Higher order approximations of stochastic rational expectations models

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Abstract

We describe algorithm to find higher order approximations of stochastic rational expectations models near the deterministic steady state. Using matrix representation of function derivatives instead of tensor representation we obtain simple expressions of matrix equations determining higher order terms.

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1 Introduction

We describe analyze a perturbation method for computing asymptotic approximation of solution to general stochastic rational expectations models around their steady state described for example in Jin, Judd, (2002).

The problem we analyze can be informally written as $0 = E_t f(x_{t-1}, x_t, x_{t+1}, \sigma\epsilon_{t+1}; E)$, where x_t is a vector of endogenous variables, ϵ_t is a random shock, σ is a small parameter, and E denotes expectation operator. We are looking for solution $x_t = R(u_t, \sigma\epsilon_t, \sigma)$ where u_t is a state variable with dynamics given by $u_{t+1} = P(u_t, \sigma\epsilon_{t+1}, \sigma)$. This approach is more general than specifying a priori state variables, since in general case state variables cannot be easily determined or there may exist no solutions dependent on given state variables. The basic idea of the perturbation method is to find asymptotic expansion of functions R and P around known solution of simpler problem, generally around steady state. We analyze only regular perturbation problems in which solution to the model does not qualitatively changes when σ approaches 0. Many macroeconomic models can be solved using regular perturbation methods but not all, for example models with incomplete asset markets.

The procedure of finding solution is quite standard and is based on successive differentiation of function f with respect to state variable u and parameter σ and then using implicit function theorem. The method proposed by Judd is based on tensor calculus and requires huge amount of symbolic computation in order to find matrix equations determining solution. Moreover complexity of these symbolic computations growth exponentially with perturbation order.

In this paper we propose using matrix representation of higher order derivatives. This allows us to find simple expressions for function derivatives and matrix equations determining solution irrespective of perturbation order. In this paper we only derive matrix equations determining approximation of functions R and P up to any order.

The rest of the paper is organized as follows. Section 2 states the problem. Section 2 presents function differentiation rules using matrix representation of derivatives, introduces generalized Kronecker products and summarize their basic properties. This section

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is based on Kowal (2007). Section 3 describe the model we analyze. In section 4 we find conditions determining approximation of solutions R and P. In sections 5-6 we represent these conditions as matrix equations. Finally section 8 concludes.

2 The flattered tensor calculus

Let as consider smooth functions $\Omega \ni \theta \mapsto X(\theta) \in \mathbb{R}^{m\times n}$, $\Omega \ni \theta \mapsto Y(\theta) \in \mathbb{R}^{p\times q}$, where $\Omega \subset \mathbb{R}^k$ is an open set. Functions X, Y associate a $m \times n$ and $p \times q$ matrix for a given vector of parameters, $\theta = \mathrm{col}(\theta_1, \theta_2, \ldots, \theta_k)$. Let the differential of the function X with respect to θ is defined as

$$\frac{\partial X}{\partial \theta} = \left[\begin{array}{cccc} \frac{\partial X}{\partial \theta_1} & \frac{\partial X}{\partial \theta_2} & \cdots & \frac{\partial X}{\partial \theta_k} \end{array} \right]$$

for $\partial X/\partial \theta_i \in \mathbb{R}^{m\times n}$, $i = 1, 2, \ldots, k$.

Proposition 2.1. *The following equations hold*

1. $\frac{\partial}{\partial \theta}(\alpha X) = \alpha \frac{\partial X}{\partial \theta}$
2. $\frac{\partial}{\partial \theta}(X + Y) = \frac{\partial X}{\partial \theta} + \frac{\partial Y}{\partial \theta}$
3. $\frac{\partial}{\partial \theta}(X \times Y) = \frac{\partial X}{\partial \theta} \times (I_k \otimes Y) + X \times \frac{\partial Y}{\partial \theta}$

where $\alpha \in \mathbb{R}$ and I_k is a $k \times k$ dimensional identity matrix, assuming that differentials exist and matrix dimensions coincide.

Proof. See Kowal (2007). □

Let us now turn to differentiating tensor products of matrices. Let for any matrices X, Y, where $X \in R^{\imath\times q}$ is a matrix with elements $x_{ij} \in \mathbb{R}$ for $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, q$. The Kronecker product, $X \otimes Y$ is defined as

$$X \otimes Y \doteq \left[\begin{array}{ccc} x_{11}Y & \cdots & x_{1q}Y \\ \vdots & \ddots & \vdots \\ x_{p1}Y & \cdots & x_{pq}Y \end{array} \right]$$

Let introduce the generalized Kronecker products

Definition 2.2. Let $X = [X_1, X_2, \ldots X_m]$, where $X_i \in \mathbb{R}^{p\times q}$, $i = 1, 2, \ldots, m$ is a $p \times q$ matrix is a partition of $p \times mq$ dimensional matrix X. Let $Y = [Y_1, Y_2, \ldots Y_n]$, where $Y_i \in \mathbb{R}^{r\times s}$, $i = 1, 2, \ldots, n$ is a $r \times s$ matrix is a partition of $r \times ns$ dimensional matrix Y. Then

$$X \otimes_n^1 Y \doteq [X \otimes Y_1, \ldots, X \otimes Y_n]$$
$$X \otimes_n^m Y \doteq [X_1 \otimes_n^1 Y, \ldots, X_m \otimes_n^1 Y]$$
$$X \otimes_{n_1,n_2,\ldots,n_s}^{1,m_2,\ldots,m_s} Y \doteq [X \otimes_{n_2,\ldots,n_s}^{m_2,\ldots,m_s} Y_1, \ldots, X \otimes_{n_2,\ldots,n_s}^{m_2,\ldots,m_s} Y_{n_1}]$$
$$X \otimes_{n_1,n_2,\ldots,n_s}^{m1,m_2,\ldots,m_s} Y \doteq [X_1 \otimes_{n_1,n_2,\ldots,n_s}^{1,m_2,\ldots,m_s} Y, \ldots, X_{m_1} \otimes_{n_1,n_2,\ldots,n_s}^{1,m_2,\ldots,m_s} Y]$$

assuming that appropriate matrix partitions exist.

Proposition 2.3. *The following equations hold*

1. $\frac{\partial}{\partial \theta}(X \otimes Y) = \frac{\partial X}{\partial \theta} \otimes Y + X \otimes_k^1 \frac{\partial Y}{\partial \theta}$
2. $\frac{\partial}{\partial \theta}(X \otimes_{n_1,\ldots,n_s}^{m_1,\ldots,m_s} Y) = \frac{\partial X}{\partial \theta} \otimes_{1,n_1,\ldots,n_s}^{k,m_1,\ldots,m_s} Y + X \otimes_{k,n_1,\ldots,n_s}^{1,m_1,\ldots,m_s} \frac{\partial Y}{\partial \theta}$

Proof. See Kowal (2007). □

Proposition 2.4. *Let α is a scalar function of θ and X is a matrix valued function of θ, $X(\theta) \in \mathbb{R}^{p\times q}$. Then*

$$\frac{\partial}{\partial\theta}(\alpha X) = \alpha \times \frac{\partial X}{\partial\theta} + \frac{\partial\alpha}{\partial\theta} \otimes X$$

Proof. See Kowal (2007). □

Let f is a matrix valued function given by

$$R^p \ni x \mapsto f(x) \in R^{m\times n}$$

and g is a vector valued function $\Omega \ni \theta \mapsto g(\theta) \in R^p$. We can define a function composition $\Omega \ni \theta \mapsto f(g(\theta)) \equiv f(x) \circ g(\theta) \in R^{m\times n}$.

Proposition 2.5. *The following condition holds*

$$\frac{\partial}{\partial\theta} f(x) \circ g(\theta) = \Big[\frac{\partial f(x)}{\partial x} \circ g(\theta)\Big] \times \Big(\frac{\partial g(\theta)}{\partial\theta} \otimes I_n\Big)$$

Proof. See Kowal (2007). □

We need additional properties of the generalized Kronecker products

Proposition 2.6. *For any matrices A, B, C, D*

$$(AB) \otimes^{m_1,\dots,m_s}_{n_1,\dots,n_s} (CD) = (A \otimes C) \times (B \otimes^{m_1,\dots,m_s}_{n_1,\dots,n_s} D)$$

assuming that products AB and CD, as well as Kronecker products exist.

