1}(x).$$

This is a variability ordering, in the sense that it requires the difference of quantiles of F to be smaller than the difference of the corresponding quantiles of G. Intuitively, when the quantiles of a distribution are more spread out, there is greater variance in the upper tail, and hence a larger difference between the first- and second-order statistics.

5. Markovian Values

Throughout, we have assumed that objects are stochastically equivalent; in effect, this implies that history is irrelevant except for its role in determining the current number of market participants. While this is a convenient assumption for the sake of tractability—implying, for instance, that we need not worry about issues of learning and the standard results of the second-price sealed-bid auction apply—it is somewhat limiting in its scope. We therefore generalize the model and consider a world in which history *does* matter, although in a manner that still allows for the same form of analysis. In particular, we consider a model in which buyers' values are drawn from one of two different distributions, and the distributions are "persistent" in the sense that they are chosen according to a Markov process. In particular, if the values are drawn from one distribution today, then they are likely to be drawn from the same distribution again tomorrow. The impetus behind such a modeling choice is the idea of a "buyer's" or "seller's" market; one distribution corresponds to the case in which, for some unmodeled exogenous reason, demand (and hence willingness to pay) is higher, independent of the number of current competitors, whereas the second corresponds to the case in which demand is relatively low. As the external forces driving value distributions typically do not change overnight, a buyer's market is likely to persist for some time.

Thus, we consider the case in which there are two states of the world $\{\omega_1, \omega_2\} \in \Omega$. In state ω_i, values are drawn from the distribution F_i (with corresponding density f_i). The state of the world is assumed to be commonly knowledge in each period. In addition, the (symmetric) probability of transitioning from one state to the other is $\tau = \pi\Delta, \pi > 0$.[7]

[7]The model is easily generalized to a greater number of states or asymmetric transitions; doing so would, however, greatly complicate notation and explication while providing little in the way of additional insight.

We will denote by $W(n, \omega_i)$ the expected payoff to a buyer when the state of the world is $\omega_i \in \Omega$ and there are k total buyers present in the market; recall that this is an *ex ante* payoff, as this is before a seller has arrived on the market in the period and the buyers do not yet know their valuations. We will denote by $V(v_i^t, n, \omega)$ the expected payoff to a buyer when a seller has arrived, and hence the buyer knows her value v_i^t, and, slightly abusing notation, we will write $\mathbb{E}[V(v_i^t, n, \omega_i)]$ to denote the expected value to a buyer who does not know her value yet but when a seller is present; the expectation is taken with respect to the distribution F_i.

Thus, for $i \in \{1, 2\}$, we have

$$W(n, \omega_i) = pq\mathbb{E}[V(v_i^t, n+1, \omega_i)] + p(1-q)\mathbb{E}[V(v_i^t, n, \omega_i)] \\ + \delta(1-p)\begin{bmatrix}(1-\tau)qW(n+1, \omega_i) + \tau qW(n+1, \omega_{-i}) \\ + (1-\tau)(1-q)W(n, \omega_i) + \tau(1-q)W(n, \omega_{-i})\end{bmatrix}.$$

Furthermore, it is relatively straightforward (using the same methods as in previous sections) to see that Lemma 1 again applies in this setting, implying that

$$\mathbb{E}[V(v_i^t, k, \omega_i)] = \widehat{Y}(k, \omega_i) + \delta\left[(1-\tau)W(k-1, \omega_i) + \tau W(k-1, \omega_{-i})\right].$$

Combining these two equations leads to a system of coupled second-order difference equations. Once again, the arithmetic becomes quite cumbersome due to the interaction of the various of parameters; therefore, we pass to the continuous time limit as Δ approaches zero. This yields

$$W(n+2, \omega_i) = \frac{r+\pi+\lambda+\rho}{\rho}W(n+1, \omega_i) - \frac{\pi}{\rho}W(n+1, \omega_{-i}) \\ - \frac{\lambda}{\rho}\widehat{Y}(n+1, \omega_i) - \frac{\lambda}{\rho}W(n, \omega_i) \text{ for all } n > 1. \tag{12}$$

Notice that this is a difference equation that is second-order in $W(\cdot, \omega_i)$. However, $W(\cdot, \omega_{-i})$ also appears in the equation, implying that we have a *coupled* system of difference equations. As in the previous section, we are interested in finding a bounded solution to this system of difference equations which also satisfies the appropriate boundary conditions. These boundary conditions may be found in a manner analogous to those discussed previously. Specifically, a buyer who is alone on the market may choose to wait, taking into account the possibility of the state switching before another seller arrives on the market. This leads to

$$(r + \lambda(1 - F_i(W(1, \omega_i))) + \rho + \pi)\, W(1, \omega_i) \\ = \lambda\,(1 - F_i(W(1, \omega_i)))\,\mathbb{E}[v|v > W(1, \omega_i)] + \rho W(2, \omega_i) + \pi W(1, \omega_{-i}). \tag{13}$$

