Bargaining Set Solution Concepts in Dynamic Cooperative Games

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Abstract. This paper is concerned with the question of defining the bargaining set, a cooperative game solution, when cooperation takes place in a dynamic setting. The focus is on dynamic cooperative games in which the players face (finite or infinite) sequences of exogenously specified TU-games and receive sequences of imputations against those static cooperative games in each time period. Two alternative definitions of what a 'sequence of coalitions' means in such a context are considered, in respect to which the concept of a dynamic game bargaining set may be defined, and existence and non-existence results are studied. A solution concept we term 'subgame-stable bargaining set sequences' is also defined, and sufficient conditions are given for the non-emptiness of subgame-stable solutions in the case of a finite number of time periods.

1. Introduction and Review of Literature

The study of repeated non-cooperative games – that is, the study of games whose structure is given by a discrete finite or infinite temporal framework in which at each time period a non-cooperative game is played and payoffs are determined accordingly – is one of the most richly studied topics in game theory. It has a history stretching back over half a century – the celebrated Folk Theorem of repeated non-cooperative game theory, to take just one example, was proved in the 1950s – and has influenced theories in several different disciplines, including political science, philosophy and evolutionary theory.

In contrast to the abundance of research in repeated non-cooperative games, to date the study of the analogous situation, in which the game played in each time period is a cooperative game, has been relatively sparse and comparatively quite recent. This is perhaps surprising, because the study of repeated cooperative games can be motivated just as readily as that of repeated non-cooperative game – many, if not most, cooperative endeavours occur more than once, or repeatedly over time. Examples can be easily adduced, such as multi-year profit-sharing arrangements, cost-sharing agreements, supply relationships, labour contracts, renewable treaty negotiations, and so forth. The insights gained from further progress in this topic should be expected to have broad implications.

To the best of our knowledge, the first papers devoted entirely to the systematic study of cooperative games played iteratively appeared in 2000, independently by [Oviedo; 2000] and [Kranich, Perea, Peters; 2001]. To those pioneering efforts have been added contributions by [Kranich, Perea, Peters; 2005], [Predtetchinski et al; 2002, 2004, 2006] and [Predtetchinski; 2007] and [Breden; 2007].

The above papers in the main concentrate on extensions, to the repeated setting, of the cooperative solution concepts the core and the Shapley value. It is the intention of this paper to contribute to the literature by considering to the bargaining set solution concept in the repeated setting, largely inspired by the frameworks for studying the core in repeated situations appearing in [Oviedo; 2000] and [Kranich, Perea, Peters; 2005].

Aside from the concentration on the bargaining set, as opposed to the core, this paper also differs from these other papers in the following way: [Oviedo; 2000] assumes throughout that paper that the underlying stage-games are super-additive – we study both super-additive and non-super-additive games; [Kranich, Perea, Peters; 2005] restrict their study to finite numbers of time periods, and to what is defined in our paper as 'repeated coalitions', as opposed to 'dynamic coalitions'. On the other hand, [Kranich, Perea, Peters; 2005] work with general time-dependent utility functions. In this paper, however, similar to the case in [Oviedo; 2000], the change in the utilities of payoffs over time is determined by a constant and universal discount factor δ^t , $0 < \delta < 1$.

2. Preliminaries

A *(static) cooperative transferable utility (TU) game* consists of a pair (N, v) such that N is a set of n elements, termed players, where n is a positive integer and $v : 2^N \to \mathbb{R}$, $v(\varnothing) = 0$ is termed the *characteristic function* of the game. A *coalition* is a subset of N. For any coalition S, $\mathbb{R}^S$ denotes the $|S|$–dimensional Euclidean space in which the dimensions are indexed by the members of S. Given any n-tuple x and coalition $S \subseteq N$, $x(S) \equiv \sum_{i \in S} x_i$.

If the characteristic function satisfies, for all coalitions $S, T \subseteq N$,

$$v(S \cup T) \geq v(S) + v(T) \quad \text{if } S \cap T = \varnothing$$

then the game is *superadditive*. Superadditivity will be assumed here only when explicitly noted. On the other hand, it will be assumed without loss of generality that $v(S) \geq 0$ for all functions and all characteristic coalitions.

A *coalition structure for* $S \subseteq N$ is a partition of S. We will denote by $\mathrm{C}(S)$, where $S \subseteq N$, the set of all coalition structures over S. If R is a coalition structure for N, we will write, by a slight abuse of notation, $\mathrm{R}(i)$ to stand for the element $Q \in \mathrm{R}$ such that $i \in Q$.

If (N,v) is a game and R is a coalition structure for *N*, the triple (N,v,R) is a *game with coalition structure*. For any (N,v,R),

$$I(N,v,\mathrm{R}) = \{x \in \mathbb{R}^N \mid x(S) \le v(S) \text{ for every } S \in \mathrm{R}, \text{and } x_i \ge v(\{i\}) \text{ for all } i \in N\}$$

denotes the set of *imputations* of (N,v,R). Given a coalition structure R for *S*, two players $i,j \in S$ will be said to be *partners* with respect to R, denoted $i \sim_{\mathrm{R}} j$, if both $i \in P$ and $j \in P$ for some $P \in \mathrm{R}$.

Given $k,l \in N$, with $k \ne l$, denote $T_{kl}(N) = T_{kl} = \{S \subseteq N \setminus \{l\} \mid k \in S\}$. Then an *objection* of *k* against *l* at $x \in I(N,v,\mathrm{R})$ is a pair (P,y) satisfying

$$P \in T_{kl}, y \in \mathbb{R}^P,\ y_i \ge x_i \text{ for all } i \in P,\ y_k > x_k \text{ and } y(P) \le v(P).$$

A *counter-objection* to an objection (P,y) of *k* against *l* at *x* is a pair (Q,z) satisfying

$$Q \in T_{lk},\ z \in \mathbb{R}^Q,\ z_i \ge x_i \text{ for all } i \in Q,\ z_i \ge y_i \text{ for all } i \in P \cap Q \text{ and } z(Q) \le v(Q).$$

An imputation $x \in I(N,v,\mathrm{R})$ is *stable* if for every objection at *x* there exists a counter-objection. The *(static) bargaining set* $M(N,v,\mathrm{R})$ is defined by

$$M(N,v,\mathrm{R}) = \{x \in I(N,v,\mathrm{R}) \mid x \text{ is stable}\}.$$

A game *without transferable utility* (*NTU game*) is a pair (N,V) where $V(S) \subseteq \mathbb{R}^S$ for each coalition *S*, and $V(\varnothing) = \varnothing$, along with the following additional conditions:

i. for all $S \ne \varnothing$, $V(S)$ is non-empty and closed
ii. if $x \in V(S)$ and $y_i \le x_i$ for all $i \in S$, then $y \in V(S)$
iii. for every $i \in N$ there is an $m_i \in \mathbb{R}$ with $V(\{i\}) = \{x \in \mathbb{R} \mid x_i \le m_i\}$, and, in addition $V(N) \cap \{x \in \mathbb{R}^N \mid x_i \ge m_i \text{ for all } i \in N\}$ is compact.

An NTU game with coalition structure R is denoted (N,V,R). If $S \subseteq N$ and $A \subseteq \mathbb{R}^S$, A_e is the set of weakly Pareto optimal elements of *A*, that is

$$A_e = \{x^S \in A \mid \text{for all } y \in A \text{ there exists } i \in S \text{ such that } x_i \ge y_i\}$$

Denote $X(N,V,\mathrm{R}) = \Pi_{R \in \mathrm{R}} V(R)_e$, $I(N,V,\mathrm{R}) = \{x \in X \mid x^i \ge v^i \text{ for all } i \in N\}$ and denote by $IX(N,V,\mathrm{R}) = X(N,V,\mathrm{R}) \cap I(N,V,\mathrm{R})$.

The definition of bargaining set for NTU games is as follows: Let $x \in IX(N,V,\mathrm{R})$ and let $k,l \in R$, $k \neq l$, for some $R \in \mathrm{R}$. An objection of *k* against *l* is a pair (P,y) such that

$$P \in T_{kl}, y \in V(P), \text{ and } y_i \geq x_i \text{ for all } i \in P, \text{ with } y_k > x_k.$$

A counter-objection to an objection (P,y) is a pair (Q,z) such that

$$Q \in T_{lk}, z \in V(Q), \; z^{Q/P} \geq x^{Q/P} \text{ and } z^{P \cap Q} \geq y^{P \cap Q}.$$

An objection (P,y) is justified if there is no counter-objection to (P,y). A vector $x \in IX(N,V,\mathrm{R})$ is stable if there is no justified objection at *x*, and the bargaining set of (N,V,R) is the set of stable vectors.

