INFORMATION : PRICE AND IMPACT ON GENERAL WELFARE AND OPTIMAL INVESTMENT. AN ANTICIPATIVE STOCHASTIC DIFFERENTIAL GAME MODEL.

Christian-Oliver Ewald and Yajun Xiao

University of St.Andrews, School of Economics and Finance

2007

University of St.Andrews Economics Preprints (8 May 2007)

INFORMATION : PRICE AND IMPACT ON GENERAL WELFARE AND OPTIMAL INVESTMENT. AN ANTICIPATIVE STOCHASTIC DIFFERENTIAL GAME MODEL.

CHRISTIAN-OLIVER EWALD,* *Department of Economics, University of St.Andrews*
YAJUN XIAO,** *Department of Economics , University of Frankfurt*

Abstract

We consider a continuous time market model, in which agents influence asset prices. The agents are assumed to be rational and maximizing expected utility from terminal wealth. They share the same utility function but are allowed to possess different levels of information. Technically our model represents a stochastic differential game with anticipative strategy sets. We derive necessary and sufficient criteria for the existence of Nash-equilibria and characterize them for various levels of information asymmetry. Furthermore we study in how far the asymmetry in the level of information influences Nash-equilibria and general welfare. We show that under certain conditions in a competitive environment an increased level of information may in fact lower the level of general welfare. This effect can not be observed in representative agent based models, where information always increases welfare. Finally we extend our model in a way, that we add prior stages, in which agents are allowed to buy and sell information from each other, before engaging in trading with the market assets. We determine equilibrium prices for particular pieces of information in this setup.

Keywords: information, financial markets, stochastic differential games
JEL Subject Classification: Primary C730,G110,G140

* Postal address: Department of Economics, University of St.Andrews, St Salvator's College, St Andrews, Fife KY16 9AL, Scotland, UK (e-mail : ce16@st-andrews.ac.uk)
** Postal address: Department of Economics, University of Frankfurt, Mertonstr. 17-21, 60054 Frankfurt/Main Germany, (e-mail: xiao@finance.uni-frankfurt.de)

1. Introduction

In the classical Black-Scholes model and in fact in most other continuous time financial market models it is assumed that agents behavior does not influence asset-prices and all agents possess the same level of information. These models have been very successfully applied to classical questions such as the pricing of options and derivatives as well as optimal asset allocation. However these models are not suitable to explain how the level of information influences the general performance of the stock market or how and for what price information may be exchanged between individual agents on the market. There is no doubt about that in real world markets agents possess different levels of information and that it is important to understand what value particular pieces of information have and how general welfare is affected by these. In the first part of this article, we set up a continuous time market model in which agents are assumed to influence asset-prices and are exposed to different information flows. The framework is the one of a stochastic differential game with anticipative strategy sets. We derive necessary and sufficient conditions for the existence of Nash-equilibria for this game and characterize these for various levels of information asymmetry. Furthermore we study the consequences, an increased level of information has on general welfare. Information asymmetry is not a contradiction to the efficient market hypotheses, as the agents may indeed learn all the necessary information by carefully studying the market, if they invest enough effort to do so. However the emphasize here lies on effort. Different agents invest different amounts of effort or capital in order to obtain information, that may enable them to trade more successfully. Information is costly, and in the second part of this article we study the aspect of pricing information within a competetive market. To illustrate this, let us consider the following three scenarios. In scenario 1 a private investors may start buying stocks or funds without knowing anything about the market and not intending to learn anything about it, simply because he has read an advertisement in his local bank. In scenario 2 a private investor is strongly engaged in monitoring his individual portfolio, buying financial newspapers, investing time to watch business channels etc. Finally, in scenario 3, a big company who represents a significant market player may invest large sums in hiring a consulting company, which essentially provides it with key information on the market. All three scenarios present different aspects under which information may be traded between market players. In the first scenario the exchange of information is costless for the investor while the bank bears costs due to the advertisement campaign. Obviously there is a matter of trust here, but we leave this issue aside. In scenario 2 the investor invests money and time to obtain information on the market, essentially to trade more successfully and outperform other market participants. The exchange of information in this case is costly for the private investor, while the seller of information, i.e. financial press, media, but also financial institutions, which often provides media with important information on their business strategies, gain from this sale. The situation in the third scenario is very similar as in the second one, as long as the investor is concerned. Here again, the investor invests to obtain information from the consulting company, the exchange of information is costly. There is a significant difference in this scenario however, and this reflects the point of view of the seller of information. While in the second example the private investor is assumed to be a price taker and has no influence on market prices, a major company which owns large portfolios represents a trader, which has market impact and may influence market prices due to the mechanisms of supply and demand. The consulting company, which is also engaged on the market, must bear in mind the consequences that giving information to a large trader may change prices and therefore affect the value of their own portfolios, when considering whether or not and for which price to sell information. The first example more or less represents a problem in finding the right advertisement strategy for the bank and can be addressed in the general context of advertisement. A situation as described in scenario 2 has been addressed in continuous time diffusion-type market models, complete and incomplete, by various authors, for example Karatzas and Pikovsky (1996), Corcuera (2004), Imkeller (2003), Leon *et al.* (2003) and Ewald (2005). In this context the value of additional information is determined from the point of view of a representative uninformed agent. This agent would buy the information for any price P such that

$$u(x, \hat{\pi}^*) \leq u(x - P, \pi^*)$$

where π^* and $\hat{\pi}^*$ denote the optimal portfolios under additional resp. no additional information. Here u denotes expected utility from terminal wealth while x resp. $x - P$ denote the different levels of initial wealth. The owner of the information would sell for any price, as he does not fear for any consequences on the market. Any positive price offer for the information would give an incentive to sell, and in the presence of many possible information providers bring the sellers-price down to zero. In the third example, and this is the situation on which we focus in this article, the situations is more complex, as the seller of the information must take the buyers market impact into account. Selling information comes with the risk that the buyer may use the information in a way, that market prices change to the disadvantage of the seller. This situation has not been studied before in a continuous time diffusion-type financial market model. The framework we consider is most general. We study a market model in which two agents use information flows modeled by filtrations $\mathcal{G}^1$ and $\mathcal{G}^2$ in order to buy or sell assets, whose prices they may influence depending on their current position in the market. These filtrations are assumed to satisfy the usual conditions, see Karatzas-Schreve (1988) page 10. The technical framework of the second part of this article is an extension of the market game studied in the first part, including two initial stages in which information can be traded for monetary units. We solve for the Nash-equilibria of this game and in this way determine competitive prices for the pieces of information sold. In both cases, extended and original market game, agents face continuous time investment decisions. Trading strategies need to be integrated with respect to price processes in order to compute returns. In diffusion type models integration with respect to price processes is essentially the same as integration with respect to Brownian motion. The standard stochastic integral, which is the Itô-integral, does not allow the integrand to depend on more information than revealed by the Brownian motion itself. In our framework, where agents have asymmetric information, which may exceed the level of information revealed by the underlying Brownian motion, the Itô integral is to restrictive. In order to avoid these problems we use an anticipative stochastic calculus which has been developed in the last two decades. We use the technical framework based on the forward integral as found in Kohatsu-Higa and Sulem (2006). In order to provide analytically tractable examples we also make use of the classical technique of enlargement of filtration, developed originally by Jacod (1985) which is nowadays used throughout the literature.

The paper is organized as follows. We give a short introduction on anticipative stochastic calculus in section 2 while in section 3 we set up our market model and compute Nash-equilibrium strategies. In this section we also study the question, how these equilibria change with respect to changes in the information level and how general welfare is affected by this. In section 4 we extend the game with a pre-stage in which information may be exchanged in return for monetary units and determine equilibrium prices for the information. Section 5 contains the main conclusions from the paper.

2. A brief review of anticipative stochastic calculus

In this section we introduce some preliminaries about the anticipative stochastic calculus, which in fact is strongly related to what is called the Malliavin calculus. A standard reference for this is Nualart (1995). Let us consider the set $\mathcal{S}$ of cylindrical functionals $F : \Omega \to \mathbb{R}$, given by $F = f\left(\mathbb{W}(t_1), ..., \mathbb{W}(t_l)\right)$ where $f \in C_b^\infty\left((\mathbb{R}^n)^l\right)$ is a smooth function with bounded derivatives of all orders and $(\mathbb{W}(t))$ denotes an n-dimensional Brownian motion on Ω. We define the Malliavin derivative operator on $\mathcal{S}$ via

$$D_s F \;\; := \;\; \sum_{i=1}^{l} \frac{\partial f}{\partial x_i}\left(\mathbb{W}_{t_1}(\omega), ..., \mathbb{W}_{t_l}(\omega)\right) \cdot \mathbf{1}_{[0,t_i]}(s),$$

were $\frac{\partial f}{\partial x_i}$ denotes the gradient of f with respect to its i-th n-dimensional argument. This operator and the iterated operators D^k are closable and unbounded from $L^p(\Omega)$ into $L^p\left(\Omega \times [0,T]^k, \mathbb{R}^n\right)$, for all $k \geq 1$. Their respective domains are denoted by $\mathbb{D}^{k,p}$ and obtained as the closure of $\mathcal{S}$ with respect to the norms defined by $\|F\|_{k,p}^p = \|F\|_{L^p(\Omega)}^p + \sum_{j=1}^k \|D^j F\|_{L^p\left(\Omega \times [0,T]^j, \mathbb{R}^n\right)}^p$. The adjoint of the Malliavin derivative operator $D : \mathbb{D}^{1,2} \to L^2(\Omega \times [0,T], \mathbb{R}^n)$ is called the Skorohod integral and denoted with δ. This operator has the property that its domain contains the class $L^2_a(\Omega \times [0,T], \mathbb{R}^n)$ of square integrable adapted stochastic processes and its restriction to this class coincides with the Itô-integral. We will make use of the notation $\delta(u) = \int_0^T u_t d\mathbb{W}_t$. Malliavin derivative operator and Skorohod integral are related by the following integration by parts formula

$$\mathbb{E}\left(\delta(u)F\right) = \mathbb{E}\left(\int_0^T D_t F \cdot u(t) dt\right), \quad \text{for any } F \in \mathbb{D}^{1,2}. \tag{1}$$

The following proposition is used to calculate the logarithmic derivative, often called information drift in information theory. It will prove particularly useful in our examples in the next section. The result is well known in the case where the underlying process X is a Brownian motion. Even though this is precisely the case which we refer to in our application ,we include a more general result here, where X is assumed to be general time-homogeneous diffusion. This proves to be useful in the framework of stochastic volatility models, where additional information is determined by the level of volatility in the future, see for example Ewald (2005) .

Proposition 1. *Suppose that* $X = X(T_0), T_0 \geq T$ *where* X *solves the stochastic differential equation*

$$dX(t) = b(X(t))dt + \sigma(X(t))d\mathbb{W}(t).$$

where $\mathbb{W}(t)$ *is a* 1*-dim Brownian motion. We assume that the transition density* $p(t,u,x,y)$ *is two times continuously differentiable with respect to* x *and one* *time continuously differentiable with respect to* t.* *Then*

$$\tilde{\mathbb{W}}(t) = \mathbb{W}(t) - \int_0^t \sigma(X(u))\partial_x \log(p(u, T_0, X(u), X(T_0)))du$$

is a Brownian motion w.r.t. $\mathcal{G} = (\mathcal{G}_t)$ *with* $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(X(T_0))$ *for* $t \leq T$.

