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Abstract

We revisit the analysis of subscription equilibria in a full fledged general equilibrium model with public goods. We study the case of a non-profit, or public, firm that produces the public good using private goods as inputs, which are to be financed by voluntary contributions (subscriptions) of households. We prove existence and generic regularity of subscription equilibria.

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1 Introduction

In this paper we prove the existence and generic regularity of subscription equilibria in a pure public good economy where the public good is produced by a non-profit, or public, firm.

In a series of path-breaking articles that have set forth a definitive theory of public goods, Samuelson (1954, 1969) presented the first modern analyses of public goods within a general equilibrium context. His main concern being the normative one, Samuelson provided a characterization of welfare optima in public good economies, but he did not elaborate on the process through which the level of a public good is to be determined. For a positive theory of equilibrium, one has to specify how the level of a public good is to be determined, and, owing to the distinctive nature of public goods, this is typically going to be a collective (political) decision-making process that goes beyond the standard pure market equilibrium notion.

Starting with Foley (1967), there have been various attempts to provide theories of politico-economic equilibrium with public goods in a general equilibrium context. The problem from the view point of economic theory, however, is the fact that one has to provide precise institutional details of how such collective decisions are to be made, an area of inquiry that perhaps intersects more with political science than standard economic theory.

To provide an analysis of the public good problem from pure economic theory point of view, as well as to serve as a benchmark extension of an analysis of completely decentralized private good economies to public good economies, a useful starting point is to study which equilibria will be established in the absence of a central authority or mutual agreement among the agents. Towards this end, within a general equilibrium context Malinvaud (1972, p. 213) proposed to study the system whereby the public good is financed by subscription, with each household making a contribution to increase the production of public good. The contributions are to be voluntary and contribution decisions are to be made by each household independently of other households, the complete autonomy of households thus being fully respected.

Many studies have investigated properties of subscription equilibria with voluntary contributions.[1] However, most of them adopt what is essentially a partial equilibrium framework. In a model which is now the canonical reference in the voluntary contribution literature, Bergstrom, Blume, and Varian (1986) assume the presence of only one private good, and of a linear production technology for a single public good. With these assumptions there is no need to model production explicitly, since, up to a rescaling of the unit of measure, one unit of the private good is transformed into one unit of the public good. This, in turn, implies that there will be no (relative) prices to be determined and that profit maximizing firms will have zero profits in a competitive equilibrium. Thus, the model can be reduced to a simple game in which each player has one

[1] See Myles (1995), pp. 279-290, for a review of the applications of the concept to questions in public economics.

variable to choose (the amount of private contribution) in the closed interval whose endpoints are zero and the endowment of the private good.

The large literature that follows Bergstrom et al. (1986) mainly focuses on questions other than existence of equilibria, such as neutrality of equilibrium outcomes with respect to redistribution of income. Papers on existence are limited to the case of one public good, one private good - see Cornes, Hartley and Sandler (1999) and papers quoted there. The only exception is a companion paper, Villanacci and Zenginobuz (2005), where we show existence in a model with many private goods and where the public good is produced through a non-linear production technology by competitive firms.

When more than one private good and a non-linear production technology are allowed, modeling of how and by whom the public good is produced becomes a crucial preliminary issue to be resolved. If a profit-maximizing (private) firm is assumed to produce the public good under a non-constant returns to scale technology, then how the (non-zero) profits of the firm are apportioned among its shareholders will have an impact on equilibrium outcomes. An alternative is to consider the production of the public good as being carried out by a non-profit, or a public firm subject to a balanced budget constraint.

In the present paper, we study a model with many private goods where a public good is produced through a non-linear production technology by a non-profit, or public, firm subject to a balanced budget constraint. The amount of public good to be produced by the non-profit firm is taken as the maximum amount that can be produced with the amount of subscriptions collected from consumers. Observe that this is in line with the one private good and linear production technology model by Bergstrom et al. (1986), with the firm(s)having no profits motive, and using efficiently the available technology, which allows to transform one unit of private good in one unit of public good.