3. Dynamic Games

Turning to the intertemporal context, assume that time is divided into discrete time periods. Let *m* be either a non-negative integer or ω. If *m* is a finite integer, the relevant time periods are taken from $T = \{0,1,...,m\}$. If *m* is ω, *T* is $\{0,1,...\}$. To enable infinite and finite sequences to be dealt with in a unified manner as far as possible in this paper, a sequence of numbers written here as $(x^0, x^1,..., x^m)$ will be understood to stand for the infinite sequence $(x^0, x^1,...)$ if *m* is ω.

Fix *N*, a sequence of characteristic functions $\mathbf{v} = (v^0, v^1,..., v^m)$, a coalition structure R, and a discount factor $0 < \delta < 1$. Then $(N, \mathbf{v}, \mathrm{R}, m+1, \delta)$ will be termed a *dynamic cooperative game.* The special case in which there is a single characteristic function *v* such that $v^t = v$ for all time periods *t* can, in analogy to what is customary in the non-cooperative case, be called a *repeated cooperative game* based on the *stage-game* (N, v, R). In any case, for each integer $0 \leq t \leq m$, (N, w^t, R, t) will refer to the static cooperative game defined by the characteristic function

$$w^t(S) = (1-\delta)\delta^t v^t(S)$$

over *N* and R, and will be called the *stage-game played at time t*. The set of stage-game imputations at time *t* is defined[1] by

$$I(N, w^t, \mathrm{R}, t) = \{x \in \mathbb{R}^N \mid x(Q) \leq w^t(Q) \text{ for every } Q \in \mathrm{R}\}$$

[1] Note that we do not demand that each stage-game imputation satisfy individual rationality in its respective time period, thus enabling greater flexibility in the choice of stage-game imputations. Over-all individual rationality relative to the dynamic game, however, is required.

A sequence of vectors $\mathbf{x} = (x^0, x^1, ..., x^m)$ such that x^t is an imputation vector of the stage-game (N, w^t, R, t) for each t and $\sum_{t=0}^{m} x_i^t \geq \sum_{t=0}^{m} w^t(\{i\})$ for each player i, is an *imputation sequence* of the dynamic game $(N, \mathbf{v}, \mathrm{R}, m+1, \delta)$. The set of imputation sequences of the dynamic game $(N, \mathbf{v}, \mathrm{R}, m+1, \delta)$ will be denoted by $\mathbf{I}(N, \mathbf{v}, \mathrm{R}, m+1, \delta)$. Given $\mathbf{x} = (x^0, x^1, ..., x^m)$, we will let $\mathbf{x}_i$ refer to the sequence of real numbers $(x_i^0, x_i^1, ..., x_i^m)$, where x_i^t is the payoff given to player $i \in N$ according to the imputation x^t at time *t*. We will also denote $\bar{\mathbf{x}}_i = \sum_t x_i^t$, and $\bar{\mathbf{x}} = (\bar{\mathbf{x}}_0, \bar{\mathbf{x}}_1, ..., \bar{\mathbf{x}}_n)$.

Intuitively, a dynamic cooperative game is intended to model a situation in which a group of players are to play a sequence of cooperative games $m+1$ times. At each time period, a stage-game imputation determines how much each player receives from that round of play.

Analogous to the case of static cooperative games with coalition structures, it will be assumed here, at least intuitively, that within each coalition in the coalition structure R the players will contend with each other regarding their shares of the imputations, and that they will do so by presenting each other with potential objections and counter-objections. In the dynamic game, however, we assume that each player cares only about the *sum total* of payoffs he or she receives over time, rather than particular imputations in each period – in other words, each player prefers an imputation sequence $\mathbf{y}$ to $\mathbf{x}$ precisely when $\bar{\mathbf{y}}_i > \bar{\mathbf{x}}_i$.

4. Dynamic and Repeated Coalitions and Bargaining Sets

Again, in analogy with the static case, we assume that players communicate openly with each other and sign binding and enforceable contracts specifying coalition formation and accompanying imputations. But in dynamic games, the contracts are assumed to cover all the $m+1$ time periods. In a static cooperative game, objections and counter-objections are defined against all possible coalitions containing one player but not another, but in the dynamic game setting one needs to consider *sequences of coalitions*, because an objection in the dynamic game raised by a player to a sequence of imputations might involve different coalitions in each time period.

This requires new definitions. We consider here two different possibilities for what a 'sequence of coalitions' may mean, and show that they have very different implications for solutions of dynamic-games.

Denote by $\mathrm{R}^t \in \mathrm{C}(S)$ a coalition structure over $S \subseteq N$ at time *t*, and then define a *coalition structure sequence* over *S* by $\mathbf{R}(S) = (\mathrm{R}^0, \mathrm{R}^1, ..., \mathrm{R}^m)$, over the $m+1$ time periods, where each $\mathrm{R}^t \in \mathrm{C}(S)$.

The main definitions focus on particular subsets of the set of all coalition structure sequences. Given a coalition structure sequence $\mathbf{R}(S)=(\mathrm{R}^0,\mathrm{R}^1,...,\mathrm{R}^m)$, two players $i,j\in S$ are *dynamic partners* with respect to $\mathbf{R}(S)$ if there exists a sequence of players $(p_1,...,p_k)$ and a set of coalition structures $(\mathrm{G}_1,\mathrm{G}_2,...,\mathrm{G}_{k+1})$, where each $\mathrm{G}_l\in(\mathrm{R}^0,\mathrm{R}^1,...)$ for $l\le k+1$, such that $i\sim_{\mathrm{G}_1}p_1$, $p_1\sim_{\mathrm{G}_2}p_2$, …, $p_{k-1}\sim_{\mathrm{G}_k}p_k$, $p_k\sim_{\mathrm{G}_{k+1}}j$. A coalition structure sequence $\mathbf{R}(S)$ is a *dynamic coalition* over S if for each pair $i,j\in S$, i is a dynamic partner of j with respect to $\mathbf{R}(S)$.

The intuitive reasoning for this definition is as follows. A static coalition S is a group of players who are conceived of as signing a single-period contract to achieve together as partners the value $v(S)$ and divide it amongst themselves. A dynamic coalition $\mathbf{R}(S)$ should intuitively be conceived of as a group of players S who sign a multi-period contract that determines a coalition structure in each period – i.e. it determines for each period who partners with whom in a standard coalition in that period. If $\mathbf{R}(S)$ is not a dynamic coalition, then there are players in S who ‘do not need’ the other players in S in the sense that they can sign a multi-period contract amongst themselves without affecting the others in any way, neither with respect to the coalitions they form in each period nor with respect to their payoffs.

The second type of coalition sequence is a subset of the set of dynamic coalitions. A *repeated coalition* over S is $\mathbf{R}(S)=(\mathrm{R}^0,\mathrm{R}^1,...,\mathrm{R}^m)$ such that for each time period t, $\mathrm{R}^t=\{S\}$ – in simpler words, in the repeated coalition the single-period coalition S is formed again and again in each and every time period. In this situation, S can be termed the *base coalition* of the repeated coalition $\mathbf{R}(S)$. This perhaps corresponds more closely to the naïve view of what a multi-period coalition means – a group of players who agree in each time-period to cooperate together in the same coalition.

Given $k,l\in R\in\mathrm{R}$ and $\mathbf{x}\in\mathbf{I}(N,\mathbf{v},\mathrm{R},m+1,\delta)$, a *dynamic* [respectively *repeated*] *coalition objection* of k against l at $\mathbf{x}$ is a triple $(P,\mathbf{R}(P)=(\mathrm{D}^0,\mathrm{D}^1,...),\mathbf{y}=(y^0,y^1,...,y^m))$ such that $\mathbf{R}(P)$ is a dynamic [repeated] coalition, satisfying

$$P\in T_{kl}$$
$$y^t\in\mathbb{R}^P \text{ for all } t\in\{0,...,m\}$$
$$\vec{\mathbf{y}}_i\ge\vec{\mathbf{x}}_i \text{ for all } i\in P \text{ and } \vec{\mathbf{y}}_k>\vec{\mathbf{x}}_k$$
$$\vec{\mathbf{y}}_i\ge\sum_{t=0}^m w^t(\{i\}) \text{ for all } i\in P$$
$$\text{for each } t\in\{0,...,m\}\text{, for each } D\in\mathrm{D}^t,\ y^t(D)\le w^t(D)$$