Once again, this system of difference equations has a continuum of solutions; however, similar to the previous case, we can show that there exists a unique bounded solution that satisfies the boundary conditions of Equation 13. To characterize this solution, define

$$\xi_1 = \frac{r+\lambda+\rho+2\pi - \sqrt{(r+\lambda+\rho+2\pi)^2 - 4\lambda\rho}}{2\rho},$$
$$\xi_2 = \frac{r+\lambda+\rho+2\pi + \sqrt{(r+\lambda+\rho+2\pi)^2 - 4\lambda\rho}}{2\rho},$$
$$\xi_3 = \frac{r+\lambda+\rho - \sqrt{(r+\lambda+\rho)^2 - 4\lambda\rho}}{2\rho}, \text{ and}$$
$$\xi_4 = \frac{r+\lambda+\rho + \sqrt{(r+\lambda+\rho)^2 - 4\lambda\rho}}{2\rho}.$$

Note that $0 < \xi_1, \xi_3 < 1 < \xi_2, \xi_4$, and that $\xi_1\xi_2 = \xi_3\xi_4 = \frac{\lambda}{\rho}$. We then have

PROPOSITION 2 (Equilibrium payoffs with Markovian values)**.**
The unique symmetric equilibrium with bounded payoffs of this infinite-horizon sequential auction game is determined by the ex ante expected payoff functions given by, for $i = 1, 2$,

$$\begin{aligned} W(n,\omega_i) = {} & \xi_1^{n-1}\left(\frac{W(1,\omega_i) - W(1,\omega_{-i})}{2}\right) \\ & + \frac{\xi_1^n\xi_2}{\xi_2-\xi_1}\left(\sum_{k=1}^{n-1}\left(\xi_1^{-k} - \xi_2^{-k}\right)\frac{\widehat{Y}(k+1,\omega_i) - \widehat{Y}(k+1,\omega_{-i})}{2}\right) \\ & + \frac{\xi_1\xi_2^n - \xi_1^n\xi_2}{\xi_2-\xi_1}\left(\sum_{k=n}^{\infty}\xi_2^{-k}\frac{\widehat{Y}(k+1,\omega_i) - \widehat{Y}(k+1,\omega_{-i})}{2}\right) \\ & + \xi_3^{n-1}\left(\frac{W(1,\omega_i) + W(1,\omega_{-i})}{2}\right) \\ & + \frac{\xi_3^n\xi_4}{\xi_4-\xi_3}\left(\sum_{k=1}^{n-1}\left(\xi_3^{-k} - \xi_4^{-k}\right)\frac{\widehat{Y}(k+1,\omega_i) + \widehat{Y}(k+1,\omega_{-i})}{2}\right) \\ & + \frac{\xi_3\xi_4^n - \xi_3^n\xi_4}{\xi_4-\xi_3}\left(\sum_{k=n}^{\infty}\xi_4^{-k}\frac{\widehat{Y}(k+1,\omega_i) + \widehat{Y}(k+1,\omega_{-i})}{2}\right), \end{aligned} \tag{14}$$

where $W(1,\omega_i)$ and $W(1,\omega_{-i})$ solve the boundary conditions derived from Equation 13.

Note that when $\pi = 0$ or when the distributions F_1 and F_2 are identical, the solution in Equation 14 collapses to becomes identical to Equation 10; in particular, we have the case studied in earlier sections with only one state of the world. Moreover, the difference in expected payoffs between the two states is determined solely by the difference in sample spacings between the distributions in the two states. Note that

$$\begin{aligned} W(n,\omega_i) - W(n,\omega_{-i}) = \xi_1^{n-1}\left(W(1,\omega_1) - W(1,\omega_{-i})\right) \\ + \frac{\xi_1^n \xi_2}{\xi_2 - \xi_1}\left(\sum_{k=1}^{n-1}\left(\xi_1^{-k} - \xi_2^{-k}\right)\left(\widehat{Y}(k+1,\omega_i) - \widehat{Y}(k+1,\omega_{-i})\right)\right) \\ + \frac{\xi_1\xi_2^n - \xi_1^n\xi_2}{\xi_2 - \xi_1}\left(\sum_{k=n}^{\infty}\xi_2^{-k}\left(\widehat{Y}(k+1,\omega_i) - \widehat{Y}(k+1,\omega_{-i})\right)\right). \end{aligned}$$

