Information Revelation and Random Entry in Sequential Ascending Auctions

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INFORMATION REVELATION AND RANDOM ENTRY IN SEQUENTIAL ASCENDING AUCTIONS

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ABSTRACT. We examine a model in which multiple buyers with single-unit demand are faced with an infinite sequence of auctions. New buyers arrive on the market probabilistically, and are each endowed with a constant private value. Moreover, objects also arrive on the market at random times, so the number of competitors and the degree of informational asymmetry among them may vary across from one auction to the next. We demonstrate by way of a simple example the inefficiency of the second-price sealed-bid auction in this setting, and therefore assume that each object is sold via ascending auction.

We then characterize an efficient and fully revealing periodic ex post incentive compatible equilibrium for the game in which the objects are sold via ascending auctions. We show that each buyer's bids and payoffs depend only upon their rank amongst their competitors and the (revealed) values of those with lower values. Furthermore, strategies are memoryless—bids depend only upon the information revealed in the current auction, and not on any information that may have been revealed in earlier periods. We then demonstrate that the sequential ascending auction serves as an indirect mechanism that is equivalent—in our setting—to the dynamic marginal contribution mechanism introduced by Bergemann and Välimäki (2007) and generalized in Cavallo, Parkes, and Singh (2007).

KEYWORDS: Sequential auctions, Ascending auctions, Random arrivals, Information revelation, Dynamic Vickrey-Clarke-Groves mechanism, Pivotal mechanism, Marginal contribution

JEL CLASSIFICATION: C73, D44, D83.

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1. INTRODUCTION

Many markets, most notably internet markets such as eBay, sell multiple objects via sequential auctions in which a single object is sold at a time. In this paper, we examine a model of such markets in which new buyers arrive on the market at random times. Each bidder has an independently drawn private value for purchasing an object. In contrast to much of the literature that makes use of sealed-bid auctions, we focus on the ascending auction. Although the various auction formats are in many respects equivalent in a static private-values setting, this equivalence does not hold in a dynamic environment, primarily due to the information revelation inherent in the ascending auction format. The difference between the two formats is further exacerbated in the sequential auction setting when we allow for dynamically changing populations of buyers. In particular, the entry of a new buyer introduces an additional informational asymmetry. We show, however, that this asymmetry may be easily resolved by employing ascending auctions. In equilibrium, each buyer's bids and payoffs depend only on the buyer's rank amongst their current competitors and the (revealed) values of those opponents with lower values. Furthermore, these strategies have the remarkable property of being memoryless—in each auction conducted, bids are independent of the information revealed in previous periods, despite the fact that *all* private information is revealed during every auction.

We feel that this model serves as a useful abstraction of online auction sites such as eBay or uBid, especially when considering the extensive market on these sites for individual units of brand-new homogenous goods. Typically, a variety of auctions for identical items are open simultaneously, but may be ordered by their closing time. Thus, abstracting away from intra-auction dynamics, a sequential auction model yields a good approximation.[1] With this in mind, many authors (see Sailer (2006) or Zeithammer (2006), for instance) make use of the second-price sealed-bid auction, citing evidence from Roth and Ockenfels (2002) and Bajari and Hortaçsu (2003) about the prevalence of "sniping" (last-second bidding) in online auctions in defense of their modeling choice. However, as shown by Cai, Wurman, and Chao (2007), pure-strategy symmetric equilibria do *not* exist in sequential sealed-bid auctions when buyer values are fixed across time and bids are made publicly observable after each auction. As most online auctions bear a close resemblance to English auctions in regards to intra-auction dynamics as well as the visibility of submitted bids (both during an auction and after an auction has closed), we believe that the ascending auction is better-suited than the sealed-bid second-price auction for modeling online auction markets.[2]

[1]Nekipelov (2007) examines a different aspect of these markets, choosing to ignore the sequential nature of these auctions in order to study within-auction dynamics.

[2]Either choice is, of course, a compromise, abstracting away important features of the real-world environment for the sake of tractability.

What is more, we feel that the sequential ascending auction is important for another, independent, reason. Bergemann and Välimäki (2007) demonstrate the suitability of sequential ascending auctions as a simple way to provide for the truthful implementation of the socially efficient allocation in a task scheduling problem. In particular, they provide an example in which sequential ascending auctions are equivalent to their dynamic generalization of the classic Vickrey-Clarke-Groves mechanism.[3] Cavallo, Parkes, and Singh (2007) generalize this mechanism to settings in which agents may be "inaccessible" for periods of time. The present work complements these papers, as we show that the sequential ascending auction serves as an (easily implemented and understood) indirect mechanism that is equivalent to their direct mechanisms in a complex environment, and is therefore an incentive compatible mechanism for inducing socially efficient choices.

The present work is closely related to several papers in the sequential auctions literature. Milgrom and Weber (2000) examine the properties of a variety of auction formats for the (simultaneous or sequential) sale of multiple objects with a fixed set of buyers and objects. In regards to the ascending auction with private values, they show that, in equilibrium, buyers bid exactly their values. However, they allow for neither discounting nor the entry of new buyers, features that play a central role our model. The vast majority of the literature following that work has chosen to focus on sealed-bid auctions; for example, the previously referenced Sailer (2006) and Zeithammer (2006) conduct empirical studies of eBay auctions making use of sequential second-price sealed-bid auctions and assumptions of an effectively static environment. Kittsteiner, Nikutta, and Winter (2004) examine the role of discounting in sequential sealed-bid auctions, and prove a revenue equivalence result for auctions in which the only information revealed is the valuations of bidders who have already left the market, while Jeitschko (1998) considers a model of first-price sealed-bid auctions in which winner's bids are revealed, allowing the remaining buyers to update their beliefs about their opponents' valuations. On the other hand, Cai, Wurman, and Chao (2007) demonstrate the nonexistence of pure-strategy symmetric equilibria in sealed-bid sequential auction models in which all bids are revealed. The only paper that we are aware of that examines sequential ascending auctions is that of Caillaud and Mezzetti (2004), who examine reserve prices in a model with a sequence of only two auctions.

Certain elements within the bargaining with incomplete information literature are also related to our model. Inderst (2008) considers a bargaining model in which a seller is randomly visited by heterogeneous buyers. If the seller is currently engaged in bargaining with one agent when another arrives, she may choose to switch from one buyer to the other. However, this switch is permanent, implying that the arrival of a new buyer

[3]Their example differs substantially from the present work, as it does not take into account the complications of random buyer and seller arrivals, and hence may be viewed as an essentially static problem.

either "restarts" the game or is completely irrelevant. Fuchs and Skrzypacz (2008) take a different approach: they consider an incomplete information bargaining problem between a buyer and a seller, and allow for the possibility of the arrival of "events" which end the game and yield a particular expected payoff to each agent. Their interpretation is that these events may be viewed as triggers for some sort of multi-lateral mechanism involving new entrants (a second-price auction, for example) for which the expected payoffs are a reduced-form representation. Thus, while both works are primarily concerned with characterizing the endogenous option value that results from the potential arrival of additional participants to the market, they do this in a framework of bilateral bargaining which fails to capture the dynamic nature of competition among several current and potential market participants. On the other hand, Nekipelov (2007) studies the role of entry *during* a *single* online ascending auction, while Said (2008) examines the role of buyer entry *between* periods in a model of sequential second-price auctions in which objects are stochastically equivalent. By way of comparison, the present work incorporates buyer entry between ascending auctions in a more standard private values framework, and further demonstrates the relationship between the endogenous option value arising from participating in future auctions with the marginal contribution to social welfare.

Finally, we would be remiss in not noting the relationship between our model and that of Peters and Severinov (2006). Also primarily motivated by auction markets such as eBay, their work considers a setting with multiple buyers and sellers interacting simultaneously. They find a perfect Bayesian equilibrium that supports efficient trade at Vickrey prices; moreover, if the numbers of buyers and sellers are sufficiently large, then trade is also ex post efficient. While their model has the advantage of considering the effects of competing auctions on the strategic behavior of buyers and sellers, it does not take into account what we believe are two key features of the markets in question (and the two key features of our model): auctions are conducted asynchronously, and new agents arrive on the market at random times.

The paper is organized as follows. We present our model in Section 2, and then provide a simple example demonstrating some of the advantages of the ascending auction format over the second-price sealed-bid auction in a dynamic setting with buyer entry in Section 3. Section 4 solves for the equilibrium in our model with buyer entry and demonstrates some of its desirable properties. In Section 5, we discuss the relationship between our model and the dynamic Vickrey-Clarke-Groves mechanism and generalize our setting to allow for the random arrival of objects. Finally, Section 6 concludes.

2. THE MODEL

We consider a market in which time is discrete; periods are indexed by $t \in \mathbb{N}$. There is a finite number n_t of risk-neutral buyers with single-unit demand in the market in any given period t. Each buyer $i \in \{1, \ldots, n_t\}$ has a valuation $v_i \in \mathbb{R}_+$, where v_i is drawn from the distribution F with corresponding density f. We assume that valuations are private information, and are independently and identically distributed across buyers. Moreover, additional buyers may arrive on the market in each period. We will assume that at most one new buyer arrives in any given period, and that this arrival occurs with some exogenously given probability $q \in [0, 1]$. Finally, we assume that buyers discount the future exponentially with discount factor $\delta \in (0, 1)$.

In each period, there is exactly one object available for sale via an ascending (English) auction. The auction begins with the price at zero and all bidders participating in the auction. Each bidder may choose any price at which to drop out of the auction. This exit decision is irreversible (in the current period), and is observable by all agents currently present in the market. Finally, the auction ends whenever exactly 1 active bidder remains, and the price paid by this winning bidder is the price at which the last exit occurred. Note that we assume that the number of active bidders is commonly known throughout the auction.[4] With this in mind, each bidder's decision problem *within* a given period is not to choose a single bid, but rather a sequence of functions, each of which is a exit price contingent on the (observed) exit prices of the bidders who have already dropped out of the current auction.

Throughout, we will denote by $\hat{v}$ the *ordered* vector of realized values for those buyers currently present on the market, where

$$\hat{v}_1 > \hat{v}_2 > \cdots > \hat{v}_{n_t}.$$

Furthermore, for any $k, n \in \mathbb{N}$ such that $1 \leq k \leq n$, we will denote by $V_{k,n}(\hat{v})$ the expected payoff of the buyer with the k-th highest of n values. For example, if there are three bidders present, with $v_2 > v_3 > v_1$, then $\hat{v} = \{v_2, v_3, v_1\}$, bidder 1's payoff is $V_{3,3}(\hat{v})$, bidder 2's payoff is $V_{1,3}(\hat{v})$, and bidder 3's payoff is $V_{2,3}(\hat{v})$.

3. A MOTIVATING EXAMPLE

Suppose that there are two buyers on the market with values $v_1, v_2 \in [0, 1]$, where, without loss of generality, we assume that $v_1 > v_2$. In addition, a third potential buyer with value $v_3 \sim F$, where F is the uniform distribution on $[0, 1]$ may enter the market with probability $q \in [0, 1]$. Each of these buyers wishes to purchase exactly one unit of some

[4]This assumption is a consequence of the standard Milgrom and Weber (1982) "button auction" approach to modeling English auctions.

object which is being sold via a sequence of three auctions. All buyers discount time with a common discount factor $\delta \in (0, 1)$. Furthermore, we make the assumption that v_1 and v_2 are commonly known amongst all buyers, which may be viewed as the result of information being revealed via bidding behavior in some (unmodeled) previous periods. The new entrant's value, however, is her own private information. We will consider two variants of this example; first, we will assume that objects are sold via second-price auctions in which the buyers' bids are revealed after each round, and then we will assume that objects are sold via ascending auctions.

We begin with the second-price auction. Note that in any round in which there is only one bidder present, that bidder receives the object at a price of zero, regardless of her bid. Therefore, if there are two bidders present in the second period, each bidder i has an option value of δv_i from losing. Thus, regardless of the information that each bidder has about the other, it is weakly dominant for each bidder to submit a bid of their true value less their option value—the optimal bid for each bidder i is $(1-\delta)v_i$. Thus, denoting the payoff of a bidder in the second round when there are two bidders present as $U(v_i, v_j)$, we have

$$U(v_i, v_j) = \begin{cases} v_i - (1-\delta)v_j, & \text{if } v_i > v_j; \\ \delta v_i, & \text{if } v_i \leq v_j. \end{cases} \tag{3.1}$$

Note that, using this expression, we may write the payoff of a lone bidder with value v_i as $U(v_i, 0)$.

Now consider the third bidder (when present). Under the assumption that bidder 1 bids a greater amount than bidder 2 (that is, that $b_1 > b_2$), the third bidder faces a choice between winning the auction and receiving a payoff of $v_3 - b_1$ or losing the auction and facing bidder 2 in the next period, yielding a payoff of $\delta U(v_3, v_2)$. Thus, bidder 3 prefers to win if, and only if, $v_3 - b_1 \geq \delta U(v_3, v_2)$, or, equivalently, $b_1 \leq v_3 - \delta U(v_3, v_2)$. She can then win the auction if, and only if, it is optimal for her to do so by bidding

$$b_3(v_3) = v_3 - \delta U(v_3, v_2) = \begin{cases} (1-\delta)v_3 + \delta(1-\delta)v_2, & \text{if } v_3 > v_2; \\ (1-\delta^2)v_3, & \text{if } v_3 \leq v_2. \end{cases} \tag{3.2}$$

Note that b_3 is strictly increasing in v_3, and hence fully identifies bidder 3's valuation in the next period when bids are revealed. For convenience, we will denote by u_1 and u_2 the values of bidder 3 that submit bids equal to those of bidders 1 and 2, respectively; that is,

$$u_1 = b_3^{-1}(b_1) \text{ and } u_2 = b_3^{-1}(b_2).$$

Now consider the case of bidder 2's bid in the first period of the game. If she submits a winning bid in the first period, she receives a payoff of $v_2 - b^*$, where b^* is the highest competing bid that she faces. On the other hand, if she loses the first-round auction, she receives a payoff of $\delta\mathbb{E}[U(v_2, v^*)]$, where

$$v^* = \begin{cases} 0, & \text{with probability } 1-q; \\ v_3, & \text{with probability } qF(u_1); \\ v_1, & \text{with probability } q\left(1-F(u_1)\right). \end{cases}$$

Thus, bidder 2 prefers to win if, and only if, $v_2 - b^* \geq \delta\mathbb{E}[U(v_2, v^*)]$. She may then guarantee that she wins only when it is desirable to do so by bidding

$$\begin{aligned} b_2 &= v_2 - \delta\mathbb{E}[U(v_2, v^*)] \\ &= v_2 - \delta\left[\begin{array}{c} (1-q)v_2 + \delta q\left(1-F(u_1)\right)v_2 \\ +q\int_0^{v_2}\left(v_2-(1-\delta)v'\right)dF(v') + q\int_{v_2}^{u_1}\delta v_3\,dF(v') \end{array}\right] \\ &= (1-\delta)(1+\delta q)v_2 - (1-\delta)\delta q\frac{v_2^2}{2}. \end{aligned} \tag{3.3}$$