Proof. Let f be a smooth function and M be an $\mathcal{F}_s$-adapted random variable.

$$\begin{aligned}
\mathbb{E}((\mathbb{W}(t) - \mathbb{W}(s))Mf(X(T_0))) &= \mathbb{E}\left(\mathbb{E}((\mathbb{W}(t) - \mathbb{W}(s))Mf(X(T_0))|\mathcal{F}_t)\right) \\
&= \mathbb{E}\left((\mathbb{W}(t) - \mathbb{W}(s))M\int f(y)p(t, T_0, X(t), y)dy\right) \\
&= \mathbb{E}\left(\delta(1_{(s,t]}(u))M\int f(y)p(t, T_0, X(t), y)dy\right) \\
&= \mathbb{E}\left(\int_s^t D_u\left(M\int f(y)p(t, T_0, X(t), y)dy\right)du\right).
\end{aligned}$$

Because M is adapted to $\mathcal{F}_s$, we have$D_uM = 0$ for $u > s$. Applying first the product rule to to $M\int f(y)p(u, T_0, X(u), y)dy)$ and then Fubini's theorem we obtain

$$\begin{aligned}
\mathbb{E}((\mathbb{W}(t) - \mathbb{W}(s))Mf(X(T_0))) &= \mathbb{E}\left(\int_s^t M\int f(y)D_up(t, T_0, X(t), y)dydu\right) \\
&= \mathbb{E}\left(\int f(y)M\int_s^t D_up(t, T_0, X(t), y)dudy\right).
\end{aligned}$$

It follows from the Itô formula that

$$\begin{aligned}
D_up(t, T_0, X(t), y) &= D_u\Bigg\{p(s, X(s), y) + \int_s^t\Bigg[\frac{\partial p(v, T_0, X(v), y)}{\partial v} \\
&\quad + b(X(v))\frac{\partial p(v, T_0, X(v), y)}{\partial x} + \frac{1}{2}\sigma^2(X(v))\frac{\partial^2 p(v, T_0, X(v), y)}{\partial x^2}\Bigg]dv \\
&\quad + \int_s^t \sigma(X(v))\frac{\partial p(v, T_0, X(v), y)}{\partial x}d\mathbb{W}(v)dudy\Bigg\}.
\end{aligned}$$

* Condition for this in terms of the Malliavin derivative are given in Theorem 2.1.4 and Corollary 2.1.2 in Nualart (1995)

From the Kolmogorov backward equation we can conclude that the expression in the square brackets is zero. Furthermore $D_u p(s, X(s), y) = 0$ for the reason that $u \geq s$ implies that $p(s, X(s), y)$ is $\mathcal{F}_u$ adapted. We therefore obtain that $\mathbb{E}((\mathbb{W}(t) - \mathbb{W}(s))Mf(X(T_0)))$ is given by the expression

$$\mathbb{E}\left(\int f(y)M\int_s^t D_u\left[\int_s^t \sigma(X(v))\frac{\partial p(v,T_0,X(v),y)}{\partial x}d\mathbb{W}(v)\right]dudy\right)$$

Applying the Malliavin derivative operator on the expression in the square brackets leads according to Nualart (1995) Lemma 1.3.4 to

$$\begin{aligned}\mathbb{E}\left((\mathbb{W}(t) - \mathbb{W}(s))Mf(X(T_0))\right) &= \mathbb{E}\left(\int f(y)M\int_s^t\left(\sigma(X(u))\frac{\partial p(u,T_0,X(u),y)}{\partial x}\right.\right.\\ &\quad \left.\left.+\int_u^t D_u\left[\sigma(X(v))\frac{\partial p(v,T_0,X(v),y)}{\partial x}\right]d\mathbb{W}(v)\right)dudy\right)\end{aligned}$$

Using the Fubini theorem to interchange the order of integration and taking expectations inside the integral and furthermore realizing that the expectation of an Itô integral with respect to Brownian motion is always zero, we obtain that

$$\mathbb{E}\left((\mathbb{W}(t) - \mathbb{W}(s))Mf(X(T_0))\right) = \mathbb{E}\left(\int f(y)M\int_s^t\sigma(X(u))\frac{\partial p(u,T_0,X(u),y)}{\partial x}dudy\right)$$

Another application of Fubini's theorem and the fact that for a positive differentiable function $\alpha(x)$ we have $\frac{\partial\log(\alpha(x))}{\partial x}\cdot\alpha(x) = \frac{\partial\alpha(x)}{\partial x}$ leads us to

$$\begin{aligned}\mathbb{E}\left((\mathbb{W}(t) - \mathbb{W}(s))Mf(X(T_0))\right) &= \mathbb{E}\left(\int_s^t\left(\int f(y)M\sigma(X(u))\frac{\partial\log p(u,T_0,X(u),y)}{\partial x}\right.\right.\\ &\quad \times p(u,T_0,X(u),y)dy)\,du)\end{aligned}$$

By definition of the transition density function we conclude that

$$\mathbb{E}\left((\mathbb{W}(t) - \mathbb{W}(s))Mf(X(T_0))\right) = \mathbb{E}\left(f(X(T_0))M\int_s^t\sigma(X(u))\frac{\partial\log p(u,T_0,X(u),X(T_0))}{\partial x}du\right)$$

A density argument then establishes that

$$\mathbb{E}\left(\mathbb{W}(t)-\mathbb{W}(s)-\int_s^t \sigma(X(u))\frac{\partial \log p(u,T_0,X(u),X(T_0))}{\partial x}du\,\middle|\,\mathcal{G}_s\right)=0 \qquad (2)$$

Now, by definition of $\tilde{\mathbb{W}}(t)$ the last equality is equivalent to

$$\mathbb{E}(\tilde{\mathbb{W}}(t)-\tilde{\mathbb{W}}(s)|\mathcal{G}_s)=0$$

and $\left(\tilde{\mathbb{W}}(t)\right)_{[0,T]}$ is therefore a continuous martingale with respect to the filtration $\mathcal{G}$. Its quadratic variation is given by $<\tilde{\mathbb{W}}(t)>=t$ for $t\in[0,T]$. Hence, by Lévy's theorem we have that $\left(\tilde{\mathbb{W}}(t)\right)_{[0,T]}$ is a Brownian motion w.r.t. $\mathcal{G}_t$.

Example 1. Assume that $X_i(T_0)=\mathbb{W}_i(T_0)$, $i=1,\dots,n$ such that $\mathcal{G}_t=\mathcal{F}_t\vee\sigma(X_1(T_0))\vee\cdots\vee\sigma(X_n(T_0))$. We can then explicitly write down the transition probability density of $X_i(T_0)$ conditional on $\mathcal{F}_t$

$$p(t,T_0,X_i(t),y)=\frac{1}{\sqrt{2\pi(T_0-t)}}\exp\left(-\frac{(y-X_i(t))^2}{2(T_0-t)}\right).$$

Then

$$\sigma(X_i(u))\partial_x\log(p(u,T_0,X_i(u),X(T_0))=\frac{X_i(T_0)-X_i(u)}{T_0-u}=\frac{\mathbb{W}_i(T_0)-\mathbb{W}_i(u)}{T_0-u},$$

and $\tilde{\mathbb{W}}_i(t)=\mathbb{W}_i(t)-\int_0^t\frac{\mathbb{W}_i(T_0)-\mathbb{W}_i(u)}{T_0-u}du$ is a $\mathcal{G}$-Brownian motion, noticing $E(X_i(t)-X_i(s)|\mathcal{G}_s)=E(X_i(t)-X_i(s)|\mathcal{F}_s\vee\sigma(X_i(s)))$.

In the following section we will use the so called forward integral which allows us more flexibility in the choice of stochastic integrands. For details see for example Russo and Valois (1993).

Definition 1. Let $\varphi:[0,T]\times\Omega\to\mathbb{R}^n$ be a measurable process. The forward integral of φ with respect to $\mathbb{W}(t)$ is defined by

$$\int_0^T\varphi(t)\cdot d^-\mathbb{W}(t)=\lim_{\epsilon\to0}\int_0^T\varphi^\top(t)\cdot\frac{\mathbb{W}(t+\epsilon)-\mathbb{W}(t)}{\epsilon}dt, \qquad (3)$$

if the limit exists in probability.

The forward integral is related to the Skorohod integral in the following way. Suppose that φ is $\mathbb{R}^n$-valued as in Definition 1 with $\varphi \in \mathbb{D}^{1,2}$ satisfying

$$\mathbb{E}\left(\int_0^T |\varphi(t)|^2\right) dt + \mathbb{E}\left(\int_0^T \int_0^T \|D_s\varphi(t)\|^2\right) dsdt \leq \infty$$

where $\|\cdot\|$ denotes the euclidean matrix norm. Moreover, assume that $Tr(D_{t^+}\varphi(t))$ $:= \lim_{s \to t^+} Tr(D_s\varphi(t))$ exists in $L^2([0,T] \times \Omega)$. Then φ is forward integrable and

$$\int_0^T \varphi(t) \cdot d^-\mathbb{W}(t) = \delta(\varphi(t)) + \int_0^T Tr(D_{t^+}\varphi(t)dt. \qquad (4)$$

A proof of this result can be found for example Russo and Valois (1993) or Kohatsu-Higa and Sulem (2006). Taking into account that the expectation of a Skorohod integral always vanishes, we obtain

$$\mathbb{E}\left(\int_0^T \varphi(t) \cdot d^-\mathbb{W}(t)\right) = \mathbb{E}\left(\int_0^T Tr(D_{t^+}\varphi(t)dt\right). \qquad (5)$$

Furthermore it can be shown that if φ is forward integrable and c$\grave{a}$gl$\grave{a}$d (i.e. left continuous with left limits) and $\Delta := \{0 = t_0 < t_1 < \cdots < t_n = T\}$ is a sequence of partitions such that $\Delta_n := \sup_{i=0,\cdots,n-1} \{t_{i+1} - t_i\}$ goes to zero when $n \to \infty$, then

$$\int_0^T \varphi(t) \cdot d^-\mathbb{W}(t) = \lim_{\Delta_n \to 0} \sum_{i=0}^{n-1} \varphi^\top(t_i) \cdot (\mathbb{W}(t_{i+1}) - \mathbb{W}(t_i)) \qquad (6)$$

if the limit exists in probability. Taken the latter into account one can indeed argue that the forward integral is predestined to model financial markets in continuous time when allowing trading strategies to depend on a more general information structure. It also follows from the latter equation, that in case that $\mathbb{W}$ remains a semi-martingale when changing the filtration, then the forward integral coincides with the Itô-integral for semi-martingales.