Whenever this difference is positive, buyers strictly prefer to be in state ω_i than in state ω_{-i}; the converse of this is, of course, that bidding is more aggressive in state ω_i than in state ω_{-i} in the sense of absolute magnitude of bid shading away from the true values.[8] Figure 3 demonstrates an example of this in the case that F_1 dominates F_2 in terms of the dispersive order as discussed earlier. Note that $W(n,\omega_1) > W(n,\omega_2)$ for all n. Moreover, as may be seen in Figure 4, buyer's

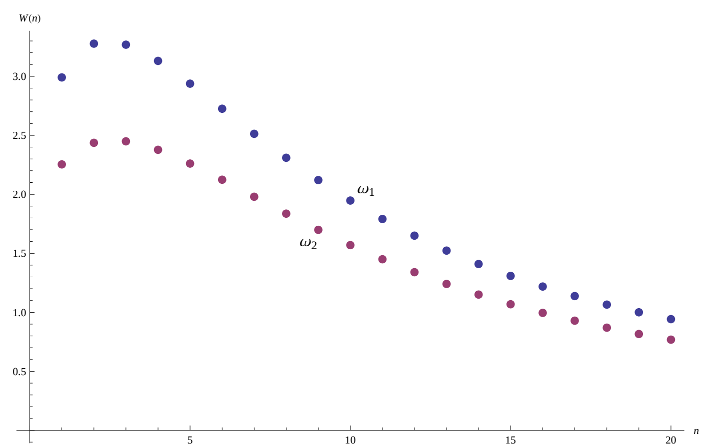


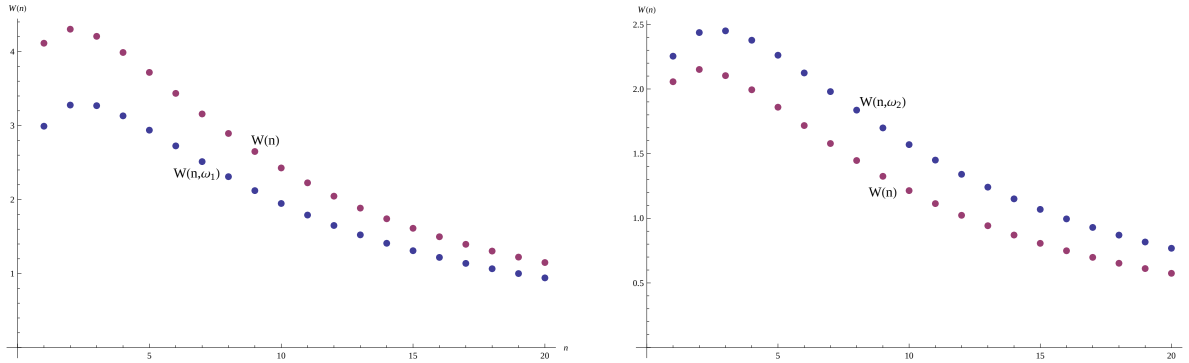
Figure 3: $W(n,\omega_i)$ with $F_1(v) = 1 - e^{-v}$, $F_2(v) = 1 - e^{-2v}$, $r = 0.05$, $\lambda = 1$, $\rho = 1$, and $\pi = 0.05$.

payoffs in state ω_1 are lower than they would be if there were no transitions to state ω_2, while the payoffs in state ω_2 are higher than they would otherwise be in a one-state model. In essence, the possibility of transitioning to an unambiguously better state improves buyer utility in state ω_2, while the possibility of transitioning to an unambiguously worse state decreases buyer utility in state ω_1.

6. Conclusion

This paper characterizes the manner in which current market conditions, as well as anticipated future conditions, create an endogenous option value for bidders in a dynamic market. In particular, the expected payoff to a buyer in such a market is the discounted sum of the potential payoffs from individual transactions in the infinite sequence, each differentiated by the potential number of buyers present on the market at that time. When the trading institution is an auction

[8]Note that since $\partial\xi_1/\partial\pi < 0 < \partial\xi_2/\partial\pi$, as the rate of state-to-state transitions increases without bound, there is enough churning between the two states that the differential between the state-contingent payoffs goes to zero.



(a) State ω_1 (b) State ω_2

Figure 4: $W(n, \omega_i)$ and $W(n)$ with distribution F_i, $r = 0.05,\ \lambda = 1$, $\rho = 1$, and $\pi = 0.05$.

mechanism, buyers are therefore willing to bid their true values less the discounted option value of participating in an auction in the following period.