Finally, let us consider buyer 1's bidding behavior in the first period of the game. Note first that $u_2 < v_2 < v_1$, implying that if bidder 1 loses today, she will definitely win the auction in the next period. To see this, note that if bidder 3 enters and wins the first round, bidder 1 faces $v_2 < v_1$ in the next period. On the other hand, if bidder 2 is the high bidder in the first round, then bidder 1 is either alone or faces $v_3 < u_2 < v_1$ in the second round. Thus, when the high opponent bid is b^*, winning yields bidder 1 a payoff of $v_1 - b^*$, while losing yields a payoff of $\delta U(v_1, v^*)$, where

$$v^* = \begin{cases} 0, & \text{with probability } 1-q; \\ v_3, & \text{with probability } qF(u_2); \\ v_2, & \text{with probability } q\left(1-F(u_2)\right). \end{cases}$$

Thus, similar to the cases of bidders 2 and 3, bidder 1 may guarantee that she wins only when it is desirable for her to do so by bidding

$$b_1 = v_1 - \delta\mathbb{E}[U(v_1, v^*)] = v_1 - \delta\left[\begin{array}{c} (1-q)v_1 + \delta q\left(1-F(u_2)\right)\left(v_1-(1-\delta)v_2\right) \\ +q\int_0^{u_2}\left(v_1-(1-\delta)v'\right)dF(v') \end{array}\right].$$

Recall that $u_2 = b_3^{-1}(b_2) < v_2$, implying that $u_2 = b_2/(1-\delta^2)$. Combining this with the assumption that $F(x) = x$ implies that

$$u_2 = \frac{b_2}{1-\delta^2} = \frac{1+\delta q}{1+\delta}v_2 - \frac{\delta q}{1+\delta}\frac{v_2^2}{2}.$$

Thus, we may conclude that

$$\begin{aligned} b_1 &= (1-\delta)v_1 + (1-\delta)\delta q v_2 \\ &\quad - \frac{\delta(1-\delta)(1+\delta q)(1+\delta(2-q))qv_2^2}{2(1+\delta)^2} + \frac{\delta^3(1-\delta)(1-q)q^2v_2^3}{2(1+\delta)^2} + \frac{\delta^3(1-\delta)q^3v_2^4}{8(1+\delta)^2}. \end{aligned} \tag{3.4}$$

For clarity, Figure 1 plots the bids of all three buyers for fixed parameter values.[5] The key features to note are that $u_1 < v_1$ and $u_2 < v_2$; use of the second-price auction in this context may lead to inefficient outcomes, as "low" values of bidder 3 may outbid bidders 1 and 2 despite their having higher values. This result is driven by two main features of our setting: first, agents discount the future and hence the order in which objects are allocated matters; and second, there is a fundamental asymmetry in information—bidder 3's value is private information, while the values of bidders 1 and 2 are commonly known. Thus, in addition to the nonexistence of symmetric equilibria in sequential second-price

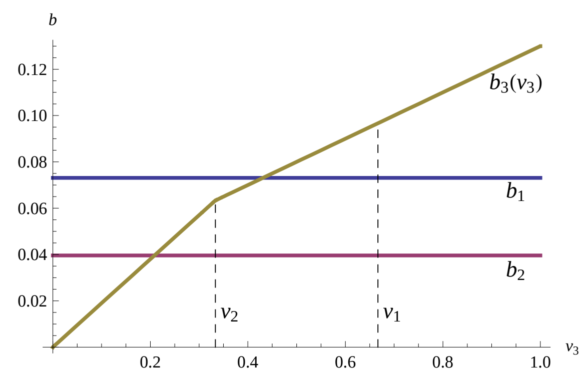

Figure 1: Initial bids when $v_1 = \frac{2}{3}$, $v_2 = \frac{1}{3}$, $v_3 \sim U[0,1]$, $\delta = \frac{9}{10}$, and $q = \frac{1}{4}$.

sealed-bid auctions as demonstrated by Cai, Wurman, and Chao (2007), allowing for the entry of new buyers may induce inefficient outcomes, even in the asymmetric equilibria of the sequential auction game.

We now demonstrate that the ascending auction does not share the inefficiency of the second-price auction in this setting. Note that when there are only two bidders present, the losing bidder is guaranteed a payoff of δv_i in the next period. Therefore, bidders are willing to remain active in an auction until the price reaches $(1-\delta)v_i$. Thus, the expected payoff of a bidder when she has only one opponent present on the market is given by $U(v_i, v_j)$ from Equation 3.1.

When there are three bidders present, matters are slightly different. In particular, the very nature of an ascending auction immediately reveals to all bidders the number of

[5]The qualitative features of the equilibrium do not depend on these parameter values. Moreover, the result remains true even if the presence of the new entrant is made common knowledge or contingent bidding is used. This may be easily seen by examing the bids of buyers 1 and 2 in the cases where $q = 0$ and $q = 1$.

participants. Thus, bidder 3 is unable to keep private her presence on the market. This implies that the first bidder to drop out of the auction knows that they have the lowest value among three bidders, and hence will receive an expected payoff of $\delta^2 v_i$. Thus, each of the three bidders remains active until the price reaches

$$(1-\delta^2)v_i.$$

Denoting by $\hat{v}_3$ the lowest of the three values, the two remaining bidders now know that they are guaranteed a payoff of $U(v_i, \hat{v}_3)$ in the following period, and are hence willing to remain active until they are indifferent between winning at the current price and winning the object in the following period; that is, until the price reaches

$$(1-\delta)v_i + \delta(1-\delta)\hat{v}_3.$$

Notice that these cutoff prices are strictly increasing in each bidder's value, and hence are both efficient and fully revealing.[6] Thus, we have established that the ascending auction does not suffer from the same shortcomings as the second-price auction in this relatively simple setting. We will therefore focus exclusively on the ascending auction from this point forward.

4. Equilibrium Analysis

4.1. **Preliminaries and Equilibrium Strategies.** One of the most remarkable features of the equilibrium that we construct in this model is that buyer's bids and payoffs do not depend upon the valuations of higher-ranked bidders (neither in expectation nor realization), even if that information is publicly available. Recall that $\hat{v}$ is the *ordered* vector of realized buyer valuations, where

$$\hat{v}_1 > \cdots > \hat{v}_n,$$

and that we denote by $V_{k,n}(\hat{v})$ the expected payoff of the buyer with the k-th highest of n values. To show the property described above, we will show that (abusing notation slightly) we may write

$$V_{k,n}(\hat{v}_1, \ldots, \hat{v}_n) = V_{k,n}(\hat{v}_k, \ldots, \hat{v}_n).$$

A formal statement of this result may be found in the subsequent section; in the meantime, we will describe the equilibrium taking this property as a given.

Suppose that an auction is in progress with n bidders with (ordered) values $\hat{v}$. When all bidders are still active, a bidder with valuation v_i who drops out of the bidding learns (and reveals) that, in equilibrium, she has the lowest value; that is, that $\hat{v}_n = v_i$. Therefore, her expected payoff in the next period is $V_{n-1,n-1}(v_i)$, as at the beginning of the next period,

[6] In addition, it is straightforward to verify that these strategies do, in fact, constitute an equilibrium. Conditional on participation, no bidder wishes to deviate from these strategies. Furthermore, no bidder wishes to postpone their participation to a future period.

there will be $n-1$ bidders remaining (the current periods n bidders less the winning buyer) and she will have the lowest value. Therefore, each bidder i should remain in the auction until the current price p is such that

$$v_i - p = \delta V_{n-1,n-1}(v_i).$$

At this price, bidder i is indifferent between purchasing the object today and waiting until the next period when i will be the lowest-valued buyer. Thus, when no one has dropped out, bidder i will remain in the auction until the price reaches

$$\beta_{n,n}(v_i) := v_i - \delta V_{n-1,n-1}(v_i). \tag{4.1}$$

Once someone drops out of the auction, the remaining $n-1$ bidders learn the realization of $\hat{v}_n$ and that they are not the lowest-valued competitor.[7] Therefore, the next bidder (with value v_j) to drop out reveals herself to be the second-lowest of the n bidders; therefore, her expected payoff in the next period is $V_{n-2,n-1}(v_j, \hat{v}_n)$, as she will be the second-lowest of the $n-1$ buyers remaining in the following period. Thus, each bidder j who has not already dropped out should remain in the auction until the current price p is such that she is indifferent between purchasing the object in the present period and waiting until the next period—that is, when

$$v_j - p = \delta V_{n-2,n-1}(v_j, \hat{v}_n).$$

Thus, when no one has dropped out, bidder j remains in the auction until the price reaches

$$\beta_{n-1,n}(v_i, \hat{v}_n) := v_i - \delta V_{n-2,n-1}(v_i, \hat{v}_n). \tag{4.2}$$

Proceeding inductively, we define for each $k = 2, \ldots, n$ the bidding function

$$\beta_{k,n}(v_i, \hat{v}_{k+1}, \ldots, \hat{v}_n) := v_i - \delta V_{k-1,n-1}(v_i, \hat{v}_{k+1}, \ldots, \hat{v}_n). \tag{4.3}$$

These bidding functions define the drop-out points for a bidder with value v_i when there are k buyers still active in the auction. Notice that this implies that the final price in this auction will be

$$\beta_{2,n}(\hat{v}_2, \ldots, \hat{v}_n) = \hat{v}_2 - \delta V_{1,n-1}(\hat{v}_2, \ldots, \hat{v}_n).$$

Keep in mind, however, that we must verify that these bid functions are invertible (so that values are revealed), and also that these bidding strategies indeed form an equilibrium. This requires a characterization of the expected payoff functions $V_{k,n}$.

4.2. **The Payoff Functions.** As a preview of our results, consider first the case of a lone buyer present on the market at the beginning of a period with valuation v_1, and that

[7] This of course requires $\beta_{n,n}$ to be invertible, something that we will verify in short order.

a second buyer may arrive with probability q. Once the price clock starts rising, it is immediately revealed whether there are one or two bidders present. Thus, there is no asymmetric information regarding the number of active bidders.

Note that if the second bidder does not arrive, the lone bidder receives the object for free. In the case of two bidders present, however, each bidder $i = 1, 2$ will stay in the auction until the price rises to $\beta_{2,2}(v_i) = v_i - \delta V_{1,1}(v_i)$. Thus,

$$V_{1,1}(v_1) = (1-q)v_1 + q\left[\int_0^{v_1} (v_1 - \beta_{2,2}(v'))\, dF(v') + \int_{v_1}^{\infty} \delta V_{1,1}(v_1)\, dF(v')\right].$$

The first term in this expression is bidder 1's payoff if she is alone on the market. The second term is her expected payoff if a second bidder arrives, and is the sum of her expected winnings if the second bidder has a lower value than her and her expected continuation payoff if she loses the auction. Differentiation of this expression with respect to v_1 and then substituting for $\beta_{2,2}(v_1)$ yields

$$\begin{aligned} V_{1,1}'(v_1) &= (1-q) + q\left(f(v_1)v_1 + F(v_1) - f(v_1)\beta_{2,2}(v_1)\right) \\ &\qquad - \delta q\left((1-F(v_1))V_{1,1}'(v_1) - f(v_1)V_{1,1}(v_1)\right) \\ &= \frac{1-q(1-F(v_1))}{1-\delta q(1-F(v_1))}. \end{aligned}$$

Note that we may rewrite this expression as

$$\sum_{t=0}^{\infty} (\delta q(1-F(v_1)))^t\, [1-q(1-F(v_1))],$$

which is the summation of the expected per-period gain from a marginal increase in v_1, discounted by the probability of that gain being realized in any given period.

Furthermore, note that $V_{1,1}(0) = 0$, implying that

$$V_{1,1}(v_i) = V_{1,1}(0) + \int_0^{v_i} V_{1,1}'(v')\, dv' = \int_0^{v_i} \frac{1-q(1-F(v'))}{1-\delta q(1-F(v'))}\, dv'. \tag{4.4}$$

Note that $0 < V_{1,1}'(v') < 1$ for all $v' \in \mathbb{R}_+$. Hence, $V_{1,1}$ is strictly increasing, as is $\beta_{2,2}$.

We will now proceed to characterize $V_{k,n}$ inductively for all $k \in \mathbb{N}$ and all $j \in \{1, \ldots, k\}$ via a series of propositions.

PROPOSITION 1 (Existence and uniqueness of $V_{k,n}$).
Fix any $n > 1$, and suppose that the expected payoff to a buyer when a period starts with $n-1$ bidders present depends only on the rank of that bidder and the values of those with values lower than her; that is, given (known) values $\hat{v} \in \mathbb{R}_+^{n-1}$, the k-th highest of the $n-1$ bidders receives expected payoff $V_{k,n-1}(\hat{v}_k, \ldots, \hat{v}_{n-1})$. Then the expected payoff of the k-th highest of n bidders, for all $k = 1, \ldots, n$, is given by $V_{k,n}(\hat{v}_k, \ldots, \hat{v}_n)$. Furthermore, given $\{V_{k,n-1}\}_{k=1}^{n-1}$, the functions $\{V_{k,n}\}_{k=1}^{n}$ are uniquely determined.

Proof. The proof may be found in Appendix A. ∎

Thus, the strategies in Equation 4.3 lead to well-defined and unique value functions for the buyers. In addition, following these strategies implies that these expected payoffs are not dependent upon history—they do not depend upon the values or prices paid in previous periods—but rather depend only upon the values of those buyers ranked below a bidder.

We may also use the indifference inherent in the definition of our conjectured equilibrium strategy in order to illustrate the link between the various payoff functions. In particular, we have the following

PROPOSITION 2 (Relationship between $V_{k,n}$ and $V_{1,n}$)**.**
Fix any $n \in \mathbb{N}$. Then for all $k = 1, \ldots, n$, the expected payoff to the k-th ranked of n buyer is equal to that of the highest-ranked buyer when she is tied with $k-1$ of her opponents; that is,

$$V_{k,n}(\hat{v}_k, \ldots, \hat{v}_n) = V_{1,n}(\hat{v}_k, \ldots, \hat{v}_k, \hat{v}_{k+1}, \ldots, \hat{v}_n).$$

Proof. The proof may be found in Appendix A. ∎

As mentioned above, this result makes heavy use of the indifference conditions built into the bidding strategies described in Equation 4.3, and in particular the indifference of the buyer with the second-highest value. This bidder drops out at a price at which she is indifferent between winning immediately or waiting one period. Unsurprisingly, when the top two buyers have the same value, they must receive the same payoff, regardless of the tie-breaking rule used to determine which one of the two should receive the object when they drop out simultaneously. The intuition behind the relationship between lower-ranked buyers' payoff functions is analogous. Moreover, Proposition 2 implies that knowledge of the functions $\{V_{1,n}\}_{n=1}^{\infty}$ is sufficient to determine the remaining value functions. Thus, define the function $\lambda : \mathbb{R}_+ \to [0,1]$ by

$$\lambda(v) := \delta \frac{1 - q(1 - F(v))}{1 - \delta q(1 - F(v))}. \tag{4.5}$$

We have the following

THEOREM 1 (Characterization of $V_{k,n}$)**.**
For all $n \in \mathbb{N}$ and all $k = 1, \ldots, n$,

$$V_{k,n}(\hat{v}) = \delta^{-1} \sum_{j=k}^{n} \int_{\hat{v}_{j+1}}^{\hat{v}_j} \lambda^j(v')\, dv', \tag{4.6}$$

where we take $\hat{v}_{n+1} := 1$.