A *dynamic* [*repeated*] *coalition counter-objection* to an objection $(P,\mathbf{R}(P),\mathbf{y})$ of k against l at $\mathbf{x}\in I(N,\mathbf{v},\mathrm{R},m+1,\delta)$ is a triple $(Q,\mathbf{R}(Q)=(\mathrm{B}^0,\mathrm{B}^1,...),\mathbf{z}=(z^0,z^1,...,z^m))$ such that $\mathbf{R}(Q)$ is a dynamic [repeated] coalition, satisfying

$$Q \in T_{lk}$$
$$z^t \in \mathbb{R}^Q \text{ for all } t \in \{0,...,m\}$$
$$\mathbf{z}_i \geq \mathbf{x}_i \text{ for all } i \in Q$$
$$\mathbf{z}_i \geq \mathbf{y}_i \text{ for all } i \in P \cap Q$$
$$\bar{z}_i \geq \sum_{t=0}^{m} w^t(\{i\}) \text{ for all } i \in Q$$
$$\text{for each } t \in \{0,...,m\}, \text{ for each } B \in \mathrm{B}^t,\ z^t(B) \leq w^t(B)$$

A dynamic [repeated] coalition objection of player *i* for which player *j* has no dynamic [repeated] coalition counter-objection is a *justified* dynamic [*repeated*] coalition objection. A sequence $\mathbf{x} \in I(N, \mathbf{v}, \mathrm{R}, m+1, \delta)$ is *dynamic* [*repeated*] *coalition stable* if for each dynamic [repeated] coalition objection at $\mathbf{x}$ there is a dynamic [repeated] coalition counter-objection. The *dynamic* [*repeated*] *coalition bargaining set*, is the set of all dynamic [repeated] coalition stable members of $\mathbf{I}(N, \mathbf{v}, \mathrm{R}, m+1, \delta)$.

We now proceed to show by a series of examples that contemplation of cooperative dynamic games adds new and interesting considerations beyond those encountered in static cooperative games[2]:

Example 1. Let $n = 4$ and the set of players be denoted by $\{I, A, P, Q\}$. Consider a cooperative dynamic game with the stage game defined by $\mathrm{R} = \{IA, P, Q\}$ and $v(IA) = v(AP) = v(AQ) = 48$, $v(IP) = v(IQ) = 74$. The value of every other possible coalition, including single-player coalitions and the grand-coalition, is equal to zero.

The stage-game imputation $x = (24,24;0;0)$ is in the single-period stage-game bargaining set: any objection by *I* must necessarily involve either the coalition IP or IQ, but in either case, *A* can form a counter-objection by way of a coalition with whichever player was excluded in *I*'s objection; *A* is in an even weaker position than *I* with respect to justifiable objections.

However, the 2-period imputation sequence $\mathbf{x} = (x^0, x^1)$, where $x^0 = (1-\delta)(24,24;0;0)$ and $x^1 = (1-\delta)\delta(24,24;0;0)$ is not in the dynamic coalition bargaining set of the dynamic game $(N, \mathbf{v}, \mathrm{R}, 2, \delta)$ when $\delta > 12/13$ – despite the fact that each single-period imputation is in the (standard) bargaining set when attention is limited to one period alone. Player *I* has the following justified dynamic coalition objection: for $\varepsilon < 2$, let the objection be the triple $\{S, \mathbf{R}(S), \mathbf{y}\}$ where $S = (IPQ)$, $\mathbf{R}(S) = (\{IP, Q\}, \{IQ, P\})$ and $\mathbf{y} = \{y^0, y^1\} \in \mathbb{R}^{S \times 2}$ is given by $y^0 = (1-\delta)(24+\varepsilon, 50-\varepsilon, 0)$ with $y^1 = (1-\delta)\delta(24, 0, 50)$. Player *A* can offer no counter-objection. Any proposed counter-objection can give to either player *Q* or *P* at most the sum $(1-\delta)(24+24\delta)$, which is insufficient to persuade either one to give up the greater reward offered by player *I* in his objection.

[2] Note that the dynamic game in each of these three examples is actually a repeated game, because the same characteristic function is used in each time period.

Example 2. Let $n = 6$, with the set of players *N* denoted by $\{1,2,3,4,5,6\}$. Again consider a dynamic game with the stage game defined by $\mathrm{R} = \{12,3,4,5,6\}$ and $v(12) = 2$, $v(1,3,4) = 5$, $v(1,5,6) = 3$, $v(2,5,6) = 5$, $v(2,3,4) = 3$, $v(3,4) = v(5,6) = 20$. The value of every other possible coalition, including single-player coalitions and the grand-coalition, is equal to zero.

Consider a 2-period dynamic game based on the stage-game (N, v, R) with δ extremely close to 1, and disregard the $1-\delta$ multiplier, so that in effect we can think of $(N, \mathbf{v}, \mathrm{R}, 2, \delta)$ as (N, v, R) repeated twice.

We now show this dynamic game has an empty dynamic coalition bargaining set.

Let $\mathbf{x}$ be an imputation sequence in $I(N, \mathbf{v}, \mathrm{R}, 2, \delta)$. Because $v(12) = 2$, $\mathbf{x}_1 + \mathbf{x}_2 \le 4$. Suppose that $\mathbf{x}_1 \le 2$ and let $\varepsilon < 1/2$. Then player 1 has the objection $\{P, \mathbf{R}(P), \mathbf{y}\}$ where $P = (1,3,4,5,6)$, $\mathbf{R}(P) = (\{134,56\}, \{156,34\})$, and $\mathbf{y} = \{y^0, y^1\} \in \mathbb{R}^{P\times 2}$ is given by $y^0 = (1+\varepsilon, 2-0.5\varepsilon, 2-0.5\varepsilon, 10, 10)$ with $y^1 = (1,10,10,1,1)$. Player 2 has no feasible counter-object – he or she cannot offer players 5 and 6 enough to counter-object by forming a coalition with them in both periods, and therefore needs to include players 3 and 4 in a dynamic coalition, but there is no way that those player can be guaranteed an amount equal to what they are receiving under $\mathbf{y}$.

On the other hand, if $\mathbf{x}_2 \le 2$, player 2 has a symmetric justified objection $\{Q, \mathbf{R}(Q), \mathbf{z}\}$ where $Q = (1,3,4,5,6)$, $\mathbf{R}(Q) = (\{256,34\}, \{234,56\})$, and $\mathbf{z} = \{z^0, z^1\} \in \mathbb{R}^{Q\times 2}$ is given by $z^0 = (1+\varepsilon, 10, 10, 2-0.5\varepsilon, 2-0.5\varepsilon)$ with $z^1 = (1,1,1,10,10)$. The conclusion is that the dynamic coalition bargaining set is empty.

Example 3. Let $n = 3$, with the set of players *N* denoted by $\{1,2,3\}$, and let the dynamic game be defined by the stage game as given by the coalition structure $\mathrm{R} = \{12,3\}$, with $v(1) = v(2) = v(3) = v(123) = 0$, $v(12) = 100$, $v(13) = 100$, $v(23) = 50$. The bargaining set of the one-period stage game consists of a single imputation, $(75,25;0)$.

Repeat the game over three periods, with an arbitrary δ. Consider the imputation sequence given by $\mathbf{x} = (x^0, x^1, x^2)$, where $x^0 = (1-\delta)(80,20;0)$, $x^1 = (1-\delta)\delta(72,28;0)$, and $x^2 = (1-\delta)\delta^2(61,39;0)$. Then it is the case that $x^t \notin I(N, w^t, \mathrm{R}, t)\}$ for all *t* = 0, 1, 2, but never the less, $\mathbf{x} \in \mathbf{I}(N, \mathbf{v}, \mathrm{R}, 3, \delta)$, because the sum total produced by players 1 and 2 over the three periods is 175 units, which are then divided amongst themselves by granting player 1 three-quarters and player 2 one-quarter – exactly reflecting their relative 'bargaining strengths' vis-à-vis objections and counter-objections.

These three examples show that, in general, the dynamic coalition bargaining set may be empty, and that even when it is non-empty it is possible for the every element in an imputation sequence to be in the stage-game bargaining set for its respective time period without the sequence itself being in the repeated coalition bargaining set, while conversely, even if every element in an imputation sequence fails to be in the in the stage-game bargaining set, the sequence itself might still be in the repeated coalition bargaining set.