There are several directions for extending our analysis. One possibility is dropping the assumption of stochastic equivalence and endowing each buyer with a fixed private value for obtaining an object. There are several technical difficulties in conducting such an analysis in a model with sealed-bid auctions. These complications are discussed in Said (2008), who instead examines a model in which objects are sold using the ascending auction format. Another potentially interesting line of research involves allowing for multiple simultaneous auctions, or allowing sellers to remain on the market for several periods and overlapping with one another. Additional possibilities include endogenizing the entry behavior of buyers and sellers in response to market conditions and dynamics, or allowing for the setting of reserve prices by sellers; in particular, considering the limit behavior of a model with a cap on the number of market participants may be particularly useful. These extensions are, however, left for future work.

APPENDIX

Proof of Lemma 1. Note that, since

$$\Pr(b_i^t < \max_{j\neq i}\{b_j^t\}) = 1 - \Pr(b_i^t > \max_{j\neq i}\{b_j^t\}),$$

we may rewrite $V(v_i^t, n)$ as

$$V(v_i^t, n) = \max_{b_i^t}\left\{ \begin{array}{c} \Pr(b_i^t > \max_{j\neq i}\{b_j^t\})\mathbb{E}\left[v_i^t - \delta W(n-1) - \max_{j\neq i}\{b_j^t\} | b_i^t > \max_{j\neq i}\{b_j^t\}\right] \\ +\delta W(n-1) \end{array} \right\}.$$

Since the trailing $\delta W(n-1)$ in the above expression is merely an additive constant, the maximization problem above is corresponds to that of a second-price auction with n bidders in which each bidder i's valuation is given by $v_i^t - \delta W(n-1)$. Thus, the standard dominance argument for static second-price auctions allows us to conclude that bidding $v_i^t - \delta W(n-1)$ is optimal. □

Proof of Lemma 2. Recall that Equation 3 provides an expression for $V(v_i^t, n)$. Taking the expectation of this expression with respect to v_i^t yields

$$\begin{aligned} \mathbb{E}\left[V(v_i^t, n)\right] &= \int_{-\infty}^{\infty}\left(x - \mathbb{E}\left[Y_2^{(n)} | Y_1^{(n)} = v_i^t\right]\right) f(x) G_1^{(n-1)}(x)\, dx + \int_{-\infty}^{\infty} \delta W(n-1) f(x)\, dx \\ &= \frac{1}{n}\left(\int_{-\infty}^{\infty} x g_1^{(n)}(x)\, dx - \int_{-\infty}^{\infty}\mathbb{E}\left[Y_2^{(n)} | Y_1^{(n)} = v_i^t\right] g_1^{(n)}(x)\, dx\right) + \delta W(n-1) \\ &= \frac{1}{n}\left(\mathbb{E}\left[Y_1^{(n)}\right] - \mathbb{E}\left[Y_2^{(n)}\right]\right) + \delta W(n-1). \end{aligned}$$

Moreover, it is straightforward (see Krishna (2002, Appendix C), for instance) to show that

$$\frac{1}{n}\left(\mathbb{E}\left[Y_1^{(n)}\right] - \mathbb{E}\left[Y_2^{(n)}\right]\right) = \mathbb{E}\left[Y_1^{(n)}\right] - \mathbb{E}\left[Y_1^{(n-1)}\right].$$

This is exactly the quantity previously defined as $\widehat{Y}(n)$, implying that

$$\mathbb{E}\left[V(v_i^t, n)\right] = \widehat{Y}(n) + \delta W(n-1),$$

as desired. □

Proof of Proposition 1. Define $w_m := (W(m+1), W(m))'$ and $y_m := \left(-\zeta_1\zeta_2\widehat{Y}(m+1), 0\right)'$ for all $m \in \mathbb{N}$. Then Equation 9 becomes

$$w_{n+1} = Aw_n + y_n,$$

where A is the matrix in Equation 9. Applying Elayedi (2005, Theorem 3.17) yields the general solution

$$w_n = A^{n-1}w_1 + \sum_{k=1}^{n-1} A^{n-k-1} y_k.$$

Recalling that ζ_1 and ζ_2 are the eigenvalues of the matrix A, it is straightforward to show that

$$A^k = \frac{1}{\zeta_2 - \zeta_1}\left[\begin{array}{cc} \zeta_2^{k+1} - \zeta_1^{k+1} & \zeta_1^{k+1}\zeta_2 - \zeta_1\zeta_2^{k+1} \\ \zeta_2^k - \zeta_1^k & \zeta_1^k\zeta_2 - \zeta_1\zeta_2^k \end{array}\right],$$

implying that the general solution to this second-order system may be (after some rearrangement) written as

$$\begin{aligned} W(n) = \frac{\zeta_2^{n-1}}{\zeta_2 - \zeta_1} \left(W(2) - \zeta_1 W(1) - \zeta_1\zeta_2 \sum_{k=1}^{n-1} \zeta_2^{-k} \widehat{Y}(k+1) \right) \\ - \frac{\zeta_1^{n-1}}{\zeta_2 - \zeta_1} \left(W(2) - \zeta_2 W(1) - \zeta_1\zeta_2 \sum_{k=1}^{n-1} \zeta_1^{-k} \widehat{Y}(k+1) \right). \end{aligned} \tag{A.1}$$