Proof. Note that we may write $V_{1,n}(\hat{v})$ as

$$\begin{aligned} V_{1,n}(\hat{v}) &= (1-q)\left[\hat{v}_1 - \beta_{2,n}(\hat{v}_2,\ldots,\hat{v}_n)\right] \\ &\quad + q\left[\sum_{j=0}^{n-1}\int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} \left(\hat{v}_1 - \beta_{2,n+1}(\hat{\hat{v}}_{-1}(v'))\right)\, dF(v') + \int_{\hat{v}_1}^{\infty} \delta V_{1,n}(\hat{v})\, dF(v')\right], \end{aligned} \tag{4.7}$$

where $\hat{\hat{v}}(v')$ is the ordered vector that arises from adding v' to $\hat{v}$. We will denote by $V_{1,n}^{(j)}$ the partial derivative of $V_{1,n}$ with respect to its j-th argument. Differentiation with respect to $\hat{v}_1$ then implies that

$$V_{1,n}^{(1)}(\hat{v}) = 1 - q(1-F(\hat{v}_1)) + \delta q(1-F(\hat{v}_1))V_{1,n}^{(1)}(\hat{v}) = \delta^{-1}\lambda(\hat{v}_1).$$

Notice that this result is independent of n. Furthermore, note that this implies that $V_{1,n}^{(j)}(\hat{v})$ does not depend on $\hat{v}_1$ for any $j \neq 1$; equivalently, for all $n \in \mathbb{N}$,

$$V_{1,n}^{(1,j)}(\hat{v}) = 0 \text{ for all } \hat{v} \in \mathbb{R}_+^n \text{ and } j \neq 1.$$

Differentiating Equation 4.7 with respect to $\hat{v}_2$ now leads to

$$\begin{aligned} V_{1,n}^{(2)}(\hat{v}) &= -\left(1 - q(1-F(\hat{v}_2)\right) + \delta(1-q)V_{1,n-1}^{(1)}(\hat{v}_2,\ldots,\hat{v}_n) + \delta q(1-F(\hat{v}_1))V_{1,n}^{(2)}(\hat{v}) \\ &\quad + \delta q \sum_{j=0}^{n-2}\int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} V_{1,n}^{(1)}(\hat{\hat{v}}_{-1}(v'))\, dF(v') + \delta q \int_{\hat{v}_2}^{\hat{v}_1} V_{1,n}^{(2)}(v',\hat{v}_2,\ldots,\hat{v}_n)\, dF(v'). \end{aligned}$$

Note first that

$$\int_{\hat{v}_2}^{\hat{v}_1} V_{1,n}^{(2)}(v',\hat{v}_2,\ldots,\hat{v}_n)\, dF(v') = \left(F(\hat{v}_1) - F(\hat{v}_2)\right) V_{1,n}^{(2)}(\hat{v})$$

since $V_{1,n}^{(1,2)} = 0$. Moreover,

$$\sum_{j=0}^{n-2}\int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} V_{1,n}^{(1)}(\hat{\hat{v}}_{-1}(v'))\, dF(v') = \int_0^{\hat{v}_2} \delta^{-1}\lambda(\hat{v}_2)\, dF(v') = \delta^{-1}\lambda(\hat{v}_2)F(\hat{v}_2).$$

Thus, we have

$$\begin{aligned} V_{1,n}^{(2)}(\hat{v}) &= \frac{-\left(1-q(1-F(\hat{v}_2))\right) + \left(1-q(1-F(\hat{v}_2))\right)\lambda(\hat{v}_2)}{1-\delta q(1-F(\hat{v}_2))} \\ &= \frac{1-q(1-F(\hat{v}_2))}{1-\delta q(1-F(\hat{v}_2))}\left(\lambda(\hat{v}_2) - 1\right) = \delta^{-1}\left(\lambda^2(\hat{v}_2) - \lambda(\hat{v}_2)\right). \end{aligned}$$

Note that, similar to the case of $V_{1,n}^{(1)}$, $V_{1,n}^{(2)}$ depends only on the second argument of $V_{1,n}$. Thus, for all $n \in \mathbb{N}$,

$$V_{1,n}^{(2,j)}(\hat{v}) = 0 \text{ for all } \hat{v} \in \mathbb{R}_+^n \text{ and } j \neq 2.$$

Proceeding inductively, fix any $k \in \{3, \ldots, n\}$ for arbitrary $n \in \mathbb{N}$, and suppose that

$$V_{1,n}^{(j)}(\hat{v}) = \delta^{-1}\left(\lambda^j(\hat{v}_j) - \lambda^{j-1}(\hat{v}_j)\right)$$

for all $j = 2, \ldots, k-1$. Differentiating Equation 4.7 with respect to $\hat{v}_k$ yields

$$\begin{aligned} V_{1,n}^{(k)}(\hat{v}) &= \delta(1-q)V_{1,n-1}^{(k-1)}(\hat{v}_2, \ldots, \hat{v}_n) + \delta q \sum_{j=0}^{n-k} \int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} V_{1,n}^{(k-1)}(\hat{\hat{v}}_{-1}(v'))\, dF(v') \\ &\quad + \delta q \sum_{j=n-k+1}^{n-1} \int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} V_{1,n}^{(k)}(\hat{\hat{v}}_{-1}(v'))\, dF(v') + \delta q(1 - F(\hat{v}_1))V_{1,n}^{(k)}(\hat{v}). \end{aligned}$$

Since $V_{1,n}^{(k-1)}$ does not depend on any of its arguments but the $(k-1)$-th,

$$\sum_{j=0}^{n-k} \int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} V_{1,n}^{(k-1)}(\hat{\hat{v}}_{-1}(v')\, dF(v') = F(\hat{v}_k)V_{1,n-1}^{(k-1)}(\hat{v}_{-1}).$$

In addition, $V_{1,n}^{(j,k)} = 0$ for all $j < k$ implies that

$$\sum_{j=n-k+1}^{n-1} \int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} V_{1,n}^{(k)}(\hat{\hat{v}}_{-1}(v'))\, dF(v') = \left(F(\hat{v}_1) - F(\hat{v}_k)\right) V_{1,n}^{(k)}(\hat{v}).$$

Thus, we have

$$\begin{aligned} V_{1,n}^{(k)}(\hat{v}) &= \delta(1-q)V_{1,n-1}^{(k-1)}(\hat{v}_{-1}) + \delta q F(\hat{v}_k)V_{1,n-1}^{(k-1)}(\hat{v}_{-1}) + \delta q(1 - F(\hat{v}_k))V_{1,n}^{(k)}(\hat{v}) \\ &= \delta \frac{1 - q(1 - F(\hat{v}_k))}{1 - \delta q(1 - F(\hat{v}_k))} V_{1,n-1}^{(k-1)}(\hat{v}_{-1}) = \delta^{-1}\left(\lambda^k(\hat{v}_k) - \lambda^{k-1}(\hat{v}_k)\right). \end{aligned}$$

By induction, the above expression holds for all $k = 2, \ldots, n$, where $n \in \mathbb{N}$ is arbitrary.

We may then apply the boundary condition $V_{1,n}(0, \ldots, 0) = 0$ in order to show that

$$V_{1,n}(\hat{v}) = V_{1,n}(0, \ldots, 0) + \sum_{j=1}^{n} \int_0^{\hat{v}_j} V_{1,n}^{(j)}(v')\, dv'_j = \delta^{-1} \sum_{j=1}^{n} \left[\int_0^{\hat{v}_j} \lambda^j(v')\, dv' - \int_0^{\hat{v}_{j+1}} \lambda^j(v')\, dv' \right],$$

where we take $\hat{v}_{n+1} := 0$. Applying Proposition 2 and some arithmetic manipulation to the above expression then yields the desired result that

$$V_{k,n}(\hat{v}) = \delta^{-1} \sum_{j=k}^{n} \left[\int_0^{\hat{v}_j} \lambda^j(v')\, dv' - \int_0^{\hat{v}_{j+1}} \lambda^j(v')\, dv' \right]. \tag{4.8}$$ ■

To better understand this result, let us consider two "corner" cases. In particular, notice that if $q = 0$ (that is, if no new buyers ever arrive on the market), then

$$V_{k,n}(\hat{v}) = \sum_{j=l}^{n} \delta^{j-1}(\hat{v}_j - \hat{v}_{j+1}) \text{ for all } k = 1, \ldots, n \text{ and any } n \in \mathbb{N}.$$

Thus, the expected payoff to a buyer in this case is the discounted difference between consecutively ranked valuations. Note that this is also exactly the externality imposed by the k-th highest buyer on all those ranked below her when there is no entry, as she postpones each one's receipt of an object by exactly one period. On the other hand, if $\delta = 1$ and buyers are "infinitely patient," then for any q, we have

$$V_{k,n}(\hat{v}) = \hat{v}_k \text{ for all } k = 1, \ldots, n \text{ and any } n \in \mathbb{N}.$$

In this case, buyers care only about their eventual receipt of an object, but not about the timing of that event. Therefore, their bids are all equal to zero, and any random assignment of objects leaves the buyers equally well off.

4.3. **Equilibrium.** With the characterization derived in Theorem 1, we may now reformulate the bidding strategies from Equation 4.3 as

$$\beta_{k,n}(v_i, \hat{v}_{k+1}, \ldots, \hat{v}_n) = v_i - \int_{\hat{v}_{k+1}}^{v_i} \lambda^{k-1}(v')\,dv' - \sum_{j=k+1}^{n} \int_{\hat{v}_{j+1}}^{\hat{v}_j} \lambda^{j-1}(v')\,dv'. \tag{4.9}$$

This expression allows us to demonstrate the properties of bids in the following

PROPOSITION 3 (Information revelation and sequential consistency of $\beta_{l,k}$)**.**
The buyers' bids $\beta_{k,n}$, where $n \in \mathbb{N}$ and $k \in \{1, \ldots, n\}$, are strictly increasing in each buyers' own valuation. Furthermore, when the buyers use these bidding functions, the exit of a lower-ranked bidder does not *induce the immediate exit of any higher-ranked bidders.*

Proof. The proof may be found in Appendix A. ■

Note that this proposition verifies our previous assumption that buyers' values are revealed after each round—since the bidding functions are strictly increasing in each buyers' own private valuation, the price at which they drop out of the auction is an invertible function, thereby allowing the inference of their value by their competitors. Furthermore, since the bidding functions are "sequentially consistent," a higher-ranked bidder remains in the auction instead of immediately exiting after a lower-ranked bidder drops out, thereby allowing the other buyers to (eventually) deduce their value.

Finally, it remains to be shown that the bidding strategies described are, in fact, an equilibrium of this model. We demonstrate this in the following

THEOREM 2 (Equilibrium verification)**.**
Suppose that in each period, buyers bid according to the cutoff strategies given in Equation 4.9. This strategy profile forms a perfect Bayesian equilibrium of the sequential auction game.

Proof. Consider any period with $n \in \mathbb{N}$ buyers on the market, and fix an arbitrary bidder i. Suppose that all bidders other than i are using the conjectured strategy. We must show that bidder i has no incentive to make a one-shot deviation from the collection of bidding functions $\{\beta_{k,n}\}_{k=2}^{n}$.

Note first that if $v_i < \hat{v}_1$, dropping out of the auction early has no bearing on expected future payoffs due to the memorylessness of the bidding strategies—in each period, the process of information revelation is repeated, and hence a one-shot deviation to an early exit will not affect the bidding behavior in future periods. On the other hand, suppose that $v_i = \hat{v}_1$; that is, bidder i has the highest realized valuation among those bidders present on the market. Following the conjectured equilibrium leads to a payoff of $\hat{v}_1 - \beta_{2,n}(\hat{v}_{-1})$, while deviating and exiting at a lower price leads to the second-ranked bidder winning and a payoff to i of $\delta V_{1,n-1}(\hat{v}_{-2})$. Letting $\hat{w} := (\hat{v}_3, \dots, \hat{v}_n)$, we then have

$$\begin{aligned}\hat{v}_1 - \beta_{2,n}(\hat{v}_2, \hat{w}) - \delta V_{1,n-1}(\hat{v}_1, \hat{w}) &= \hat{v}_1 - \hat{v}_2 + \delta\left(V_{1,n-1}(v_2, \hat{w}) - V_{1,n-1}(v_1, \hat{w})\right) \\ &= \hat{v}_1 - \hat{v}_2 + \left(\int_0^{\hat{v}_2} \lambda(v')\, dv' - \int_0^{\hat{v}_1} \lambda(v')\, dv'\right) \\ &= \int_{\hat{v}_2}^{\hat{v}_1} \left(1 - \lambda(v')\right)\, dv' \geq 0\end{aligned}$$

since $\hat{v}_1 > \hat{v}_2$ and $0 \leq \lambda(v') \leq 1$ for all $v' \in \mathbb{R}_+$. Thus, deviating and exiting the auction early leads to a strict decrease in utility if the realized values are such that bidder i has the highest value, and does not affect payoffs otherwise.

On the other hand, bidder i also has the option of remaining active beyond the cutoffs specified in the conjectured equilibrium. If the realized values are such that $v_i = \hat{v}_1$, delaying exit will have no effect, as the other bidders will have already dropped out of the auction earlier than i. If, on the other hand, $v_i = \hat{v}_k$ for some $k > 1$, then delaying exit may have an effect on i's payoffs. To be precise, if i exits before the eventual winner, her payoff will remain unchanged as behavior in future periods does not depend upon information already revealed. Thus, in order to influence her payoff, i must win the auction, remaining present in the auction until all other bidders have dropped out. Winning the auction yields a payoff of $\hat{v}_k - \beta_{2,n}(\hat{v}_{-k})$, while following the strategy in Equation 4.9 leads to an expected payoff of $\delta V_{k-1,n-1}$. Letting $\hat{w} := (\hat{v}_{k+1}, \dots, \hat{v}_n)$, we have

$$\begin{aligned}\hat{v}_k - \beta_{2,n}(\hat{v}_{-k}) - \delta V_{k-1,n-1}(\hat{v}_k, \hat{w}) &= \hat{v}_k - \hat{v}_1 + \delta\left(V_{1,n-1}(\hat{v}_1, \dots, \hat{v}_{k-1}, \hat{w}) - V_{k-1,n-1}(\hat{v}_k, \hat{w})\right) \\ &= \hat{v}_k - \hat{v}_1 + \sum_{j=1}^{k-1} \int_{\hat{v}_{j+1}}^{\hat{v}_j} \lambda^j(v')\, dv' \\ &\leq \hat{v}_k - \hat{v}_1 + \sum_{j=1}^{k-1} \left(\hat{v}_j - \hat{v}_{j+1}\right) = 0,\end{aligned}$$

where the second line follows from from Theorem 1, and the third from the fact that $0 \leq \lambda(v') \leq 1$ for all $v' \in \mathbb{R}_+$. Hence, deviating and exiting the auction later than prescribed has no effect if i has the highest value, but may leads to a decrease in utility if the realized values are such that $v_i < \hat{v}_1$.

Thus, we may conclude that bidder i has no incentive to make a one-shot deviation from the collection of bidding functions $\{\beta_{k,n}\}_{k=2}^{n}$ regardless of the realized values. Furthermore, the choice of n throughout was arbitrary, implying that bidding according to Equation 4.9 is optimal along the equilibrium path.