The reason that the dynamic coalition bargaining set may be empty stems from the following fact (a similar observation appears in (Kranich, Perea, Peters; 2005)): every cooperative dynamic game $(N, \mathbf{v}, \mathrm{R}, m+1, \delta)$ can be associated with a static non-transferable utility coalitional game:

Given $S \subseteq N$, m, $\boldsymbol{v}$, and δ as above, and $\mathbf{R}(S) = (\mathrm{R}^0, \mathrm{R}^1, ..., \mathrm{R}^m)$, a coalition structure sequence with each $\mathrm{R}^t \in \mathrm{C}(S)$, define

$\mathbf{I}(\mathbf{R}(S)) =$
$\{(x^0, x^1, ..., x^m) \mid \text{for all } t, x^t \in \mathbb{R}^S, x^t(P) \le w^t(P) \text{ for every } P \in \mathrm{R}^t, \text{and} \sum_t x_i^t \ge \sum_t w^t(\{i\}) \text{ for all } i\}$

Definition. The *static NTU-game associated with a cooperative dynamic game* $(N, \mathbf{v}, \mathrm{R}, m+1, \delta)$ is given by (N, V), with

$$V(S) = \{(\bar{\mathbf{x}}_{i_1}, \bar{\mathbf{x}}_{i_2}, ..., \bar{\mathbf{x}}_{i_{|S|}}) \in \mathbb{R}^S \mid \text{there is a } \mathbf{R}(S) \text{ with } \mathbf{x} \in \mathbf{I}(\mathbf{R}(S))\}$$

where $\bar{\mathbf{x}}_i = \sum_t x_i^t$ for each $i \in S$.

The following observation is nearly immediate from the definitions:

Proposition 1. *The dynamic coalition bargaining set of the dynamic cooperative game* $(N, \mathbf{v}, \mathrm{R}, m+1, \delta)$ *is non-empty if and only if the bargaining set of its associated static NTU game is non-empty.*

Proof. Let $\mathbf{x} \in I(N, \mathbf{v}, \mathrm{R}, m+1, \delta)$ be in the dynamic coalition bargaining set of $(N, \mathbf{v}, \mathrm{R}, m+1, \delta)$. Unravelling definitions, it follows that the vector $\bar{\mathbf{x}} = (\bar{\mathbf{x}}_{i_1}, \bar{\mathbf{x}}_{i_2}, ..., \bar{\mathbf{x}}_{i_{|N|}}) \in \mathbb{R}^N$ is contained in $IX(N, V, \mathrm{R})$ of the associated NTU game. Suppose there is a justified NTU-objection $(P, \bar{\mathbf{y}})$ of player *k* against player *l* at $\bar{\mathbf{x}}$. Then there is a coalition structure sequence over *P*, $\mathbf{R}(P)$, such that $P \in T_{kl}$, there is a $\mathbf{y} \in \mathbf{I}(\mathbf{R}(S))$ corresponding to $\bar{\mathbf{y}}$, and $\bar{\mathbf{y}}_i \ge \bar{\mathbf{x}}_i$ for all $i \in P$, against which there is no counter-objection. But then $(P, \mathbf{R}(P), \mathbf{y})$ is a justified objection in the sense of the dynamic coalition bargaining set of the TU-game $(N,\mathbf{v},\mathrm{R},m+1,\delta)$, which is a contradiction. The proof in the other direction is similar. *QED*

Given that NTU games in general do not have non-empty bargaining sets (see [Peleg; 1963]), it is not surprising that the dynamic bargaining set of repeated TU-games may also be empty.

In contrast to the dynamic coalition bargaining set, the repeated coalition bargaining set is guaranteed to be non-empty.

Definition. Given $(N,\mathbf{v},\mathrm{R},m+1,\delta)$ and $t<m$, define $q^t(S)=\sum_{l=t}^{m} w^l(S)$.

Definition. The *static TU-game associated with a dynamic game* $(N,\mathbf{v},\mathrm{R},m+1,\delta)$ is given by (N,q,R), with

$$q(S):=q^0(S)=\sum_{t=0}^{m} w^t(S)$$

for every $S\subseteq N$.

Definition. Given (N,q,R) associated with $(N,\mathbf{v},\mathrm{R},m+1,\delta)$ and an element $x\in M(N,q,\mathrm{R})$, define each player *i*'s *relative share with respect to x* as $\alpha_i(x)=\dfrac{x_i}{q(\mathrm{R}(i))}$. The *monotonic imputation sequence with respect to x*, $\mathbf{a}(x)=(a^0,a^1,\ldots,a^m)$, is defined by setting $a_i^t=\alpha_i(x)w^t(\mathrm{R}(i))$.

When the context of $x\in M(N,q,\mathrm{R})$ is clear, $\alpha_i(x)$ will sometimes be written here simply as α_i.

Proposition 2. *If $I(N,q,\mathrm{R})$ is not empty, the repeated coalition bargaining set of a dynamic cooperative game $(N,\mathbf{v},\mathrm{R},m+1,\delta)$ is not empty – for each $x\in M(N,q,\mathrm{R})$, the associated static TU-game, every feasible imputation sequence $\mathbf{c}=(c^0,c^1,\ldots,c^m)$ such that $\bar{\mathbf{c}}_i=\bar{\mathbf{a}}(x)_i$ for each player i, including $\bar{\mathbf{a}}(x)$ itself, is in the repeated coalition bargaining set.*

Proof. By well-known results, the associated static TU-game (N,q,R) has a non-empty bargaining set – i.e. there exists at least one vector $x=(x_1,\ldots,x_n)\in M(N,q,\mathrm{R})$, such that no player *k* has a justified objection against another player *l* at *x* relative to the characteristic function *q*. Let *x* be an arbitrary such vector.

Writing $\alpha_i := \alpha_i(x)$, trivially, for each $S \in \mathrm{R}$, $\sum_{i \in S} \alpha_i \leq 1$ because $\sum_{i \in S} x_i \leq q(S)$. Let $\mathbf{a} = (a^0, a^1, ..., a^m)$ be the monotonic imputation sequence with respect to *x*, as defined above. The sequence is feasible for each $S \in \mathrm{R}$, because $a^t(S) = (\sum_{i \in S} \alpha_i) w^t(S) \leq a^t(S)$. We have in addition that for each $i \in N$, $\bar{\mathbf{a}}_i = \sum_t a_i^t = \alpha_i \sum_t w^t(\mathrm{R}(i)) = \alpha_i q(\mathrm{R}(i)) = x_i$, so that $\mathbf{a} = (a^0, a^1, ..., a^m)$ represents a way of granting each player an amount in each time period in such a way that the sum total over all time periods is exactly equal to the vector $(x_1, ..., x_n)$. By definition, the same applies to any feasible imputation sequence $\mathbf{c} = (c^0, c^1, ..., c^m)$ such that $\vec{\mathbf{c}}_i = \vec{\mathbf{a}}(x)_i$ for each player *i*.

Suppose that $(P, \mathbf{y})$ is a repeated coalition objection of player *k* against player *l* at $\mathbf{c}$. Then $\vec{\mathbf{y}}_i \geq \vec{\mathbf{c}}_i$ for all $i \in P$ and $\vec{\mathbf{y}}_k > \vec{\mathbf{c}}_k$. By the definition of repeated coalitions, it must be the case that for each time period *t*, $y^t(P) \leq w^t(P)$, hence $\sum_t y^t(P) \leq \sum_t w^t(P) = q(P)$. But, because $\vec{\mathbf{c}}_i = x_i$, this means that the pair $(P, \vec{\mathbf{y}})$ is an objection of player *k* against player *l* at *x* in the game (N, q, R). As *x* is in the bargaining set of (N, q, R), there is by definition a counter-objection (Q, z) to $(P, \vec{\mathbf{y}})$ of *l* against *k*.

Defining the fractions $\beta_i = \dfrac{z_i}{q(\mathrm{R}(i))}$ and setting $\mathbf{b} = (b^0, b^1, ..., b^m)$, $b_i^t = \beta_i w^t(\mathrm{R}(i))$, it follows that $\vec{\mathbf{b}}_i = z_i$, so $\vec{\mathbf{b}}_i \geq \vec{\mathbf{c}}_i$ for all $i \in Q$, $\vec{\mathbf{b}}_i \geq \vec{\mathbf{y}}_i$ for all $i \in P \cap Q$, and hence $(Q, \mathbf{b})$ is a repeated coalition counter-objection to $(P, \mathbf{y})$ at $\mathbf{a}$ in $(N, v, \mathrm{R}, m+1, \delta)$.

The conclusion is that $\mathbf{c}$ is in the repeated coalition bargaining set of $(N, v, \mathrm{R}, m+1, \delta)$.
QED

A similar line of proof shows that if the stage-games are superadditive, there is no point to distinguishing between dynamic coalitions and repeated coalitions:

Proposition 3. *If the stage games of a dynamic cooperative game* $(N, \mathbf{v}, \mathrm{R}, m+1, \delta)$ *are superadditive, then the dynamic coalition bargaining set of* $(N, \mathbf{v}, \mathrm{R}, m+1, \delta)$ *is equal to the repeated coalition bargaining set.*

Proof. Again, form the associated single-stage TU-game (N, q, R) by setting, for every $S \subseteq N$, $q(S) = \sum_t w^t(S)$. Assuming $\mathbf{I}(N, \mathbf{v}, \mathrm{R}, m+1, \delta)$ is not empty, select arbitrarily an imputation sequence $\mathbf{x} = (x^0, x^1, ..., x^m)$ in $\mathbf{I}(N, \mathbf{v}, \mathrm{R}, m+1, \delta)$.