Proof. See Kowal (2007). □

Proposition 2.7. *For any matrices A, B, C*

$$A \otimes^{m_1,\dots,m_s}_{n_1,\dots,n_s} (B \otimes C) = (A \otimes^{m_1,\dots,m_s}_{n_1,\dots,n_s} B) \otimes C$$

assuming that Kronecker products exist.

Proof. See Kowal (2007). □

Proposition 2.8. *Let A is $m \times n$ matrix. Let B is $p \times q$ matrix. Then*

$$A \otimes^1_q B = (I_m \otimes^1_p I_p) \times (B \otimes A)$$

Proof. See Kowal (2007). □

Proposition 2.9. *Let A is $m \times n$ matrix. Let B is $p \times q$ matrix. Then*

$$A \otimes B = (I_m \otimes^1_p I_p) \times (B \otimes A) \times (I_q \otimes^1_n I_n)$$

Proof. See Kowal (2007). □

Proposition 2.10. *Let A is $m \times n$ matrix. Then*

$$I_p \otimes^1_n (A \otimes I_q) = (I_p \otimes^1_n I_{nq}) \times (A \otimes I_{pq})$$

Proof.

$$\begin{aligned} I_p \otimes^1_n (A \otimes I_q) &= (I_p \otimes^1_n A) \otimes I_q = \Big((I_p \otimes^1_n I_n) \times (A \otimes I_p)\Big) \otimes I_q \\ &= (I_p \otimes^1_n I_n) \otimes I_q \times (A \otimes I_p \otimes I_q) = (I_p \otimes^1_n I_{nq}) \times (A \otimes I_{pq}) \end{aligned}$$

□

3 The problem

Consider a model

$$0 = E_t f(\tilde{x}_{t-1}, \tilde{x}_t, \tilde{x}_{t+1}, \sigma\epsilon_{t+1}; E) \tag{1}$$

where $\tilde{x}_t$ is a vector of endogenous variables, $\epsilon_t \in \mathbb{R}^k$ is i.i.d. random variable, $\sigma \in \mathbb{R}$ is a small parameter, and E denotes expectation operator. In this way we allow for existence terms containing expectations of functions of variables under information sets in any period. Suppose that $\tilde{x}^*$ satisfies $0 = E_t f(\tilde{x}^*, \tilde{x}^*, \tilde{x}^*, 0, E)$ and suppose that we have expansion of $E_t f(\tilde{x}_{t-1}, \tilde{x}_t, \tilde{x}_{t+1}, \sigma\epsilon_{t+1}, E)$ in the Taylor series around $\tilde{x}_{t-1} = \tilde{x}_t = \tilde{x}_{t+1} = \tilde{x}^*$ and $\sigma = 0$ up to order q. Observe that we can represent this Taylor series as

$$0 = A_1 x_t + A_2 x_{t+1} + A_3 E_t x_{t+1} + \sigma V \epsilon_t + B_1 x_t \otimes x_t + o_q \tag{2}$$

after appropriate redefinition and expansion of the set of endogenous variables. The term o_q contains derivatives of the function f of order higher than q. We are looking for solution in the form

$$x_t = R(u_t, \sigma\epsilon_t, \sigma) \qquad\qquad u_t = P(u_{t-1}, \sigma\epsilon_t, \sigma) \tag{3}$$

where $u_t \in \mathbb{R}^n$ is a state variable. Additionally we require

$$R(0,0,0) = 0 \qquad\qquad P(0,0,0) = 0$$

We approximate functions R and P expanding these functions in asymptotic series

$$R(u_t, \sigma\epsilon_t, \sigma) \sim \sum_{i=1}^{p} R_i \times \bigotimes_i v_t$$
$$P(u_{t-1}, \sigma\epsilon_t, \sigma) \sim \sum_{i=1}^{p} P_i \times \bigotimes_i w_t$$

where $v_t = \mathrm{col}(u_t, \sigma\epsilon_t, \sigma)$, $w_t = \mathrm{col}(u_{t-1}, \sigma\epsilon_t, \sigma)$ around $u \to 0$ and $\sigma \to 0$. If we are looking for approximation of order $p \leq q$, then the term o_q can be ignored.

4 Matrix equations

We can express functions R, P as

$$R(u_t, \sigma\epsilon_t, \sigma) = R(v) \circ f_{\epsilon_t}(u_t, \sigma)$$
$$P(u_t, \sigma\epsilon_t, \sigma) = P(v) \circ f_{\epsilon_t}(u_t, \sigma)$$

where $v = \mathrm{col}(u_t, \sigma\epsilon_t, \sigma)$ and

$$f_{\epsilon_t}(u_t, \sigma) = \begin{bmatrix} u_t \\ \sigma\epsilon_t \\ \sigma \end{bmatrix}$$

Let $q_t = \mathrm{col}(u_t, \sigma)$. Then

$$\frac{\partial}{\partial q_t} f_{\epsilon_t}(u_t, \sigma) = \begin{bmatrix} I_n & 0_{n,1} \\ 0_{k,n} & \epsilon_t \\ 0_{1,n} & 1 \end{bmatrix} \equiv M(\epsilon_t)$$

and for any $k > 0$ $\partial^k / \partial q_t^k f_{\epsilon_t}(u_t, \sigma) = 0$.

Let $Q(u_t, \sigma\epsilon_t, \sigma) \equiv R(u_t, \sigma\epsilon_t, \sigma) \otimes R(u_t, \sigma\epsilon_t, \sigma)$. Then $Q(u_t, \sigma\epsilon_t, \sigma) = [R(v) \otimes R(v)] \circ f_{\epsilon_t}(q_t)$. Let $S(u_t, \sigma\epsilon_{t+1}, \sigma) \equiv R(u_{t+1}, \sigma\epsilon_{t+1}, \sigma) = R(P(u_t, \sigma\epsilon_{t+1}, \sigma), \sigma\epsilon_{t+1}, \sigma)$. Observe that

$$S(u_t, \sigma\epsilon_{t+1}, \sigma) = R(v) \circ g(w) \circ f_{\epsilon_{t+1}}(q_t) \equiv S(w) \circ f_{\epsilon_{t+1}}(q_t)$$

where $w = \mathrm{col}(u_t, \sigma\epsilon_{t+1}, \sigma)$ and

$$g(u_t, \sigma\epsilon_{t+1}, \sigma) = \begin{bmatrix} P(u_t, \sigma\epsilon_{t+1}, \sigma) \\ \sigma\epsilon_{t+1} \\ \sigma \end{bmatrix}$$

Substituting (3) to (2) yields $0 = T(u_t, \epsilon_t, \epsilon_{t+1}, \sigma)$, where

$$\begin{aligned} T(u_t, \epsilon_t, \epsilon_{t+1}, \sigma) \equiv A_1 R(u_t, \sigma\epsilon_t, \sigma) + A_2 S(u_t, \sigma\epsilon_{t+1}, \sigma) + A_3 E_t S(u_t, \sigma\epsilon_{t+1}, \sigma) + \sigma V \epsilon_t \\ + B_1 Q(u_t, \sigma\epsilon_t, \sigma) \end{aligned} \tag{4}$$

Differentiating function T m times with respect to q_t yields

$$\begin{aligned} T_m(u_t, \epsilon_t, \epsilon_{t+1}, \sigma) \equiv A_1 \frac{\partial^m}{\partial q_t^m} R(u_t, \sigma\epsilon_t, \sigma) + A_2 \frac{\partial^m}{\partial q_t^m} S(u_t, \sigma\epsilon_{t+1}, \sigma) \\ + A_3 E_t \frac{\partial^m}{\partial q_t^m} S(u_t, \sigma\epsilon_{t+1}, \sigma) + V^m \times \bigotimes_m M(\epsilon_t) + B_1 \frac{\partial^m}{\partial q_t^m} Q(u_t, \sigma\epsilon_t, \sigma) \end{aligned} \tag{5}$$

where $T_m = \frac{\partial^m}{\partial q^m} T$, $V^m = 0$ for $m \geq 2$ and $V^1 = [0_{n,n}, V, 0_{n,1}]$.