In order to determine optimality *off* the equilibrium path, we need to consider the behavior of bidders after a deviation. We have already shown, however, that deviations will be zero probability events, and hence we are free to choose arbitrary off-equilibrium beliefs—Bayes' rule has no bite in this situation. In particular, we will suppose that after a deviation, buyers ignore the history of the game and believe that the deviator is *currently* truthfully revealing her value in accordance with the bidding functions $\{\beta_{k,n}\}_{k=2}^{n}$. The arguments used above therefore imply that continuing to bid according to this strategy remains optimal for all agents, including any that may have deviated in the current or previous periods. Thus, bidding according to $\{\beta_{k,n}\}_{k=2}^{n}$ as defined in Equation 4.9 is optimal along the entirety of the game tree, and hence is a perfect Bayesian equilibrium of the sequential auction game. ■

One should note that this equilibrium is by no means unique, especially given the multiplicity of equilibria possible in a one-shot auction game. In particular, the results of Bikhchandani, Haile, and Riley (2002) may be applied in this setting to show that there exists a continuum of symmetric perfect Bayesian equilibria with full information revelation in every period. These equilibria, however, are all characterized by identical bidding behavior when there are only two active buyers remaining in any period—the values of lower-ranked bidders may be revealed in a variety of ways, but this information is used identically across all equilibria of this kind. Thus, we have payoff and outcome equivalence across all symmetric separating equilibria of this game.

5. DYNAMIC VICKREY-CLARKE-GROVES MECHANISM

Bergemann and Välimäki (2007) develop the dynamic pivotal mechanism (also referred to as the dynamic marginal contribution or dynamic Vickrey-Clarke-Groves mechanism), a direct mechanism that implements the socially efficient allocation in a dynamic private value environment in which agents receive private information over time. In the mechanism that they propose, agents receive in each period their marginal contribution to the social welfare in a dynamic generalization of the standard Vickrey-Clarke-Groves

Mechanism. In this mechanism, the truthtelling strategy is periodic ex post individually rational and incentive compatible.[8] Moreover, the authors show that the sequential ascending auction yields an identical implementation in the case of a scheduling problem with a fixed set of independent tasks. Cavallo, Parkes, and Singh (2007) take the model one step further, demonstrating that dynamic VCG truthfully implements the socially efficient allocation in more general dynamic settings. In this section, we show that the equilibrium in the sequential ascending auction discussed above is equivalent to the truthtelling equilibrium of the dynamic VCG mechanism. In addition, we use the result of Cavallo, Parkes, and Singh (2007) to characterize equilibrium in the sequential ascending auction when objects are no longer available with certainty in every period, and hence there may be (effectively) multiple new entrants participating in a given auction.

5.1. **Constant Availability of Objects.** We first consider the model examined above in which exactly one object is available for sale in every period. In this setting, the socially efficient policy is to allocate each object to the buyer with the highest valuation present on the market.[9]

Let us define W_0 to be the expected value to the social planner at the beginning of a period in which no buyers are present on the market. Then, letting $\bar{v}$ denote the expected value of the distribution F, we may write

$$W_0 = q\int_0^\infty (v' + \delta W_0)\, dF(v') + (1-q)\delta W_0 = \frac{q\bar{v}}{1-\delta}.$$

Denote by $W_n(\hat{v})$ the expected value to the social planner at the beginning of a period when there are n buyers with values $\hat{v}_1 > \cdots > \hat{v}_n$, *before* the realization of the new buyer arrival process. We may recursively solve for this function; in particular, we have the following

PROPOSITION 4 (Planner's payoff function)**.**
The social planner's expected value at the beginning of a period in which there are n buyers present on the market with values $\hat{v}_1 > \cdots > \hat{v}_n$ is given by

$$W_n(\hat{v}) = W_0 + \delta^{-1}\sum_{j=1}^{n}\int_0^{\hat{v}_j} \lambda^j(v')\, dv'. \tag{5.1}$$

[8]In dynamic settings with the arrival of new information, the concepts of *periodic* ex post individual rationality and incentive compatibility are the natural counterparts of ex post individual rationality and incentive compatibility, as they account for the possibility of the arrival of additional information in the future.

[9]Intuitively, since a new object arrives in every period and future entrants' values are independent of the current state, there is no benefit to not allocating the object in any particular period. Moreover, allocating an object to a lower-valued buyer is inefficient due to the fact that the common discount factor δ is smaller than one; therefore, postponing a higher-valued buyer for the benefit of a lower-valued one is costly. A more formal exposition of this argument may be found in Appendix B.

Proof. The proof, which is similar to that of Theorem 1, may be found in Appendix A. ■

Effectively, this proposition yields an analogue to the social planner's payoff in the case of a fixed number of buyers without any entry. With n buyers whose values are given by $v_1 > \cdots > v_n$, the efficient allocation yields a value to the planner given by $\sum_{j=1}^{n} \delta^{j-1} v_j$. In our setting, however, potential entrants can and do rearrange the ordering of agents in the efficient allocation, postponing the time at which buyers with lower valuations receive an object. Thus, their contribution to social welfare must take this effect into account. So, consider the buyer with the highest valuation $\hat{v}_1$. If we increase his valuation by an infintesimal amount, the planner gains an equal amount with probability $1-q(1-F(\hat{v}_1))$, the probability that no higher-valued new entrant arrives. On the other hand, with the complementary probability $q(1-F(\hat{v}_1))$, assignment of the object to our buyer (and the realization of the planner's gain) is postponed. Thus, the benefit from the increase in $\hat{v}_1$ is

$$\begin{aligned}(1-q(1-F(\hat{v}_1))) &+ q(1-F(\hat{v}_1))\delta\left[(1-q(1-F(\hat{v}_1))) + q(1-F(\hat{v}_1))\delta\cdots\right] \\ &= \sum_{m=0}^{\infty} \left(\delta q(1-F(\hat{v}_1))\right)^m (1-q(1-F(\hat{v}_1)) = \frac{1-q(1-F(\hat{v}_1))}{1-\delta q(1-F(\hat{v}_1))} = \delta^{-1}\lambda(\hat{v}_1).\end{aligned}$$

Integrating this ratio therefore captures the total contribution (relative to assigning the object to a buyer with value 0) of the high-value buyer. Analogous reasoning follows for the remaining (lower-ranked) buyers.

We may then use this result to provide an interpretation for the buyer payoff functions characterized in Theorem 1 by relating the expression for buyer payoffs in Equation 4.6 to the planner's payoff function given in Equation 5.1. In particular, we have the following

THEOREM 3 (Relationship between V and W).
For any $n \in \mathbb{N}$ and any $k \in \{1,\ldots,n\}$, the expected payoff of the k-th ranked buyer in the sequential auction game is equal to her marginal contribution to the social welfare; that is,

$$V_{k,n}(\hat{v}) = W_n(\hat{v}) - W_{n-1}(\hat{v}_{-k}). \tag{5.2}$$

Proof. The proof proceeds via straightforward arithmetic:

$$\begin{aligned}W_n(\hat{v}) - W_{n-1}(\hat{v}_{-k}) &= \left[W_0 + \delta^{-1}\sum_{j=1}^{n}\delta^{j-1}\int_0^{\hat{v}_j}\lambda^j(v')\,dv'\right] \\ &\qquad - \left[W_0 + \delta^{-1}\sum_{j=1}^{k-1}\int_0^{\hat{v}_j}\lambda^j(v')\,dv' + \delta^{-1}\sum_{j=k+1}^{n}\int_0^{\hat{v}_j}\lambda^{j-1}(v')\,dv'\right] \\ &= \delta^{-1}\left[\sum_{j=k}^{n}\int_0^{\hat{v}_j}\lambda^j(v')\,dv' - \sum_{j=k+1}^{n}\int_0^{\hat{v}_j}\lambda^{j-1}(v')\,dv'\right].\end{aligned}$$

Rearranging the terms of the summation yields $V_{k,n}(\hat{v})$, as desired. ■

Thus, the marginal contribution of a buyer, and hence their expected payoff in the equilibrium of the sequential ascending auction game, is exactly the buyer's marginal contribution to the social welfare, which is determined by the difference in the scheduling of object assignments to those bidders who have lower values. Moreover, this demonstrates the equivalence between the dynamic marginal contribution mechanism and the sequential ascending auction in this setting. Not only are continuation payoffs identical in the two settings, but the timing of payments and object allocations are also the same.

5.2. **Random Arrival of Objects.** We now consider a generalization of the setting of the previous sections. Instead of a single object arriving with certainty in every period, we now allow the arrival of objects to be probabilistic. In particular, at most one object arrives on the market in each period with probability $p \in (0,1)$. Thus, the number of buyers present on the market may increase between auctions, as it is possible for multiple new buyers to arrive before another object becomes available for sale.[10] Notice that the socially efficient policy remains unchanged from that of the previous setting—when available, objects should be allocated to the highest-valued buyer currently on the market.[11]

Once again, we denote by $W_n(\hat{v})$ the expected value to the social planner when there are n buyers present at the beginning of a period with (ordered) values $\hat{v}_1 > \cdots > \hat{v}_n$. W_n must satisfy the relationship given by

$$\begin{aligned} W_n(\hat{v}) = pq &\left[\sum_{j=0}^{n-1} \int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} \left(\hat{v}_1 + \delta W_n(\hat{\hat{v}}_{-1}(v'))\right) dF(v') + \int_{\hat{v}_1}^{\infty} \left(v' + \delta W_n(\hat{v})\right) dF(v')\right] \\ &+ p(1-q)\left[\hat{v}_1 + \delta W_{n-1}(\hat{v}_{-1})\right] + (1-p)q\left[\sum_{j=0}^{n} \delta \int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} W_{n+1}(\hat{\hat{v}}(v'))\, dF(v')\right] \\ &+ (1-p)(1-q)\left[\delta W_n(\hat{v})\right]. \end{aligned} \tag{5.3}$$

Define $\mu : \mathbb{R}_+ \to [0,1]$ by

$$\begin{aligned} \mu(v) := &\frac{1 - \delta\left[pq(1-F(v)) + (1-p)(1-q(1-F(v)))\right]}{2\delta(1-p)q(1-F(v))} \\ &- \frac{\sqrt{1 - 2\delta\left[pq(1-F(v)) + (1-p)(1-q(1-F(v)))\right] + \delta^2\left(1-p-q(1-F(v))\right)^2}}{2\delta(1-p)q(1-F(v))}. \end{aligned} \tag{5.4}$$

Analogous to Proposition 4, we may then show the following

[10]Note that this assumption not only leads to a more general competitive environment amongst buyers, but also leads to a greater expected delay before the assignment of an object to any given market participant.
[11]This argument is made explicit in Appendix B.

PROPOSITION 5 (Planner's payoffs with random object arrivals)**.**
The social planner's expected value at the beginning of a period in which there are n buyers present on the market with values $\hat{v}_1 > \cdots > \hat{v}_n$ and objects arrive randomly is given by

$$W_n(\hat{v}) = W_0 + \delta^{-1} \sum_{j=1}^{n} \int_0^{\hat{v}_j} \mu^j(v')\, dv', \tag{5.5}$$

where W_0 is a constant equal to the planner's payoff when no buyers are present on the market.

Proof. The proof may be found in Appendix A. ■

Thus, as in the case in which objects arrive deterministically in every period, the planner's expected payoff is an additively separable sum of contributions from each buyer present on the market, where the magnitude of each contribution is increasing in both the buyer's value and rank amongst her competitors. Although μ is not as straightforward to interpret as λ (defined in Equation 4.5), it reflects the anticipated marginal benefit from increasing the value of the highest-ranked buyer, taking into account the fact that multiple new entrants may arrive before the next object arrives and thereby delaying the realization of the increase to the social welfare from the increase in $\hat{v}_1$. [12]

Notice that our setting satisfies the conditions discussed by Cavallo, Parkes, and Singh (2007) for the ex post periodic incentive compatible and truthful implementation of the socially efficient policy by the dynamic Vickrey-Clarke-Groves mechanism.[13] Thus, using the social planner's payoffs from Proposition 5, we may construct contingent transfers such that truthful revelation of private values is ex post periodic incentive compatible in the sequential auction game. In particular, at every point in time, each buyer receives a transfer equal to their "flow marginal contribution" in that period. This is defined as the social welfare excluding the player in question when making the allocation that is efficient given her existence, less the total welfare of all other players when acting as though the player in question is not present on the market.

To make this clearer, consider a buyer i with value v_i who is present on the market, and suppose that there are n other buyers present on the market with (ordered) values $\hat{v}_1 > \cdots > \hat{v}_n$. Let us suppose first that $v_i > \hat{v}_1$. Thus, if an object is available, the efficient policy involves assigning the object to buyer i, yielding her a flow payoff of v_i. Furthermore, i should receive a transfer equal to her marginal contribution to the social

[12] The expression for μ is (slightly) less unwieldy if we consider the continuous-time limit of this model. If we set $\delta = e^{-r\Delta}$, $p = \rho_S\Delta$, and $q = \rho_B\Delta$ and consider the limit as Δ approaches zero, we have

$$\mu(v) \to \frac{r + \rho_S + \rho_B(1 - F(v)) - \sqrt{(r + \rho_S + \rho_B(1 - F(v)))^2 - 4\rho_S\rho_B(1 - F(v))}}{2\rho_B(1 - F(v))}.$$

Moreover, note that $\lim_{p\to 1} \mu(v) = \lambda(v)$ for all $v \in \mathbb{R}_+$.

[13] In particular, the model satisfies the conditions of their Theorem 6.

welfare, less the current-period utility she receives; this transfer may be written as

$$\underbrace{(0+\delta W_n(\hat{v}))}_{\text{Others' welfare with } i} - \underbrace{(\hat{v}_1+\delta W_{n-1}(\hat{v}_{-1}))}_{\text{Social welfare excluding } i} = -\hat{v}_1 + \delta\left(W_n(\hat{v}) - W_{n-1}(\hat{v}_{-1})\right). \tag{5.6}$$

On the other hand, if an object is not available, i cannot be allocated an object in the current period. Therefore, i receives a flow payoff of zero and a transfer of

$$\underbrace{(0+\delta W_n(\hat{v}))}_{\text{Others' welfare with } i} - \underbrace{(0+\delta W_n(\hat{v}))}_{\text{Social welfare excluding } i} = 0.$$

Suppose on the other hand that i does *not* have the highest value. If an object is available, she will not receive it; moreover, she will receive a transfer of

$$\underbrace{(\hat{v}_1+\delta W_{n-1}(\hat{v}_{-1}))}_{\text{Others' welfare with } i} - \underbrace{(\hat{v}_1+\delta W_{n-1}(\hat{v}_{-1}))}_{\text{Social welfare excluding } i} = 0,$$

as i's presence does not change the optimal policy. Similarly, if no object is available, i's presence, or lack thereof, still does not affect the socially efficient policy. In this case, she therefore is given a transfer of

$$\underbrace{(0+\delta W_n(\hat{v}))}_{\text{Others' welfare with } i} - \underbrace{(0+\delta W_n(\hat{v}))}_{\text{Social welfare excluding } i} = 0.$$

In future periods, i (again) receives no transfers until she is eventually the highest-ranked buyer, as her flow marginal contribution remains zero until that point.

Thus, a buyer's expected continuation payoff at every point in time in this dynamic Vickrey-Clarke-Groves mechanism is given by her marginal contribution to the social welfare. We may therefore construct an equilibrium of the sequential auction game by appropriately choosing bidding functions that yield continuation payoffs equal to those in the direct mechanism.