As for the behavior of households, each household starts with endowments in private good only. There is no public good initially. For given private good prices, initial endowments of private goods, as well as other households' choices of subscriptions, each household chooses a vector of private good consumption levels and a subscription level (in the amount of the numeraire good) so as to maximize utility.

Note that in choosing their subscription levels the households will take into consideration relative prices. Observe that household behavior described above amounts to assuming that the prices of private goods are taken as given by households in their maximization problem, while there is strategic interaction among them regarding the subscription levels chosen. This type of behavior is plausible when the set of prospective consumers of the public good are 'small' with respect to the economy in which they are embedded. For example, consider donations to a large agency that is involved in projects to eradicate poverty in Africa. Eradication of poverty in Africa is the public good for those who care about it in this case, and their contributions (amounts they subscribe) to the agency are not going to affect the prices of goods that go into the activities of the agency.

We provide a proof of existence in this model using a homotopy argument.

In addition to proving existence, we also show generic regularity of equilibria. Regularity is an indispensable tool for carrying out comparative statics analyses. We show that for any vector of utility and production functions, in an open and full measure subset of the endowments, there is a finite number of equilibria and a local smooth dependence of the equilibrium variables on the exogenous variables.

Section 2 presents the setup of the model. In Section 3 the existence of equilibrium is proved using an homotopy argument. Section 4 contains the proof of generic regularity of the equilibria. Section 5 provides some concluding remarks.

2 Set-up of the Model

We consider a general equilibrium model with private provision of a public good. There are C, $C \geq 1$, private commodities, labelled by $c = 1, 2, ..., C$. There are H households, $H > 1$, labelled by $h = 1, 2, ..., H$. Let $\mathcal{H} = \{1, ..., H\}$ denote the set of households. Let x_h^c denote consumption of private commodity c by household h; e_h^c embodies similar notation for the endowment in private goods.

The following standard notation is also used:

- $x_h \equiv (x_h^c)_{c=1}^C$, $x \equiv (x_h)_{h=1}^H \in \mathbb{R}_{++}^{CH}$.
- $e_h \equiv (e_h^c)_{c=1}^C$, $e \equiv (e_h)_{h=1}^H \in \mathbb{R}_{++}^{CH}$.
- p^c is the price of private good c, with $p \equiv (p^c)_{c=1}^C$. Prices are expressed in units of the numeraire good C, whose price is therefore normalized to 1. Define $p^{\setminus} \equiv (p^c)_{c=1}^{C-1}$ and $p \equiv (p^{\setminus}, 1)$.
- $g_h \in \mathbb{R}_+$ is the amount of resources (measured in units of the numeraire good) that consumer h provides. Let $g \equiv (g_h)_{h=1}^H$, $G \equiv \sum_{h=1}^H g_h$, and $G_{\setminus h} \equiv G - g_h$.
- y^g is the amount of public good produced in the economy.

The preferences over the private goods and the public good of household h are represented by a utility function

$$u_h : \mathbb{R}_{++}^C \times \mathbb{R}_{++} \to \mathbb{R}, \qquad u_h : (x_h, y^g) \mapsto u_h(x_h, y^g)$$

Assumption 1 *$u_h(x_h, y^g)$ is a smooth, differentiably strictly increasing (i.e., for every $(x_h, y^g) \in \mathbb{R}_{++}^{C+1}$, $Du_h(x_h, y^g) \gg 0$)[2], differentiably strictly quasiconcave function (i.e.,$\forall (x_h, y^g) \in \mathbb{R}_{++}^{C+1}$,$\forall v \in \mathbb{R}^{C+1} \setminus \{0\}$, if $Du_h(x_h, y^g) v = 0$, then $v D^2 u_h(x_h, y^g) v < 0$) and for each $\underline{u} \in \mathbb{R}$, the set $\{(x_h, y^g) \in \mathbb{R}_{++}^{C+1} : u_h(x_h, y^g) \geq \underline{u}\}$ is closed (in the standard topology of $\mathbb{R}^{C+1}$).*

[2] For vectors y, z, $y \geq z$ (resp. $y \gg z$) means every element of y is not smaller (resp. strictly larger) than the correponding element of z; $y > z$ means that $y \geq z$ but $y \neq z$.