3. Continuous time market games with heterogeneous information

We consider a market with a finite time horizon $[0, T]$ and agents which are heterogeneously informed. For simplicity we restrict the number of agents to two. Our analysis however can easily be modified to model the case of arbitrary many agents. Assets include one riskless asset, which we call bond and denote with $B(t)$, as well as n risky assets, which we think of stocks and denote with $S^i(t)$. The different levels of information are modeled by using four different filtrations consecutively throughout the remaining of this paper. These are $\mathcal{G}^1 = (\mathcal{G}^1_t)$ for agent number one, $\mathcal{G}^2 = (\mathcal{G}^2_t)$ for agent number two, $\mathcal{F} = (\mathcal{F}_t)$ the σ-algebra generated by the underlying noise process, which we assume to be a Brownian motion $\mathbb{W}(t)$, and finally the filtration $\mathcal{G} = (\mathcal{G}_t)$ for the coefficients of the underlying model. We assume $\mathcal{F}_t \subseteq \mathcal{G}^p_t \subseteq \mathcal{F}_T$ for $p = 1, 2$ and $\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{G}^1_t \cap \mathcal{G}^2_t$ for $t \in [0, T]$. The latter relationship guarantees that even though agents may have different level of information, they both understand how the market works and how other agents behavior affects the market. At the current moment we do not impose any further relationships between $\mathcal{G}^1$ and $\mathcal{G}^2$. The agents investments are described by their individual portfolio processes $\pi_p = \left(\pi^i_p(t)\right)_{1 \leq i \leq n}$, where $\pi^i_p(t)$ denotes the proportion of wealth of agent p which at time t is invested in stock $i = 1, \ldots k$, while the remaining portion $\pi^0_p(t)$ is assumed to be invested in the bond. We will later model the process of selling pieces of information from one agent to the other. In order that the selling agent is not indifferent to giving away information to the other agent for free, we need to assume that the agents behavior affects asset-prices. More precisely we assume the following dynamics for assets:

$$\begin{aligned} dB(t) &= r(t)B(t)dt, \;\; B(t) = 1, \\ dS(t) &= diag(S(t))\left\{\mu(t,\pi_1(t),\pi_2(t))dt + \sigma(t)d^-\mathbb{W}(t)\right\}, \;\; S(0) > 0, \end{aligned} \tag{7}$$

with $diag(S(t))$ the $n \times n$-matrix with $S^i(t)$ as diagonal elements and zeros elsewhere. We assume that the following conditions hold for the coefficients:

1. $\mu(t,x,y) = (\mu_i(t,x,y)_{1\leq i\leq n})$ is a $\mathcal{G}$-adapted process with values in $C(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $r(t)$ is a $\mathcal{G}$-adapted and real-valued stochastic process, $\sigma(t) = (\sigma_{ij}(t))_{1\leq i,j\leq n}$ is a $\mathcal{G}$-adapted and $\mathbb{R}^{n\times n}$-valued stochastic process.
2. $\int_0^T (|r(t)| + |\mu(t,x,y)| + \|\sigma(t)\sigma^\top(t)\|)dt < \infty$ a.s. for all $x, y \in \mathbb{R}^n$
3. $\sigma(t)$ is forward integrable and càglàd.

The chosen dynamics (7) incorporates a supply and demand feature, in which agents current positions influence the drift term of the asset-prices. A similar dynamic for the case of a representative agent has been used in Kohatsu-Higa and Sulem (2006). We denote with $X_p(t,\pi_1,\pi_2)$ the discounted wealth process corresponding to agent p given chosen investment strategies π_1 and π_2. The wealth processes also depend on the initial endowments of the agents, but for the moment we omit this from the notation. The stochastic differential equation governing the evolution of the wealth processes is given by

$$dX_p(t,\pi_1,\pi_2) = X_p(t,\pi_1,\pi_2)\left(\pi_p^\top(t)(\mu(t,\pi_1(t),\pi_2(t)) - r(t))dt + \pi_p^\top(t)\sigma(t)d^-\mathbb{W}(t)\right), \tag{8}$$

with initial condition $X_p(0) = x_p$. Note that this equation presents a stochastic differential equation with anticipating coefficients. Nevertheless, the Itô formula for the forward integral (see Russo and Vallois (2000) implies that 8 is satisfied by

$$\begin{aligned} X_p\left(t,\pi_1,\pi_2\right) = \quad & x_p \exp\left(\int\limits_0^T ((\pi_p^\top(s)\left(\mu(s,\pi_1(s),\pi_2(s)) - r(s)\right)\right. \\ & \left. -\tfrac{1}{2}\pi_p^\top(s)\sigma(s)\sigma^\top(s)\pi_p(s))ds + \int\limits_0^T \pi_p^\top(s)\sigma(s)d^-\mathbb{W}(s))\right), \end{aligned} \tag{9}$$

For technical reasons we have to impose certain restriction on our portfolio strategies which guarantee that the solution above is well defined.

Definition 2. We call a pair of portfolio strategies (π_1, π_2) admissible and write $(\pi_1, \pi_2) \in \mathcal{A}$ if the following conditions are satisfied:

1. $X_p(t, \pi_1, \pi_2) > 0$ for all $t \in [0, T]$,
2. $\pi_p(t)$ is c$\grave{a}$gl$\grave{a}$d and $\pi_p^\top(t)\sigma(t)$ is forward integrable. Moreover, $\int\limits_0^T \left\{|\pi_p^\top(t)\mu(t,\pi_1(t),\pi_2(t)) - r(t)| + |\pi_p^\top(t)\sigma(t)\sigma^\top(t)\pi_p(t)|\right\} dt < \infty$
3. For any bounded c$\grave{a}$gl$\grave{a}$d process $\tilde{\pi}$ s.t. $\tilde{\pi}^\top(t)\sigma(t)$ is forward integrable there exists $\gamma > 0$ such that the families $\{|M_1(T, \pi_1 + \epsilon\tilde{\pi}, \pi_2)|\}_{0\le\epsilon\le\gamma}$ and $\{|M_2(T, \pi_1, \pi_2 + \epsilon\tilde{\pi})|\}_{0\le\epsilon\le\gamma}$ are uniformly integrable where

$$\begin{aligned} M_p(t,\pi_1,\pi_2) := \quad & \mathbb{E}\left(\int\limits_0^t \left(\mu(s,\pi_1(s),\pi_2(s)) - r(s) + J_\mu^{\pi_p}(s)\pi_p(s)\right.\right. \\ & \left.\left. -\sigma(s)\sigma^\top(s)\pi_p(s)\right)ds + \int\limits_0^t \sigma(s)d^-\mathbb{W}(s)\,\middle|\, \mathcal{G}_t^p\right). \end{aligned} \tag{10}$$

Here $J_\mu^{\pi_p}(s)$ is the Jacobian matrix of μ with respect to π_p evaluated at time s.

We assume that our agents are risk averse and that their objective is to maximize expected utility from discounted terminal wealth. In order to obtain analytically tractable results we use logarithmic utility. Taking this into account, the payoffs for the agents in our market game are given by

$$u_p(\pi_1, \pi_2) := \mathbb{E}(\ln(X_p^{\pi_1,\pi_2}(T))) \tag{11}$$

for $p = 1, 2$. We note that the payoff's u_p also depend on the initial endowments x_1 and x_2 but omit this in our notation. In this setup the optimization objective for both agents is identical and therefore asymmetry effects concerning the level of risk averseness are left out in our discussion. The asymmetry arising in our model comes from the fact that the strategies of the individual players rely on different information and that they may effect the market in different ways. We consider the market to be in equilibrium if the strategy pair $(\pi_1^*, \pi_2^*) \in \mathcal{A}$ constitutes a Nash-equilibrium, i.e.

$$\begin{aligned} u_1(\pi_1^*, \pi_2^*) &= \sup_{\pi_1 \in \mathcal{A}_1(\pi_2^*)} \mathbb{E}\left(\ln(X_1^{\pi_1,\pi_2^*}(T))\right) \\ u_2(\pi_1^*, \pi_2^*) &= \sup_{\pi_2 \in \mathcal{A}_2(\pi_1^*)} \mathbb{E}\left(\ln(X_1^{\pi_1^*,\pi_2}(T))\right) \end{aligned}$$

with $A_1(\pi_2^*) = \{\pi_1 | (\pi_1, \pi_2^*) \in \mathcal{A}\}$ and $A_2(\pi_1^*) = \{\pi_2 | (\pi_1^*, \pi_2) \in \mathcal{A}\}$. The following theorem provides necessary and sufficient conditions on the existence of a Nash-equilibrium for the market game above in terms of a martingale condition.

Theorem 1. *Under the assumptions stated in the preceding paragraph we have that*

1. *if (π_1^*, π_2^*) constitutes a Nash-equilibrium for the market game, then $M_p(t, \pi_1^*, \pi_2^*)$ for $t \in [0, T]$ is a martingale with respect to the filtration $\mathcal{G}^p$ for $p = 1, 2$.*

2. *If $(\pi_1^*, \pi_2^*) \in \mathcal{A}$ and $M(t, \pi_1^*, \pi_2^*), t \in [0, T]$ is a martingale with respect to the filtration $\mathcal{G}^p$ and $u_p(\pi_1, \pi_2)$ is concave with respect to π_p for $p = 1, 2$ resp., then (π_1^*, π_2^*) constitutes a Nash-equilibrium of the market game.*

Proof. 1. If (π_1^*, π_2^*) constitutes a Nash-equilibrium for the market game, then for bounded θ_1 as in Definition 2 part 3, we have

$$u_1(\pi_1^*, \pi_2^*) \geq u_1(\pi_1^* + \epsilon\theta_1, \pi_2^*), \tag{12}$$

for all ϵ in an open neighborhood of 0. This implies that the partial directional derivative of u_1 along the direction θ_1 evaluated at π_1^* is zero, i.e.,

$$\begin{aligned} 0 &= \tfrac{d}{d\epsilon} u_1(\pi_1^* + \epsilon\theta_1, \pi_2^*)|_{\epsilon=0} \\ &= \mathbb{E}\left(\int_0^T \theta_1^\top(t)\left(\mu(t, \pi_1^*(t), \pi_2^*(t)) - r(t) + J_\mu^{\pi_1^*}(t)\pi_1(t) - \sigma(t)\sigma^\top(t)\pi_1^*(t)\right) dt \right. \\ &\quad \left. + \int_0^T \theta_1^\top(t)\sigma(s) d^-\mathbb{W}(s)\right). \end{aligned} \tag{13}$$

We notice that the differentiation and the integral can be interchanged because our admissibility definition implies that $\{|M_1(T, \pi_1^* + \epsilon\theta_1, \pi_2^*)|\}_{0 \leq \epsilon \leq \gamma}$ is uniformly integrable. Let us now consider the particular process $\theta_1(u) = \theta(t)\mathbf{1}_{(t,t+h]}(u)$, $h > 0, 0 \leq t \leq T$, where $\theta(t)$ is a bounded, $\mathbb{R}^n$-valued and $\mathcal{G}_t^1$-measurable random variable. Thus, (13) can be written as

$$\begin{aligned} 0 &= \mathbb{E}\left(\theta^\top(t)\left(\int_t^{t+h} (\mu(u, \pi_1^*(u), \pi_2^*(u)) - r(u) + J_\mu^{\pi_1^*}(u)\pi_1(u) \right.\right. \\ &\quad \left.\left. - \sigma(u)\sigma^\top(u)\pi_1(u))du + \int_t^{t+h} \sigma(u) d^-\mathbb{W}(u)\right)\right) \end{aligned} \tag{14}$$

Since (14) holds for all such θ we conclude

$$\begin{aligned} 0 &= \mathbb{E}\left(\int_t^{t+h} \left(\mu(u, \pi_1^*(u), \pi_2^*(u)) - r(u) + J_\mu^{\pi_1^*}(u)\pi_1(u) - \sigma(u)\sigma^\top(u)\pi_1(u)\right) du \right. \\ &\quad \left. + \int_t^{t+h} \sigma(u) d^-\mathbb{W}(u) \,\middle|\, \mathcal{G}_t^1\right) \end{aligned} \tag{15}$$

Using the definition of $M_p(t,\pi_1,\pi_2)$ we obtain:

$$\mathbb{E}\left(M_1(t+h,\pi_1^*,\pi_2^*)-M_1(t,\pi_1^*,\pi_2^*)|\,\mathcal{G}_t^1\right)=0 \tag{16}$$

An analogous argumentation using $u_2(\pi_1^*,\pi_2^*)\geq u_2(\pi_1^*,\pi_2^*+\epsilon\theta_2)$ establishes:

$$\mathbb{E}\left(M_2(t+h,\pi_1^*,\pi_2^*)-M_2(t,\pi_1^*,\pi_2^*)|\,\mathcal{G}_t^1\right)=0 \tag{17}$$

From (16) and (17), we infer that $M_p(t,\pi_1^*,\pi_2^*)$ is a $\mathcal{G}^p$-martingale for $p=1,2$.