In particular, we will (abusing notation slightly) let $V_{k,n}$ denote the expected payoff to a buyer with the k-th highest of the n values present at the beginning of a period, where

$$V_{k,n}(\hat{v}) := W_n(\hat{v}) - W_{n-1}(\hat{v}_{-k}) = \delta^{-1}\sum_{j=k}^{n}\int_{\hat{v}_{j+1}}^{\hat{v}_j} \mu^j(v')\,dv'. \tag{5.7}$$

Similarly, denoting by $\beta_{k,n}$ the drop-out point of a bidder with value v_i when there are k active bidders in an auction with n buyers, define

$$\beta_{k,n}(v_i,\hat{v}_{k+1},\ldots,\hat{v}_n) := v_i - \delta V_{k-1,n-1}(v_i,\hat{v}_{k+1},\ldots,\hat{v}_n), \tag{5.8}$$

where, as before, $\hat{v}_j$ for $j=k+1,\ldots,n$ are the revealed values of those bidders that have already exited the auction. With these definitions in mind, we may prove the following

THEOREM 4 (Auction equilibrium with random object arrivals)**.**
Suppose that in each period in which an object is available, buyers bid according to the cutoffs

$$\beta_{k,n}(v_i, \hat{v}_{k+1}, \ldots, \hat{v}_k) = v_i - \int_{\hat{v}_{k+1}}^{v_i} \mu^{k-1}(v')\, dv' - \sum_{j=k+1}^{n} \int_{\hat{v}_{j+1}}^{\hat{v}_j} \mu^{j-1}(v')\, dv', \tag{5.9}$$

where $n \in \mathbb{N}$ is the number of buyers present on the market, and $k \in \{2, \ldots, n\}$ is the number of active bidders remaining in the current period, and $\hat{v}_{n+1} = 0$. Then this strategy profile forms a fully revealing ex post periodic incentive compatible equilibrium of the sequential auction game.

Proof. Note first that $\beta_{k,n}$ is strictly increasing in v_i for all k and n; in particular, since $\mu(v') \in (0, 1)$ for all $v' \in \mathbb{R}_+$,

$$\frac{\partial}{\partial v_i} \beta_{k,n}(v_i, \hat{v}_{k+1}, \ldots, \hat{v}_n) = 1 - \mu^{k-1}(v_i) > 0.$$

Furthermore, notice that

$$\begin{aligned} &\beta_{k,n}(v_i, \hat{v}_{k+1}, \ldots, \hat{v}_n) - \beta_{k+1,n}(\hat{v}_{k+1}, \ldots, \hat{v}_n) \\ &\qquad = v_i - \hat{v}_{k+1} - \int_{\hat{v}_{k+1}}^{v_i} \mu^{k-1}(v')\, dv' = \int_{\hat{v}_{k+1}}^{v_i} \left(1 - \mu^{k-1}(v')\right)\, dv', \end{aligned}$$

which is strictly positive whenever $v_i > \hat{v}_{k+1}$. Thus, a lower-ranked bidder's exit from an auction does not induce the immediate exit of a higher-ranked bidder, implying that following the bidding strategies in Equation 5.9 is efficient and fully revealing of all private information.

Note that in the sequential ascending auction game, buyers do not make any payments unless they win an auction. Moreover, when they do win, they make a payment equal to the drop-out point of their last remaining opponent in that auction. Following the bidding strategies defined in Equation 5.9 implies that a bidder with value $v_i > \hat{v}_1 > \cdots > \hat{v}_n$, when engaged in an auction, will win the auction and make a payment of

$$\beta_{2,n+1}(\hat{v}) = \hat{v}_1 - \delta V_{1,n}(\hat{v}) = \hat{v}_1 - \delta \left(W_n(\hat{v}) - W_{n-1}(\hat{v}_{-1})\right),$$

where the second equality comes from the definition of $V_{k,n}$ in Equation 5.7. Notice that this payment is exactly the dynamic Vickrey-Clarke-Groves transfer described in Equation 5.6. Thus, following the bidding strategies of Equation 5.9 yields exactly the payoffs of truthful reporting in the direct mechanism.

Finally, suppose that some arbitrary player i with value v_i has an incentive to deviate from the bidding strategy. Since the cutoff bids are fully revealing, this is equivalent to i bidding as though her value were $v_i' \neq v_i$. Since the bidding strategies yield the same payoffs as the dynamic Vickrey-Clarke-Groves mechanism, this is equivalent to player i having an incentive to misreport her type to the social planner in the direct mechanism.

However, this contradicts the ex post periodic incentive compatibility of truthtelling in the direct mechanism. Thus, no player has an incentive to deviate. ■

Not only does Theorem 4 characterize an efficient, fully revealing, and symmetric equilibrium of the sequential ascending auction game, it also generalizes the equivalence result of the previous subsection; in a setting with random arrivals of both buyers and objects, the sequential ascending auction serves as a straightforward and intuitive indirect mechanism that is equivalent to the Vickrey-Clarke-Groves mechanism.[14]

6. Discussion

This paper solves for an equilibrium in a model of online auctions. In particular, we show that in sequential ascending auctions, objects are allocated efficiently in a manner that employs the truthful revelation of private information. Moreover, the bidding strategy employed by buyers in this equilibrium has the striking property of being robust to the random entry of new buyers whose valuations are private information—in each period, *all* private information is revealed anew, and hence there is no incentive for new entrants to attempt to manipulate the outcome of future periods by altering the information that they (truthfully) reveal upon their entry. Furthermore, we show that the sequential ascending auction in this setting is equivalent to the dynamic Vickrey-Clarke-Groves mechanism developed and characterized by Bergemann and Välimäki (2007) and Cavallo, Parkes, and Singh (2007). It should be pointed out, however, that in this indirect mechanism, the burden of sophistication and information processing is shifted from a social planner or mechanism designer to the buyers. While in some settings this may be problematic, in dynamic marketplaces with a variety of buyers and sellers, it may in fact be desirable.

There are several interesting avenues for future research in this area. For example, it would be desirable to have a fully developed model of seller behavior and competition in "overlapping" auctions, perhaps applying some of the insights of Peters and Severinov (2006) in a setting with multiple simultaneously running auctions. Such a setting also allows for the introduction and study of endogenous arrival and entry deterrence in a manner similar to Nekipelov (2007) but while accounting for the endogenously determined option value of participating in future auctions.

Another important question regards the usefulness of sequential ascending auctions as an indirect mechanism that implements socially efficient policies when agents are not

[14]Notice that we have constructed equilibrium strategies in the sequential auction game by using the payoffs of the dynamic marginal contribution mechanism. A simple adaptation of this argument may serve as an alternate proof of Theorem 2.

constrained to have single-unit demand. Bergemann and Välimäki (2007) provide an example that demonstrates the failure of the sequential ascending auction in implementing the efficient policy in one such setting; it would be useful to understand how this example may be generalized so as to better understand when indirect implementation via an auction mechanism is possible. These questions are, however, left for future work.

Appendix A. Proofs

In order to prove Proposition 1, we will make repeated use of the following result regarding closed subsets of $\mathbf{C}(\mathbb{R}^n_+)$, the set of continuous real-valued functions on $\mathbb{R}^n_+$ endowed with the sup-metric topology. Furthermore, for any $k < n$, let $\mathbf{C}_k(\mathbb{R}^n_+) \subseteq \mathbf{C}(\mathbb{R}^n_+)$ denote the subset of such functions that do *not* depend on their first k arguments. We have the following

Lemma 1 ($\mathbf{C}_k(\mathbb{R}^n_+)$ is closed)**.**
For any $k \leq n$, $\mathbf{C}_k(\mathbb{R}^n_+)$ is a closed in $\mathbf{C}(\mathbb{R}^n_+)$.

Proof. Fix any convergent sequence $\{f_m\}_{m=1}^{\infty}$ in $\mathbf{C}_k(\mathbb{R}^n_+)$, and let $f^* \in \mathbf{C}(\mathbb{R}^n_+)$ denote the limit of this sequence. Suppose that there exist distinct $x, y \in \mathbb{R}^n_+$ such that $x_i = y_i$ for $i = k+1, k+2, \ldots, n$, but

$$\epsilon := |f^*(x) - f^*(y)| > 0.$$

Since uniform convergence implies point-wise convergence and f_m converges to f^*, there exists $M_x \in \mathbb{N}$ such that $|f_m(x) - f^*(x)| < \frac{\epsilon}{2}$ for all $m > M_x$. SImilarly, there exists $M_y \in \mathbb{N}$ such that $|f_m(y) - f^*(y)| < \frac{\epsilon}{2}$ for all $m > M_y$. Therefore, for any $m > \max\{M_x, M_y\}$,

$$\begin{aligned}\epsilon = |f^*(x) - f^*(y)| &\leq |f^*(x) - f_m(x)| + |f_m(x) - f_m(y)| + |f_m(y) - f^*(y)| \\ &< \frac{\epsilon}{2} + 0 + \frac{\epsilon}{2} = \epsilon,\end{aligned}$$

a contradiction. The first inequality above follows from the triangle inequality, and the second is due to the fact that $f_m \in \mathbf{C}_k(\mathbb{R}^n_+)$ implies $f_m(x) = f_m(y)$. Thus, we must have $f^*(x) = f^*(y)$; that is, $f^* \in \mathbf{C}_k(\mathbb{R}^n_+)$. ■

Proof of Proposition 1. Let $\hat{v} \in \mathbb{R}^n_+$ denote the ordered vector of values of those bidders present at the beginning of the period, and suppose that they are commonly known. Furthermore, suppose that all buyers use the bidding strategies described in Equation 4.3. If there are no entrants, then the highest-valued buyer (without loss of generality, bidder 1) wins the object, and pays the price

$$\beta_{2,n}(\hat{v}_2, \ldots, \hat{v}_n) = \hat{v}_2 - \delta V_{1,n-1}(\hat{v}_2, \ldots, \hat{v}_n).$$

On the other hand, if a new entrant enters with value v', bidder 1 may no longer win the object. Furthermore, even if she does win, the price she pays will depend upon the realization of v'. In particular, we may write the expected payoff of bidder 1 as

$$\begin{aligned}V_{1,n}(\hat{v}) &= (1-q)\left[\hat{v}_1 - \beta_{2,n}(\hat{v}_2, \ldots, \hat{v}_n)\right] \\ &\quad + q\left[\sum_{j=0}^{n-1} \int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} \left(\hat{v}_1 - \beta_{2,n+1}(\hat{\hat{v}}_{-1}(v'))\right) dF(v') + \int_{\hat{v}_1}^{\infty} \delta V_{1,n}(\hat{v})\, dF(v')\right],\end{aligned} \tag{A.1}$$

where $\hat{\hat{v}}(v')$ is the ordered vector of values including the new entrant, and we define $\hat{v}_{n+1} := 0$. The first term (multiplied by $1-q$) is bidder 1's payoff when no entrant arrives, while the second term is the (probability-weighted) sum of the payoffs for each possible realized ranking of the new entrant.

Substituting the definition of $\beta_{2,n}$ and $\beta_{2,n+1}$ from Equation 4.3 and simplifying, we see that $V_{1,n}$ is given by the fixed point of the operator $T_{1,n} : \mathbf{C}(\mathbb{R}^n_+) \to \mathbf{C}(\mathbb{R}^n_+)$ defined by

$$\begin{aligned}[T_{1,n}(W)](\hat{v}) := {} & (1-q)\left[\hat{v}_1 - \hat{v}_2 + \delta V_{1,n-1}(\hat{v}_2, \ldots, \hat{v}_n)\right] \\ & + q\Bigg[\int_0^{\hat{v}_1} \hat{v}_1\, dF(v') - \int_0^{\hat{v}_2} \hat{v}_2\, dF(v') - \int_{\hat{v}_2}^{\hat{v}_1} v'\, dF(v') \\ & \quad + \delta \sum_{j=0}^{n-1} \int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} W(\hat{\hat{v}}_{-1}(v'))\, dF(v') + \int_{\hat{v}_1}^{\infty} \delta W(\hat{v})\, dF(v')\Bigg].\end{aligned} \tag{A.2}$$

Fix any $W, W' \in \mathbf{C}(\mathbb{R}^n_+)$ such that $W \geq W'$. Then

$$[T_{1,n}(W) - T_{1,n}(W')](\hat{v}) = \delta q \begin{bmatrix} \sum_{j=0}^{n-1} \int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} [W - W'](\hat{\hat{v}}_{-1}(v'))\, dF(v') \\ +(1 - F(\hat{v}_1))[W - W'](\hat{v}) \end{bmatrix} \geq 0.$$

In addition, for any $W \in \mathbf{C}(\mathbb{R}^n_+)$ and any $\alpha \in \mathbb{R}_{++}$,

$$[T_{1,n}(W + \alpha)](\hat{v}) = [T_{1,n}(W)](\hat{v}) + \delta q \alpha.$$

Thus, $T_{1,n}$ satisfies the monotonicity and discounting conditions of Blackwell's Contraction Lemma, and hence we may apply the Banach Fixed Point Theorem to show that $V_{1,n}$ is the unique fixed point of $T_{1,n}$.

Now consider $V_{2,n}$. Suppose (again without loss of generality) that bidder 1 has the second-highest of the n values; that is, that $v_1 = \hat{v}_2$. If there are no new entrants, then bidder 1 loses the auction, but has the highest value in the next period. On the other hand, if a new entrant arrives, bidder 1 will still lose the auction. However, in the next period, her ranking depends on the realization of the new entrant's value. Thus, we may write her payoff as the fixed point of the operator $T_{2,n} : \mathbf{C}(\mathbb{R}^n_+) \to \mathbf{C}(\mathbb{R}^n_+)$ defined by

$$\begin{aligned}[T_{2,n}(W)](\hat{v}) := {} & \delta(1-q)V_{1,n-1}(\hat{v}_2, \ldots, \hat{v}_n) + q\Bigg[\sum_{j=0}^{n-2} \int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} \delta V_{1,n}(\hat{\hat{v}}_{-1}(v'))\, dF(v') \\ & + \int_{\hat{v}_2}^{\hat{v}_1} \delta W(\hat{\hat{v}}_{-1}(v'))\, dF(v') + \int_{\hat{v}_1}^{\infty} \delta W(\hat{v})\, dF(v')\Bigg].\end{aligned} \tag{A.3}$$

Applying exactly the same technique and steps as with $T_{1,n}$, we see that $T_{2,n}$ is a contraction mapping on $\mathbf{C}(\mathbb{R}^n_+)$. Notice that $T_{2,n}$ in fact maps elements of $\mathbf{C}_1(\mathbb{R}^n_+)$ into $\mathbf{C}_1(\mathbb{R}^n_+)$ itself; thus, applying Lemma 1, the unique fixed point of $T_{2,n}$ does not depend upon its first argument. We may therefore, with a slight abuse of notation, write $V_{2,n}(\hat{v}) = V_{2,n}(\hat{v}_2, \dots, \hat{v}_n)$.

Now consider any arbitrary k such that $1 < k \leq n$, and suppose that $V_{k-1,n} \in \mathbf{C}_{k-2}(\mathbb{R}^n_+)$. Then $V_{k,n}$ is given by a fixed point of the operator $T_{k,n} : \mathbf{C}(\mathbb{R}^n_+) \to \mathbf{C}(\mathbb{R}^n_+)$, where $T_{k,n}$ is defined by

$$\begin{aligned}[T_{k,n}(W)](\hat{v}) := \delta(1-q)V_{k-1,n-1}(\hat{v}_k, \dots, \hat{v}_n) + q\Bigg[\sum_{j=0}^{n-k} \int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} \delta V_{k-1,n}(\hat{\hat{v}}_{-1}(v'))\, dF(v') \\ + \sum_{j=n-k+1}^{n-1} \int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} \delta W(\hat{\hat{v}}_{-1}(v'))\, dF(v') + \int_{\hat{v}_1}^{\infty} \delta W(\hat{v})\, dF(v')\Bigg].\end{aligned} \tag{A.4}$$

We may again apply Blackwell's Contraction Lemma and the Banach Fixed Point Theorem to show that $V_{k,n}$ is the unique fixed point of $T_{k,n}$. Furthermore, it is straightforward to show that $T_{k,n}$ maps elements of $\mathbf{C}_{k-1}(\mathbb{R}^n_+)$ into $\mathbf{C}_{k-1}(\mathbb{R}^n_+)$. Therefore, again using Lemma 1, we may write $V_{k,n}(\hat{v}) = V_{k,n}(\hat{v}_k, \dots, \hat{v}_n)$.