Proposition 4.1. *For any $p \geq 1$ and any differentiable function $F : R^{n+k+1} \to R^{m_2 \times n_2}$, where m_2, n_2 are any positive integers*

$$\frac{\partial^p}{\partial q_t^p}(F(v) \circ f_\epsilon(q_t)) = \Big[\frac{\partial^p}{\partial v^p} Q(v)\Big] \circ f_\epsilon(q_t) \times \bigotimes_p M(\epsilon)$$

Let $R_k(v)$, $P_k(v)$ denotes respectively $\partial^k/\partial v^k R(v)$, $\partial^k/\partial v^k P(v)$. Proposition 4.1 implies

Proposition 4.2. *For any $p \geq 1$*

$$\frac{\partial^p}{\partial q_t^p} R(u_t, \sigma\epsilon_t, \sigma) = [R_p(v) \circ f_{\epsilon_t}(q_t)] \times \bigotimes_p M(\epsilon_t)$$

We are going to find derivatives of $Q(u_t, \sigma\epsilon_t, \sigma)$ with respect to q_t. Let for any $p, q \geq 0$, $Q_{p,q}(v) = R_p(v) \otimes R_q(v)$. Then $Q(u_t, \sigma\epsilon_t, \sigma) = Q_{0,0}(v) \circ f_\epsilon(q_t)$. We have

Proposition 4.3. *For any $p \geq 1$*

$$\frac{\partial^p}{\partial q_t^p} Q(u_t, \sigma\epsilon_t, \sigma) = [\frac{\partial^p}{\partial v^p} Q_{0,0}(v)] \circ f_{\epsilon_t}(q_t) \times \bigotimes_p M(\epsilon_t)$$

$$\frac{\partial^p}{\partial v^p} Q_{0,0}(v) = \sum_{i=0}^{p} Q_{i,p-i}(v) \times \Gamma_i^p$$

where $\Gamma_0^1 = \Gamma_1^1 = I_{n+k+1}$, for any $p \geq 1$, $\Gamma_{-1}^p = \Gamma_{p+1}^p = 0$, and for any $m \geq 1$, $0 \leq p \leq m+1$, $q \geq 0$

$$\begin{aligned} \Gamma_p^{m+1} &= I_{n+k+1} \otimes \Gamma_{p-1}^m + \Lambda_{p,m-p} \times (I_{n+k+1} \otimes \Gamma_p^m) \\ \Lambda_q^p &= I_{(n+1+k)^p} \otimes_{n+1+k}^1 I_{(n+1+k)^{q+1}} \end{aligned}$$

and $\Lambda_{-1}^p = 0$.

Similarly for derivatives of the function S

Proposition 4.4. *For any $p \geq 1$*

$$\frac{\partial^p}{\partial q_t^p} S(u_t, \sigma\epsilon_{t+1}, \sigma) = [\frac{\partial^p}{\partial w^p} S(w)] \circ f_{\epsilon_{t+1}}(q_t) \times \bigotimes_p M(\epsilon_{t+1})$$

$$\frac{\partial^p}{\partial w^p} S(w) = \sum_{i=1}^{p} [R_i(v) \circ g(w)] \times \Delta_i^p(w)$$

where $\Delta_1^1(w) = P^*(w)$, for any $p \geq 1$, $\Delta_{p+1}^p(w) = \Delta_0^p(w) = 0$, and for any $m \geq 1$, $1 \leq p \leq m+1$

$$\Delta_p^{k+1}(w) = P^*(w) \otimes \Delta_{p-1}^k(w) + \frac{\partial}{\partial w} \Delta_p^k(w)$$

with

$$P^*(w) = \begin{bmatrix} \partial/\partial v_1 P(w) & \partial/\partial v_2 P(w) & \partial/\partial v_3 P(w) \\ 0_{k,n} & I_k & 0 \\ 0_{1,n} & 0 & 1 \end{bmatrix}$$

and any $k \geq 1$.

$$\frac{\partial^k}{\partial w^k} P^*(w) = \begin{bmatrix} I_n \\ 0_{k+1,n} \end{bmatrix} \times P_{k+1}(w)$$

Proof. The first equation results from proposition 4.1. Observe that $\partial g(w)/\partial w = P^*$. For $k = 1$ we have

$$\frac{\partial}{\partial w} S(w) = [\tilde{R}_1(v) \circ g(w)] \times \frac{\partial}{\partial w} g(w) = [\tilde{R}_1(v) \circ g(w)] \times \Delta_1^1(w)$$

Let the proposition holds for any $1 \leq p \leq m$, where $m \geq 1$. Then

$$\begin{aligned} \frac{\partial^{m+1}}{\partial w^{m+1}} S(w) &= \sum_{i=0}^{m} [R_{i+1}(v) \circ g(w)] \times (P^*(w) \otimes \Delta_i^m(w)) + \sum_{i=1}^{m+1} [R_i(w) \circ g(w)] \times \frac{\partial}{\partial w} \Delta_i^m(w) \\ &= \sum_{i=1}^{m+1} [R_i(v) \circ g(w)] \times \Delta_i^{m+1}(w) \end{aligned}$$

□

Let υ is a finite sequence containing elements $0, 1, k$, let $[]$ denotes empty sequence, let $[\upsilon_1, \upsilon_2]$ denotes concatenation of two sequences, let $|\upsilon|$ denotes length of the sequence υ. Let for a function $F(w_0, w_1, \ldots, w_k)$, where $w_i \in R^{n_i}$ is a vector, and for a sequence υ of length p, $F_\upsilon(v)$ denotes

$$F_\upsilon(w_0, w_1, \ldots, w_k) = \frac{\partial}{\partial q_1} \cdots \frac{\partial}{\partial q_p} F(w_0, w_1, \ldots, w_k)$$

where q_i denotes variable w_i if i-th element of υ is i.

Proposition 4.5. *Let υ is a sequence of length p containing elements $0, 1, \ldots, k$. Then*

$$F_\upsilon(w_0, \ldots, w_k) = \frac{\partial^p}{\partial w^p} F(w_0, \ldots, w_k) \times J_\upsilon$$

and

$$\sum_{|\upsilon|=p} F_\upsilon(w_0,\dots,w_k)\times J'_\upsilon = \frac{\partial^p}{\partial w^p}F(w_0,\dots,w_k)$$

where $w=\mathrm{col}(w_0,\dots,w_k)$*,* $J_{[]}=1$*, and for* $i=0,\dots,k$

$$J_{[i,\upsilon]} = J^i \otimes J_\upsilon$$

where J^i *is a matrix selecting variable* w_i *from* w*, given by* $J^i=\mathrm{col}(0_{n_0,n_i},\dots,I_{n_i},\dots,0_{n_k,n_i})$*.*

Proof. Let F_p denotes $\frac{\partial^p}{\partial w^p}F(w)$. For sequence of length 0 $F_{[]}=F=F_0$, hence proposition holds for $p=0$. Let the proposition holds for sequences of length $m\le p$, $p\ge 1$. Let υ denotes any sequence of length m and i takes value from $\{0,\dots,k\}$. Then

$$F_{[i,\upsilon]} = \frac{\partial}{\partial w_i}F_\upsilon = \frac{\partial}{\partial w_i}(F_m\times J_\upsilon) = (\frac{\partial}{\partial w_i}F_m)\times(I_{n_i}\otimes J_\upsilon)$$

Since $\partial F_m/\partial w = [\partial F_m/\partial w_0,\dots,\partial F_m/\partial w_k]$, $\partial F_m/\partial w_i = F_{m+1}\times(J^i\otimes I_{n^m})$, where $n=\sum_i n_i$. Hence

$$F_{[i,\upsilon]} = F_{m+1}\times(J_i\otimes I_{n^m})\times(I_{n_i}\otimes J_\upsilon) = F_{m+1}\times J_i\otimes J_\upsilon$$

since $J_\upsilon = I_{n^m}\times J_\upsilon$.