Let $\mathcal{U}$ be the set of utility functions u_h satisfying Assumption 1.

The production technology available to produce the public good is described by the following production function.

$$f : \mathbb{R}^C_{++} \to \mathbb{R}_{++}, \qquad f : y \longmapsto f(y)$$

Assumption 2 f is C^2, differentiably strictly increasing, differentiably strictly concave (i.e., $\forall y \in \mathbb{R}^C_{++}$, $D^2 f$ is negative definite), and $\forall \underline{f} \in \mathbb{R}_{++}$, $cl_{\mathbb{R}^C} \{y \in \mathbb{R}^C_{++} : f(y) \geq \underline{f}\} \subseteq \mathbb{R}^C_{++}$.

Let $\mathcal{F}$ be the set of production functions f satisfying Assumption 2. Note that Assumption 2 rules out constant returns to scale and increasing returns to scale production functions.[3]

The government collects resources from the contributors, and maximizes the production of public goods, given the constraint to balance the budget, i.e., it solves the following problem. For given $p^{\setminus} \in \mathbb{R}^{C-1}_{++}$ and $G \in \mathbb{R}_{++}$,

$$\max_{y \in \mathbb{R}^C_{++}} f(y) \quad s.t \quad -py + G = 0 \qquad (\alpha) \tag{1}$$

with $G = \sum_h g_h$ and where we follow the convention of writing associated Lagrange or Kuhn-Tucker multipliers next to the constraint.

For given $p^{\setminus} \in \mathbb{R}^{C-1}_{++}$ and $G \in \mathbb{R}_{++}$ a solution to problem (1) is characterized by Lagrange conditions.

Define

$$\widehat{f} : \mathbb{R}^{C-1}_{++} \times \mathbb{R}_{++} \to \mathbb{R}_{++}, \quad \left(p^{\setminus}, G\right) \mapsto \max\ (1)$$

Remark 1 *As an application of the envelope and the implicit function theorems, we have that* $\forall \left(p^{\setminus}, G\right) \in \mathbb{R}^C_{++}$, $D_G \widehat{f}\left(p^{\setminus}, G\right) = \alpha > 0$ *and* $D_{GG} \widehat{f}\left(p^{\setminus}, G\right) = \left(p \left(D^2 f \left(y \left(p^{\setminus}, G\right)\right)\right)^{-1} p\right)^{-1} < 0$, *where* $y\left(p^{\setminus}, G\right) = \arg\max(1)$.

Household's problem is the following one. For given $p^{\setminus} \in \mathbb{R}^{C-1}_{++}, G_{\setminus h} \in \mathbb{R}_+, e_h \in \mathbb{R}^C_{++}$,

$$\max_{(x_h, g_h) \in \mathbb{R}^C_{++} \times \mathbb{R}} u_h \left(x_h, \widehat{f}\left(p, g_h + G_{\setminus h}\right)\right) \quad s.t. \quad \begin{array}{l} -px_h + pe_h - g_h = 0 \\ g_h \geq 0 \end{array}$$

Equivalently, we can write the household's problem as follows. For given $p^{\setminus} \in \mathbb{R}^{C-1}_{++}, G_{\setminus h} \in \mathbb{R}_+, e_h \in \mathbb{R}^C_{++}$,

$$\max_{\left(x_h, g_h, y^g_h\right) \in \mathbb{R}^C_{++} \times \mathbb{R}_{++}} u_h\left(x_h, y^g_h\right) \quad s.t. \quad \begin{array}{ll} -px_h + pe_h - g_h \geq 0 & \lambda_h \\ g_h \geq 0 & \mu_h \\ -y^g_h + \widehat{f}\left(p, g_h + G_{\setminus h}\right) \geq 0 & \eta_h \end{array} \tag{2}$$

[3] The analysis of linear, constant and increasing returns to scale production technologies are surely important, in general, and in the case of public good production, in particular. On the other hand, they lead to multiple solutions and/or severe discontinuities in the firm supply map, preventing the use of the smooth approach we adopt in this paper.