Thus, by induction, the bidding strategies in Equation 4.3 lead to unique value functions $V_{k,n}$ such that, for all n and all $k = 1, \dots, n$, $V_{k,n} \in \mathbf{C}_{k-1}(\mathbb{R}^n_+)$. ∎

Proof of Proposition 2. Recall from Equation A.2 in the proof of Proposition 1 that $V_{1,n}$ is defined as the unique fixed point of $T_{1,n}$. Letting $\hat{w} := \hat{v}_{-1} = (\hat{v}_2, \hat{v}_3, \dots, \hat{v}_n)$, we have

$$\begin{aligned}V_{1,n}(\hat{v}_2, \hat{w}) = [T_{1,n}(V_{1,n})](\hat{v}_2, \hat{w}) &= \delta(1-q)V_{1,n-1}(\hat{w}) \\ &+ \delta q\Bigg[\sum_{j=0}^{n-2} \int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} V_{1,n}(\hat{\hat{w}}(v'))\, dF(v') + \int_{\hat{v}_2}^{\infty} \delta V_{1,n}(\hat{w})\, dF(v')\Bigg].\end{aligned}$$

However, this is identical to the definition of $T_{2,n}$ given in Equation A.3, implying that, for all $n \in \mathbb{N}$,

$$V_{2,n}(\hat{w}) = V_{1,n}(\hat{v}_2, \hat{w}).$$

Fix $k > 1$, and suppose that $V_{k,n}(\hat{v}_k, \hat{v}_{k+1}, \dots, \hat{v}_n) = V_{k-1,n}(\hat{v}_k, \hat{v}_k, \hat{v}_{k+1}, \dots, \hat{v}_n)$ for all $n \geq k$. Redefine $\hat{w} := (\hat{v}_{k+1}, \hat{v}_{k+2}, \dots, \hat{v}_n)$, and consider $V_{k,n}(\hat{v}_{k+1}, \hat{w})$. Recalling from Equation A.4 the definition of $T_{k,n}$, we have

$$\begin{aligned}V_{k,n}(\hat{v}_{k+1}, \hat{w}) = [T_{k,n}(V_{k,n})](\hat{v}_{k+1}, \hat{w}) &= \delta(1-q)V_{k-1,n-1}(\hat{w}) \\ &+ q\Bigg[\sum_{j=0}^{n-k} \int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} \delta V_{k-1,n}(\hat{\hat{w}}(v'))\, dF(v') \\ &+ \sum_{j=n-k+1}^{n-1} \int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} \delta V_{k,n}(\hat{\hat{w}}(v'))\, dF(v') + \int_{\hat{v}_1}^{\infty} \delta V_{k,n}(\hat{w})\, dF(v')\Bigg].\end{aligned}$$

Taking into account the fact that the fixed point of this operator lies in $\mathbf{C}_k(\mathbb{R}_+^n)$ allows us to rewrite the above as

$$V_{k,n}(\hat{v}_{k+1},\hat{w}) = \delta(1-q)V_{k,n-1}(\hat{w})+q\left[\sum_{j=0}^{n-k-1}\int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}}\delta V_{k,n}(\hat{\hat{w}}(v'))\,dF(v')+\int_{\hat{v}_{k+1}}^{\infty}\delta V_{k,n}(\hat{w})\,dF(v')\right].$$

Notice that the above is a reformulation of the expression for $T_{k+1,n}$. Since $V_{k,n}(\hat{w})$ is a fixed point of the operator, the uniqueness result from Proposition 1 implies that

$$V_{k+1,n}(\hat{w}) = V_{k,n}(\hat{v}_{k+1},\hat{w})$$

for all $n \geq k+1$. Thus, by induction on k, we have established that, for arbitrary $n \in \mathbb{N}$ and for all $k = 1,\dots,n$,

$$V_{k,n}(\hat{v}_k,\dots,\hat{v}_n) = V_{1,n}(\hat{v}_k,\dots,\hat{v}_k,\hat{v}_{k+1},\dots,\hat{v}_n). \qquad \blacksquare$$

Proof of Proposition 3. To prove the first part of this proposition, it suffices to simply differentiate the bid function $\beta_{k,n}$ with respect to the bidder's own value v_i. In particular, we have, for all $n \in \mathbb{N}$ and $k = 1,\dots,n$,

$$\frac{\partial}{\partial v_i}\beta_{k,n}(v_i,\hat{v}_{k+1},\dots,\hat{v}_n) = 1-\lambda^{l-1}(v_i).$$

However, $\lambda(v) \in (0,1)$ for all $v \in \mathbb{R}_+$, and so $\frac{\partial}{\partial v_i}\beta_{k,n}(v_i,\hat{v}_{k+1},\dots,\hat{v}_n) > 0$.

As for the second part of this proposition, let $\hat{w} := (\hat{v}_{k+1},\dots,\hat{v}_n)$ and note that

$$\begin{aligned}
\beta_{k,n}(v_i,\hat{w}) - \beta_{k+1,n}(\hat{w}) &= v_i - \delta V_{k-1,n-1}(v_i,\hat{w}) - \hat{v}_{k+1} + \delta V_{k,n-1}(\hat{w}) \\
&= v_i - \hat{v}_{k+1} - \delta\left(V_{1,n-1}(v_i,\dots,v_i,\hat{w}) - V_{1,n-1}(\hat{v}_{k+1},\dots,\hat{v}_{k+1},\hat{w})\right) \\
&= v_i - \hat{v}_{k+1} - \int_{\hat{v}_{k+1}}^{v_i}\lambda^k(v')\,dv' = \int_{\hat{v}_{k+1}}^{v_i}\left(1-\lambda^k(v')\right)\,dv'.
\end{aligned}$$

Since $\lambda(v') \in (0,1)$ for all $v' \in \mathbb{R}_+$, this expression is positive if, and only if, $v_i > \hat{v}_{k+1}$. Thus, the exit of a lower-ranked bidder does not induce the immediate exit of a higher-ranked bidder who is using the bidding strategy given in Equation 4.9. $\blacksquare$

Proof of Proposition 4. We begin by showing that W_1 has the desired form and then proceed inductively. Note that W_1 is a fixed point of the operator $\hat{T}_1 : \mathbf{C}(\mathbb{R}_+) \to \mathbf{C}(\mathbb{R}_+)$ defined by

$$[\hat{T}(g)](x) := (1-q)(x+\delta W_0) + q\left[\int_0^x (x+\delta g(y))\,dF(y) + \int_x^\infty (y+\delta g(x))\,dF(y)\right]. \tag{A.5}$$

This operator is clearly a self-map from $\mathbf{C}(\mathbb{R}_+)$ into itself. Furthermore, it is straightforward to see that $\hat{T}_1$ is a contraction mapping. Fix any $g, g' \in \mathbf{C}(\mathbb{R}_+)$ such that $g' > g$.

Then

$$[\hat{T}_1(g'-g)](x) = \delta q \left[\int_0^x (g'(y)-g(y))\,dF(y) + (1-F(x))(g'(x)-g(x))\right] > 0.$$

Furthermore, for any $g \in \mathbf{C}(\mathbb{R}_+)$ and any $\alpha \in \mathbb{R}_{++}$,

$$[\hat{T}_1(g+\alpha)](x) = [\hat{T}_1(g)](x) + \delta q \alpha.$$

Since $\delta q < 1$, we may apply Blackwell's Contraction Lemma and the Banach Fixed Point Theorem, implying that $\hat{T}_1$ has a unique fixed point W_1 such that

$$W_1(\hat{v}_1) = (1-q)(\hat{v}_1 + \delta W_0) + q\left[\int_0^{\hat{v}} (\hat{v} + \delta W_1(v'))\,dF(v') + \int_{\hat{v}}^{\infty} (v' + \delta W_1(\hat{v}))\,dF(v')\right].$$

Differentiating this expression with respect to $\hat{v}_1$ yields

$$W_1'(\hat{v}_1) = (1-q) + qF(\hat{v}_1) + \delta q(1-F(\hat{v}))W_1'(\hat{v}) = \delta^{-1}\lambda(\hat{v}_1).$$

Finally, note that $W_1(0) = W_0$, since a buyer with value zero adds nothing to the social welfare. Therefore, by the Fundamental Theorem of Calculus, we have

$$W_1(\hat{v}_1) = W_0 + \delta^{-1}\int_0^{\hat{v}_1} \lambda(v')\,dv'. \tag{A.6}$$

Now consider $W_n(\hat{v})$ for arbitrary $n > 1$, and suppose that W_{n-1} takes the desired form.[15] W_n is defined to be a fixed point of the operator $\hat{T}_n : \mathbf{C}(\mathbb{R}^n_+) \to \mathbf{C}(\mathbb{R}^n_+)$ given by

$$\begin{aligned}
[\hat{T}_n(g)](x) := {} & (1-q)(x_1 + \delta W_{n-1}(x_{-1})) \\
& + q\Bigg[\sum_{j=0}^{n-1} \int_{x_{n-j+1}}^{x_{n-j}} (x_1 + \delta g(x_2,\ldots,x_{n-j},y,x_{n-j+1},\ldots,x_n))\,dF(y) \\
& \qquad + \int_{x_1}^{\infty} (y + \delta g(x))\,dF(y)\Bigg].
\end{aligned} \tag{A.7}$$

Note that for any $g, g' \in \mathbf{C}(\mathbb{R}^n_+)$ such that $g' > g$, we have

$$\begin{aligned}
[\hat{T}_n(g'-g)](x) = \delta q\Bigg[& \sum_{j=0}^{n-1} \int_{x_{n-j+1}}^{x_{n-j}} [g'-g](x_2,\ldots,x_{n-j},y,x_{n-j+1},\ldots,x_n)\,dF(y) \\
& + (1-F(x_1)(g'(x)-g(x))\Bigg] > 0.
\end{aligned}$$

Furthermore, for any $g \in \mathbf{C}(\mathbb{R}^n_+)$ and any $\alpha \in \mathbb{R}_{++}$,

$$[\hat{T}_n(g+\alpha)](x) = [\hat{T}_n(g)](x) + \delta q \alpha.$$

Since $\delta q < 1$, Blackwell's monotonicity and discounting conditions are satisfied. Thus, Blackwell's Contraction Lemma and the Banach Fixed Point Theorem imply that $\hat{T}_n$ has a

[15]Notice that this implies that all of the cross-derivatives of W_{n-1} are identically zero.

unique fixed point W_n such that

$$\begin{aligned} W_n(\hat{v}) &= (1-q)(\hat{v}_1 + \delta W_{n-1}(\hat{v}_{-1})) \\ &\quad + q\Bigg[\sum_{j=0}^{n-1}\int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} (\hat{v}_1 + \delta W_n(\hat{v}_2,\dots,\hat{v}_{n-j},v',\hat{v}_{n-j+1},\dots,\hat{v}_n))\,dF(v') \\ &\quad + \int_{\hat{v}_1}^{\infty}(v' + \delta W_n(\hat{v}))\,dF(v')\Bigg]. \end{aligned} \tag{A.8}$$

Differentiating this expression implicitly with respect to $\hat{v}_1$ yields

$$W_n^{(1)}(\hat{v}) = (1-q) + qF(\hat{v}_1) + \delta q(1-F(\hat{v}_1))W_n^{(1)}(\hat{v}) = \delta^{-1}\lambda(\hat{v}_1).$$

Note that this expression is independent of n and of $\hat{v}_j$ for $j \neq 1$, implying that $W_n^{(1,j)}$ is identically zero for all $j \neq 1$.

Similarly, implicit differentiation with respect to $\hat{v}_2$ yields

$$\begin{aligned} W_n^{(2)}(\hat{v}) &= \delta(1-q)W_{n-1}^{(1)}(\hat{v}_{-1}) + \delta q\Bigg[\sum_{j=0}^{n-2}\int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} W_n^{(1)}(\hat{\hat{v}}_{-1}(v'))\,dF(v') \\ &\quad + \int_{\hat{v}_2}^{\hat{v}_1} W_n^{(2)}(v',\hat{v})\,dF(v') + (1-F(\hat{v}_1))W_n^{(2)}(\hat{v})\Bigg], \end{aligned}$$

where $\hat{\hat{v}}(\hat{v})$ is the re-ordering of $\hat{v}$ and v'. Since $W_n^{1,j}$ is identically zero,

$$\sum_{j=0}^{n-2}\int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} W_n^{(1)}(\hat{\hat{v}}_{-1}(v'))\,dF(v') = \delta^{-1}\lambda(\hat{v}_2)F(\hat{v}_2).$$

Furthermore, $W_n^{(2,1)} = 0$ implies that

$$\int_{\hat{v}_2}^{\hat{v}_1} W_n^{(2)}(v',\hat{v})\,dF(v') + (1-F(\hat{v}_1))W_n^{(2)}(\hat{v}) = (1-F(\hat{v}_2))W_n^{(2)}(\hat{v}).$$

Thus, making use of the fact that $W_{n-1}^{(1)}(\hat{v}_{-1}) = \delta^{-1}\lambda(\hat{v}_2)$, we may conclude that

$$W_n^{(2)} = (1-q)\lambda(\hat{v}_2) + qF(\hat{v}_2)\lambda(\hat{v}_2) + \delta q(1-F(\hat{v}_2))W_n^{(2)}(\hat{v}) = \delta^{-1}\lambda^2(\hat{v}_2). \tag{A.9}$$

Once again, note that this expression is independent of n and of $\hat{v}_j$ for $j \neq 2$, implying that $W_n^{2,j}$ is identically zero for all $j \neq 2$.

Proceeding inductively, consider the derivative of W_n with respect to its k-th argument, where $k \leq n$. We have

$$\begin{aligned} W_n^{(k)}(\hat{v}) &= (1-q)\delta W_{n-1}^{(k-1)}(\hat{v}_{-1}) + \delta q\Bigg[\sum_{j=0}^{n-k}\int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} W_n^{(k-1)}(\hat{\hat{v}}_{-1}(v'))\,dF(v') \\ &\quad + \sum_{j=n-k+1}^{n-1}\int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} W_n^{(k)}(\hat{\hat{v}}_1(v'))\,dF(v') + (1-F(\hat{v}_1))W_n^{(k)}(\hat{v})\Bigg]. \end{aligned}$$

Applying the same simplifications as above and the inductive hypothesis that $W_{n-1}^{(k-1)}(\hat{v}_{-1}) = \delta^{-1}\lambda^{k-1}(\hat{v}_k)$, we have

$$\begin{aligned} W_n^{(k)}(\hat{v}) &= (1-q)W_{n-1}^{(k-1)}(\hat{v}_{-1}) + qF(\hat{v}_k)W_{n-1}^{(k-1)}(\hat{v}_{-1}) + \delta q(1-F(\hat{v}_k))W_n^{(k)}(\hat{v}) \\ &= \delta^{-1}\lambda^k(\hat{v}_k). \end{aligned} \tag{A.10}$$

Finally, note that $W_n(0,\ldots,0) = W_0$ since, as with one agent with value zero, assigning an object to a "null" agent yields no increase in social welfare. By induction on n, we may then conclude that, for all $n \in \mathbb{N}$ and all $\hat{v} \in \mathbb{R}_+^n$,

$$W_n(\hat{v}) = W_0 + \delta^{-1}\sum_{j=1}^{n}\int_0^{\hat{v}_j} \lambda^j(v')\,dv'. \tag{A.11}$$ ∎

In order to prove Proposition 5, we will need to make use of the following result.