Let $\upsilon_i,\dots,\upsilon_k$ are distinct sequences of length p, where k is a number of sequences of length p, let $J=[J_{\upsilon_1},\dots,J_{\upsilon_k}]$ and $\bar F=[F_{\upsilon_1},\dots,F_{\upsilon_k}]\circ w$. Then $\bar F(w)=\frac{\partial^p}{\partial w^p}F(w)\times J$. The matrix J is orthogonal and square, hence $\frac{\partial^p}{\partial w^p}F(w)=\bar F(w)\times J' = F_{\upsilon_1}(w)\times J'_{\upsilon_1}+\dots+F_{\upsilon_k}(w)\times J'_{\upsilon_k} = \sum_{|\upsilon|=p}F_\upsilon(w)\times J'_\upsilon$. □

Now let υ denotes any sequence containing elements 0 and 1. Let $n(\upsilon)$, $m(\upsilon)$ are respectively number of zeros and nonzeros elements in the sequence υ. Let element 0 in the sequence υ denotes variable u_t and element 1 denotes variable σ.

Equation $T(u_t,\epsilon_t,\epsilon_{t+1},\sigma)=0$ must hold for any u_t, ϵ_t, ϵ_{t+1} and σ. Hence we obtain a set of equations

$$0 = T_m(0,\epsilon_t,\epsilon_{t+1},0)\times J_\upsilon \tag{6}$$

for any ϵ_t, ϵ_{t+1}, any sequence υ of length m and for any m which is equivalent to

$$\begin{aligned}0 =& \Big(A_1R_m(0)+B_1Q_m(0)+V^m\Big)\times\bigotimes_m M(\epsilon_t)\times J_\upsilon\\ &+A_2S_m(0)\times\bigotimes_m M(\epsilon_{t+1})\times J_\upsilon + A_3S_m(0)\times E\{\bigotimes_m M(\epsilon_{t+1})\}\times J_\upsilon\end{aligned}$$

Proposition 4.6. *For any sequence* υ *of length* m

$$\bigotimes_m M(\epsilon_t)\times J_\upsilon = M_\upsilon\times\bigotimes_{n(\upsilon)} I_n\otimes\bigotimes_{m(\upsilon)} V_1(\epsilon_t)$$

with $M_{[]}=1$ *and*

$$M_{[0,\upsilon]} = V_0\otimes M_\upsilon \qquad\qquad M_{[1,\upsilon]} = I_{n+k+1}\otimes^1_{q(\upsilon)} M_\upsilon$$

where $q(\upsilon)=n^{n(\upsilon)}\times(n+k+1)^{m(\upsilon)}$*,* $V_0=\mathrm{col}(I_n,0_{k+1,n})$*, and* $V_1(\epsilon_t)=\mathrm{col}(0_{n,1},\epsilon_t,1)$*.*

Proposition 4.7. *Let* $N_0=\mathrm{col}(0_{n+k,1},1)$*,* $N_1=\mathrm{col}(0_{n,k},I_k,0_{1,k})$*. Then*

$$\bigotimes_p V_1(\epsilon_t) = \sum_{i=0}^{p}\sum_{|\upsilon|=p,m(\upsilon)=i}\Pi^p_\upsilon\times\bigotimes_i\epsilon_t$$

where υ *is a sequence containing elements* 0 *and* 1*,* $\Pi^1_{[0]}=N_0$*,* $\Pi^1_{[1]}=N_1$*, and for any sequence* υ *of length* p

$$\Pi^{p+1}_{[0,\upsilon]} = N_0\otimes\Pi^p_\upsilon \qquad\qquad \Pi^{p+1}_{[1,\upsilon]} = N_1\otimes\Pi^p_\upsilon$$

5 Deterministic terms

Assume that $\epsilon_t = \epsilon_{t+1} = 0$. Then we obtain a sequence of problems

$$\begin{aligned} 0 = \Big(A_1 R_m(0) + B_1 Q_m(0) + (A_2 + A_3) S_m(0)\Big) \times M_\upsilon \times \bigotimes_{n(\upsilon)} I_n \otimes \bigotimes_{m(\upsilon)} V_1(0) \\ + A_3 S_m(0) \times M_\upsilon \times \Big(\bigotimes_{n(\upsilon)} I_n \otimes E\{\bigotimes_{m(\upsilon)} V_1(\epsilon_{t+1})\} - \bigotimes_{n(\upsilon)} I_n \otimes \bigotimes_{m(\upsilon)} V_1(0)\Big) \end{aligned} \tag{7}$$

If υ is a sequence containing only elements 1, then $M_\upsilon \bigotimes_{m(\upsilon)} V_1(\epsilon) = \bigotimes_{m(\upsilon)} I_{n+k+1} \times \bigotimes_{m(\upsilon)} V_1(\epsilon)$. Let us concern only sequences of the form $[0, \ldots, 0, 1, \ldots, 1]$, called basic sequences. For such sequences $M_\upsilon = \bigotimes_{n(\upsilon)} V_0 \otimes \bigotimes_{m(\upsilon)} I_{n+k+1}$. Matrix $S_m(0)$ for $m = 1$ takes the form

$$S_1(0) = \sum_{i=1}^{1} R_i(0) \times \Delta_i^m(0) = R_1(0) \times P^*(0)$$

and for $m \geq 2$

$$S_m(0) = \sum_{i=1}^{m} R_i(0) \times \Delta_i^m(0) = R_m(0) \times \bigotimes_m P^*(0) + R_1(0) V_0 \times P_m(0) + \sum_{i=2}^{m-1} R_i(0) \times \Delta_i^m(0)$$

Proposition 5.1. *For any basic sequence υ of length m*

$$R_m(0) \times \bigotimes_m P^*(0) \times M_\upsilon = \frac{\partial^{n(\upsilon)}}{\partial v_1^{n(\upsilon)}} R_{m(\upsilon)}(0) \times \bigotimes_{n(\upsilon)} (\partial / \partial v_1 P(0)) \otimes \bigotimes_{m(\upsilon)} P^*(0)$$

Proposition 5.1 implies that we need to know derivative of $R(v_1, v_2, v_3)$ with respect to v_2 to find solution to (7) for sequences υ containing at least one element 1.

Assume that $m(\upsilon) = 0$. Then equation (7) for $m = 1$ reduces to

$$0 = A_1 \times R_1(0) K_{\eta_0} + (A_2 + A_3) \times R_1(0) K_{\eta_0} \times P_1(0) K_{\eta_0}$$

and for $m \geq 2$

$$\begin{aligned} 0 = A_1 R_m(0) K_{\eta_0} + (A_2 + A_3) R_m(0) K_{\eta_0} \bigotimes_m P_1(0) K_{\eta_0} + (A_2 + A_3) R_1(0) V_1 \times P_m(0) K_{\eta_0} \\ + B_1 Q_m(0) \times K_{\eta_0} + \sum_{i=2}^{m-1} (A_2 + A_3) R_i(0) \times \Delta_i^m(0) \times K_{\eta_0} \end{aligned}$$

where K_{η_0} denotes matrix selecting derivative of functions of v with respect to sequence η_0 of length m containing only zeros. This is an equation with respect to $R_m K_{\eta_0} \equiv X$ and $P_m K_{\eta_0} \equiv Y$. Observe that last two terms are already known. Hence, we obtain matrix equation

$$\begin{aligned} 0 &= A_1 X + (A_2 + A_3) XY, && \text{for } m = 1; \\ 0 &= A_1 X + (A_2 + A_3) X C_1 + C_2 Y + C_3, && \text{for } m \geq 2. \end{aligned}$$

where $C_1 = \bigotimes_m P_1(0) K_{\eta_0}$, $C_2 = (A_2 + A_3) R_1(0) V_0$, and $C_3 = B_1 Q_m(0) K_{\eta_0} + \sum_{i=2}^{m-1} (A_2 + A_3) R_i(0) \Delta_i^m(0) K_{\eta_0}$.