Observe that in the latter formulation, in equilibrium it must be the case that for every h, $y_h^g = y^g$.

Remark 2 *For given* $p^\backslash \in \mathbb{R}_{++}^{C-1}, G_{\backslash h} \in \mathbb{R}_+, e_h \in \mathbb{R}_{++}^C$, *a solution to problem (2) is characterized by Kuhn-Tucker conditions.*

Definition 3 *An economy is an element* $\pi \equiv (e, u, f)$ *in* $\Pi \equiv \mathbb{R}_{++}^{CH} \times \mathcal{U}^H \times \mathcal{F}$.

Definition 4 *A vector* $(y, x, y^g, g, p^\backslash)$ *is an equilibrium for an economy* $\pi \in \Pi$ *if:*

1. *the public firm maximizes, i.e., it solves problem (1) at* $\left(p, \sum_{h=1}^H g_h\right)$*;*
2. *households maximize, i.e., for each* h *,* (x_h, y_h^g, g_h) *solves problem (2) at* $p^\backslash \in \mathbb{R}_{++}^{C-1}, \sum_{h' \neq h} g_h \in \mathbb{R}_+, e_h \in \mathbb{R}_{++}^C$*; and*
3. *markets clear , i.e.,* (x, y) *solves*

$$-\sum_{h=1}^H x_h^\backslash - y^\backslash + \sum_{h=1}^H e_h^\backslash \quad = 0$$

where for each h*,* $x_h^\backslash \equiv (x_h^c)_{c \neq C}$ *,* $e_h^\backslash \equiv (e_h^c)_{c \neq C} \in \mathbb{R}_{++}^{C-1}$ *and* $y^\backslash \equiv (y^c)_{c \neq C} \in \mathbb{R}^{C-1}$*.*[4]

In the remainder of the paper we are going to use several equivalent equilibrium systems. System (3) below simply lists Kuhn-Tucker conditions of the agent's maximization problems and market clearing conditions. System (5) is used to prove existence and it is the closest one to the fictitious economy used to construct the needed homotopy. System (13) is used to show generic regularity.

Define

$$\Xi' \equiv \mathbb{R}_{++}^C \times \mathbb{R}_{++} \times \mathbb{R}_{++} \times \left(\mathbb{R}_{++}^C \times \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}\right)^H \times \{g \in \mathbb{R}^H : \Sigma_{h=1}^H g_h > 0\} \times \mathbb{R}_{++}^{C-1}$$

$$\xi' \equiv \left(y, \alpha, y^g, (x_h, y_h^g, \lambda_h, \eta_h, \mu_h)_{h=1}^H, g, p^\backslash\right)$$

and

$$F_1 : \Xi' \times \mathbb{R}_{++}^{CH} \to \mathbb{R}^{\dim \Xi'}, \quad F_1 : (\xi', \pi) \mapsto \text{left hand side of (3) below}$$

[4] Clearly, the Walras' law applies in this model.

$$
\begin{array}{lll}
(1) & Df(y) - \alpha p & = 0 \\
(2) & -py + \sum_h g_h & = 0 \\
(3) & y^g - f(y) & = 0 \\
(h.1) & D_{x_h} u_h(x_h, y_h^g) - \lambda_h p & = 0 \\
(h.2) & D_{y^g} u_h(x_h, y_h^g) - \eta_h & = 0 \\
(h.3) & -\lambda_h + \mu_h + \eta_h D_G \widehat{f}\left(p, g_h + G_{\backslash h}\right) & = 0 \\
(h.4) & -px_h + pe_h - g_h & = 0 \\
(h.5) & -y_h^g + \widehat{f}\left(p, g_h + G_{\backslash h}\right) & = 0 \\
(h.6) & \min\{g_h, \mu_h\} & = 0 \\
(M) & -\sum_{h=1}^H x_h^{\backslash} - y^{\backslash} + \sum_{h=1}^H e_h^{\backslash} & = 0
\end{array}
\tag{3}
$$

Note that in the above system we in fact have

$$
y^g = f(y) = \widehat{f}\left(p, g_h + G_{\backslash h}\right) = y_h^g \qquad \text{for all } h \tag{4}
$$

where we used the definition of $\widehat{f}$ and equations (1) and (2) in the above system.