Consider the second term in the above expression. Since $0 < \zeta_1 < 1$, the first two parts of it are clearly bounded; in particular, we have $\zeta_1^{n-1}(\zeta_2 - \zeta_1)^{-1}(W(2) - \zeta_2 W(1)) \to 0$ as $n \to \infty$. The third part of this term may be rewritten as

$$\begin{aligned} \frac{\zeta_1^n \zeta_2}{\zeta_2 - \zeta_1} \sum_{k=1}^{n-1} \zeta_1^{-k} \widehat{Y}(k+1) &= \frac{\zeta_2}{\zeta_2 - \zeta_1} \sum_{k=1}^{n-1} \zeta_1^{k} \widehat{Y}(n-k+1) \\ &\le \frac{\zeta_2}{\zeta_2 - \zeta_1} \sum_{k=1}^{n-1} \zeta_1^{k}\sigma < \frac{\zeta_2}{\zeta_2 - \zeta_1} \sum_{k=1}^{\infty} \zeta_1^{k}\sigma < \frac{\zeta_1\zeta_2\sigma}{(\zeta_2 - \zeta_1)(1-\zeta_1)}, \end{aligned}$$

where σ^2 is the (assumed finite) variance of the distribution F. This follows from Arnold and Groeneveld (1979), who show that

$$\mathbb{E}\left[Y_1^{(m)}\right] - \mathbb{E}\left[Y_2^{(m)}\right] \le \frac{m\sigma}{\sqrt{m-1}} \text{ for all } m > 1.$$

Recalling the definition of $\widehat{Y}$, we then have

$$\widehat{Y}(m) \le \frac{\sigma}{\sqrt{m-1}} < \sigma$$

for all $m > 1$, implying that the second term in Equation A.1 is bounded.

The first term in Equation A.1, however, is multiplied by positive powers of $\zeta_2 > 1$, implying that an appropriate choice of $W(2)$ is crucial for ensuring the boundedness of our solution. One such choice is to let

$$W(2) = \zeta_1 W(1) + \zeta_1\zeta_2 \sum_{k=1}^{\infty} \zeta_2^{-k} \widehat{Y}(k+1). \tag{A.2}$$

Note that, for any $W(1) \in \mathbb{R}$, $W(2)$ is well-defined by the expression above, as $\zeta_2 > 1$ and $\{\widehat{Y}(m)\}$ is a bounded sequence. The first term in Equation A.1 then becomes

$$\frac{\zeta_1\zeta_2^n}{\zeta_2 - \zeta_1} \sum_{k=n}^{\infty} \zeta_2^{-k} \widehat{Y}(k+1) = \frac{\zeta_1}{\zeta_2 - \zeta_1} \sum_{k=n}^{\infty} \zeta_2^{n-k} \widehat{Y}(k+1) = \frac{\zeta_1}{\zeta_2 - \zeta_1} \sum_{k=0}^{\infty} \zeta_2^{-k} \widehat{Y}(n+k+1) < \infty,$$

where we again rely on the boundedness of the sequence $\{\widehat{Y}(m)\}$ when F has finite variance. Thus, for any choice of $W(1)$, choosing $W(2)$ in accordance with Equation A.2 leads to a bounded solution of the difference equation.

To show that this is, in fact, the *unique* bounded solution, consider any fixed $W(1)$, and denote by $\bar{c}$ the choice of $W(2)$ that accords with Equation A.2. Fix any arbitrary $\alpha \in \mathbb{R}$, and let $W(2) = \alpha\bar{c}$.