Suppose that $(P, \mathbf{R}(P) = (\mathrm{D}^0, \mathrm{D}^1, ...), \mathbf{y} = (y^0, y^1, ..., y^m))$ is a dynamic coalition objection of *k* against *l* at $\mathbf{x}$. Denote by $(d_1^t, d_2^t, ..., d_{E(t)}^t)$ the partition of *P* given by each D^t. For each t, super-additivity implies $w^t(\bigcup_{j=1}^{E(t)} d_j^t) \geq \sum_{j=1}^{E(t)} w^t(d_j^t)$. Since the objection imputation at each time t must be feasible, for each $d_j^t \in \mathrm{D}^t$, $y^t(d_j^t) \leq w^t(d_j^t)$. By definition, $P = \bigcup_{j=1}^{E(t)} d_j^t$, so it follows that $w^t(P) \geq y^t(P)$ and therefore $\sum_t y^t(P) \leq \sum_t v^t(P) = q(P)$.

Forming the sequence $\mathbf{a} = (a^0, a^1, ..., a^m)$ by setting $\alpha_i = \frac{y_i}{q(P)}$ and $a_i^t = \alpha_i w^t(P)$ for each $i \in P$, it follows that $\bar{\mathbf{a}}_i = \bar{\mathbf{y}}_i$, and hence the net effect of the repeated coalition objection $(P, \mathbf{a})$ is equivalent to the net effect of the dynamic coalition objection $(P, \mathbf{R}(P), \mathbf{y})$. Similarly, any dynamic coalition counter-objection $(Q, \mathbf{R}(Q), \mathbf{z})$ can be achieved equally well by the repeated coalition counter-objection $(Q, \mathbf{b})$ where $\mathbf{b} = (b^0, b^1, ..., b^m)$ is derived by setting $\beta_i = \frac{y_i}{q(Q)}$ and $b_i^t = \beta_i w^i(Q)$ for each $i \in Q$. The conclusion is that under the assumptions of the proposition, any $\mathbf{x} \in \mathbf{I}(N, \mathbf{v}, \mathrm{R}, m+1, \delta)$ is dynamic coalition stable if and only if it is repeated coalition stable. *QED*

These results make clear that the reason dynamic coalitions in non-supperadditive situations can lead to dynamic game bargaining set solutions that diverge from solutions derived directly from stage game solutions – as in example 1 presented above – is because in that situation, the associated TU-game does not reflect accurately what can be attained in the dynamic game. A dynamic coalition over $S \subseteq N$ can, in certain cases, attain for its members a total pay-off greater than $q(S)$ by cleverly arranging different coalition structures over S in different time-periods – but that total pay-off may not necessarily be freely transferable between the members of S.

5. Credit Sequences

Assume in this section that in any imputation sequence $\mathbf{x} = (x^0, x^1, ...)$, $x^t(S) = w^t(S)$ for all time periods t.

By Proposition 2 of the previous section, the set of bargaining set solutions for a dynamic game $(N, \mathbf{v}, \mathrm{R}, m+1, \delta)$ is at least as large as the set of bargaining set solutions of its associated static TU-game (N, q, R) – for each solution $x \in M(N, q, \mathrm{R})$, the monotonic imputation sequence $\mathbf{a}(x)$ is a solution of $(N, \mathbf{v}, \mathrm{R}, m+1, \delta)$. But this by no means begins to exhaust the set of solutions of the bargaining set of $(N, \mathbf{v}, \mathrm{R}, m+1, \delta)$, as the same proposition extends that set to any feasible imputation sequence $\mathbf{c} = (c^0, c^1, ..., c^m)$ such that $\bar{\mathbf{c}}_i = \bar{\mathbf{a}}(x)_i$ for each player i.

Consider the repeated cooperative game based on the stage-game (N, v, R) of example 3 above. The associated static TU-game of that example, for any $\delta < 1$, has a single bargaining set solution given by $\alpha_1 = 3/4$, $\alpha_2 = 1/4$, $\alpha_3 = 1$, in terms of player share. In sharp contrast, the bargaining set of $(N, v, \mathrm{R}, 3, \delta)$ has an infinite number of solutions. Example 3 exhibits one such solution, which deviates from the monotonic imputation sequence in each time period. But clearly a solution in the bargaining set of the dynamic cooperative game cannot allow *every* feasible imputation in *every* period – for example, any imputation sequence $\mathbf{x} = (x^0, x^1, x^2)$ with $x^0 = (1-\delta)(25,75;0)$ must lie outside the bargaining set of $(N, v, \mathrm{R}, 3, \delta)$, even if x^1 and x^2 are each feasible imputations in their respective time periods.

The intention of this section is to give a finer characterisation of the possible solutions of the bargaining set of a dynamic cooperative game by establishing bounds on the payoffs that can be granted to each player in each time period within the context of a repeated coalition bargaining set solution.

Returning to a consideration of example 3, with the solution given by $\mathbf{x} = (x^0, x^1, x^2)$, $x^0 = (1-\delta)(80,20;0)$, $x^1 = (1-\delta)\delta(72,28;0)$, and $x^2 = (1-\delta)\delta^2(61,39;0)$, one way to regard the solution is to interpret it as if player 1 'justifies' the first-period imputation of $(80,20;0)$, which deviates from the static bargaining set solution of $(75,25;0)$, by 'borrowing' 5 units from player 2. The debt is then re-paid, with interest implicit in the discount-factor (which reduces the value of the total payoff that can be divided), over the next two time periods.

This is, of course, only an anthropomorphic story that is overlaid over a particular solution to a mathematical construct – and in fact, there are many such 'stories' that can be told relative to each solution – but it so enhances intuitive insight that we present here a formalisation of the idea of players borrowing and repaying debts over time. Within the context of studying the core in dynamic situations, [Kranich, Perea, Peters; 2001] and [Breden; 2007] consider what they term inter-temporal transfers, in which players receive more in some time periods at the expense of less in other time periods. This takes the form of postulating a sequence $\{c_i^t\}$ for each player such that $\sum_{t=0}^{m} c_i^t = 0$ which changes the player's payoff in each time period relative to an imputation sequence $\mathbf{x} = (x^0, x^1, ..., x^m)$ from x_i^t to $x_i^t + c_i^t$. We expand here on this idea, explicitly taking into account the need for one player to borrow from another player in each time period in order to increase his personal payoff, which then also imposes a requirement on the debtor to repay the creditors in a later time period.

This motivates the following:

The players first select a 'goal' vector $g \in M(N, q, \mathrm{R})$, from the bargaining set of the associated TU-game. This is intended to be interpreted as an agreement between them that the sum-total of the imputation sequence of the $m+1$-period game they are to play will be the vector g. The vector g, along with the array of associated relative shares per player $\alpha_i(g)$, can be regarded as determining an over-all canonical 'income distribution'. The monotonic imputation sequence given by these relative shares with respect to g, $\mathbf{a}(g) = (a^0, a^1, ..., a^m)$, also serves as a 'baseline' against which deviations in imputations in particular time-periods are interpreted as credits and debits.