Observe that $(y, x, y^g, g, p^{\backslash})$ is an equilibrium associated with an economy π if and only if there exists $(\alpha, \lambda, \eta, \mu)$ such that $F_1\left(y, \alpha, y^g, (x_h, y_h^g, \lambda_h, \eta_h, \mu_h)_{h=1}^H, g, p^{\backslash}, \pi\right) = 0$. With innocuous abuse of terminology, we will call ξ'an equilibrium.

3 Existence of equilibria

To show existence we consider an arbitrary $(u, f) \in \mathcal{U} \times \mathcal{F}$, which remains fixed throughout the analysis. Hence the relevant exogenous variable space is the endowment space $\mathbb{R}_{++}^{CH}$.

We first define a new equilibrium system which is equivalent to system (3) above[5].

$$
\begin{array}{lll}
(fh.1) & Df(y_h) - \alpha_h p & = 0 \\
(fh.2) & -py_h + g_h + G_{\backslash h} & = 0 \\
(h.1) & D_{x_h} u_h(x_h, y_h^g) - \lambda_h p & = 0 \\
(h.2) & D_{y^g} u_h(x_h, y_h^g) - \eta_h & = 0 \\
(h.3) & -\lambda_h + \mu_h + \eta_h \alpha_h & = 0 \\
(h.4) & -px_h + pe_h - g_h & = 0 \\
(h.5) & -y_h^g + \widehat{f}\left(p, g_h + G_{\backslash h}\right) & = 0 \\
(h.6) & \min\{g_h, \mu_h\} & = 0 \\
(M) & -\sum_{h=1}^H x_h^{\backslash} - y_1^{\backslash} + \sum_{h=1}^H e_h^{\backslash} & = 0
\end{array}
\tag{5}
$$

Define

$$
\Xi \equiv \left(\mathbb{R}_{++}^C \times \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{++}^C \times \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}\right)^H \times \{g \in \mathbb{R}^H : \Sigma_{h=1}^H g_h > 0\} \times \mathbb{R}_{++}^{C-1}
$$

[5] The differences between the two systems are the following: We identified y with $y_1$and α with α_1; we introduced the new variables $(y_h, \alpha_h)_{h \neq 1}$; we basically repeated H times the first two equations.

$$\xi \equiv \left(\left(y_h, \alpha_h, y_h^g, x_h, y_h^g, \lambda_h, \eta_h, \mu_h \right)_{h=1}^H, g, p^{\backslash} \right)$$

and

$$F : \Xi \times \mathbb{R}_{++}^{CH} \to \mathbb{R}^{\dim \Xi}, \quad F : (\xi, e) \mapsto \text{left hand side of (5) below}$$

If $F(\xi, e) = 0$, we still call ξ an equilibrium associated with e. We now want to apply the following theorem to prove existence of equilibria.[6]

Theorem 5 *Let M and N be two C^2 boundaryless manifolds of the same dimension, $y \in N$ and $f, g : M \to N$ be such that f is C^0 and g is C^0 and C^1 in an open neighborhood of $g^{-1}(y)$, y is a regular value for g, $\# g^{-1}(y)$ is odd, there exists a continuous homotopy H from f to g such that $H^{-1}(y)$ is compact. Then $f^{-1}(y) \neq \emptyset$.*

In the remaining part of the present section we verify the assumptions of Theorem 5. The basic idea of the proof is as follows. Construct a model where every households produces her own amount of the public good "at home" using the available technology f (and household 1 is treated in a slightly asymmetric way). Equilibria in that model are zeros of the function g mentioned in the above theorem. Then, find an homotopy which "links" those fictitious equilibria with the "true" ones, which are zeros of the function f.

The first step toward the construction of the homotopy is to construct a so-called test economy.