Lemma 2 (W is Additively Separable)**.**
The social planner's expected payoff function in the case of random object arrivals is an additively separable function of the (ordered) values of the the buyers present on the market.

Proof. Recall that the socially optimal policy in this setting is for the planner to assign an object—whenever it is available—to the highest-valued buyer on the market. Since the arrival process of new buyers and the realized valuations of these new entrants are independent of the number of agents present on the market, a marginal increase in the value of the highest-valued buyer present on the market does not affect the planner's expectations of future realized values, nor does it impact the anticipated plan of object assignments. Therefore, for any values $\hat{v}_1 > \cdots > \hat{v}_n > \hat{v}_{n+1}$, we must have

$$\frac{\partial}{\partial \hat{v}_1}W_n(\hat{v}_1,\ldots,\hat{v}_n) = \frac{\partial}{\partial \hat{v}_1}W_{n+1}(\hat{v}_1,\ldots,\hat{v}_{n+1}).$$

Similarly, for any $k = 2,\ldots,n$, a marginal increase in $\hat{v}_k$, the value of the k-th highest-ranked buyer, affects neither future arrivals and valuations nor the planner's optimal plan of action. Therefore, we must have

$$\frac{\partial}{\partial \hat{v}_k}W_n(\hat{v}_1,\ldots,\hat{v}_n) = \frac{\partial}{\partial \hat{v}_k}W_{n+1}(\hat{v}_1,\ldots,\hat{v}_{n+1}).$$

Thus, for any $n \in \mathbb{N}$ and any $k = 1,\ldots,n$,

$$W_n^{(k)}(\hat{v}_1,\ldots,\hat{v}_n) = W_{n+1}^{(k)}(\hat{v}_1,\ldots,\hat{v}_n,\hat{v}_{n+1}), \tag{A.12}$$

where $\hat{v}_1 > \cdots > \hat{v}_n > \hat{v}_{n+1}$.

Now consider any $n \in \mathbb{N}$ and ordered values $\hat{v}_1 > \cdots > \hat{v}_n$. Equation A.12 implies that

$$W_n^{(1)}(\hat{v}_1,\ldots,\hat{v}_n) = W_{n-1}^{(1)}(\hat{v}_1,\ldots,\hat{v}_{n-1}) = W_{n-2}^{(1)}(\hat{v}_1,\ldots,\hat{v}_{n-2}) = \cdots = W_1^{(1)}(\hat{v}_1).$$

Therefore, it must be the case that

$$W_2^{(1,2)}(\hat{v}_1, \hat{v}_2) = \frac{\partial}{\partial \hat{v}_2} W_2^{(1)}(\hat{v}_1, \hat{v}_2) = \frac{\partial}{\partial \hat{v}_2} W_1^{(1)}(\hat{v}_1) = 0.$$

Straightforward induction therefore yields

$$W_k^{(1,k)}(\hat{v}_1, \ldots, \hat{v}_k) = \frac{\partial}{\partial \hat{v}_k} W_k^{(1)}(\hat{v}_1, \ldots, \hat{v}_k) = \frac{\partial}{\partial \hat{v}_k} W_{k-1}^{(1)}(\hat{v}_1, \ldots, \hat{v}_{k-1}) = 0$$

for all $k = 2, \ldots, n$. Hence, it must be the case that, for any $k \neq 1$, $W_k^{(1)}(\hat{v}_1, \ldots, \hat{v}_k)$ is a function of $\hat{v}_1$ alone.

Since Equation A.12 implies that

$$W_n^{(2)}(\hat{v}_1, \ldots, \hat{v}_n) = W_{n-1}^{(2)}(\hat{v}_1, \ldots, \hat{v}_{n-1}) = W_{n-2}^{(2)}(\hat{v}_1, \ldots, \hat{v}_{n-2}) = \cdots = W_2^{(2)}(\hat{v}_1, \hat{v}_2),$$

we may perform a similar exercise as above. In particular,

$$W_3^{(2,3)}(\hat{v}_1, \hat{v}_2, \hat{v}_3) = \frac{\partial}{\partial \hat{v}_3} W_3^{(2)}(\hat{v}_1, \hat{v}_2, \hat{v}_3) = \frac{\partial}{\partial \hat{v}_3} W_2^{(2)}(\hat{v}_1, \hat{v}_2) = 0.$$

The same inductive reasoning again leads to the conclusion that

$$W_k^{(2,k)}(\hat{v}_1, \ldots, \hat{v}_k) = \frac{\partial}{\partial \hat{v}_k} W_k^{(2)}(\hat{v}_1, \ldots, \hat{v}_k) = \frac{\partial}{\partial \hat{v}_k} W_{k-1}^{(2)}(\hat{v}_1, \ldots, \hat{v}_{k-1}) = 0$$

for all $k = 3, \ldots, n$. Combining this with the fact that $W_k^{(1)}$ is independent of its second argument yields the conclusion that $W_k^{(2)}(\hat{v}_1, \ldots, \hat{v}_k)$ is a function of $\hat{v}_2$ alone. Proceeding inductively in this manner, we may conclude that

$$W_n^{(j,k)}(\hat{v}_1, \ldots, \hat{v}_n) = 0 \text{ for all } j \neq k, \tag{A.13}$$

where $j, k \in \{1, \ldots, n\}$.

Thus, Equation A.13 implies that, for any $n \in \mathbb{N}$ and any $k = 1, \ldots, n$, we may write

$$W_n^{(k)}(\hat{v}_1, \ldots, \hat{v}_n) = g_{k,n}(\hat{v}_k)$$

for some real-valued function $g_{k,n}$. Combining this with Equation A.12 immediately implies that

$$g_{k,n} = g_{k,m} \text{ for all } m \geq n.$$

Therefore, there exists a sequence of real-valued functions $\{g_k\}_{k=1}^{\infty}$ such that

$$W_n^{(k)}(\hat{v}_1, \ldots, \hat{v}_n) = g_k(\hat{v}_k)$$

for arbitrary $n \in \mathbb{N}$ and $k = 1, \ldots, n$. Therefore, applying the Fundamental Theorem of Calculus allows us to conclude that, W_n is additively separable; in particular,

$$W_n(\hat{v}_1, \ldots, \hat{v}_n) = W_n(0, \ldots, 0) + \sum_{k=1}^{n} \int_0^{\hat{v}_k} g_k(v')\, dv'. \tag{A.14}$$

■

Proof of Proposition 5. Note that we may rewrite Equation 5.3 as

$$W_n(\hat{v}) = pq\left[F(\hat{v}_1)\hat{v}_1 + \int_{\hat{v}_1}^{\infty} v'\,dF(v') + \delta\sum_{j=0}^{n-1}\int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} W_n(\hat{\hat{v}}_{-1}(v'))\,dF(v') + \delta(1-F(\hat{v}_1))W_n(\hat{v})\right]$$
$$+ p(1-q)\left[\hat{v}_1 + \delta W_{n-1}(\hat{v}_{-1})\right] + (1-p)q\left[\sum_{j=0}^{n}\delta\int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} W_{n+1}(\hat{\hat{v}}(v'))\,dF(v')\right]$$
$$+ (1-p)(1-q)\left[\delta W_n(\hat{v})\right].$$

Differentiating this expression with respect to $\hat{v}_1$ yields

$$W_n^{(1)}(\hat{v}) = p[1-q(1-F(\hat{v}_1))] + \delta[pq(1-F(\hat{v}_1)) + (1-p)(1-q)]W_n^{(1)}(\hat{v})$$
$$+ \delta(1-p)q\sum_{j=0}^{n-1}\int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} W_{n+1}^{(1)}(\hat{\hat{v}}(v'))\,dF(v') + \delta(1-p)q\int_{\hat{v}_1}^{\infty} W_{n+1}^{(2)}(v',\hat{v})\,dF(v').$$

Similarly, differentiating $W_n(\hat{v})$ with respect to $\hat{v}_k$ for $k>1$ yields

$$W_n^{(k)}(\hat{v}) = \delta pq\left[\sum_{j=0}^{n-k-1}\int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} W_n^{(k-1)}(\hat{\hat{v}}_{-1}(v'))\,dF(v') + \sum_{j=n-k}^{n-1}\int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} W_n^{(k)}(\hat{\hat{v}}_{-1}(v'))\,dF(v')\right]$$
$$+ \delta pq(1-F(\hat{v}_1))W_n^{(k)}(\hat{v}) + \delta(1-p)q\left[\sum_{j=0}^{n-k}\int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} W_{n+1}^{(k)}(\hat{\hat{v}}(v'))\,dF(v')\right]$$
$$+ \delta(1-p)q\left[\sum_{j=n-k+1}^{n}\int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} W_{n+1}^{(k+1)}(\hat{\hat{v}}(v'))\,dF(v')\right] + \delta p(1-q)W_{n-1}^{(k-1)}(\hat{v}_{-1})$$
$$+ \delta(1-p)(1-q)W_n^{(k)}(\hat{v}).$$

Applying Lemma 2 allows us to denote $W_n^{(m)}(\hat{v})$ by $g_m(\hat{v}_m)$, where $m,n\in\mathbb{N}$ and $m\leq n$. Thus, we may rewrite the two expressions above as

$$\begin{aligned} g_1(v) &= \delta\left[pq(1-F(v)) + (1-p)(1-q(1-F(v)))\right]g_1(v) \\ &\quad + \delta(1-p)q(1-F(v))g_2(v) + p\left[1-q(1-F(v))\right], \end{aligned} \tag{A.15}$$

$$\begin{aligned} g_k(v) &= \delta\left[pq(1-F(v)) + (1-p)(1-q(1-F(v)))\right]g_k(v) \\ &\quad + \delta(1-p)q(1-F(v))g_{k+1}(v) + \delta p\left[1-q(1-F(v))\right]g_{k-1}(v), \end{aligned} \tag{A.16}$$

where $k>1$. Note, however, that Equation A.16 also holds for $k=1$ if we define

$$g_0(v) := \frac{1}{\delta}\text{ for all } v\in\mathbb{R}_+.$$

Thus, the partial derivatives of W_n are determined by a second-order difference equation. Defining

$$y_m(v) := \begin{bmatrix} g_{m+1}(v) \\ g_m(v) \end{bmatrix} \text{ and } A(v) := \begin{bmatrix} a(v) & b(v) \\ 1 & 0 \end{bmatrix},$$

where

$$a(v) := \frac{1-\delta\left[pq(1-F(v))+(1-p)(1-q(1-F(v)))\right]}{\delta(1-p)q(1-F(v))} \text{ and } b(v) := -\frac{\delta p\left[1-q(1-F(v))\right]}{\delta(1-p)q(1-F(v))},$$

we may rewrite this difference equation as the first-order system

$$y_{m+1}(v) = A(v)y_m(v).$$

Induction immediately yields the solution

$$y_m(v) = \left[A(v)\right]^m y_0(v). \tag{A.17}$$

We denote by $\mu_1(v)$ and $\mu_2(v)$ the eigenvalues of $A(v)$; in fact, these can be shown to take the form

$$\mu_1(v) = \frac{a(v)-\sqrt{a^2(v)+4b(v)}}{2} \text{ and } \mu_2(v) = \frac{a(v)+\sqrt{a^2(v)+4b(v)}}{2}.$$

Furthermore, since $p, q, \delta \in (0,1)$ and $F(v) \in [0,1]$ for all $v \in \mathbb{R}_+$, (tedious) arithmetic manipulation yields

$$\mu_2(v) > 1 > \mu_1(v) > 0 \text{ for all } v \in \mathbb{R}_+.$$

Finally, note that $A(v)$ is diagonalizable, and, moreover,

$$[A(v)]^m = \frac{1}{\mu_2(v)-\mu_1(v)}\begin{bmatrix} \mu_2^{m+1}(v)-\mu_1^{m+1}(v) & \mu_1^{m+1}(v)\mu_2(v)-\mu_1(v)\mu_2^{m+1}(v) \\ \mu_2^m(v)-\mu_1^m(v) & \mu_1^m(v)\mu_2(v)-\mu_1(v)\mu_2^m(v) \end{bmatrix}.$$

Thus, Equation A.17 may be rewritten as

$$\begin{aligned} g_m(v) &= \frac{1}{\mu_2(v)-\mu_1(v)}\left[\left(\mu_2^m(v)-\mu_1^m(v)\right)g_1(v)+\left(\mu_1^m(v)\mu_2(v)-\mu_1(v)\mu_2^m(v)\right)g_0(v)\right] \\ &= \frac{\mu_2^m(v)}{\mu_2(v)-\mu_1(v)}\left(g_1(v)-\mu_1(v)g_0(v)\right)+\frac{\mu_1^m(v)}{\mu_2(v)-\mu_1(v)}\left(\mu_2(v)g_0(v)-g_1(v)\right). \end{aligned} \tag{A.18}$$

Since $\mu_2(v) > 1$, the first term in Equation A.18 is divergent unless

$$g_1(v) = \mu_1(v)g_0(v).$$

Such a divergence would, of course, be contradictory; the marginal impact of an increase of the m-th highest value on the social welfare must be bounded above by δ^{m-1}, as the benefit of this increase is not realized for a minimum of m periods. Thus,

$$g_m(v) = \mu_1^m(v)g_0(v). \tag{A.19}$$

Recalling that $W_n^m(\hat{v}) = g_m(\hat{v}_m)$ and that $g_0(v) = \delta^{-1}$, application of Lemma 2 allows us to conclude that

$$W_n(\hat{v}) = W_n(0) + \delta^{-1}\sum_{j=1}^{n}\int_0^{\hat{v}_j}\mu_1^j(v')\,dv'. \tag{A.20}$$

Finally, note that any agent with a "null" value adds nothing to the social welfare, and hence the planner's payoff when they are not present is given by $W_n(0) = W_0$, where

$$\begin{aligned} W_0 &= pq\left[\int_0^\infty (v' + \delta W_0)\, dF(v')\right] + p(1-q)\left[\delta W_0\right] \\ &\qquad + (1-p)q\left[\int_0^\infty \delta W_1(v')\, dF(v')\right] + (1-p)(1-q)\left[\delta W_0\right] \\ &= \frac{q}{1-\delta}\left[p\int_0^\infty v'\, dF(v') + (1-p)\int_0^\infty \left(\int_0^{v''} \mu_1(v')\, dv'\right)\, dF(v'')\right]. \end{aligned}$$

In addition, it is easily verified that $\mu_1(v)$ is, in fact, equal to the definition provided in Equation 5.4 for $\mu(v)$. ■

APPENDIX B. THE SOCIALLY EFFICIENT POLICY

In this appendix, we demonstrate that the socially efficient policy in the setting with random object availability is that which allocates an object to the highest-valued buyer currently present on the market whenever an object is available.[16]

B.1. **Preliminaries.** Denote the state space of the planner's problem by $\Omega := \Omega_1 \times \Omega_2$, where

$$\Omega_1 := \{S \subset \mathbb{Z} \times \mathbb{R}_+ : |S| < \infty\} \text{ and } \Omega_2 := \{0, 1\}. \tag{B.1}$$

A state $\omega = (\omega_1, \omega_2)$ is interpreted as follows: buyer $i \in \mathbb{Z}$, with value $v_i \in \mathbb{R}_+$, is present on the market if, and only if, $(i, v_i) \in \omega_1$; and an object is available if, and only if, $\omega_2 = 1$. We will be indexing buyers by their period of arrival, so that buyer t is the buyer who may arrive in period t.[17]

Given a state ω, the planner must choose an allocation $a \in A(\omega)$, where

$$A(\omega) := \begin{cases} \omega_1 \bigcup \{\diamond\}, & \text{if } \omega_2 = 1; \\ \{\diamond\}, & \text{if } \omega_2 = 0. \end{cases}$$

We interpret the allocation $a = \diamond$ as the null action of not allocating an object. Note that we will abuse notation slightly when $a = \diamond$. When taking a set difference, we will treat $\{\diamond\}$ as the empty set; that is, $S \setminus \{\diamond\} := S$ for all $S \subseteq \mathbb{Z}$.