Definition. A *credit sequence* relative to a dynamic cooperative game $(N, \mathbf{v}, \mathrm{R}, m+1, \delta)$ and a vector $g \in M(N, q, \mathrm{R})$ is composed of real numbers $\{d^t_{i,j}\}$, $\{p^t_{i,j}\}$ defined inductively at each time period for each pair of players $i, j \in N$, subject to the following list of constraints:

i. $d^t_{i,j} \geq 0$; $d^t_{i,j} = 0$ whenever i and j are not in the same partition of R; $d^t_{i,i} = 0$ for all $i \in N$ and times $t \geq 0$. We also define $d^{-1}_{i,j} = 0$ to initiate the induction, for all $i, j \in N$

ii. $p^t_{i,j} \geq 0$; $p^t_{i,j} = 0$ whenever i and j are not in the same partition of R; $p^t_{i,i} = 0$ for all $i \in N$ and times $t \geq 0$. We also define $p^{-1}_{i,j} = 0$ to initiate the induction, for all $i, j \in N$

iii. $p^t_{i,j} \leq \sum_{l=0}^{t-1} d^l_{i,j} - p^l_{i,j}$ for all $t \geq 0$, for all $i, j \in N$

iv. For each $i \in N$ and time period t, defining $c^t_{i,j} := c^{t-1}_{i,j} + d^t_{i,j} - p^t_{i,j} - d^t_{j,i} + p^t_{j,i}$ for each j and $c^t_i := \sum_{j \in N} c^t_{i,j}$, constrain c^t_i to be $c^t_i \leq \sum_{l=t+1}^{m} a^l_i$ (with the understanding that $\sum_{l=m+1}^{m} a^l_i = 0$), where a^t_i is given at each t by the monotonic imputation sequence $\mathbf{a}(g)$

v. $\sum_{j \in N} d^t_{i,j} - p^t_{i,j} - d^t_{j,i} + p^t_{j,i} \leq a^t_i$ for each time period t and each player $i \in N$

Given a credit sequence $\{\{d^t_{i,j}\}, \{p^t_{i,j}\}\}$ relative to $g \in M(N, q, \mathrm{R})$, a feasible imputation sequence $\mathbf{x} = (x^0, x^1, ..., x^m)$ will be said to be *derivable from* $\{\{d^t_{i,j}\}, \{p^t_{i,j}\}\}$ if for each $i \in N$ and time period t, $x^t_i = a^t_i + \sum_{j \in N} d^t_{i,j} - p^t_{i,j} - d^t_{j,i} + p^t_{j,i}$.

These constraints are justified by intuitive interpretations. Each pair $d^t_{i,j}$ is to be interpreted as saying that 'i is indebted to (or borrows from) j in period t the amount $d^t_{i,j}$'

or equivalently that '*j* has an IOU written by *i* in period *t* for $d_{i,j}^t$ units', and each pair $p_{i,j}^t$ is intended to represent a (possibly partial) re-payment of a debt made by player *i* to *j* in period *t*. Constraint (i) then says that debts are always counted in positive units, that players only borrow from their partners relative to the coalition structure, and that a player can never borrow from himself. Constraint (ii) says much the same about debt re-payments.

The term $\sum_{l=0}^{t-1} d_{i,j}^l - p_{i,j}^l$ represents 'total outstanding debt' owed by player *i* to player *j* in time period *t* – it sums up all IOUs given over all previous time periods by *i* to *j*, and subtracts all repayments made against them. Constraint (iii) then states that in time period *t* player *i* can not give player *j* more in debt re-payments than the total outstanding debt he owes to *j* (this does not prevent *i* from giving *j* more than $\sum_{l=0}^{t-1} d_{i,j}^l - p_{i,j}^l$ – but any transfer from *i* to *j* greater than that sum will be counted as a loan from *i* to *j*).

In each time period *t*, c_i^t represents the 'cumulative debt portfolio' held by player *i*, as it takes into account all loans given to other players, all loans taken and all the respective re-payments to date. Note that although each $d_{i,j}^t$ is greater than or equal to zero, $c_{i,j}^t$ may be positive or negative – if it is positive, then player *i* is a net debtor with respect to *j*, and if it is negative, *i* is a net creditor with respect to *j*. It also follows from the definitions that $c_{i,j}^t = -c_{j,i}^t$, and hence that in any single time period and for any single $S \in \mathrm{R}$, $\sum_{i \in S} c_i^t = 0$.

Under that interpretation, constraint (iv) establishes an important 'credit limit' for each player *i*, in the following sense. The vector *g* determines the 'total income' for player *i* as g_i, which by definition equals $\bar{\mathbf{a}}_i = \sum_t a_i^t$. At each time period *t*, therefore, $\sum_{l=t+1}^{m} a_i^l$ represents player *i*'s future income stream. Constraint (iv) is intuitively a 'no-default' condition: at no time is a player permitted to have outstanding positive cumulative debt which is greater than his future income stream – total debt in this system is always staked against future income.

Note, however, that because constraint (iv) applies to the total debt portfolio of a player, there is an implication under this system that a player can borrow both against future income and against previously issued IOUs he holds. In effect, 'debt securities' which are tradable and negotiable instruments arise naturally from the system.

Constraint (v) exists to ensure that under any imputation sequence $\mathbf{x} = (x^0, x^1, ..., x^m)$ derivable from a credit sequence, no player transfers to others so much in loans and re-payments that he receives less than zero.

Proposition 4. *In the context of a dynamic cooperative game* $(N, \mathbf{v}, \mathrm{R}, m+1, \delta)$, *each imputation sequence* $\mathbf{x} = (x^0, x^1, ..., x^m)$ *derivable from a credit sequence* $\{\{d_{i,j}^t\}, \{p_{i,j}^t\}\}$

relative to a vector $g \in M(N,q,\mathrm{R})$ *is located within the repeated coalition bargaining set. Conversely, for each* $\mathbf{x} = (x^0, x^1, ..., x^m)$ *in the repeated coalition bargaining set, there is at least one credit sequence* $\{\{d_{i,j}^t\},\{p_{i,j}^t\}\}$ *relative to* $g = \bar{\mathbf{x}} = (\bar{\mathbf{x}}_0, \bar{\mathbf{x}}_1, ..., \bar{\mathbf{x}}_n)$ *such that* $\mathbf{x}$ *is derivable from* $\{\{d_{i,j}^t\},\{p_{i,j}^t\}\}$.

The proof appears in the appendix.

Proposition 5. *In a dynamic cooperative game* $(N, \mathbf{v}, \mathrm{R}, m+1, \delta)$*, if the players agree on an over-all goal vector* $g \in M(N,q,\mathrm{R})$*, then in any imputation sequence* $\mathbf{x} = (x^0, x^1, ..., x^m)$*, at time period t, the largest value received by any player is bound by* $x_i^t \le \min(w^t(\mathrm{R}(i)), g_i)$ *and the smallest by* $x_i^t \ge \max(0, a_i^t - \sum_{j \ne i}(\sum_{l=0}^{t-1} a_j^l + \sum_{l=t+1}^{m} a_j^l))$.

Proof. Obviously, player *i* cannot receive at time *t* more than the total produced by the coalition to which he belongs, hence not more than $w^t(S)$. What he receives, however, is also limited by the fact that $\mathbf{x} = (x^0, x^1, ..., x^m)$ must be derivable from a credit sequence $\{\{d_{i,j}^t\},\{p_{i,j}^t\}\}$, as per the previous proposition. Under any credit sequence, at time *t* he cannot borrow more than his 'future income stream', given by $\sum_{j \ne i} d_{i,j}^t \le \sum_{l=t+1}^{m} a_i^l$, and the most he can receive in debt-repayment is limited by the most he could have lent in past periods, given by $\sum_{l=0}^{t-1} a_i^l$, so the most he can pocket in time t is $\sum_{l=t+1}^{m} a_i^l + \sum_{l=0}^{t-1} a_i^l + a_i^t = g_i$. The maximal value of x_i^t is then the smaller of $w^t(S)$ or g_i.

For calculating the least value of x_i^t, clearly player *i* cannot receive less than 0. Again, what he receives is also limited by the fact that $\mathbf{x} = (x^0, x^1, ..., x^m)$ must be derivable from a credit sequence $\{\{d_{i,j}^t\},\{p_{i,j}^t\}\}$. As a lender, he cannot give the other players in his coalition more than their 'credit limit' at time *t*, which is represented by $\sum_{j \ne i} \sum_{l=t+1}^{m} a_j^l$, their 'future income streams'. On the other hand, as a (former) borrower the most he can now repay is limited by the most they could have lent him, which is given by their *past* income streams $\sum_{j \ne i} \sum_{l=0}^{t-1} a_j^l$. This means the greatest downward deviation from a_i^t possible is limited by $\sum_{j \ne i}(\sum_{l=0}^{t-1} a_j^l + \sum_{l=t+1}^{m} a_j^l)$. *QED*

6. Subgame Stable Sequences

The paradigm in which the players negotiate a target goal vector $g \in M(N,q,\mathrm{R})$ relative to a dynamic cooperative game $(N,\mathbf{v},\mathrm{R},m+1,\delta)$, against which they then negotiate a contract establishing the detailed imputation sequence $\mathbf{x} = (x^0, x^1, ..., x^m)$ they will share 'once and for all', enfolds within it implicit assumptions regarding enforcement. Cooperative game theory itself, of course, leans on an implicit enforcement postulation – even in the single-stage case, the players negotiate an imputation of the payoff they will receive for forming a coalition.

In the multi-stage case, the assumption of an enforcement mechanism is even more critical, especially given the interpretation presented in the previous section of the imputation sequence as encoding 'inter-temporal' borrowing from one player to another: a player who has 'borrowed heavily' in the earlier rounds and in later rounds is expected to 'repay' the loans by accepting imputations outside the bargaining set, has a strong incentive to defect to another coalition, thus 'defaulting' on his debt to the detriment of the other players.