Definition 6 *An allocation $(x, y^g, y) \in \mathbb{R}_{++}^{CH} \times \mathbb{R}_{++}^{H} \times \mathbb{R}_{++}^{CH}$ is feasible at total resources $r \in \mathbb{R}_{++}^{C}$ for an economy where good g can only be "privately home produced" if*

$$\begin{array}{ll} -\sum_{h=1}^{H} (x_h - y_h) + r \geq 0 & \\ -y_h^g + f(y_h) \geq 0 & \text{for each } h \in \mathcal{H} \end{array}$$

A feasible allocation (x, y^g, y) is Pareto optimal in the model where the public good is "privately home produced", if there is no other feasible allocation $\left(x', y^{g'}, y'\right)$ such that $\left(u_h\left(x'_h, y_h^{g'}\right)\right)_{h=1}^H > \left(u_h\left(x_h, y_h^g\right)\right)_{h=1}^H$.

We will refer to Pareto optimal allocations presented in Definition 6 simply as Pareto optimal allocations. Define

$$U^r \equiv \{ (\underline{u}_h)_{h=1}^H \in \mathbb{R}^H : \text{ there exists a feasible allocation } (x, y_h^g) \text{ at } r \text{ such that for each } h,\ u_h(x_h, y_h^g) - \underline{u}_h = 0 \}$$

and

$$\underline{U}_{\backslash 1}^r \equiv \{ (\underline{u}_h)_{h \neq 1} \in \mathbb{R}^{H-1} : \text{ there exists } \underline{u}_1 \in \mathbb{R} \text{ such that } \left(\underline{u}_1, (\underline{u}_h)_{h \neq 1}\right) \in \underline{U}^r \}$$

[6] The theorem is a well-known result in degree thoery; for a self-contained proof, see, for example, Villanacci and others (2002), Chapter 7.

Given our smooth and strictly concave framework, we can easily characterize Pareto optimal allocations in terms of the solutions to the following maximization problem. For given $\left(r, \underline{u}_{\setminus 1}\right) \in \mathbb{R}^C_{++} \times \underline{U}^r_{\setminus 1}$,

$$\begin{array}{lll} \underset{(x,y^g,y)\in\mathbb{R}^{CH}_{++}\times\mathbb{R}^{H}_{++}\times\mathbb{R}^{CH}_{++}}{\text{Max}} & u_1\left(x_1, y_1^g\right) & \\ s.t & & \\ (1) & u_h\left(x_h, y_h^g\right) - \underline{u}_h \geq 0 & \forall h \neq 1 \\ (2) & -y_h^g + f\left(y_h\right) \geq 0 & \forall\ h \in \mathcal{H} \\ (3) & -\sum_{h=1}^{H}\left(x_h + y_h\right) + r \geq 0 & \end{array} \qquad (6)$$

The following lemmas are standard results in general equilibrium theory. Define $\theta_1 \equiv 1$.

Lemma 7 *There exists a unique solution* (x^*, y^{g*}, y^*) *to problem (6) at* $\left(r, \underline{u}_{\setminus 1}\right)$, *and it is Pareto optimal. Moreover, there exists* $(\theta^*, \gamma^{g*}, \gamma^*)$ *such that*

$$(x^*, y^{g*}, y^*, \theta^*, \gamma^{g*}, \gamma^*)$$

is the unique solution to the system of first order conditions for that problem, i.e., to the system below:

$$\begin{array}{lll} (h.1) & \theta_h D_{x_h} u_h\left(x_h, y_h^g\right) - \gamma & = 0 \\ (h.2) & \theta_h D_{y_h^g} u_h\left(x_h, y_h^g\right) - \gamma_h^g & = 0 \\ (fh.1) & \gamma_h^g D f\left(y_h\right) - \gamma & = 0 \\ (5) & \left(u_h\left(x_h, y_h^g\right) - \underline{u}_h\right)_{h\neq 1} & = 0 \\ (6) & -y_h^g + f\left(y_h\right) & = 0 \\ (7) & -\sum_{h=1}^{H}\left(x_h + y_h\right) + r & = 0 \end{array}$$