Then, denoting by $\bar{W}$ the solution when $W(2) = \bar{c}$, Equation A.1 becomes

$$\begin{aligned} W(n) &= \frac{\zeta_2^{n-1}}{\zeta_2 - \zeta_1}\left(\alpha\bar{c} - \zeta_1 W(1) - \zeta_1\zeta_2 \sum_{k=1}^{n-1} \zeta_2^{-k}\widehat{Y}(k+1)\right) \\ &\quad - \frac{\zeta_1^{n-1}}{\zeta_2 - \zeta_1}\left(\alpha\bar{c} - \zeta_2 W(1) - \zeta_1\zeta_2 \sum_{k=1}^{n-1} \zeta_1^{-k}\widehat{Y}(k+1)\right) \\ &= \frac{\zeta_2^{n-1}}{\zeta_2 - \zeta_1}\left(\bar{c} + (\alpha-1)\bar{c} - \zeta_1 W(1) - \zeta_1\zeta_2 \sum_{k=1}^{n-1} \zeta_2^{-k}\widehat{Y}(k+1)\right) \\ &\quad - \frac{\zeta_1^{n-1}}{\zeta_2 - \zeta_1}\left(\bar{c} + (\alpha-1)\bar{c} - \zeta_2 W(1) - \zeta_1\zeta_2 \sum_{k=1}^{n-1} \zeta_1^{-k}\widehat{Y}(k+1)\right) \\ &= \bar{W}(n) + \left(\zeta_2^{n-1} - \zeta_1^{n-1}\right)\frac{(\alpha-1)}{\zeta_2 - \zeta_1}\bar{c}. \end{aligned}$$

Since $\zeta_2 > 1 > \zeta_1 > 0$, the above expression remains bounded if, and only if, $\alpha = 1$.[9]

Thus, for any choice of $W(1)$, choosing $W(2)$ in accordance with Equation A.2 leads to a bounded solution. The only remaining free variable is then $W(1)$, which is then determined by the boundary condition derived from single-buyer behavior; that is, $W(1)$ may be found by combining Equation 8 and Equation A.2, leading to the condition found in Equation 11, as desired. □

Proof of Proposition 2. Letting $a := \frac{r+\pi+\lambda+\rho}{\rho}$, $b := -\frac{\pi}{\rho}$, and $c := -\frac{\lambda}{\rho}$, we may then write the coupled system defined by Equation 12 as

$$\begin{pmatrix} W_1(n+2) \\ W_2(n+2) \\ W_1(n+1) \\ W_2(n+1) \end{pmatrix} = \begin{bmatrix} a & b & c & 0 \\ b & a & 0 & c \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} W_1(n+1) \\ W_2(n+1) \\ W_1(n) \\ W_2(n) \end{pmatrix} + \begin{pmatrix} c\widehat{Y}_1(n+1) \\ c\widehat{Y}_2(n+1) \\ 0 \\ 0 \end{pmatrix}, \tag{A.3}$$

where the subscripts on W and $\widehat{Y}$ denote the state of the world. Writing A for the matrix above, $w_k = (W_1(k+1), W_2(k+1), W_1(k), W_2(k))'$, and $y_k = (c\widehat{Y}_1(k+1), c\widehat{Y}_2(k+1), 0, 0)'$, this can be written more compactly as

$$w_{n+1} = Aw_n + y_n.$$

Applying Elayedi (2005, Theorem 3.17), we then may conclude that the general solution to this system is

$$w_n = A^{n-1}w_1 + \sum_{k=1}^{n-1} A^{n-k-1}y_k. \tag{A.4}$$

[9] Note that we also have boundedness for arbitrary α if $\bar{c} = 0$. However, this would imply that $W(1) = -\zeta_2 \sum_{k=1}^{\infty} \zeta_2^{-k}\widehat{Y}(k+1) < 0$, contradicting the boundary condition in Equation 8.

In particular, if we denote by $a_{ij}^{(k)}$ the ij-th element of A^k, this may be rewritten as

$$\begin{pmatrix} W_1(n) \\ W_2(n) \end{pmatrix} = \begin{bmatrix} a_{31}^{(n-1)} & a_{32}^{(n-1)} & a_{33}^{(n-1)} & a_{34}^{(n-1)} \\ a_{41}^{(n-1)} & a_{42}^{(n-1)} & a_{43}^{(n-1)} & a_{34}^{(n-1)} \end{bmatrix} \begin{pmatrix} W_1(2) \\ W_2(2) \\ W_1(1) \\ W_2(1) \end{pmatrix} + c\sum_{k=1}^{n-1} \begin{bmatrix} a_{31}^{(n-k-1)} & a_{32}^{(n-k-1)} \\ a_{41}^{(n-k-1)} & a_{42}^{(n-k-1)} \end{bmatrix} \begin{pmatrix} \widehat{Y}_1(k+1) \\ \widehat{Y}_2(k+1) \end{pmatrix}. \tag{A.5}$$