6 Stochastic terms

Collecting terms containing ϵ_t and ϵ_{t+1} we obtain two sets of equations

$$0 = \Big(A_1 R_m(0) + B_1 Q_m(0) + V^m\Big) \times \bigotimes_m M(\epsilon_t) \times J_\upsilon$$
$$0 = A_2 S_m(0) \times \bigotimes_m M(\epsilon_{t+1}) \times J_\upsilon$$

for any sequence υ of length m containing at least one element 1. Let υ is a basic sequence. Then

$$0 = \Big(A_1 R_m(0) + B_1 Q_m(0) + V^m\Big) \times \bigotimes_{n(\upsilon)} V_0 \otimes \bigotimes_{m(\upsilon)} V_1(\epsilon_t)$$
$$0 = A_2 S_m(0) \times \bigotimes_{n(\upsilon)} V_0 \otimes \bigotimes_{m(\upsilon)} V_1(\epsilon_{t+1})$$

Let us concern the second equation. We have

$$0 = \sum_{i=1}^{m(\upsilon)} \sum_{|\mu|=m(\upsilon), m(\mu)=i} A_2 S_m(0) \times (\bigotimes_{n(\upsilon)} V_0 \otimes \Pi_\mu^{m(\upsilon)}) \times (\bigotimes_{n(\upsilon)} I_n \otimes \bigotimes_i \epsilon_t)$$

where μ is any sequence containing elements 0 and 1. Since this equation must hold for any ϵ_{t+1}, we obtain a set of equations

$$0 = \sum_{|\mu|=m(\upsilon), m(\mu)=i} A_2 S_m(0) \times (\bigotimes_{n(\upsilon)} V_0 \otimes \Pi_\mu^{m(\upsilon)}) \times (\bigotimes_{n(\upsilon)} I_n \otimes \bigotimes_i \epsilon_t) \tag{8}$$

for $1 \le i \le m(\upsilon)$.

There are many terms of $S_m(w)$ which contribute to the same derivative with respect to σ and u_t, since changing of order of differentiation with respect to w_2 and w_3 contribute to the same derivative of $S(w) \circ f_{\epsilon_t}(q_t)$ with respect to q_t. Equations (6) do not uniquely determine $R_m(0)$ and $P_m(0)$ since $R_m(0)$ and $P_m(0)$ contain respectively $q \times (n + k + 1)^m$ and $n \times (n + k + i)^m$, where q is a number of endogenous variables, but there are only $q \times (n + 1)^m$ equations. We need to impose additional restrictions, which guaranties symmetry of derivatives P_m and R_m. These restrictions can be imposed assuming that not only conditions (8) are fulfilled but also

$$0 = A_2 S_m(0) \times (\bigotimes_{n(\upsilon)} V_0 \otimes \Pi_\mu^{m(\upsilon)}) \times (\bigotimes_{n(\upsilon)} I_n \otimes \bigotimes_i \epsilon_t)$$

for $1 \le i \le m(\upsilon)$ and any sequence μ satisfying $|\mu| = m(\upsilon)$, $m(\mu) = i$ and for any ϵ_{t+1}. In this way we obtain set of matrix equations

$$0 = A_2 S_m(0) \times (\bigotimes_{n(\upsilon)} V_0 \otimes \Pi_\mu^{m(\upsilon)}) \tag{9}$$

for any basic sequence υ of length m, $m(\upsilon) > 0$, any sequence μ satisfying $|\mu| = m(\upsilon)$, and $m(\mu) > 0$.

We need to restrict the set of matrix equations, since we do not know derivatives of $R(w)$ and $P(w)$ with respect to w_3.

Proposition 6.1. *Consider expression*

$$Q = S_m(0) \times (\bigotimes_{n(\upsilon)} V_0 \otimes \Pi_\mu^{m(\upsilon)})$$

where υ is a basic sequence of length m and μ is a sequence of length $m(\upsilon)$. If sequence μ contains at most p elements 0, then expression Q does not depend on derivatives of $R_\eta(0)$, $P_\eta(0)$ for sequences η of length m containing more than p elements 2.

Let concern set of equations of the form $0 = A_2 S_m(0) \times \Omega_\upsilon$, where

$$\Omega_\upsilon = M_\upsilon \times \bigotimes_{n(\upsilon)} I_n \otimes \bigotimes_{m(\upsilon)} N_1$$

for any sequence υ of length m, $m(\upsilon) \geq 1$. This set of equations contains all equations for sequences described by proposition 6.1 and its permutations. By the proposition 6.1 $S_m(0) \times \Omega_\upsilon$ does not depend on derivatives of $R(w)$ and $P(w)$ with respect to w_3. Let Ω is a matrix obtained by horizontally concatenating all matrices Ω_υ. Finally we obtain two matrix equations for $m = 1$

$$0 = A_1 \frac{\partial}{\partial w_2} R(0) + V$$
$$0 = A_2 \frac{\partial}{\partial w_1} R(0) \times \frac{\partial}{\partial w_2} P(0) + A_2 \frac{\partial}{\partial w_2} R(0)$$

and for $m \geq 2$

$$0 = A_1 R_m(0) \times \Omega + B_1 Q_m(0) \times \Omega$$
$$0 = A_2 R_m(0) \times \bigotimes_m P^*(0) \times \Omega + A_2 R_1(0) V_0 \times P_m(0) \times \Omega + \sum_{i=2}^{m-1} A_2 R_i(0) \times \Delta_i^m(0) \times \Omega$$

Observe that Ω is an orthogonal matrix. Let $\tilde{\Omega}$ is an orthogonal matrix, such that $[\Omega, \tilde{\Omega}]$ is an invertible matrix. Then $[\Omega, \tilde{\Omega}]^{-1} = \mathrm{col}(\Omega', \tilde{\Omega}')$. We have $R_m(0) = R_m(0)[\Omega, \tilde{\Omega}]\,\mathrm{col}(\Omega', \tilde{\Omega}') = R_m(0)\Omega\Omega' + R_m(0)\tilde{\Omega}\tilde{\Omega}'$. The matrix $R_m(0)\tilde{\Omega}$ contains derivatives other than selected by the matrix Ω, i.e. deterministic terms found in previous step and derivatives with respect to σ, which by the proposition 6.1 can be ignored. Let $\bar{R}$ denotes matrix of the same dimension as $R_m(0)$ containing all found derivatives and with other derivatives set to zero. Then

$$0 = A_1 X + B_1 Q_m(0) \times \Omega$$
$$0 = A_2 X \times \Omega' \bigotimes_m P^*(0)\Omega + A_2 R_1(0) V_0 \times Y + A_2 \bar{R} \bigotimes_m P^*(0)\Omega + \sum_{i=2}^{m-1} A_2 R_i(0) \times \Delta_i^m(0)\Omega$$

where $X = R_m(0) \times \Omega$, $Y = P_m(0) \times \Omega$ are matrices to find.

Now we have found all derivatives of $R(w)$ and $P(w)$ with respect to w_1 and w_2.

7 Other terms

Suppose that we already know derivatives of $R(w)$ and $P(w)$ with respect to w_1, w_2, as well as with respect to w_3 up to q times, where $q \geq 0$. We are going to find derivatives of $R(w)$ and $P(w)$ with respect to w_3 $q+1$ times. Let us analyze set of equations (7). Let υ is a base sequence of length m containing elements 0 and 1. We have

$$T \equiv A_3 S_m(0) \times M_\upsilon \times \Big(\bigotimes_{n(\upsilon)} I_n \otimes E\{\bigotimes_{m(\upsilon)} V_1(\epsilon_{t+1})\} - \bigotimes_{n(\upsilon)} I_n \otimes \bigotimes_{m(\upsilon)} V_1(0) \Big)$$
$$= \sum_{i=1}^{m(\upsilon)} \sum_{|\mu|=m(\upsilon), m(\mu)=i} A_3 S_m(0) \times \bigotimes_{n(\upsilon)} V_0 \otimes \Pi_\mu^{m(\upsilon)} \times \bigotimes_{n(\upsilon)} I_n \otimes E\{\bigotimes_i \epsilon_t\}$$

Let υ is any basic sequence, such that $m(\upsilon) = q + 1$. Then for any $1 \le i \le q+1$ and sequence μ satisfying $|\mu| = q + 1$ and $m(\mu) = i$, we have $n(\mu) = q + 1 - i \le q$. Hence, by the proposition 6.1 expression T contains derivatives of $R(w)$ and $P(w)$ with respect to w_3 of order at most q and value of expression T is already known.