2. Let us now assume that there exists a pair (π_1^*,π_2^*) such that $M_1(t,\pi_1^*,\pi_2^*)$ is a $\mathcal{G}^1$-martingale and $M_2(t,\pi_1,\pi_2)$ a $\mathcal{G}^2$-martingale. Therefore (16) and (17) hold simultaneously. Let us consider the optimization problem for agent 1. (16) implies that (15) holds, hence (14) holds for $\theta(t)$ bounded $\mathbb{R}^n$-valued and $\mathcal{G}^1$ measurable. Inductively we see that (13) holds for processes of the form

$$\tilde{\theta}_1(u)=\sum_{i=0}^{n-1}\theta_1(t_i)\mathbf{1}_{(t_i,t_{i+1}]}(u), 0=t_0<t_1\cdots<t_n=T,$$

with $\theta_1(t_i)$ bounded, $\mathbb{R}^n$-valued and $\mathcal{G}_{t_i}^1$-measurable random variables. Here we use the equality

$$\int_0^T\tilde{\theta}_1^\top(t)\sigma(t)d^-\mathbb{W}(t)=\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\theta_1^\top(t_i)\sigma(u)d^-\mathbb{W}(u),$$

We obtain that (13) is true for all simple processes $\tilde{\theta}_1(u)$ and a density argument establishes that (13) holds for all processes θ_1 as in Definition 2 part 3. This implies

$$\frac{d}{d\epsilon}u_1(\pi_1+\epsilon\theta_1,\pi_2)|_{\epsilon=0}=0. \tag{18}$$

On the other hand, using that $u_p(\pi_1,\pi_2)$ is concave in each π_p we obtain

$$\begin{aligned}
&\tfrac{1}{\epsilon}(u_1(\pi_1+\epsilon\theta_1,\pi_2)-u_1(\pi_1,\pi_2))\\
=\ &\tfrac{1}{\epsilon}(u_1((1-\epsilon)\tfrac{\pi_1}{1-\epsilon}+\epsilon\theta_1,\pi_2)-u_1(\pi_1,\pi_2))\\
\geq\ &\tfrac{1}{\epsilon}((1-\epsilon)u_1(\tfrac{\pi_1}{1-\epsilon},\pi_2)+\epsilon u_1(\theta_1,\pi_2)-u_1(\pi_1,\pi_2))\\
=\ &\tfrac{1}{\epsilon}(u_1(\tfrac{\pi_1}{1-\epsilon},\pi_2)-u_1(\pi_1,\pi_2))+u_1(\theta_1,\pi_2)-u_1(\tfrac{\pi_1}{1-\epsilon},\pi_2).
\end{aligned}$$

Taking the limit for $\epsilon\to 0$, and taking into account that $\lim_{\epsilon\to 0}\frac{1}{\epsilon}(u_1(\frac{\pi_1^*}{1-\epsilon},\pi_2^*)-u_1(\pi_1^*,\pi_2^*))=0$, as the latter is basically the directional derivative of u_1 along π_1^*, which by (18) must be zero, we obtain that $0\geq u_1(\theta_1,\pi_2^*)-u_1(\pi_1^*,\pi_2^*)$. As θ_1 can be chosen within the set $\mathcal{A}_1(\pi_2^*)$ we obtain by formally setting $\theta_1=\pi_1$ that

$$u_1(\pi_1^*,\pi_2^*)\geq u_1(\pi_1,\pi_2^*) \text{ for all } \pi_1\in\mathcal{A}_1(\pi_2^*). \tag{19}$$

Analogously we obtain

$$u_2(\pi_1^*,\pi_2^*)\geq u_2(\pi_1^*,\pi_2) \text{ for all } \pi_2\in\mathcal{A}_2(\pi_1^*). \tag{20}$$

This means (π_1^*,π_2^*) is a Nash-equilibrium for the market game.

In the following we discuss how we can use the criterion presented in Theorem 1 in order to identify Nash-equilibria for our market game with heterogeneous information.

Lemma 1. *Assume that (π_1^*,π_2^*) is a Nash-equilibrium for our market game. Then the process $\epsilon\mapsto\mathbb{E}\left(\int_t^{t+\epsilon}\sigma(u)d^-\mathbb{W}(u)|\mathcal{G}_t^p\right)$ has absolutely continuous paths for $p=1,2$ and the derivative*

$$I_p(t):=\frac{d}{d\epsilon}\mathbb{E}\left(\int_t^{t+\epsilon}\sigma(u)d^-\mathbb{W}(u)\,\middle|\,\mathcal{G}_t^p\right) \tag{21}$$

exist a.s. for $p = 1, 2$*. Furthermore the following equation holds for* $p = 1, 2$

$$\mathbb{E}\left(\mu(t, \pi_1^*(t), \pi_2^*(t)) - r(t) + J_\mu^{\pi_p^*}(t)\pi_p(t) - \sigma(t)\sigma^\top(t)\pi_p(t)\middle|\, \mathcal{G}_t^p\right) + I_p(t) = 0. \quad \textbf{(22)}$$

Under the concavity assumption for the utilities in Theorem 1 part 2 condition (22) *is a sufficient condition for a pair* $(\pi_1^*, \pi_2^*) \in \mathcal{A}$ *to constitute a Nash-equilibrium.*

Proof. These statements follow from equation (15) when dividing the latter by h and taking the limit for $h \to 0$, Theorem 1 and Definition 2.

For the choice of μ, various specifications appear to be reasonable. However to obtain a tractable dynamics and analytical results we focus on the linear form

$$\mu(t, \pi_1, \pi_2) = \mu(t) + a(t)\pi_1 + b(t)\pi_2$$

For the general case the latter should be considered as a first order approximation. In order to satisfy condition 1 on page 13 we need to assume that the processes $\mu(t)$,$a(t)$,$b(t)$ are $\mathcal{G}$-adapted. In order to apply the second part of Theorem 1 it is important to note that under our assumptions a sufficient criterion for concavity of $u_p(\pi_1, \pi_2)$ is the following:

Assumption 1. $\big(a(t) + a^\top(t)\big) - \sigma(t)\sigma^\top(t)$ *and* $\big(b(t) + b^\top(t)\big) - \sigma\sigma^\top(t)$ *take values in the set of negative definite matrices*

The latter assumption can be interpreted in a way that the influences of both agents's portfolio strategies on expected returns are embedded in the noise. Otherwise, the agents may drive the stock prices arbitrarily high just by buying and selling large volumes to obtain a high return. Obviously embedded does not mean without effect. We will assume from now on that Assumption 1 is satisfied. For notational reasons let us define the matrix valued function

$$\begin{array}{rcl}\Sigma : M^{n\times n} & \rightarrow & M^{n\times n} \\ y & \mapsto & \sigma\sigma^\top - 2y.\end{array}$$

Using this specification we obtain the following characterization of a Nash-equilibrium.

Proposition 2. *If the following system of equations*

$$\begin{array}{rcl}\pi_1^*(t) & = & \Sigma^{-1}(a(t))\left[\mu(t) - r(t) + I_1(t) + b(t)\mathbb{E}\left(\pi_2^*(t)|\,\mathcal{G}_t^1\right)\right] \\ \pi_2^*(t) & = & \Sigma^{-1}(b(t))\left[\mu(t) - r(t) + I_2(t) + a(t)\mathbb{E}\left(\pi_1^*(t)|\,\mathcal{G}_t^2\right)\right]\end{array}$$

admits a solution $(\pi_1^*, \pi_2^*) \in \mathcal{A}$ *then* (π_1^*, π_2^*) *constitutes a Nash-equilibrium of our market game.*

Proof. This follows directly from Lemma 1, noticing that $J_\mu^{\pi_1} = a(t)$, $J_\mu^{\pi_2} = b(t)$ and that all coefficients are measurable with respect to $\mathcal{G}_t^p$, $p = 1, 2$.

To better understand how the equilibrium strategies are constructed, we study how they change, while changing the complexity of the model, starting with the standard Black-Scholes model, in which we have standard information and no market impact, i.e. $\mathcal{G}_t^p = \mathcal{F}_t$ for $t \in [0, T]$ and $a(t) = b(t) = 0$. In this case $I_1(t) = I_2(t) = 0$ as the expectation of an Ito-integral is always zero and therefore the equilibrium strategies are just the Merton rule. If we allow for market impact but no non-standard information, we will still have that $I_1(t) = I_2(t) = 0$, however the equilibrium strategies now adjust for the actions of the opponent. In this case agents have complete information about the opponents strategies and the Nash-equilibrium is given by

$$\begin{array}{rcl}\pi_1^*(t) & = & \Sigma^{-1}(a(t))\left[\mu(t) - r(t) + b(t)\pi_2^*(t)\right] \\ \pi_2^*(t) & = & \Sigma^{-1}(a(t))\left[\mu(t) - r(t) + a(t)\pi_1^*(t)\right].\end{array}$$

Now, in the presence of nonstandard, possibly asymmetric information two things occur. First, the agents are no longer able to fully reflect on their opponents strategy and instead have to take expectations based on their current level of information. This accounts to taking conditional expectations in Proposition 2. Furthermore the additional terms $I_p(t)$ occur. These can be interpreted as information drifts and adjust the strategy for a differently perceived growth rate of the underlying asset.

The particular form of the equilibrium strategies in Proposition 2 is implicit and has been chosen in order to understand how agents react and adjust for their opponents strategies. Substituting the formula for $\pi_2^*(t)$ into the formula for $\pi_1^*(t)$ and vice versa it is possible to obtain an explicit form. Note that in order to obtain explicit formulas for the strategies it is only necessary to give explicit formulas for the conditional expectations on the right hand side of the expressions in Proposition 2. Such expressions are derived in the following proposition.

Proposition 3. *Assume that* (π_1^*, π_2^*) *constitutes a Nash-equilibrium of our market game. Then the conditional expectations* $\mathbb{E}\left(\pi_1^*(t)|\,\mathcal{G}_t^2\right)$ *and* $\mathbb{E}\left(\pi_2^*(t)|\,\mathcal{G}_t^1\right)$ *in Proposition 2 are explicitly given by the following formulas*

$$\begin{aligned}\mathbb{E}\left(\pi_1^*(t)|\,\mathcal{G}_t^2\right) &= \left(\mathbf{1} - \Sigma^{-1}(a(t))b(t)\Sigma^{-1}(b(t))a(t)\right)^{-1}\Big\{\Sigma^{-1}(a(t))(\mu(t) - r(t) + \tilde{I}_1(t)) + \\ &\quad \Sigma^{-1}(a(t))b(t)\Sigma^{-1}(b(t))(\mu(t) - r(t) + \tilde{I}_2(t))\Big\} \\ \mathbb{E}\left(\pi_2^*(t)|\,\mathcal{G}_t^1\right) &= \left(\mathbf{1} - \Sigma^{-1}(b(t))a(t)\Sigma^{-1}(a(t))b(t)\right)^{-1}\Big\{\Sigma^{-1}(b(t))(\mu(t) - r(t) + \tilde{I}_2(t)) + \\ &\quad \Sigma^{-1}(b(t))a(t)\Sigma^{-1}(a(t))(\mu(t) - r(t) + \tilde{I}_1(t))\Big\}\end{aligned}$$

where $\tilde{I}_1(t) = \mathbb{E}\left(I_1(t)|\,\mathcal{G}_t^2\right)$ *and* $\tilde{I}_2(t) = \mathbb{E}\left(I_2(t)|\,\mathcal{G}_t^1\right)$ *denote the information drift of the individual agents as perceived by the opposite agent.*

Proof. This follows from Proposition 2 by substituting the equation for $\pi_2^*(t)$

into the equation for $\pi_1^*(t)$ and vice versa and then take conditional expectations on $\mathcal{G}_t^1$ resp. $\mathcal{G}_t^2$.