Note that A is diagonalizable: defining $D := \operatorname{diag}[\xi_1, \xi_2, \xi_3, \xi_4]$ and $P := [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4]$ to be the diagonal matrix of eigenvalues of A the matrix formed from the corresponding eigenvectors, respectively, we may write $A = PDP^{-1}$. Therefore, $A^k = PD^kP^{-1}$, allowing for the explicit calculation of A^k for all k. In particular, we have

$$P = \begin{bmatrix} -\xi_1 & -\xi_2 & \xi_3 & \xi_4 \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

which implies that

$$a_{31}^{(k)} = a_{42}^{(k)} = \frac{\xi_2^k - \xi_1^k}{2(\xi_2 - \xi_1)} + \frac{\xi_4^k - \xi_3^k}{2(\xi_4 - \xi_3)}, \qquad a_{33}^{(k)} = a_{44}^{(k)} = \frac{\xi_1\xi_2^k - \xi_1^k\xi_2}{2(\xi_2 - \xi_1)} + \frac{\xi_3\xi_4^k - \xi_3^k\xi_4}{2(\xi_4 - \xi_3)},$$
$$a_{32}^{(k)} = a_{41}^{(k)} = \frac{\xi_1^k - \xi_2^k}{2(\xi_2 - \xi_1)} + \frac{\xi_4^k - \xi_3^k}{2(\xi_4 - \xi_3)}, \qquad a_{34}^{(k)} = a_{43}^{(k)} = \frac{\xi_1^k\xi_2 - \xi_1\xi_2^k}{2(\xi_2 - \xi_1)} + \frac{\xi_3\xi_4^k - \xi_3^k\xi_4}{2(\xi_4 - \xi_3)}.$$

Notice that (due to the symmetry detailed above), we need concentrate only on the value function from one state. Thus, we may (after some rearrangement) write

$$\begin{aligned} W_1(n) = {} & \frac{\xi_2^{n-1}}{\xi_2 - \xi_1}\left(\frac{W_1(2) - W_2(2)}{2} - \xi_1\frac{W_1(1) - W_2(1)}{2} - \xi_1\xi_2\sum_{k=1}^{n-1}\xi_2^{-k}\frac{\widehat{Y}_1(k+1) - \widehat{Y}_2(k+1)}{2}\right) \\ & - \frac{\xi_1^{n-1}}{\xi_2 - \xi_1}\left(\frac{W_1(2) - W_2(2)}{2} - \xi_2\frac{W_1(1) - W_2(1)}{2} - \xi_1\xi_2\sum_{k=1}^{n-1}\xi_1^{-k}\frac{\widehat{Y}_1(k+1) - \widehat{Y}_2(k+1)}{2}\right) \\ & + \frac{\xi_4^{n-1}}{\xi_4 - \xi_3}\left(\frac{W_1(2) + W_2(2)}{2} - \xi_3\frac{W_1(1) + W_2(1)}{2} - \xi_3\xi_4\sum_{k=1}^{n-1}\xi_4^{-k}\frac{\widehat{Y}_1(k+1) + \widehat{Y}_2(k+1)}{2}\right) \\ & - \frac{\xi_3^{n-1}}{\xi_4 - \xi_3}\left(\frac{W_1(2) + W_2(2)}{2} - \xi_4\frac{W_1(1) + W_2(1)}{2} - \xi_3\xi_4\sum_{k=1}^{n-1}\xi_3^{-k}\frac{\widehat{Y}_1(k+1) + \widehat{Y}_2(k+1)}{2}\right). \end{aligned}$$

Since $0 < \xi_1, \xi_3 < 1$ and both F and G are assumed to have finite variance, it is straightforward to verify that for any choices of $W_1(1), W_1(2), W_2(1)$, and $W_2(2)$ that the second and fourth terms in this expression are bounded. As in the main text, however, the fact that $\xi_2, \xi_4 > 1$ implies that the first and third terms may be unbounded if $W_1(2)$ and $W_2(2)$ are not chosen carefully. Therefore, let

$$W_1(2) := \frac{\xi_1+\xi_3}{2}W_1(1) + \frac{\xi_3-\xi_1}{2}W_2(1) \tag{A.6}$$
$$+\frac{\lambda}{\rho}\sum_{k=1}^{\infty}\frac{\xi_2^{-k}+\xi_4^{-k}}{2}\widehat{Y}_1(k+1) + \frac{\lambda}{\rho}\sum_{k=1}^{\infty}\frac{\xi_4^{-k}-\xi_2^{-k}}{2}\widehat{Y}_2(k+1) \text{ and}$$
$$W_2(2) := \frac{\xi_3+\xi_1}{2}W_2(1) + \frac{\xi_3-\xi_1}{2}W_1(1) \tag{A.7}$$
$$+\frac{\lambda}{\rho}\sum_{k=1}^{\infty}\frac{\xi_4^{-k}+\xi_2^{-k}}{2}\widehat{Y}_2(k+1) + \frac{\lambda}{\rho}\sum_{k=1}^{\infty}\frac{\xi_4^{-k}-\xi_2^{-k}}{2}\widehat{Y}_1(k+1).$$

Verifying that these values lead to a bounded solution for $W_1(n)$ and $W_2(n)$ for any choices of $W_1(1)$ and $W_2(1)$ follows in a manner directly analogous to that used in the proof of Proposition 1.