Let us consider set of equations (7) for sequences υ of length m satisfying $m(\upsilon) = q+1$. Let $\Omega_\upsilon^1 = M_\upsilon \times \bigotimes_{n(\upsilon)} I_n \otimes \bigotimes_{m(\upsilon)} V_1(0)$ and let $\Omega_\upsilon^2 = M_\upsilon \times \Big(\bigotimes_{n(\upsilon)} I_n \otimes E\{\bigotimes_{m(\upsilon)} V_1(\epsilon_{t+1})\} - \bigotimes_{n(\upsilon)} I_n \otimes \bigotimes_{m(\upsilon)} V_1(0) \Big)$. Let Ω^1 and Ω^2 are matrices obtained by horizontally concatenating respectively matrices Ω_υ^1, Ω_υ^2. Then we obtain equation

$$
\begin{aligned}
0 &= A_1 R_m(0)\Omega^1 + (A_2 + A_3) R_m(0) \times \bigotimes_m P^*(0)\Omega^1 + (A_2 + A_3) R_1(0) V_1 \times P_m(0)\Omega^1 \\
&\quad + B_1 Q_m(0) \times \Omega^1 + A_3 S_m(0) \times \Omega^2 + \sum_{i=2}^{m-1} (A_2 + A_3) R_i(0) \times \Delta_i^m(0) \times \Omega^1
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
0 &= A_1 X + (A_2 + A_3) X \times (\Omega^1)' \bigotimes_m P^*(0)\Omega^1 + (A_2 + A_3) R_1(0) V_1 \times Y \\
&\quad + B_1 Q_m(0)\Omega^1 + A_3 S_m(0)\Omega^2 + \sum_{i=2}^{m-1} (A_2 + A_3) R_i(0) \times \Delta_i^m(0)\Omega^1 + (A_2 + A_3)\bar{R} \bigotimes_m P^*(0)\Omega^1
\end{aligned}
$$

where $X = R_m(0)\Omega^1$, $Y = P_m(0)\Omega^1$ and other matrices are known. The matrix $\bar{R}$ denotes matrix of the same size as $R_m(0)$ containing all found derivatives in previous steps and with other derivatives set to zero. This equation determines derivatives of $R(w)$ and $P(w)$ with respect to w_1 and w_3.

Let consider again the set of equations (9). We need now to find derivatives of $R(w)$ and $P(w)$ with respect to w_2 and w_3. In this state we do not know derivatives of $R(w)$ and $P(w)$ with respect to w_3 of order at least $q+2$. Using again proposition 6.1 we choose selecting matrix as follow. Let $\Omega_{\upsilon,\mu}$ is defined as

$$
\Omega_{\upsilon,\mu} = M_\upsilon \times \bigotimes_{n(\upsilon)} I_n \otimes \Pi_\mu^{q+1}
$$

where υ is any sequence of length m, $m(\upsilon) \ge 1$, and μ is any sequence of length $m(\upsilon)$, satisfying $n(\mu) = q + 1$. Then $A_3 S_m(0) \times \Omega_{\upsilon,\mu}$ contains derivatives of $R(w)$ and $P(w)$ with respect to w_3 of order at most $q + 1$. Let Ω is a matrix obtained by horizontally concatenating matrices $\Omega_{\upsilon,\mu}$. In this way we obtain equations

$$
\begin{aligned}
0 &= A_1 R_m(0) \times \Omega + B_1 Q_m(0) \times \Omega \\
0 &= A_2 R_m(0) \times \bigotimes_m P^*(0) \times \Omega + A_2 R_1(0) V_0 \times P_m(0) \times \Omega + \sum_{i=2}^{m-1} A_2 R_i(0) \times \Delta_i^m(0) \times \Omega
\end{aligned}
$$

observe that the matrix Ω is orthogonal, thus these equations are equivalent to

$$
\begin{aligned}
0 &= A_1 X + B_1 Q_m(0) \times \Omega \\
0 &= A_2 X \times \Omega' \bigotimes_m P^*(0) \times \Omega + A_2 R_1(0) V_0 \times Y + \sum_{i=2}^{m-1} A_2 R_i(0) \times \Delta_i^m(0) \times \Omega \\
&\quad + A_2 \bar{R} \times \bigotimes_m P^*(0) \times \Omega
\end{aligned}
$$

where $X = R_m(0) \times \Omega$, $Y = P_m(0) \times \Omega$, $\bar{R}$ is a matrix of the same size as $R_m(0)$ containing all found derivatives in previous steps and with other derivatives set to zero.

Having found matrices X, Y, the $q+1$-th step is finished. Repeating this procedure m times we obtain all derivatives of $R(w)$ and $P(w)$ with respect to w_3, hence all elements of matrices $R_m(0)$ and $P_m(0)$.

8 Conclusions

We have presented a method of finding asymptotic expansion of solution to stochastic rational expectations models up to any order. We have obtained relatively simple representation of matrix equations determining solution, which allows for simple implementations.

The method presented in this paper has one important drawback. We base on second order representation of nonlinear model, in which all terms of order higher than 2 are represented as second order terms, for example $x \otimes x \otimes x$ is represented as $y \otimes x$, where $y = x \otimes x$. This simplifies equations but expands dimension of vector of endogenous variables, hence computational cost. In the worst case dimension of vector of endogenous variables may growth exponentially with order of perturbation. However in standard DSGE models this is not a serious problem, since generally higher order terms depends only on few variables. Presented method can be applied to less restrictive representations but with cost of more complicated derivation of matrix equations.

References

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A Proofs of propositions

A.1 Proof of proposition 4.1

Proposition. *For any $p \geq 1$ and any differentiable function $F : R^{n+k+1} \to R^{m_2 \times n_2}$, where m_2, n_2 are any positive integers*

$$\frac{\partial^p}{\partial q_t^p}(F(v) \circ f_\epsilon(q_t)) = \Big[\frac{\partial^p}{\partial v^p} Q(v)\Big] \circ f_\epsilon(q_t) \times \bigotimes_p M(\epsilon)$$

Proof. For $p = 1$ we have

$$\frac{\partial}{\partial q_t}(F(v) \circ f_\epsilon(q_t)) = \Big[\frac{\partial}{\partial v} F(v)\Big] \circ f_\epsilon(q_t) \times (\frac{\partial}{\partial q_t} f_\epsilon(q_t)) = \Big[\frac{\partial}{\partial v} F(v)\Big] \circ f_\epsilon(q_t) \times M(\epsilon)$$

Let the proposition holds for any $1 \leq p \leq m$, where $m \geq 1$. Then

$$\begin{aligned}
\frac{\partial^{m+1}}{\partial q_t^{m+1}}(F(v) \circ f_\epsilon(q_t)) &= \frac{\partial}{\partial q_t}\Big(\frac{\partial^m}{\partial v^m} F(v) \circ f_\epsilon(q_t) \times \bigotimes_m M(\epsilon)\Big) \\
&= \frac{\partial}{\partial q_t}\Big(\frac{\partial^m}{\partial v^m} F(v) \circ f_\epsilon(q_t)\Big) \times (I_{n+k+1} \otimes \bigotimes_m M(\epsilon)) \\
&= \Big(\frac{\partial^m}{\partial v^{m+1}} F(v) \circ f_\epsilon(q_t)\Big) \times \Big(\frac{\partial}{\partial q_t} f_\epsilon(q_t) \otimes I_{(n+k+1)^m}\Big) \times (I_{n+k+1} \otimes \bigotimes_m M(\epsilon)) \\
&= \Big(\frac{\partial^m}{\partial v^{m+1}} F(v) \circ f_\epsilon(q_t)\Big) \times \bigotimes_{m+1} M(\epsilon)
\end{aligned}$$

□

A.2 Proof of proposition 4.3

Proposition. *For any $p \geq 1$*

$$\frac{\partial^p}{\partial q_t^p} Q(u_t, \sigma\epsilon_t, \sigma) = [\frac{\partial^p}{\partial v^p} Q_{0,0}(v)] \circ f_{\epsilon_t}(q_t) \times \bigotimes_p M(\epsilon_t)$$

$$\frac{\partial^p}{\partial v^p} Q_{0,0}(v) = \sum_{i=0}^{p} Q_{i,p-i}(v) \times \Gamma_i^p$$

where $\Gamma_0^1 = \Gamma_1^1 = I_{n+k+1}$, for any $p \geq 1$, $\Gamma_{-1}^p = \Gamma_{p+1}^p = 0$, and for any $m \geq 1$, $0 \leq p \leq m+1$, $q \geq 0$

$$\begin{aligned}
\Gamma_p^{m+1} &= I_{n+k+1} \otimes \Gamma_{p-1}^m + \Lambda_{p,m-p} \times (I_{n+k+1} \otimes \Gamma_p^m) \\
\Lambda_q^p &= I_{(n+1+k)^p} \otimes_{n+1+k}^1 I_{(n+1+k)^{q+1}}
\end{aligned}$$

and $\Lambda_{-1}^p = 0$.