For the case of nonstandard homogeneous information we obtain the following Corollary.

Corollary 1. *Assume that both agents have the same level of information, i.e. $\mathcal{G}^1 = \mathcal{G}^2$. If the following system of equations*

$$\begin{aligned} \pi_1^*(t) &= \Sigma^{-1}(a(t))\,(\mu(t) - r(t) + I(t) + b(t)\pi_2^*(t)) \\ \pi_2^*(t) &= \Sigma^{-1}(b(t))\,(\mu(t) - r(t) + I(t) + a(t)\pi_1^*(t)) \end{aligned}$$

with $I(t) = I_1(t) = I_2(t)$ admits a solution $(\pi_1^*, \pi_2^*) \in \mathcal{A}$ then (π_1^*, π_2^*) constitutes a Nash-equilibrium for the corresponding market game.

Proof. Symmetry of information implies that the the conditional expectation in Proposition 2 can be replaced by the actual strategies. Furthermore from the definition it is clear that $I_1(t) = I_2(t)$.

In the case above explicit solutions can be obtained simply by substituting the expression for π_2^* resp. π_1^* into π_1^* resp. π_2^* and solving out. Let us now assume that the agents do not only have the same level of information, but also that the market impact of both agents is the same. This relates to choosing $a(t) = b(t)$. In this case we are particularly interested in symmetric Nash-equilibria.

Corollary 2. *Under symmetric information and same market impact factors $a(t) = b(t)$ a symmetric Nash-equilibrium (π^*, π^*) of our market game is given by*

$$\pi^*(t) = \left(\mathbf{1} - \Sigma^{-1}(a(t))a(t)\right)^{-1} \cdot \Sigma^{-1}(a(t))(\mu(t) - r(t) + I(t))$$

Proof. This follows by straightforward computation from Corollary 1.

In the following we study the welfare implications of information in our market game. By welfare implications we mainly mean, whether the market is better of with more information or not. Doing an analysis starting with a representative agent model such as in the classical literature Karatzas and Pikovsky (1996), Corcuera (2004), Imkeller (2003), Leon *et al.* (2003) and Ewald (2005) the answer to this question is trivial : Yes! In our framework the analysis however is different, in the way that given more information the agents may be able to outperform and in fact harm each other, with more severe consequences. We will derive explicit conditions on the model parameters which determine whether general welfare is improved or worsened by adding more information. In order to proceed with this we need the following technical lemma.

Lemma 2. *Assume that a Nash-equilibrium for our market game exists. Then* $\int_0^t \sigma(s)d^-\mathbb{W}(s)$ *is a* $\mathcal{G}^p$ *semi-martingale for* $p = 1, 2$. *Furthermore, if additionally, the matrix valued process* $\sigma(s)$ *is invertible a.s. then* $\mathbb{W}$ *is a* $\mathcal{G}^p$*-semi-martingale.*

Proof. Under our assumption that the process $\sigma(s)$ is $\mathcal{G}$-adapted we obtain that the forward integral $\int_0^t \sigma(s)d^-\mathbb{W}(s)$ is $\mathcal{G}_t^p$-adapted. From equation (10) we obtain

$$\int_0^t \sigma(s)d^-\mathbb{W}(s) = M_p(t,\pi_1,\pi_2) - \mathbb{E}\left(\int_0^t (\mu(s) + a(s)\pi_1(s) + b(s)\pi_2(s)) \right. \quad \textbf{(23)}$$

$$\left. -r(s) + a(s)\pi_p(s) - \sigma(s)\sigma^\top(s)\pi_p(s))ds \,\middle|\, \mathcal{G}_t^p\right) \quad \textbf{(24)}$$

By separating the positive and negative parts of the integrands in the conditional expectation, the latter can clearly be written as the difference of two non-decreasing $\mathcal{G}^p$-adapted processes. On the other side it follows from

Proposition 1 and (10) that $M_p(t,\pi_1,\pi_2)$ is a continuous martingale. According to Definition 3.1. in Karatzas-Schreve (1988) $\int\limits_0^t \sigma(s)d^-\mathbb{W}(s)$ is therefore a continuous semi martingale.

For the following discussion we assume $\mathcal{G} = \mathcal{F}$. Let us consider a filtration $\mathcal{H} = (\mathcal{H}_t)$ s.t $\mathcal{G}_t \subset \mathcal{H}_t$ for all $t \in [0,T]$. Denote Nash-equilibria of our market game corresponding to the setup $\mathcal{G}^1 = \mathcal{G} = \mathcal{G}^2$ with $(\hat{\pi}_1^*, \hat{\pi}_2^*)$ and Nash-equilibria corresponding to the setup $\mathcal{G}^1 = \mathcal{H} = \mathcal{G}^2$ with (π_1^*, π_2^*).

Definition 3. The information $\mathcal{H} = (\mathcal{H}_t)$ is welfare increasing, if the payoffs from (π_1^*, π_2^*) Pareto-dominate the payoffs from $(\hat{\pi}_1^*, \hat{\pi}_2^*)$. $\mathcal{H}$ is called welfare decreasing if the opposite is the case. Furthermore we define the information welfare impact of $\mathcal{H}$ as the vector

$$iwi(\mathcal{H}) = \begin{pmatrix} u_1(\pi_1^*, \pi_2^*) - u_1(\hat{\pi}_1^*, \hat{\pi}_2^*) \\ u_2(\pi_1^*, \pi_2^*) - u_2(\hat{\pi}_1^*, \hat{\pi}_2^*) \end{pmatrix}.$$

Clearly the information welfare impact does not depend on the initial wealth and information is welfare increasing if both components are positive and welfare decreasing if both components are negative. We have the following proposition, which provides necessary and sufficient conditions depending on the various parameters of the game, whether or not information is welfare increasing.

Proposition 4. *Writing* $\mathbb{W}(t) = \hat{\mathbb{W}}(t) - \int_0^t \alpha(s)ds$ *according to Lemma 2 with* $\hat{\mathbb{W}}(t)$ *an* $\mathcal{H}$*-Brownian motion and using the notation*

$$\begin{aligned} \Sigma_1 &= \Sigma(a(t)) - b(t)\Sigma(b(t))^{-1}\Sigma(a(t)) \\ \Sigma_2 &= \Sigma((t)) - b(t)\Sigma(b(t))^{-1}\Sigma(a(t)) \end{aligned}$$

the two components of the information welfare impact of $\mathcal{H}$ *are explicitly given* *as*

$$\begin{aligned}
iwi(\mathcal{H})_1 = \;& \mathbb{E}\Bigg(\int_0^T (\Sigma_1^{-1}\sigma(t)\alpha(t))^\top a\Sigma_1^{-1}\sigma(t)\alpha(t) + (\Sigma_1^{-1}\sigma(t)\alpha(t))^\top b\Sigma_2^{-1}\sigma(t)\alpha(t) \\
& -\tfrac{1}{2}(\Sigma_1^{-1}\sigma(t)\alpha(t))^\top\sigma(t)\sigma(t)^\top\Sigma_1^{-1}\sigma(t)\alpha(t) + (\Sigma_1^{-1}\sigma(t)\alpha(t))^\top\sigma^\top(t)\alpha(t)dt\Bigg) \\
iwi(\mathcal{H})_2 = \;& \mathbb{E}\Bigg(\int_0^T (\Sigma_2^{-1}\sigma(t)\alpha(t))^\top a\Sigma_1^{-1}\sigma(t)\alpha(t) + (\Sigma_2^{-1}\sigma(t)\alpha(t))^\top b\Sigma_2^{-1}\sigma(t)\alpha(t) \\
& -\tfrac{1}{2}(\Sigma_2^{-1}\sigma(t)\alpha(t))^\top\sigma(t)\sigma(t)^\top\Sigma_2^{-1}\sigma(t)\alpha(t) + (\Sigma_2^{-1}\sigma(t)\alpha(t))^\top\sigma^\top(t)\alpha(t)dt\Bigg)
\end{aligned}$$

Proof. Using Corollary 1 we can easily derive the following two equations for the equilibrium strategy (π_1^*, π_2^*) under information $\mathcal{H}$:

$$\begin{aligned}
\pi_1^* &= \Sigma_1^{-1}(\mu(t) - r(t)) + \Sigma_1^{-1}I(t), \\
\pi_2^* &= \Sigma_2^{-1}(\mu(t) - r(t)) + \Sigma_2^{-1}I(t)
\end{aligned}$$

The equilibrium strategies without additional information are given by the market impact adjusted Merton-rules:

$$\begin{aligned}
\hat{\pi}_1^* &= \Sigma_1^{-1}(\mu(t) - r(t)) \\
\hat{\pi}_2^* &= \Sigma_2^{-1}(\mu(t) - r(t)).
\end{aligned}$$

Substitution of these strategies in the utility function leads to the following expression for the first component of $iwi(\mathcal{H})$

$$\begin{aligned}
& \mathbb{E}\Bigg(\int_0^T \Sigma_1^{-1}(\mu(t) - r(t))(a(t)\Sigma_1^{-1}I(t) + b(t)\Sigma_2^{-1}I(t)) + (\Sigma_1^{-1}I(t))^\top(\mu(t) - r(t)) \\
& +(\Sigma_1^{-1}I(t))^\top(a\Sigma_1^{-1}(\mu(t) - r(t)) + a\Sigma_1^{-1}I) + (\Sigma_1^{-1}I)(b\Sigma_2^{-1}(\mu(t) - r(t)) + b\Sigma_2^{-1}I) \\
& -\tfrac{1}{2}(\Sigma_1^{-1}I)^\top\sigma(t)\sigma(t)^\top\Sigma_1^{-1}I - \tfrac{1}{2}(\Sigma_1^{-1}I)^\top\sigma(t)\sigma(t)^\top\Sigma_1^{-1}I \\
& -\tfrac{1}{2}(\Sigma_1^{-1}I)^\top\sigma(t)\sigma(t)^\top(\Sigma_1^{-1}(\mu(t) - r(t)))^\top dt + \int_0^T (\Sigma_1^{-1}I)d^-\mathbb{W}(t)\Bigg)
\end{aligned}$$

A similar expression can be derived for the second component. Under our assumptions it follows from Lemma 2 and Biagini and Oksendal (2005), page

(178) that

$$\mathbb{E}\left(\int\limits_0^t g(a(s),b(s),\mu(s)-r(s))I(s)ds\right)=0, \qquad (25)$$

for any bounded function g. Using this relationship it can be verified that the long expression above simplifies to

$$\mathbb{E}\left(\int\limits_0^T (\Sigma_1^{-1}\sigma(t)\alpha(t))^\top a\Sigma_1^{-1}\sigma(t)\alpha(t)+(\Sigma_1^{-1}\sigma(t)\alpha(t))^\top b\Sigma_2^{-1}\sigma(t)\alpha(t)\right.$$
$$\left.-\tfrac{1}{2}\,(\Sigma_1^{-1}\sigma(t)\alpha(t))^\top\sigma(t)\sigma(t)^\top\Sigma_1^{-1}\sigma(t)\alpha(t)+(\Sigma_1^{-1}\sigma(t)\alpha(t))^\top\sigma^\top(t)\alpha(t)dt\right)$$

The analysis of the second component is completely analogous.

The expressions for the information wellfare impact are quite lengthy. For the one dimensional case we are able to derive the following corollary.