A couple of examples can further elucidate this possibility:

Example 4. Going back to example 3 above, the imputation sequence in the repeated-coalition bargaining set presented there, $\mathbf{x} = (x^0, x^1, x^2)$, where $x^0 = (1-\delta)(80,20;0)$, $x^1 = (1-\delta)\delta(72,28;0)$, and $x^2 = (1-\delta)\delta^2(61,39;0)$, presents player 1 with an incentive to 'default' as early as period 1. With period 0 concluded, at period 1 player 1 can present player 2 with an objection according to which he will defect to a coalition with player 3 over the next two periods, by offering to share with player 3 the payoffs $(1-\delta)\delta(75,25)$, and $(1-\delta)\delta^2(75,25)$. Player 2 has no counter-objection against this, and is likely to agree to 're-negotiate' the contract with player 1, changing $\mathbf{x}$ to $\mathbf{x}' = (x^0, x'^1, x'^2)$ with $x'^1 = (1-\delta)\delta(75,25;0)$, and $x'^2 = (1-\delta)\delta^2(75,25;0)$. This has the effect of raising player 1's overall payoff at the expense of player 2.

Example 5. This example is similar to the above, but in contrast to it, the characteristic functions are not equal in each time period. Again there are three players and three time periods. The characteristic function for the first time period is $v^0(12) = 100$, $v^0(13) = 100$, $v^0(23) = 50$; for the second it is $v^1(12) = 100$, $v^1(13) = 50$, $v^1(23) = 100$, and in the third it is exactly equal to that of the second. With arbitrary δ, select the canonical monotonic sequence $\mathbf{x} = (x^0, x^1, x^2)$, with $x^0 = (1-\delta)(41.\overline{6},58.\overline{6};0)$, $x^1 = (1-\delta)\delta(41.\overline{6},58.\overline{6};0)$, and $x^2 = (1-\delta)\delta^2(41.\overline{6},58.\overline{6};0)$. Player 2, who is weaker in the first time period but stronger in the next two, has an incentive to default by presenting player 1 with an objection that cannot be countered in both periods 1 and 2, thus gaining more than he could by sticking to the monotonic sequence.

The idea of players mistrusting each other when multiple rounds of a game are being player appears in several early papers (such as [Gale; 1978] and [Becker, Chakribarti;

1995]) and in particular has become a theme in studies of the core in dynamic cooperative games, where concepts such as the weak and strong sequential cores have been developed to analyse such situations (see [Kranich, Perea, Peters; 2005], [Predtetchinski et al; 2002, 2004, 2006] and [Predtetchinski; 2007]). It is in that spirit that we present the following definition.

Definition. An imputation sequence $\mathbf{x} = (x^0, x^1, ..., x^m)$, relative to a dynamic game $(N, \mathbf{v}, \mathrm{R}, m+1, \delta)$, is *subgame stable* if for each time period *t*, the sub-sequence of vectors $(x^t, x^{t+1}, ..., x^m)$ is in the bargaining set of the static TU-game (N, q^t, R) defined by the characteristic function $q^t(S) = \sum_{l=t}^{m} w^l(S)$ for all $S \subseteq N$.

Subgame stability guards against player defection in later rounds by replicating the stability of the bargaining set with respect to future time periods at any point in time: any suggested defection by a player with respect to future time periods by way of an objection can be met by a counter-objection. Examples 4 and 5 show that the set of subgame stable sequences, if it exists, is generally strictly smaller than the set of repeated coalition bargaining set sequences.

Definition. A sequence of characteristic functions $\mathbf{v} = (v^0, v^1, ..., v^m)$ defined relative to a set of players *N* and a coalition structure R is *sequentially essential* if for each time period *t* and each $S \in \mathrm{R}$, $v^t(S) \geq \sum_{i \in S} v^t(\{i\})$.

Proposition 6. *If m is finite and* $\mathbf{v} = (v^0, v^1, ..., v^m)$ *is sequentially essential relative to N and* R, *the set of subgame stable sequences of* $(N, \mathbf{v}, \mathrm{R}, m+1, \delta)$ *is not empty.*

Proof. This is proved by a 'backwards induction' argument (hence the condition of finiteness of *m*). Begin the induction by selecting an arbitrary element x^m in the bargaining set of (N, q^m, R).

Suppose, for $t < m$, the sequence $(x^{t+1}, x^{t+2}, ..., x^m)$ is in the bargaining set of (N, q^{t+1}, R). Naively, it might seem that in order to define x^t it would suffice to select arbitrarily an element $\bar{x} \in M(N, q^t, \mathrm{R})$ and set $x_i^t = \bar{x}_i - \sum_{l=t+1}^{m} x_i^l$ for each player *i*. The problem is that there is no guarantee this procedure will yield a non-negative value for each $\bar{x}_i$.

This potential flaw can, however, be avoided by a tweak to the procedure. Instead of working with q^t, define the characteristic function

$$b^t(S) = \begin{cases} \sum_{l=t+1}^{m} x_i^l + w^t(\{i\}) & S = \{i\},\ i \in N \\ 0 & S = \varnothing \\ q^t(S) & \text{otherwise} \end{cases}$$

It must now be shown that the set $I(N,b^t,\mathrm{R})$ is non-empty. Select arbitrarily $S \in \mathrm{R}$, and define $r = q^t(S) - \sum_{i\in S} b^t(\{i\})$. The assumptions that $\mathbf{v} = (v^0, v^1, ..., v^m)$ is sequentially essential and that each element of $(x^{t+1}, x^{t+2}, ..., x^m)$ is a feasible vector at its respective time period implies that $q^t(S) = q^{t+1}(S) + w^t(S) \geq \sum_{i\in S}(\sum_{l=t+1}^{m} x_i^l + w^t(\{i\})))$, but the last term is equal to $\sum_{i\in S} b^t(\{i\})$, so that $r \geq 0$. Defining the $|S|$-vector $\bar{x}'$ by $\bar{x}'_i = b^t(\{i\}) + (r/|S|)$, we have $\bar{x}'(S) = q^t(S)$. As S was selected arbitrarily, it follows that $I(N,b^t,\mathrm{R})$ is not-empty.

We can therefore select a vector $\bar{x}$ in the bargaining set of (N,b^t,R), and now set $x_i^t = \bar{x}_i - \sum_{l=t+1}^{m} x_i^l$, confident that this will not lead to negative values, and that by construction $(x^t, x^{t+1}, ..., x^m)$ is in the bargaining set of (N,q^t,R). Continuing with this backward induction to time period 0, we are done with identifying a sub-game stable imputation sequence for $(N,\mathbf{v},\mathrm{R},m+1,\delta)$. *QED*

Finally, we show by an example that the contrast between the set of repeated coalition bargaining set sequences and the set of subgame stable sequences goes beyond the fact that the latter is generally a subset of the former. As shown in Proposition 2, in seeking a repeated-coalition bargaining set sequence, the players may first select any solution in the associated static TU-game bargaining set and then fit a sequence to that static solution. But if the players seek a subgame stable sequence, they might not be able to rely on first considering the associated static game and then finding a sequence that fits that, as the next example shows:

Example 6. Let $n = 5$, with the set of players N denoted by $\{1,2,3,4,5\}$. Consider a 2-period dynamic game $(N,\mathbf{v},\mathrm{R},2,\delta)$, with δ close enough to 1 to be disregarded, with coalition structure $\mathrm{R} = \{12,3,4,5\}$ and $\mathbf{v} = (v^0, v^1)$ defined by $v^0(12) = 100$, $v^0(13) = v^0(14) = v^0(15) = 200$, $v^0(2345) = 200$, $v^1(12) = 100$, $v^1(1345) = 50$, $v^1(2345) = 100$. The value of every other possible coalition at all time periods is equal to zero.

The associated TU-game includes the imputation $(150,50,0,0,0)$ in its bargaining set, and therefore sequences of imputations summing to that vector are in the repeated-coalition bargaining set of $(N,\mathbf{v},\mathrm{R},2,\delta)$. But there can be no subgame stable sequence summing to this vector, because $x^1 = (25,75,0,0,0)$ must be the second element in any subgame-stable sequence (x^0, x^1).

In conclusion, we can state the following about subgame stability: in the repeated-game setting (in which the same characteristic function holds true in each time period), a subgame stable solution always exists – even when there are an infinite number of time periods – because the canonical monotonic sequence is always subgame stable. In the dynamic game setting, the monotonic sequence might not be subgame stable – as shown in example 5. Example 6 shows that there might not be a subgame stable sequence summing to each solution of the associated static game. When there are a finite number of time periods, a subgame stable solution can be found, even in the dynamic game setting, as shown in Proposition 6. It is unclear, as of this writing, whether that result can be extended to the case of an infinite number of time periods.