Proof. The first equation results from proposition 4.1. For $p = 1$ we have

$$\frac{\partial}{\partial v} Q_{0,0}(v) = R_1(v) \otimes R(v) + R(v) \otimes_{n+k+1}^1 R_1(v) = Q_{1,0}(v) + Q_{0,1}(v) = \sum_{p=0}^{1} Q_{p,1-p}(v) \times \Gamma_p^1$$

Let the proposition holds for any $1 \leq p \leq m$, where $m \geq 1$. For any $i, j \geq 0$

$$\begin{aligned}
\frac{\partial}{\partial v} Q_{i,j}(v) &= R_{i+1}(v) \otimes R_j(v) + R_i(v) \otimes_{n+k+1}^1 R_{j+1}(v) \\
&= Q_{i+1,j} + Q_{i,j+1} \times I_{(n+1+k)^i} \otimes_{n+1+k}^1 I_{(n+1+k)^{j+1}} = Q_{i+1,j} + Q_{i,j+1} \times \Lambda_{i,j}
\end{aligned}$$

Then

$$\frac{\partial^{m+1}}{\partial \upsilon^{m+1}} Q_{0,0}(\upsilon) = \sum_{i=0}^{m} \frac{\partial}{\partial q_t} Q_{i,m-i}(\upsilon) \times (I_{n+k+1} \otimes \Gamma_i^m)$$
$$= \sum_{i=0}^{m} \Big(Q_{i+1,m-i} + Q_{i,m-i+1} \times \Lambda_{i,m-i} \Big) \times (I_{n+k+1} \otimes \Gamma_i^m)$$
$$= \sum_{i=0}^{m+1} Q_{i,m+1-i} \times (I_{n+k+1} \otimes \Gamma_{i-1}^m) + \sum_{i=0}^{m+1} Q_{i,m+1-i} \times \Lambda_{i,m-i} \times (I_{n+k+1} \otimes \Gamma_i^m)$$
$$= \sum_{i=0}^{m+1} Q_{i,m+1-i} \times \Gamma_i^{m+1}$$

□

A.3 Proof of proposition 4.6

Proposition. *For any sequence* υ *of length* m

$$\bigotimes_m M(\epsilon_t) \times J_\upsilon = M_\upsilon \times \bigotimes_{n(\upsilon)} I_n \otimes \bigotimes_{m(\upsilon)} V_1(\epsilon_t)$$

with $M_{[]} = 1$ *and*

$$M_{[0,\upsilon]} = V_0 \otimes M_\upsilon \qquad\qquad M_{[1,\upsilon]} = I_{n+k+1} \otimes^1_{q(\upsilon)} M_\upsilon$$

where $q(\upsilon) = n^{n(\upsilon)} \times (n+k+1)^{m(\upsilon)}$*,* $V_0 = \mathrm{col}(I_n, 0_{k+1,n})$*, and* $V_1(\epsilon_t) = \mathrm{col}(0_{n,1}, \epsilon_t, 1)$*.*

Proof. We have

$$\bigotimes_m M(\epsilon_t) \times J_\upsilon = (M(\epsilon_t) \times J_{\upsilon_1}) \otimes \cdots \otimes (M(\epsilon_t) \otimes J_{\upsilon_k})$$

Let $m = 1$. Then $M(\epsilon_t)J_0 = V_0 \times I_n$ and $M(\epsilon_t)J_1 = I_{n+k+1} \times V_1(\epsilon_t)$. Hence proposition holds for $m = 1$. Let the proposition holds for any $p \leq m$, where $m \geq 1$. Then for $m+1$ and any $\upsilon' = [\upsilon_1, \upsilon]$ of length $m+1$

$$X \equiv \bigotimes_{m+1} M(\epsilon_t) \times J_{\upsilon'} = M(\epsilon_t) J_{\upsilon_1} \otimes (\bigotimes_m M(\epsilon_t) \times J_\upsilon)$$
$$= M(\epsilon_t) J_{\upsilon_1} \otimes (M_\upsilon \times \bigotimes_{n(\upsilon)} I_n \otimes \bigotimes_{m(\upsilon)} V_1(\epsilon_t))$$

If $\upsilon_1 = 0$, then

$$X = V_0 \otimes (M_\upsilon \times \bigotimes_{n(\upsilon)} I_n \otimes \bigotimes_{m(\upsilon)} V_1(\epsilon_t)) = (V_0 \otimes M_\upsilon) \times \bigotimes_{n(\upsilon)+1} I_n \otimes \bigotimes_{m(\upsilon)} V_1(\epsilon_t))$$
$$= M_{[0,\upsilon]} \times \bigotimes_{n(\upsilon')} I_n \otimes \bigotimes_{m(\upsilon)'} V_1(\epsilon_t))$$

If $v_1 = 1$, then

$$X = I_{n+k+1}V_1(\epsilon_t) \otimes (M_v \times \bigotimes_{n(v)} I_n \otimes \bigotimes_{m(v)} V_1(\epsilon_t))$$
$$= (I_{n+k+1} \otimes M_v) \times (V_1(\epsilon_t) \otimes \bigotimes_{n(v)} I_n \otimes \bigotimes_{m(v)} V_1(\epsilon_t))$$
$$= (I_{n+k+1} \otimes M_v) \times (I_{n+k+1} \otimes^1_{q(v)} I_{q(v)}) \times (\bigotimes_{m(v)} I_n \otimes \bigotimes_{n(v)+1} V_1(\epsilon_t))$$
$$= (I_{n+k+1} \otimes^1_{q(v)} M_v) \times (\bigotimes_{n(v')} I_n \otimes \bigotimes_{m(v')} V_1(\epsilon_t)) = M_{[1,v]} \times (\bigotimes_{n(v')} I_n \otimes \bigotimes_{m(v')} V_1(\epsilon_t))$$

□

A.4 Proof of proposition 4.7

Proposition. *Let* $N_0 = \mathrm{col}(0_{n+k,1}, 1)$, $N_1 = \mathrm{col}(0_{n,k}, I_k, 0_{1,k})$. *Then*

$$\bigotimes_p V_1(\epsilon_t) = \sum_{i=0}^{p} \sum_{|v|=p, m(v)=i} \Pi^p_v \times \bigotimes_i \epsilon_t$$

where v *is a sequence containing elements* 0 *and* 1, $\Pi^1_{[0]} = N_0$, $\Pi^1_{[1]} = N_1$, *and for any sequence* v *of length* p

$$\Pi^{p+1}_{[0,v]} = N_0 \otimes \Pi^p_v \qquad \Pi^{p+1}_{[1,v]} = N_1 \otimes \Pi^p_v$$