Corollary 3. *Under the assumptions that there is only one stock at the market and the market parameters are given as* $\mu(t)\equiv\mu$*,* $a(t)\equiv a$ *and* $b(t)\equiv b$ *with constants* $\mu,a,b\in\mathbb{R}$ *and* $\Sigma_{\min}:=\min\{\Sigma_1,\Sigma_2\}>0$ *the information* $\mathcal{H}=(\mathcal{H}_t)$ *is wellfare increasing if*

$$1+\frac{a}{\Sigma_1}+\frac{b}{\Sigma_2}\geq\frac{\sigma^2}{2\Sigma_{\min}}$$

and welfare decreasing if

$$1+\frac{a}{\Sigma_1}+\frac{b}{\Sigma_2}\leq\frac{\sigma^2}{2\Sigma_{\max}}$$

with $\Sigma_{\max}:=\max\{\Sigma_1,\Sigma_2\}$.

Proof. Under the assumptions in the corollary it is easy to verify that the components of the information wellfare impact vector in Proposition 4 simplify to

$$iwi(\mathcal{H})=\begin{pmatrix}\frac{1}{\Sigma_1}(\frac{a}{\Sigma_1}+\frac{b}{\Sigma_2}+1-\frac{\sigma^2}{2\Sigma_1})\mathbb{E}\left(\int\limits_0^t\alpha^2(t)dt\right)\\ \frac{1}{\Sigma_2}(\frac{a}{\Sigma_1}+\frac{b}{\Sigma_2}+1-\frac{\sigma^2}{2\Sigma_2})\mathbb{E}\left(\int\limits_0^t\alpha^2(t)dt\right)\end{pmatrix}$$

The information is welfare increasing if both components of this vector are positive and welfare decreasing if both components are negative. Obviously we have that $\mathbb{E}\left(\int_0^t \alpha^2(t)dt\right) \geq 0$. The condition for positivity resp. negativity is therefore exactly the one stated in the corollary.

The corollary above specifies a certain region of the parameter space consisting of feasible parameters (a, b, σ^2) in which information is wellfare increasing. The following figure shows this region for the the example of initially enlarged filtration.

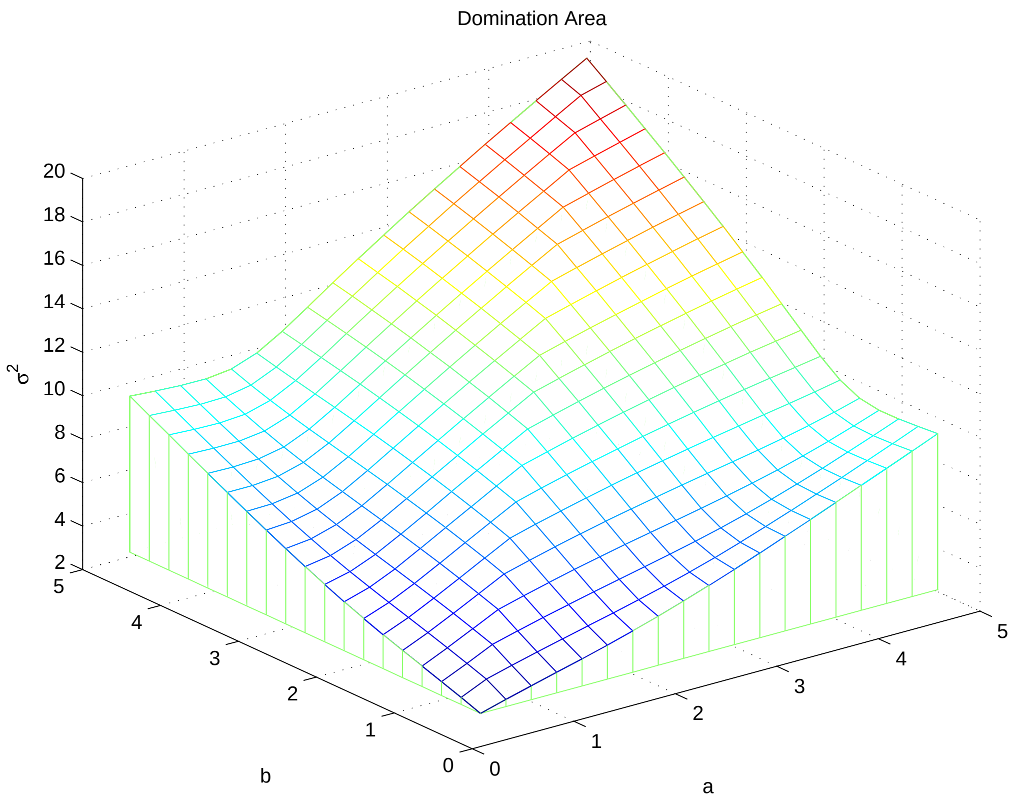

FIGURE 1: welfare increasing region

Note that some of the points in the welfare increasing region may not satisfy the concavity condition, which in this case would correspond to $2\max\{a, b\} \leq$

σ^2. However the concavity condition is not necessary for the existence of Nash-equilibria, which in our definition of welfare increasing is implicitly assumed. It is worth to mention though, that in general the intersection between those points in the welfare increasing region and those points which satisfy the concavity condition is not empty.

In order to demonstrate how our results apply to the classical case of enlarged initial filtration we include the following example.

Example 2. Let us study the implications of the various statements above for the case of initially enlarged filtration.

1. Consider for $T_0 > T$ the initially enlarged filtrations $\mathcal{G}^p = (\mathcal{G}_t^p)$ with $\mathcal{G}_t^p = \mathcal{F}_t \vee \sigma(\mathbb{W}_{\nu_1^p}(T_0)) \vee \ldots \vee \sigma(\mathbb{W}_{\nu_{k_p}^p}(T_0))$ for $t \in [0, T]$ where $\left\{\nu_1^p, \ldots, \nu_{k_p}^p\right\}$ for $p = 1, 2$ are subsets of $\{1, \ldots, n\}$. Denote the intersection $\{\nu_1^1, \ldots, \nu_{k_1}^1\} \cap \{\nu_1^2, \ldots, \nu_{k_2}^2\}$ with $\{\nu_1, \ldots, \nu_t\}$. This set may possibly be empty. We have that

$$\begin{aligned} I_p(t) &= \frac{d}{d\epsilon}\mathbb{E}\left(\int_t^{t+\epsilon} \sigma(u) d^- \mathbb{W}(u) \middle| \mathcal{G}_t^p \right)_{|\epsilon=0} \\ &= \frac{d}{d\epsilon}\mathbb{E}\left(\int_t^{t+\epsilon} \sigma(u) d\tilde{\mathbb{W}}(u) + \int_t^{t+\epsilon} \sigma(u) \frac{\mathbb{W}(T_0)-\mathbb{W}(u)}{T_0-u} du \middle| \mathcal{G}_t^p \right)_{|\epsilon=0} \\ &= \sigma(t)\alpha^p(t). \end{aligned}$$

with

$$\alpha_i^p(t) = \begin{cases} \frac{\mathbb{W}_i(T_0)-\mathbb{W}_i(t)}{T_0-t}, & i \in \{\nu_1^1, \ldots, \nu_{k_p}^p\} \\ 0, & \text{else} \end{cases}$$

This implies that $\tilde{I}_1(t) = \mathbb{E}(I_1(t)|\mathcal{G}_2(t)) = \sigma(t)\alpha(t) = \mathbb{E}(I_2(t)|\mathcal{G}_1(t)) = \tilde{I}_2(t)$ with

$$\alpha_i(t) = \begin{cases} \frac{\mathbb{W}_i(T_0)-\mathbb{W}_i(t)}{T_0-t}, & i \in \{\nu_1, \ldots, \nu_t\} \\ 0, & i \notin \{s_1, \ldots, s_t\} \end{cases}.$$

Substitution of these expressions into the corresponding expressions from Proposition 2 and Proposition 3 leads to analytical formulas for the Nash-equilibrium. If the additional information is strictly complementary, i.e. $\{\nu_1^1, \dots, \nu_{k_1}^1\} \cap \{\nu_1^2, \dots, \nu_{k_2}^2\} = \emptyset$ we find that $\alpha = 0$ and the equilibrium strategies simplify slightly. Such a case is particularly interesting to study from the point of view of cooperative game theory.

2. Consider the case where $\mathcal{G}^2 = \mathcal{F}$ and $\mathcal{G}^1$ is given as the initially enlarged filtration, i.e. $\mathcal{G}_t^1 = \mathcal{F}_t \vee \sigma(\mathbb{W}_1(T_0)) \vee \dots \vee \sigma(\mathbb{W}_n(T_0))$ for all $t \in [0, T]$ and $T_0 > T$. This distribution of information leads to the following Nash-equilibria

$$
\begin{aligned}
\pi_1^*(t) &= \Sigma(a(t))^{-1} \left(\mu(t) - r(t) + b(t)\pi_2^* + \sigma(t)\tfrac{\mathbb{W}(T_0) - \mathbb{W}(t)}{T_0 - t}\right) \\
\pi_2^*(t) &= \Lambda^{-1}(\mu(t) - r(t)) + \Lambda^{-1}a(t)\Sigma(a(t))^{-1}(\mu(t) - r(t))
\end{aligned}
$$

where $\Lambda = \Sigma(b(t)) - a(t)\Sigma(a(t))^{-1}b(t)$. In the case that agent 1 is small and does not have any market impact, i.e. $a(t) = 0$, the latter Nash-equilibrium simplifies to

$$
\begin{aligned}
\pi_1^*(t) &= \Sigma^{-1} \left(\mu(t) - r(t) + b(t)\pi_2^* + \sigma(t)\frac{\mathbb{W}(T_0) - \mathbb{W}(t)}{T_0 - t}\right) \\
\pi_2^*(t) &= \Sigma^{-1}(\mu(t) - r(t))
\end{aligned}
$$

3. Assuming symmetric information and initially enlarged filtrations $\mathcal{G}^1 = \mathcal{G}^2 = \mathcal{H}$ with $\mathcal{H}_t := \mathcal{F}_t \vee \sigma(\mathbb{W}_1(T_0)) \vee \dots \vee \sigma(\mathbb{W}_n(T_0)), T_0 \geq T$ as well as symmetric market impact $a(t) = b(t)$ we obtain from the discussion above and Corollary 4 that a symmetric Nash-equilibrium is given by

$$
\begin{aligned}
\pi_1^*(t) &= \Sigma^{-1}(a(t)) \left(\mu(t) - r(t) + \sigma(t)\frac{\mathbb{W}(T_0) - \mathbb{W}(t)}{T_0 - t} + b(t)\pi_2^*(t)\right) \\
\pi_2^*(t) &= \Sigma^{-1}(b(t)) \left(\mu(t) - r(t) + \sigma(t)\frac{\mathbb{W}(T_0) - \mathbb{W}(t)}{T_0 - t} + a(t)\pi_1^*(t)\right)
\end{aligned}
$$

As indicated before, in this case an explicit solution can be obtained by substitution of the second strategy in the first one and vice versa and solving out.