Recall that the arrival processes of both buyers and objects are history independent. In particular, the seller arrivals are independently and identically distributed according to

$$\Pr\left(\omega_2^{t+1} = 1\right) := p = 1 - \Pr\left(\omega_2^{t+1} = 0\right) \text{ for all } t.$$

On the other hand, the buyers available in the next period depends on the current period allocation. Letting $\xi^t \subset \Xi := \mathbb{N} \times \mathbb{R}_+$ denote the buyer arrival process in period t, we have

$$\xi^t := \begin{cases} \{(t, v_t)\}, & \text{with probability } q; \\ \emptyset, & \text{with probability } 1 - q, \end{cases}$$

where v_t is drawn independently from the distribution F and is unknown to the planner until buyer t is present on the market. We may therefore define a state transition function $\tau : \Omega \times A(\Omega) \times \Xi \to \Omega_1$ by

$$\omega_1^{t+1} = \tau(\omega^t, a^t, \xi_{t+1}) := \left(\omega_1^t \setminus \{a^t\}\right) \bigcup \xi_t.$$

[16] The argument for the case in which objects arrive in every period is subsumed by the present discussion, and therefore is is not explicitly considered.

[17] Buyers that are present on the market in the initial period, however, will be denoted by negative indices; for instance, if 3 buyers are present in period 0, then $\omega_1^0 = \{(-3, v_{-3}), (-2, v_{-2}), (-1, v_{-1})\}$.

We define the initial (period 0) history to be

$$\mathcal{H}^0 := \{(\omega_1^0, \omega_2^0)\}.$$

The set of period $t \in \mathbb{N}$ arrival histories is then recursively defined by

$$\mathcal{H}^t := \mathcal{H}^{t-1} \times (\Xi \times \Omega_2) \text{ for all } .$$

The set of all possible histories is then

$$\mathcal{H} := \bigcup_{t=0}^{\infty} \mathcal{H}^t.$$

We will say that a history $h \in \mathcal{H}$ precedes a history $h' \in \mathcal{H}$ if h is a prefix of h', and will denote this by $h \to h'$.

Note that given a history $h^t = ((\omega_1^0, \omega_2^0), (\xi^1, \omega_2^1), \ldots, (\xi^t, \omega_2^\tau))$ and a sequence of feasible actions $\mathbf{a}^{t-1} = (a^1, a^2, \ldots, a^{t-1})$, we may reconstruct the resultant state ω^t by repeated application of the state transition function τ. We will use the notational shorthand $\hat{\omega}(h^t, \mathbf{a}^{t-1})$ for this state. Thus, we may define an allocation policy as a function $\mathbf{a} : \mathcal{H} \to A(\Omega)$ such that, for all $t \in \mathbb{N}$ and $h^t = (h^{t-1}, (\xi^t, \omega_2^t))$,

$$\mathbf{a}(h^t) \in A\left(\hat{\omega}\left(h^t, \mathbf{a}^{t-1}(h^{t-1})\right)\right),$$

where $\mathbf{a}^{t-1}(h^{t-1})$ is the sequence of allocation decisions taken earlier in the policy.

B.2. **Planner's Problem.** Given an initial state $\omega^0 = (\omega_1^0, \omega_2^0)$, the planner's problem may be written as

$$\text{(B.2)} \qquad \max_{\{a^t\}_{t=1}^{\infty}} \left\{ \mathbb{E}_\xi \left[\sum_{t=1}^{\infty} \delta^{t-1} \omega_2^t a_2^t \right] \right\} \text{ subject to } a^t \in A(\omega_t), \omega_1^{t+1} = \tau(\omega^t, a^t, \xi^t) \text{ for all } t \in \mathbb{N}.$$

PROPOSITION 6 (Socially Efficient Policy)**.**
The policy $\mathbf{a}^*$ which always allocates an object (when available) to the highest-valued buyer present maximizes the social planner's objective function.[18]

Proof. Fix any policy $\mathbf{a_0} \neq \mathbf{a}^*$ such that $\mathbf{a_0}$ yields a the planner a strictly higher payoff than the policy $\mathbf{a}^*$, and define

$$\text{(B.3)} \qquad \hat{\mathcal{H}}_0 := \{h \in \mathcal{H} : \mathbf{a_0}(h) \neq \mathbf{a}^*(h) \text{ and } \mathbf{a_0}(h') = \mathbf{a}^*(h') \text{ for all } h' \to h\}.$$

Note that $\hat{\mathcal{H}}_0$ is the set of all histories h such that $\mathbf{a}^*$ and $\mathbf{a_0}$ disagree at h but agree on all its prefixes—it is the set of "first disagreements" between $\mathbf{a}^*$ and $\mathbf{a_0}$. Since $\mathbf{a_0}$ does strictly better than $\mathbf{a}^*$, this set must have nonzero measure (with respect to the measure induced by the arrival process ξ), as otherwise the two policies would agree almost everywhere.

[18]Thanks are due to Larry Samuelson for suggesting the method of proof used below.

For each $h \in \hat{\mathcal{H}}_0$, define

$$i_0(h) := \mathbf{a}^*(h) \text{ and } \mathcal{I}_0(h) := \{h' \in \mathcal{H} : h \to h' \text{ and } \mathbf{a_0}(h') = i(h)\}\,. \tag{B.4}$$

Thus, $\mathcal{I}_0(h)$ is the set of histories (possibly empty) at which policy $\mathbf{a_0}$ eventually allocates an object to the buyer who had the highest value at h. Letting

$$\hat{\mathcal{I}}_0 := \bigcup_{h \in \hat{\mathcal{H}}_0} \mathcal{I}_0(h),$$

we may define, for each $h \in \mathcal{I}_0$, $\hat{h}_0(h)$ to be the element of $\hat{\mathcal{H}}_0$ such that $\hat{h}_0(h) \to h$.

With these definitions in mind, we may define a new allocation policy

$$\mathbf{a_1}(h) := \begin{cases} i_0(h), & \text{if } h \in \hat{\mathcal{H}}_0; \\ \mathbf{a_0}(\hat{h}_0(h)), & \text{if } h \in \hat{\mathcal{I}}_0; \\ \mathbf{a_0}(h), & \text{otherwise.} \end{cases} \tag{B.5}$$

Thus, $\mathbf{a_1}$ is identical to $\mathbf{a_0}$ except that it "swaps" the allocation decisions at histories $h \in \hat{\mathcal{H}}_0$ with those at histories $\mathcal{I}_0(h)$. Since the value of the agent associated with $i_0(h)$ is greater than that of the agent associated with $\mathbf{a_0}(h)$ for all $h \in \hat{\mathcal{H}}_0$, this implies that $\mathbf{a_1}$ yields the planner a strictly greater payoff than $\mathbf{a_0}$. To see this, consider any $v > v'$, and $t < t'$. Since $\delta < 1$,

$$(\delta^t v + \delta^{t'} v') - (\delta^t v' + \delta^{t'} v) = (\delta^t - \delta^{t'})(v - v') > 0.$$

Such a gain is realized for every history in $\hat{\mathcal{H}}_0$. Since this set has positive measure, it must be the case that the planner's payoff increases.

Notice that if $\mathbf{a_1}$ yields the planner a payoff less than or equal to that of $\mathbf{a}^*$, transitivity of the planner's payoffs leads to a contradiction, implying that there does not exist a policy $\mathbf{a_0}$ such that $\mathbf{a_0}$ does strictly better than $\mathbf{a}^*$, and hence that $\mathbf{a}^*$ is optimal. So suppose, on the other hand, that $\mathbf{a_1}$ *does* provide a strictly higher payoff than $\mathbf{a}^*$. We may then define

$$\hat{\mathcal{H}}_1 := \{h \in \mathcal{H} : \mathbf{a_1}(h) \neq \mathbf{a}^*(h) \text{ and } \mathbf{a_1}(h') = \mathbf{a}^*(h') \text{ for all } h' \to h\}\,. \tag{B.6}$$

Since $\mathbf{a_1}$ is better than $\mathbf{a}^*$, $\hat{\mathcal{H}}_1$ must have nonzero measure. For each $h \in \hat{\mathcal{H}}_1$, define

$$i_1(h) := \mathbf{a}^*(h),\ \mathcal{I}_1(h) := \{h' \in \mathcal{H} : h \to h' \text{ and } \mathbf{a_1}(h') = i(h)\}\,, \text{ and } \hat{\mathcal{I}}_1 := \bigcup_{h \in \hat{\mathcal{H}}_1} \mathcal{I}_1(h).$$

For each $h \in \mathcal{I}_1$, let $\hat{h}_1(h)$ be the element of $\hat{\mathcal{H}}_1$ such that $\hat{h}_1(h) \to h$. Define

$$\mathbf{a_2}(h) := \begin{cases} i_1(h), & \text{if } h \in \hat{\mathcal{H}}_1; \\ \mathbf{a_1}(\hat{h}_1(h)), & \text{if } h \in \hat{\mathcal{I}}_1; \\ \mathbf{a_1}(h), & \text{otherwise.} \end{cases} \tag{B.7}$$

As before, $\mathbf{a_2}$ is identical to $\mathbf{a_1}$ except that it "swaps" the allocation decisions at histories $h \in \hat{\mathcal{H}}_1$ with those at histories $\mathcal{I}_1(h)$, leading to a gain along the path of each history. Such a gain is realized at every history in $\hat{\mathcal{H}}_1$, and since this set has positive measure, it must be the planner realizes a payoff gain by switching from $\mathbf{a_1}$ to $\mathbf{a_2}$.

If $\mathbf{a_2}$ yields the planner a payoff less than or equal to that of $\mathbf{a}^*$, transitivity of the planner's payoffs leads to a contradiction, implying that there does not exist a policy $\mathbf{a_0}$ such that $\mathbf{a_0}$ does strictly better than $\mathbf{a}^*$, and hence that $\mathbf{a}^*$ is optimal.

Proceeding inductively in this manner, we either arrive at a contradiction or we construct a sequence of policies $\{\mathbf{a_t}\}_{t=1}^{\infty}$ with strictly increasing payoffs $\{\nu_t\}_{t=1}^{\infty}$. Note, however, that for all $t \in \mathbb{N}$, each $\mathbf{a_t}$ agrees with $\mathbf{a}^*$ on all histories $h \in \mathcal{H}^{t-1}$. Since δ^t approaches zero as t becomes increasingly large, this implies that this

$$\lim_{t\to\infty} \nu_t \to \nu^*, \tag{B.8}$$

where ν^* is the (finite) expected payoff to the planner from following allocation policy $\mathbf{a}^*$. Moreover, since the sequence $\{\nu_t\}_{t=1}^{\infty}$ is increasing, this implies that

$$\nu^* \geq \nu_t \text{ for all } t \in \mathbb{N}, \tag{B.9}$$

a contradiction. Therefore, we must conclude that there does not exist any policy $\mathbf{a_0}$ that yields the planner strictly higher payoffs than $\mathbf{a}^*$, and hence that $\mathbf{a}^*$ is, in fact, a socially efficient policy. ■

Note that at the ex ante stage, other policies may do as well as $\mathbf{a}^*$ in terms of the planner's payoffs; however, using the same "swapping" argument as above, one may show that the set of histories at which such a policy disagrees with $\mathbf{a}^*$ must be of measure zero. This implies that $\mathbf{a}^*$ is the *unique* socially efficient policy when starting from *any* history.

References

BAJARI, P., AND A. HORTAÇSU (2003): "The Winner's Curse, Reserve Prices, and Endogenous Entry: Empirical Insights from eBay Auctions," *The RAND Journal of Economics*, 34(2), 329–355.

BERGEMANN, D., AND J. VÄLIMÄKI (2007): "Dynamic Marginal Contribution Mechanism," Cowles Foundation Discussion Paper 1616, Yale University.

BIKHCHANDANI, S., P. A. HAILE, AND J. G. RILEY (2002): "Symmetric Separating Equilibria in English Auctions," *Games and Economic Behavior*, 38(1), 19–27.

CAI, G., P. R. WURMAN, AND X. CHAO (2007): "The Non-Existence of Equilibrium in Sequential Auctions when Bids are Revealed," *Journal of Electronic Commerce Research*, 8(2), 141–156.

CAILLAUD, B., AND C. MEZZETTI (2004): "Equilibrium Reserve Prices in Sequential Ascending Auctions," *Journal of Economic Theory*, 117(1), 78–95.

CAVALLO, R., D. C. PARKES, AND S. SINGH (2007): "Efficient Online Mechanisms for Persistent, Periodically Inaccessible Self-Interested Agents," Unpublished manuscript, Harvard University.

FUCHS, W., AND A. SKRZYPACZ (2008): "Bargaining with Arrival of New Traders," Unpublished manuscript, Stanford University.

INDERST, R. (2008): "Dynamic Bilateral Bargaining under Private Information with a Sequence of Potential Buyers," *Review of Economic Dynamics*, 11(1), 220–236.

JEITSCHKO, T. (1998): "Learning in Sequential Auctions," *Southern Economic Journal*, 65(1), 98–112.

KITTSTEINER, T., J. NIKUTTA, AND E. WINTER (2004): "Declining Valuations in Sequential Auctions," *International Journal of Game Theory*, 33(1), 89–106.

MILGROM, P. R., AND R. J. WEBER (1982): "A Theory of Auctions and Competitive Bidding," *Econometrica*, 50(5), 1089–1122.

——— (2000): "A Theory of Auctions and Competitive Bidding, II," in *The Economic Theory of Auctions*, ed. by P. Klemperer, vol. 2, pp. 179–194. Edward Elgar Publishing, Cheltenham, UK.

NEKIPELOV, D. (2007): "Entry Deterrence and Learning Prevention on eBay," Unpublished manuscript, Duke University.

PETERS, M., AND S. SEVERINOV (2006): "Internet Auctions with Many Traders," *Journal of Economic Theory*, 130(1), 220–245.

ROTH, A., AND A. OCKENFELS (2002): "Last-Minute Bidding and the Rules for Ending Second-Price Auctions: Evidence from eBay and Amazon Auctions on the Internet," *American Economic Review*, 94(4), 1093–1103.

SAID, M. (2008): "Sequential Auctions with Random Arrivals," Unpublished manuscript, Yale University.

SAILER, K. (2006): "Searching the eBay Marketplace," CESifo Working Paper 1848, University of Munich.

ZEITHAMMER, R. (2006): "Forward-Looking Bidding in Online Auctions," *Journal of Marketing Research*, 43, 462–476.