Appendix

Proof of Proposition 4. Suppose $\mathbf{x} = (x^0, x^1, ..., x^m)$ is a feasible imputation sequence derivable from a credit sequence $\{\{d^t_{i,j}\}, \{p^t_{i,j}\}\}$ relative to vector *g*. First of all, for each *t*, x^t is feasible: because $d^t_{i,j} = 0$ and $p^t_{i,j} = 0$ whenever *i* and *j* are not partners in the same partition of R, we can write $x^t_i = a^t_i + \sum_{j \in \mathrm{R}(i)} d^t_{i,j} - p^t_{i,j} - d^t_{j,i} + p^t_{j,i}$. Given $Q \in \mathrm{R}$, $\sum_{i \in Q} \sum_{j \in Q} d^t_{i,j} - p^t_{i,j} - d^t_{j,i} + p^t_{j,i} = 0$, so that we reach the conclusion $\sum_{i \in Q} x^t_i = \sum_{i \in Q} a^t_i = w^t(Q)$.

Next, suppose there is a last period *m*. Define $\Delta^t_i := x^t_i - a^t_i = \sum_{j \in N} d^t_{i,j} - p^t_{i,j} - d^t_{j,i} + p^t_{j,i}$. By condition (iv), in time period *m*, for each player *i* $c^m_i \leq 0$. But $\sum_{i \in N} c^t_i = 0$, hence $c^m_i = 0$ must hold for each *i*. As by definition, $c^m_i = \sum_t \sum_{j \in N} d^t_{i,j} - p^t_{i,j} - d^t_{j,i} + p^t_{j,i}$, we conclude that $\sum_{t=0}^m \Delta^t_i = 0$, so $\sum_{t=0}^m x^t_i = \sum_{t=0}^m a^t_i = g_i$.

If there is no last period, then for each player *i*, $\left|c^t_i\right| \to 0$ as *t* grows, because $\sum_{l=t+1}^{\omega} a^l_i$ is continually shrinking. Reasoning similar to that in the above paragraph leads to the conclusion that $\sum_{l=0}^t \Delta^l_i \to 0$ as *t* grows, so that $\lim_t x^t_i = g_i$.

In the other direction, suppose that $\mathbf{x} = (x^0, x^1, ..., x^m)$ is an imputation sequence, with the goal of exhibiting a credit sequence from which $\mathbf{x}$ is derivable. This is done inductively, with a round of re-payments defined first in each time period, followed by a round of debt allocations. Intuitively, the construction here is rather simple: in each time-period, each player strives to re-pay as much debt as possible. After that, all other deviations from x^t are 'explained' by way of transfers undertaken through loans.

To decrease some of the clutter of symbols, define $o_{i,j}^t := \sum_{l=0}^{t-1} d_{i,j}^l - p_{i,j}^l$ and $o_i^t := \sum_{j\in N} o_{i,j}^t$. As before, $\mathbf{a}(x) = (a^0, a^1, ..., a^m)$ is the monotonic sequence defined against *g*.

Suppose that $\{\{d_{i,j}^l\},\{p_{i,j}^l\}\}$ has been defined for all time period less than *t*. In period 0, no re-payment is effected. Otherwise, the re-payment round is defined as follows: each player 're-pays as much as possible' of outstanding debt o_i^t, re-payment capped only by a_i^t, so the sum total of re-payment by player *i* in time period *t* is given by $r = \min(a_i^t, o_i^t)$. Let D_i^{t+} be the set of players such that for each player $l_j \in D_i^{t+}$, $o_{i,l_j}^t > 0$, and order them by decreasing 'debt' size, i.e. l_j comes before l_k only if $o_{i,l_j}^{t-1} \geq o_{i,l_k}^{t-1}$, with arbitrary ordering when this last semi-inequality is an equality. Next, set $n(i)$ to be the smallest integer such that $\sum_{j=1}^{n(i)} o_{i,l_j}^{t-1} \leq a_i^t$ under this ordering. For $1 \leq j \leq n(i)$, let $p_{i,j}^t = o_{i,j}^{t-1}$, and for $j = n(i)+1$, if there exists an element $l_{n(i)+1}$ in D_i^{t+}, let $p_{i,j}^t = a_i^t - \sum_{k=1}^{n(i)} o_{i,k}^{t-1}$.

After the round of re-payments has been completed, we have for each player *i* the value $f_i^t := a_i^t - \sum_{j\in N} p_{i,j}^t + \sum_{j\in N} p_{j,i}^t$, and it is against these values that the round of debt allocation is conducted. Let $\overline{\Delta}_i^t := f_i^t - a_i^t = \sum_{j\in N} p_{i,j}^t + p_{j,i}^t$, and, for an arbitrary $S \in \mathrm{R}$, order the players in S as $i_1,...,i_k$ by decreasing size of $\overline{\Delta}_i^t$. Further define $\overline{\Delta}_+^t = \{i \in S \mid \overline{\Delta}_i^t > 0\}$ and $\overline{\Delta}_-^t = \{i \in S \mid \overline{\Delta}_i^t < 0\}$. List the elements $\overline{\Delta}_+^t$ as $\{j_1,...,j_k\}$, ordered by decreasing size of $\overline{\Delta}_j^t$, and similarly list the elements of $\overline{\Delta}_-^t$ as $\{h_1,...,h_l\}$, ordered by decreasing size of $\left|\overline{\Delta}_h^t\right|$.

Define for each member of $\overline{\Delta}_+^t$ a set of 'creditors' in $\overline{\Delta}_-^t$ as follows. Set $C_{j_1} := \{h_1,...,h_{m(j_1)}\}$ such that $\sum_{i=1}^{m(j_1)} \left|\overline{\Delta}_{h_i}^t\right| \geq \overline{\Delta}_{j_1}^t$, where $m(j_1)$ is the smallest integer such that this inequality holds. Set $d_{j_1,h_i}^t = \left|\Delta_{h_i}^t\right|$ for $i < m(j_1)$, and $d_{j_1,h_{m(j_1)}}^t = \overline{\Delta}_{j_1}^t - \sum_{i=1}^{m(j_1)-1} \left|\overline{\Delta}_{h_i}^t\right|$.

For calculating C_{j_s}, for $s > 1$, first set $d_{j_s,h_{m(j_{s-1})}}^t := \left|\overline{\Delta}_{h_{m(j_{s-1})}}^t\right| - d_{j_{s-1},h_{m(j_{s-1})}}^t$ and then set $C_{j_s} = \{h_{m(j_{s-1})},...,h_{m(j_s)}\}$ such that $d_{j_s,h_{m(j_{s-1})}}^t + \sum_{i=m(j_{s-1})+1}^{m(j_s)} \left|\overline{\Delta}_{h_i}^t\right| \geq \overline{\Delta}_{j_s}^t$ and $m(j_s)$ is the smallest integer such that this inequality holds. Set $d_{j_s,h_i}^t = \left|\overline{\Delta}_{h_i}^t\right|$ for $m(j_{s-1}) < i < m(j_s)$, and $d_{j_s,h_{m(j_s)}}^t = \overline{\Delta}_{j_s}^t - \sum_{i=m(j_{s-1})+1}^{m(j_s)-1} \left|\overline{\Delta}_{h_i}^t\right|$.

It remains to be shown that following these steps leads to an admissible credit sequence $\{\{d_{i,j}^t\},\{p_{i,j}^t\}\}$. Constraints (i) and (ii) are trivially met by the constructed credit sequence. Constraint (iii), which limits the size of re-payments, is explicitly guaranteed by the construction, as is constraint (v).

To see that constraint (iv) is met, note that by the way $\{\{d_{i,j}^t\},\{p_{i,j}^t\}\}$ are constructed, for any player *i* and time period *t*, $\sum_{l=0}^{t} x_i^l = \sum_{l=0}^{t} a_i^l + c_i^t$. On the other hand, by assumption $\sum_{l=0}^{m} x_i^l = \sum_{l=0}^{m} a_i^l$. Hence, if $c_i^t > \sum_{l=t+1}^{m} a_i^l$, $\sum_{l=0}^{t} x_i^l = \sum_{l=0}^{m} a_i^l$, which would require $\sum_{l=t+1}^{m} x_i^l$ to be a negative quantity in order to ensure $\sum_{l=0}^{m} x_i^l = g_i$. This is impossible, and we conclude constraint (iv) holds. *QED*

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