4. Trading of information

In the preceding section we studied how different levels of information affect the equilibrium of the market. While it was assumed that agents have different levels of information, they were not supposed to exchange and share their private information. In this section we will extend our market model in the way that agents are allowed to sell there own private information to their opponent and/or buy the private information of their opponent. For this reason we extend our original game, which represents a continuous time sequential game, by two additional stages, which occur before agents invest into the market. For matters of simplicity we only treat the case where one of the agents, say agent 1, is better informed than the other agent, i.e. $\mathcal{G}_t^2 \subset \mathcal{G}_t^1$ for all $t \in [0, T]$. In the first stage agent one announces a price P for which he would be willing to sell the information $\mathcal{G}^1$ to his opponent. In the second stage agent 2 decides whether to buy or not to buy the information offered for the price announced by agent 1. If agent 2 decides to buy information he faces two consequences. First his initial wealth is lowered by the amount he has to pay for the information. Second in order to trade on the market and choose a portfolio, agent 2 can now make use of the increased level of information, which is then represented by the information flow $\mathcal{G}^1$ rather than $\mathcal{G}^2$. From the perspective of agent 1 the situation looks as follows: If he sells the information, his initial wealth will be raised by the amount he charges for this information, but he faces as a consequence, that his opponent is now able to use the increased level of information to decide on his individual investment strategy. In a model where agents behavior does not influence asset-prices, the latter would not really lead to consequences and agent 1 would be willing to give away information for free. However, in a model where prices are not determined exogenously, the seller of information has to fear, that the buyer of information may use this information in a way that affects asset-prices to his disadvantage. Summarizing, there are two factors which have to be taken into account in our extended market game when determining the equilibrium and the equilibrium price for the information.

1. buying information for price P lowers initial wealth from x to $x - P$ but provides the buyer with increased level of information, which he may use to improve his investment strategy and obtaining a higher expected return

2. selling information for price P increases initial wealth from x to $x + P$ but the agent has to face possible consequences on his own optimal investment strategy and expected return due to the increased level of information of his opponent

Both agents have to weigh up the benefits and losses in order to make up their decisions. In this article we assume that agents behave non-cooperatively. Alternatively, agents may be willing to share their information at the beginning, setup a mutual investment fund and then share the profits at the end. The main question, how to distribute the profits at the end, is studied by means of cooperative game theory and the Nash-bargaining approach in our working paper Ewald-Yajun (2007).

Definition 4. A price P^* is called an equilibrium price for the information

$\mathcal{H} = (\mathcal{H}_t)$, if the sequential game described above with the choice of $\mathcal{G}^1 = \mathcal{H}$ has a Nash-equilibrium of the type

$$(\{P^*, \pi_1^*\}, \{\text{"buy only if price } \leq P^*\text{"}, \pi_2^*\})$$

The definition above guarantees that if the information is offered at the equilibrium price it will indeed be traded at that price. In the following we compute equilibrium prices for the case that agent 1 is better informed but doesn't necessarily have the same market impact. This is the typical setup when a consulting company sells their information and expertise to a client which presents a major market maker. Criteria one and two above still apply for this setup. In order to solve our extended market game we apply backward induction. In the previous section we studied the third stage and have identified equilibrium strategies for general levels of information. We found that the equilibrium strategies do not depend on initial wealth. This feature depends on our choice of utility function as the logarithm, but is also observed with other utility functions such as exponential utility. Note however that even though the equilibrium strategies are unaffected by the initial wealth, the amount of utility obtained from following these strategies does. Using this fact we find that in the last stage two scenarios are possible. Scenario one occurs, if information is traded within the first two stages. In this case both players have the same level of information $\mathcal{G}^1$ in the last sage. We denote the equilibrium strategies for the corresponding sub-game starting in stage 3 with (π_1^*, π_2^*). If information is not traded within the first two periods, then agents posses asymmetric information in the third stage. In this case we denote the equilibrium for the corresponding sub-game with $(\hat{\pi}_1^*, \hat{\pi}_2^*)$. These equilibria can be computed with the methods presented in the previous section.

Proposition 5. *Let P^* be a solution of the following system*

$$\begin{aligned} u_2^{x_2-P^*}(\pi_1^*,\pi_2^*) &= u_2^{x_2}(\hat{\pi}_1^*,\hat{\pi}_2^*) \\ u_1^{x_1+P^*}(\pi_1^*,\pi_2^*) &\geq u_1^{x_1}(\hat{\pi}_1^*,\hat{\pi}_2^*). \end{aligned}$$

Here the upper indices at the utilities denote the agents initial endowment. Then P^* is an equilibrium price for the information $\mathcal{G}^1$ in the extended market game illustrated above. In particular an equilibrium price is unique.

Proof. This follows mainly from the discussion above and the definition of an equilibrium price. As maximal expected utility depends monotonically increasing on initial wealth and the price P^* of the information is added to agents 1 initial wealth x_1, agent 1 is trying to set the price as high a possible. Agent 2 has to wager whether to buy or not to buy the information for this price. Obtaining more information raises his maximum expected utility, but the price of the information P^* is subtracted from his initial wealth. The first equality in Proposition 5 sets the price in a way that agent 2 is indifferent about buying or not buying. Even though agent 2 would be willing to buy for a price which satisfies the first equality, it is not a priori clear that agent 1 would sell for this price, as he has to wager the consequence of having an opponent on the market which is better informed than original, against the immediate prospect of more initial wealth. This decision is reflected by the inequality in Proposition 5. Using these arguments it then follows from backward induction that $(\{P^*,\pi_1^*\},\{$ "buy only if price $\leq P^*$", $\pi_2^*\})$ is a sequential Nash-equilibrium and therefore that P^* is an equilibrium price.

Definition 5. The price P^* is called a feasible price if

$$\begin{aligned} u_2^{x_2-P^*}(\pi_1^*,\pi_2^*) &\geq u_2^{x_2}(\hat{\pi}_1^*,\hat{\pi}_2^*) \\ u_1^{x_1+P^*}(\pi_1^*,\pi_2^*) &\geq u_1^{x_1}(\hat{\pi}_1^*,\hat{\pi}_2^*). \end{aligned}$$

By Definition an equilibrium price is feasible. Feasibility of a price ensures that information is traded under this price, however the selling agent may perform suboptimal. In the presence of many information providers, feasible prices other than the equilibrium price may occur. It is straightforward to verify that the two inequalities in Definition 5 are equivalent to the following two inequalities

$$\begin{aligned} P^* &\geq x_1 \exp\left\{ \mathbb{E}\left(\int_0^T [\pi_1^*(t)(\mu(t,\pi_1^*,\pi_2^*) - r(t)) - \hat{\pi}_1^*(t)(\mu(t,\hat{\pi}_1^*,\hat{\pi}_2^*) - r(t)) \right.\right. \\ &\left.\left. -\frac{1}{2}((\pi_1^*(t))^2 - (\hat{\pi}_1^*(t))^2)\sigma^2(t)\right] dt + \int_0^T (\pi_1^*(t) - \hat{\pi}_1^*(t))\sigma(t) d^-\mathbb{W}(t) \right) \right\} - x_1 \end{aligned}$$

$$\begin{aligned} P^* &\leq x_2 \exp\left\{ \mathbb{E}\left(\int_0^T [\pi_2^*(t)(\mu(t,\pi_1^*,\pi_2^*) - r(t)) - \hat{\pi}_2^*(t)(\mu(t,\hat{\pi}_1^*,\hat{\pi}_2{}^*) - r(t)) \right.\right. \\ &\left.\left. -\frac{1}{2}((\pi_2^*(t))^2 - (\hat{\pi}_2^*(t))^2)\sigma^2(t)\right] dt + \int_0^T (\pi_2^*(t) - \hat{\pi}_2^*(t))\sigma(t) d^-\mathbb{W}(t) \right) \right\} - x_2. \end{aligned}$$

Note that one can evaluate the contribution of the forward integral by means of formula (4)

$$\mathbb{E}\left(\int_0^T (\pi_1^*(t) - \hat{\pi}_1^*)\sigma(t) d^-\mathbb{W}(t) \right) = \mathbb{E}\left(\int_0^T D_{t^+}((\pi_1^*(t) - \hat{\pi}_1^*(t))\sigma(t)) dt \right),$$

For an equilibrium price the second inequality has to be satisfied as an equality. The computation of the price of the information in general can now be done along the following lines. Use the formulas for the equilibrium strategies from the previous sections, substitute these in the formulas above, solve the second formula for P^* and see whether the solution verifies the first inequality. If this is the case P^* is the equilibrium price for the information specified. In the following we consider the case of initially enlarged filtration. More precisely we consider the case of a single stock and choose $\mathcal{G}_t^1 = \mathcal{F}_t \vee \sigma(W_{T_0})$ for $T_0 > T$ and $\mathcal{G}_t^2 = \mathcal{F}_t$ for $t \in [0, T]$. It follows from Example 2 parts 2 and 3 that

$$\begin{aligned}
\pi_1^*(t) &= \tfrac{\sigma^2(t)-b(t)}{\sigma^4(t)-2(a(t)+b(t))\sigma^2(t)+3a(t)b(t)} \left[\mu(t) - r(r) + \sigma(t)\tfrac{\mathbb{W}_{T_0}-\mathbb{W}_t}{T_0-t}\right], \\
\pi_2^*(t) &= \tfrac{\sigma^2(t)-a(t)}{\sigma^4(t)-2(a(t)+b(t))\sigma^2(t)+3a(t)b(t)} \left[(\mu(t) - r(r) + \sigma(t)\tfrac{\mathbb{W}_{T_0}-\mathbb{W}_t}{T_0-t}\right], \\
\hat{\pi}_1^*(t) &= \tfrac{\sigma^2(t)-b(t)}{\sigma^4(t)-2(a(t)+b(t))\sigma^2(t)+3a(t)b(t)} \left[\mu(t) - r(r)\right] + \tfrac{\sigma(t)}{\sigma^2(t)-2a(t)}\tfrac{\mathbb{W}_{T_0}-\mathbb{W}_t}{T_0-t}, \\
\hat{\pi}_2^*(t) &= \tfrac{\sigma^2(t)-a(t)}{\sigma^4(t)-2(a(t)+b(t))\sigma^2(t)+3a(t)b(t)} \left[\mu(t) - r(r)\right].
\end{aligned} \tag{26}$$

Substituting these strategies into the inequalities above, while using that for any bounded and measurable function f

$$\begin{aligned}
&\mathbb{E}\left(\int_0^T f(\mu(t), r(t), \sigma(t))\tfrac{\mathbb{W}_{T_0}-\mathbb{W}_t}{T_0-t}dt\right) \\
= \;&\mathbb{E}\int_0^T f(\mu(t), r(t), \sigma(t))d\tilde{\mathbb{W}}(t) - \mathbb{E}\left(\int_0^T f(\mu(t), r(t), \sigma(t))d\mathbb{W}(t)\right) = 0
\end{aligned}$$

we obtain

$$\begin{aligned}
P^* \;\geq\; & x_1 \exp\left\{E(\int_0^T((a(t)(\tfrac{\sigma^2(t)-b(t)}{k})^2 + b(t)\tfrac{(\sigma^2(t)-a(t))(\sigma^2(t)-b(t))}{k^2} - \tfrac{a(t)}{(\sigma^2(t)-2a(t))^2})\right. \\
&\left. -\tfrac{1}{2}((\tfrac{\sigma^2(t)-b(t)}{k})^2 - (\tfrac{1}{\sigma^2(t)-2a(t)})^2) + (\tfrac{\sigma^2(t)-b(t)}{k} - \tfrac{1}{\sigma^2(t)-2a(t)}))\sigma^2(t)(\tfrac{\mathbb{W}_{T_0}-\mathbb{W}_t}{T_0-t})^2dt)\right\} \\
& -x_1 \\
P^* \;\leq\; & x_2 \exp\left\{E(\int_0^T((a(t)\tfrac{\sigma^2(t)-b(t)}{\sigma^2(t)-a(t)} + b(t) - \tfrac{\sigma^2(t)}{2})\tfrac{(\sigma^2(t)-a(t))^2}{k^2}\right. \\
&\left. +\tfrac{\sigma^2(t)-a(t)}{k})\sigma^2(t)(\tfrac{\mathbb{W}_{T_0}-\mathbb{W}_t}{T_0-t})^2dt)\right\} - x_2.
\end{aligned} \tag{27}$$

Depending on the complexity of the model, the values on the right hand side of the equalities can either be computed in closed form, using numerical methods or Monte-Carlo valuation. The following figures present cases for which we computed feasible price areas under the assumptions above. The equilibrium prices are represented by the upper border of the feasible price areas. The ask price represents the minimum price for which the informed agent would be willing to sell the information, the offer price is the maximum price the uninformed agent would be willing to pay for the information. The feasible price region is indicated with dots. We see that for the case where the non-informed agent has no market impact, the informed agent would actually be willing to give the information away for free.