Proof. We have $V_1(\epsilon_t) = N_0 + N_1\epsilon = \Pi^1_{[0]} \times 1 + \Pi^1_{[1]} \times \epsilon_t$, thus proposition holds for $p = 1$. Let the proposition holds for any $p \geq 1$. Then

$$\bigotimes_{p+1} V_1(\epsilon_t) = (N_0 + N_1\epsilon) \otimes (\sum_{i=0}^{p} \sum_{|v|=p,m(v)=i} \Pi^p_v \times \bigotimes_i \epsilon_t)$$
$$= \sum_{i=0}^{p} \sum_{|v|=p,m(v)=i} (N_0 \otimes \Pi^p_v) \times \bigotimes_i \epsilon_t + \sum_{i=0}^{p} \sum_{|v|=p,m(v)=i} (N_1 \otimes \Pi^p_v) \times \bigotimes_{i+1} \epsilon_t$$
$$= \sum_{|v|=p,m(v)=0} \Pi^{p+1}_v \times \bigotimes_0 \epsilon_t + \sum_{|v|=p,m(v)=p+1} \Pi^{p+1}_v \times \bigotimes_{p+1} \epsilon_t$$
$$+ \sum_{i=1}^{p} \Big(\sum_{|v|=p,m(v)=i} \Pi^{p+1}_{[0,v]} + \sum_{|v|=p,m(v)=i-1} \Pi^{p+1}_{[1,v]} \Big) \times \bigotimes_i \epsilon_t$$
$$= \sum_{i=0}^{p+1} \sum_{|v|=p+1,m(v)=i} \Pi^{p+1}_v \times \bigotimes_i \epsilon_t$$

□

A.5 Proof of proposition 5.1

Proposition. *For any basic sequence* v *of length* m

$$R_m(0) \times \bigotimes_m P^*(0) \times M_v = \frac{\partial^{n(v)}}{\partial v_1^{n(v)}} R_{m(v)}(0) \times \bigotimes_{n(v)} (\partial/\partial v_1 P(0)) \otimes \bigotimes_{m(v)} P^*(0)$$

Proof. Let η denotes any sequence of length m containing elements 0, 1, 2, where element 0 denotes variable w_1, element 1 denotes variable w_2 and element 2 denotes variable w_3. Then $R_m(0) = \sum_{|\eta|=m} R_\eta(0) \times J'_\eta$. Observe that $\bigotimes_m P^*(0) \times M_\upsilon = \bigotimes_{n(\upsilon)} P^*(0)V_0 \otimes \bigotimes_{m(\upsilon)} P^*(0)$ and $P^*(0)V_0 = \mathrm{col}(\partial/\partial v_1 P(0), 0_{k,n}, 0_{1,n})$. Let $\eta = [i_1, \ldots, i_{n(\upsilon)}, \eta_2]$. We have $J_\eta = J^{i_1} \otimes \cdots \otimes J^{i_{n(\upsilon)}} \otimes J_{\eta_2}$. Hence $J'_\eta \times \bigotimes_m P^*(0) \times M_\upsilon = (J^{i_1})'P^*(0)V_0 \otimes \cdots \otimes (J^{i_{n(\upsilon)}})'P^*(0)V_0 \otimes (J'_{\eta_2} \times \bigotimes_{m(\upsilon)} P^*(0))$. Observe that $(J^{i_k})'P^*(0)V_0 \neq 0$ iff $i_k = 0$. Hence $J'_\eta \times \bigotimes_m P^*(0) \times M_\upsilon \neq 0$ iff $i_1 = \cdots = i_{n(\upsilon)} = 0$ and

$$
\begin{aligned}
R_m(0) \times \bigotimes_m P^*(0) \times M_\upsilon &= \sum_{|\eta|=m} R_\eta(0) \times J'_\eta \times \bigotimes_{n(\upsilon)} P^* V_0 \otimes \bigotimes_{m(\upsilon)} P^*(0) \\
&= \sum_{|\eta_2|=m(\upsilon)} \frac{\partial^{n(\upsilon)}}{\partial v_1^{n(\upsilon)}} R_{\eta_2}(0) \times \bigotimes_{n(\upsilon)} (\partial/\partial v_1 P(0)) \otimes (J'_{\eta_2} \times \bigotimes_{m(\upsilon)} P^*(0)) \\
&= \frac{\partial^{n(\upsilon)}}{\partial v_1^{n(\upsilon)}} (\sum_{|\eta_2|=m(\upsilon)} R_{\eta_2}(0) J'_{\eta_2}) \times \bigotimes_{n(\upsilon)} (\partial/\partial v_1 P(0)) \otimes (\bigotimes_{m(\upsilon)} P^*(0)) \\
&= \frac{\partial^{n(\upsilon)}}{\partial v_1^{n(\upsilon)}} R_{m(\upsilon)}(0) \times \bigotimes_{n(\upsilon)} (\partial/\partial v_1 P(0)) \otimes (\bigotimes_{m(\upsilon)} P^*(0))
\end{aligned}
$$

□

A.6 Proof of proposition 6.1

Proposition. *Consider expression*

$$
Q = S_m(0) \times (\bigotimes_{n(\upsilon)} V_0 \otimes \Pi_\mu^{m(\upsilon)})
$$

where υ is a basic sequence of length m and μ is a sequence of length $m(\upsilon)$. If sequence μ contains at most p elements 0, then expression Q does not depend on derivatives of $R_\eta(0)$, $P_\eta(0)$ for sequences η of length m containing more than p elements 2.

Proof. We have

$$
\begin{aligned}
T &= R_m(0) \times \bigotimes_m P^*(0) \times \bigotimes_{n(\upsilon)} V_0 \otimes \Pi_\mu^{m(\upsilon)} \\
&= \frac{\partial^{n(\upsilon)}}{\partial v_1^{n(\upsilon)}} R_{m(\upsilon)}(0) \times \bigotimes_{n(\upsilon)} (\partial/\partial v_1 P(0)) \otimes (\bigotimes_{m(\upsilon)} P^*(0) \times \Pi_\mu^{m(\upsilon)}) \\
&= \sum_{|\eta_2|=m(\upsilon)} \frac{\partial^{n(\upsilon)}}{\partial v_1^{n(\upsilon)}} R_{\eta_2}(0) \times \bigotimes_{n(\upsilon)} (\partial/\partial v_1 P(0)) \otimes (J'_{\eta_2} \times \bigotimes_{m(\upsilon)} P^*(0) \times \Pi_\mu^{m(\upsilon)})
\end{aligned}
$$

where η_2 is any sequence containing elements $0, 1, 2$. If $m(\upsilon) = 0$, then T does not depend on derivatives of $R(w)$ with respect to w_3. Let $m(\upsilon) > 0$. Let $\mu = [\mu_1, \ldots, \mu_{m(\upsilon)}]$, $\eta_2 = [\eta_1, \ldots, \eta_{m(\upsilon)}]$. Then $T = \sum_{|\eta_2|=m(\upsilon)} T_{\eta_2}$, where

$$
T_{\eta_2} = \frac{\partial^{n(\upsilon)}}{\partial v_1^{n(\upsilon)}} R_{\eta_2}(0) \times \bigotimes_{n(\upsilon)} (\partial/\partial v_1 P(0)) \otimes Q_1 \otimes \cdots \otimes Q_{\mu_{m(\upsilon)}}
$$

where $Q_i = (J^{\eta_i})'P^*(0)N_{\mu_i}$. Suppose that $\mu_i = 0$. Then $(J^2)'P^*(0)N_0 = 1$ and T may depend on derivatives of $R(w)$ with respect to w_3. We have also $(J^2)'P^*(0)N_1 = 0$. Suppose that there are at most p elements of μ equal 0 but at least $p+1$ elements of η_2 equal 2, such that T_{η_2} does not vanish. Then there must exist index i, such that $\eta_i = 2$ and $\mu_i = 0$.

But then $Q_i = 0$, and $T_{\eta_2} = 0$. Hence and T does not depend on derivatives of $R(w)$ with respect to w_3 of order higher than p.

Observe that $V_0 = J^0$ is a matrix selecting vector w_1 from vector $w = \mathrm{col}(w_1, w_2, w_3)$ and $N_1 = J^1$ is a matrix selecting vector w_2 from w. Hence $\bigotimes_{n(v)} \otimes \Pi_\mu^{m(v)} = J_\eta$, where $\eta = [0, \ldots, 0, \eta_1, \ldots, \eta_{m(v)}]$, where $\eta_i = 2$ if $\mu_i = 0$, and $\eta_i = 1$ if $\mu_i = 1$, and $P_m(0) \bigotimes_{n(v)} \otimes \Pi_\mu^{m(v)} = P_m(0) \times J_m$ does not contain derivatives with respect to w_3 of order higher than p. Rest of terms of $S_m(0)$ are constant matrices. $\square$