Finally, the values of $W_1(1)$ and $W_2(1)$ are given by the joint solution to the system of equations derived by equating the definitions of $W_1(2)$ and $W_2(2)$ above with the boundary condition from Equation 13:

$$\frac{r+\lambda(1-F_i(W(1,\omega_i))+\rho+\pi}{\rho}W(1,\omega_i) - \frac{\pi}{\rho}W(1,\omega_{-i}) - \frac{\lambda(1-F_i(W(1,\omega_i))}{\rho}\mathbb{E}[v|v>W(1,\omega_i)]$$
$$= \frac{\xi_1+\xi_3}{2}W(1,\omega_i) + \frac{\xi_3-\xi_1}{2}W(1,\omega_{-i}) \tag{A.8}$$
$$+\frac{\lambda}{\rho}\sum_{k=1}^{\infty}\frac{\xi_2^{-k}+\xi_4^{-k}}{2}\widehat{Y}(k+1,\omega_i) + \frac{\lambda}{\rho}\sum_{k=1}^{\infty}\frac{\xi_4^{-k}-\xi_2^{-k}}{2}\widehat{Y}(k+1,\omega_{-i}).$$

Uniqueness of the solution to the system of difference equations may then be shown in exactly the same manner as in the proof of Proposition 1. Thus, after some rearrangement, the unique bounded solution is given, as desired, by the expression found in Equation 14. □

References

ARNOLD, B. C., AND R. A. GROENEVELD (1979): "Bounds on Expectations of Linear Systematic Statistics Based on Dependent Samples," *Annals of Statistics*, 7(1), 220–223.

BARTOSZEWICZ, J. (1986): "Dispersive Ordering and the Total Time on Test Transformation," *Statistics and Probability Letters*, 4(6), 285–288.

BOLAND, P. J., M. SHAKED, AND J. G. SHANTHIKUMAR (1998): "Stochastic Ordering of Order Statistics," in *Handbook of Statistics*, ed. by N. Balakrishnan, and C. Rao, vol. 16, chap. 5, pp. 89–103. Elsevier.

DAVID, H., AND H. NAGARAJA (2003): *Order Statistics*. Wiley-Interscience, 3 edn.

ELAYEDI, S. (2005): *An Introduction to Difference Equations*. Springer, 3 edn.

ENGELBRECHT-WIGGANS, R. (1994): "Sequential Auctions of Stochastically Equivalent Objects," *Economics Letters*, 44, 87–90.

FUCHS, W., AND A. SKRZYPACZ (forthcoming): "Bargaining with Arrival of New Traders," *American Economic Review*.

INDERST, R. (2008): "Dynamic Bilateral Bargaining under Private Information with a Sequence of Potential Buyers," *Review of Economic Dynamics*, 11(1), 220–236.

KOCHAR, S. (1999): "On Stochastic Orderings Between Distributions and their Sample Spacings," *Statistics and Probability Letters*, 42(4), 345–352.

KRISHNA, V. (2002): *Auction Theory*. Academic Press, San Diego.

NEKIPELOV, D. (2007): "Entry Deterrence and Learning Prevention on eBay," Unpublished manuscript, Duke University.

SAID, M. (2008): "Information Revelation and Random Entry in Sequential Ascending Auctions," Unpublished manuscript, Yale University.

SAILER, K. (2006): "Searching the eBay Marketplace," CESifo Working Paper 1848, University of Munich.

SATTERTHWAITE, M., AND A. SHNEYEROV (2007): "Dynamic Matching, Two-Sided Incomplete Information, and Participation Costs: Existence and Convergence to Perfect Competition," *Econometrica*, 75(1), 155–200.

TAYLOR, C. (1995): "The Long Side of the Market and the Short End of the Stick: Bargaining Power and Price Formation in Buyers', Sellers', and Balanced Markets," *Quarterly Journal of Economics*, 110(3), 837–855.

XU, M., AND X. LI (2006): "Likelihood Ratio Order of m-spacings for Two Samples," *Journal of Statistical Planning and Inference*, 136(12), 4250–4258.

ZEITHAMMER, R. (2006): "Forward-Looking Bidding in Online Auctions," *Journal of Marketing Research*, 43, 462–476.