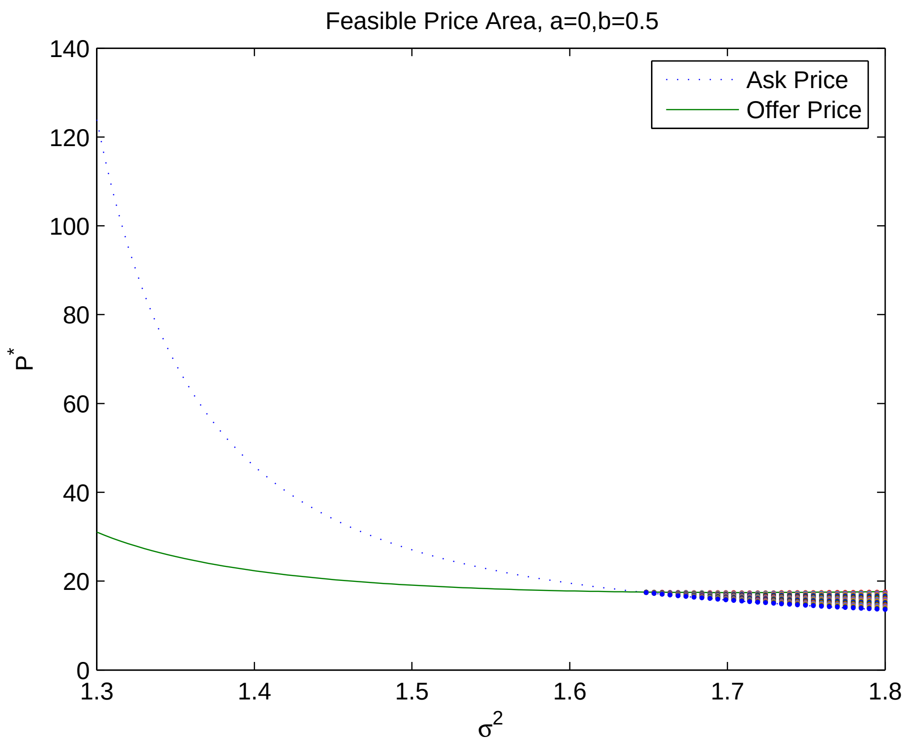

FIGURE 2: feasible price area

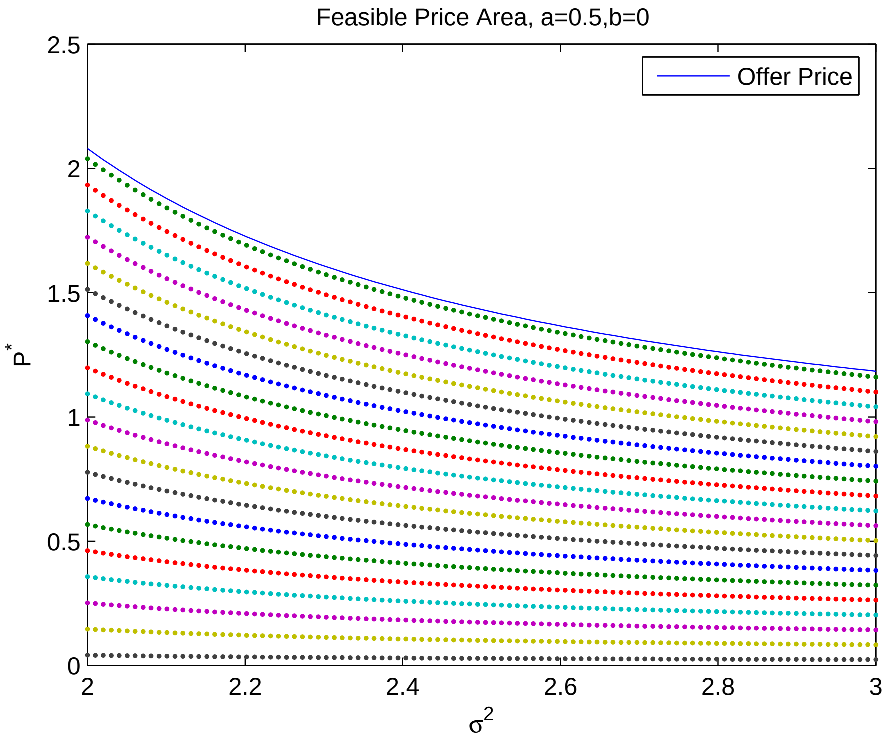

FIGURE 3: feasible price area

As one can see on figures 2-4, the volatility parameter σ^2 plays an interesting role. It can either increase or decrease the size of the range of feasible prices.

5. Conclusion

We have studied a continuous time financial market game in which agents possess different levels of information within an anticipative stochastic calculus framework. Technically our game represents an anticipative stochastic differential game. To the best of our knowledge such games have not been studied before. We derived necessary and sufficient conditions for the ex-

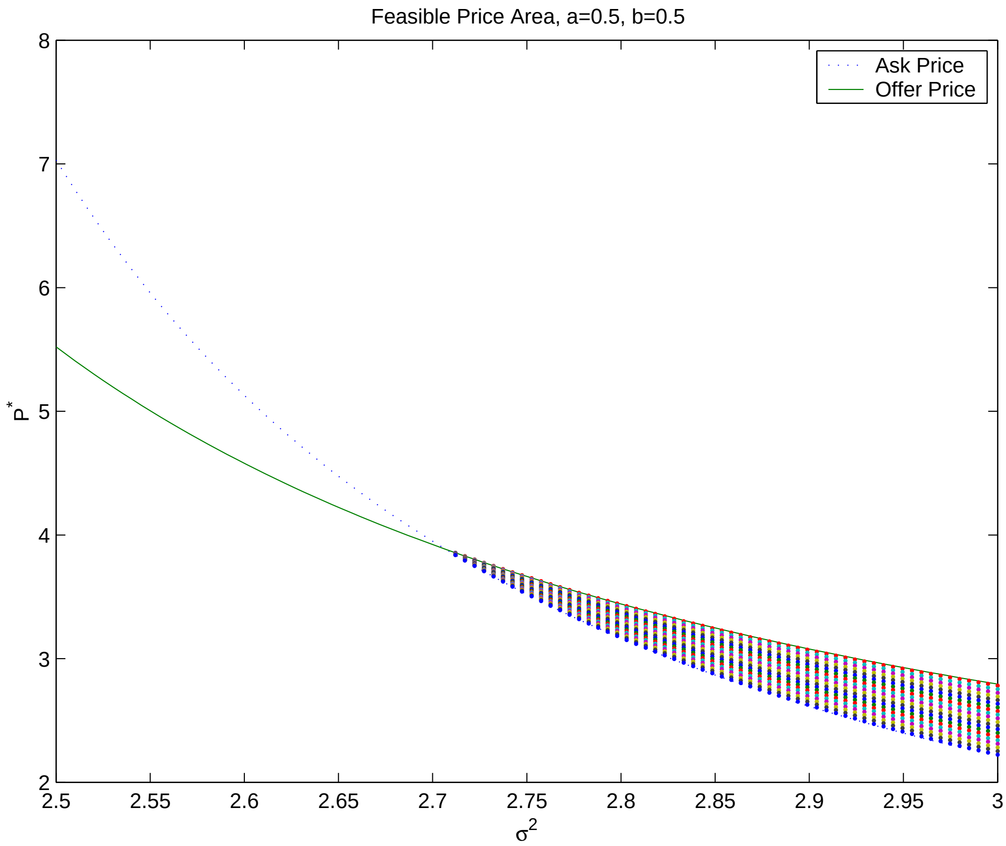

FIGURE 4: feasible price area

istence of Nash-equilibria in this game and studied the impact the level of information has on the Nash-equilibria and on general wellfare. In the second part we extended the game with two pre-stages in which information can be traded among the agents. The question of pricing information has so far only been studied in a representative agent framework. We introduced the notion of an equilibrium price for specified information and derive a certain set of inequalities which characterize it. Various examples for the case of initially enlarged filtration are given.

Acknowledgments

The first author gratefully acknowledges support from the research grant "Dependable adaptive systems and mathematical modeling", Rheinland-Pfalz Excellence Cluster. Furthermore he would like to thank Ralf Korn (University of Kaiserslautern) and David Ulph (University of St.Andrews) for many suggestions in particular with respect to the importance of the question how information levels influence general welfare. The first author gratefully acknowledges grant 529817 from the DFG and a travel grant from the Royal Society. The second author would like to thank the University of Frankfurt for generous financial support for visits to St.Andrews.

References

[1] BIAGINI, F., and ØKSENDAL, B. (2005): A General Stochastic Calculus Approach to Insider Trading. Appl. Math. Optim. 52, 167 − 81.

[2] CORCUERA, J. M., IMKELLER, P., A. KOHATSU-HIGA, and NUALART, D. (2004): Additional utility of insiders with imperfect dynamical information. Finance and Stochastics 8, 437 − 450.

[3] DI NUNNO, G., T. MEYER-BRANDIS, B. ØSENDAL, and F. PROSKE (2003): Optimal Portfolio for an Insider in a Market Driven by Lèvy Processes. Preprint, Department of Mathematics, University of Oslo.

[4] DI NUNNO, G., KOHATSUM-HIGA, T. MEYER-BRANDIS, B. ØKSENDAL, PROSKE, F., SULEM, A. 2005. Optimal Portfolio for a Large Insider in a Market driven by Levy Processes. Preprint Series in Pure Mathematics, University of Oslo

[5] EWALD, C.-O. 2005: Optimal Logarithmic Utility and Optimal Portfolios for an Insider in A Stochastic Volatility Market. Int. J. Theoretical Appl. Finance Vol. $8(3)$ $331-347$.

[6] EWALD, C.-O., XIAO, Y. 2007 Information sharing and on financial markets and its effect on welfare : A cooperative approach in continuous time. Working-Paper.

[7] KARATZAS, I., PIKOVSKY, I. (1996): Anticipative Portfolio Optimization, Adv. Appl. Probab. $28, 1095-1122$.

[8] IMKELLER, P. (2003): Malliavin's Calculus in Insiders Models: Additional Utility and Free Lunches, Math. Finance $13(1), 153-169$.

[9] JACOD, J. (1985): Groissement Initial, Hypothese (H), et Theor'eme de Girsanov; in Groissements des Filtrations: Exemples et Applications, in T. Jeulin and M. Yor eds., LNM 1118, Springer.

[10] KOHATSU-HIGA, A., SULEM, S. (2006): Utility Maximization in an Insider Influenced Market. Mathematical Finance, Vol. $16(1), 153-179$.

[11] LEON, J. A., NAVARRO R., NUALART, D. (2003): An Anticipating Calculus Approach to The Utility Maximization of an Insider, Mathematical Finance $13(1)171C185$.

[12] NUALART, D (1995): The Malliavin Calculus and Related Topics, Springer.

[13] ØKSENDAL, B. (2006): A Universal Optimal Consumption Rate for an Insider, Mathmetical Finance Vol. $16(1), 119-129$.

[14] RUSSO, F., VALLOIS, P. (1993): Forward, Backward and Symmetric Stochastic Integration, Probab. Th. Rel. Fields $97, 403 - 421$.