$(x^*, y^{g*}, y^*, \theta^*, \gamma^{g*}, \gamma^*)$ *is also the unique solution for the first order conditions to problem (6) at*

$$\left(\sum_{h=1}^{H}\left(x_h^* + y_h^*\right), \left(u_h\left(x_h^*, y_h^{g*}\right)\right)_{h\neq 1}\right) \qquad (7)$$

For each $e \in \mathbb{R}^{CH}_{++}$, and associated (x^*, y^*) described in Lemma 7, define

$$\begin{array}{lll} H & : & \Xi \times [0,1] \to \mathbb{R}^{\dim \Xi}, \\ H & : & (\xi, \tau) \mapsto \text{left hand side of system (8)} \end{array}$$

$$
\begin{array}{lll}
(fh.1) & Df(y_h) - \alpha_h p & = 0 \\
(fh.2) & -py_h + g_h + (1-\tau) G_{\setminus h} & = 0 \\
(h.1) & D_{x_h} u_h (x_h, y_h^g) - \lambda_h p & = 0 \\
(h.2) & D_{y^g} u_h (x_h, y_h^g) - \eta_h & = 0 \\
(h.3) & -\lambda_h + \mu_h + \eta_h D_G \widehat{f}\left(p, g_h + (1-\tau) G_{\setminus h}\right) & = 0 \\
(1.4) & -px_1 + p\left[(1-\tau) e_1 + \tau\left(x_1^* + y_1^*\right)\right] - g_1 & \\
(h.4), h \neq 1 & -px_h + p\left[(1-\tau) e_h + \tau\left(x_h^* + y_h^*\right)\right] - g_h - \tau(1-\tau) G_{\setminus h} & = 0 \\
(h.5) & -y_h^g + \widehat{f}\left(p, g_h + (1-\tau) G_{\setminus h}\right) & = 0 \\
(h.6) & \min\{g_h, \mu_h\} & = 0 \\
(M) & \begin{array}{l} -\sum_{h=1}^H x_h^{\setminus} - y_1^{\setminus} - \tau \sum_{h\neq 1} y_h^{\setminus} + \\ \left[(1-\tau)\sum_{h=1}^H e_h^{\setminus} + \tau \sum_{h=1}^H \left(x_h^{*\setminus} + y_h^{*\setminus}\right)\right] \end{array} & = 0
\end{array}
\tag{8}
$$

Remark 8 *Walras law holds for the above system and therefore the "missing" $C-th$ equation in (M) is indeed satisfied.*

Define $G(\xi) \equiv H(\xi, 1)$. Observe that at $\tau = 0$, we get the equilibrium system (5). At $\tau = 1$, we get

$$
\begin{array}{lll}
(fh.1) & Df(y_h) - \alpha_h p & = 0 \\
(fh.2) & -py_h + g_h & = 0 \\
(h.1) & D_{x_h} u_h (x_h, y_h^g) - \lambda_h p & = 0 \\
(h.2) & D_{y^g} u_h (x_h, y_h^g) - \eta_h & = 0 \\
(h.3) & -\lambda_h + \mu_h + \eta_h \alpha_h & = 0 \\
(h.4) & -px_h + px_h^* - g_h + py_h^* & = 0 \\
(h.5) & -y_h^g + \widehat{f}(p, g_h) & = 0 \\
(h.6) & \min\{g_h, \mu_h\} & = 0 \\
(M) & -\sum_{h=1}^H x_h^{\setminus} - y_1^{\setminus} - \sum_{h\neq 1} y_h + \sum_{h=1}^H \left(x_h^* + y_h^*\right) & = 0
\end{array}
\tag{9}
$$

Lemma 9 *There exists ξ^{**} such that $G(\xi^{**}) = 0$.*

Proof. It is enough to take the different subvectors of ξ^{**} as described below.

$$
\begin{array}{lllll}
y_h^{**} = y_h^*, & \alpha_h^{**} = \frac{\gamma^{C*}}{\gamma_h^{g*}}, & x_h^{**} = x_h^*, & y_h^{g**} = y_h^{g*}, & \lambda_h^{**} = \frac{\gamma^{C*}}{\theta_h^*}, \\
\eta_h^{**} = \frac{\gamma_h^{g*}}{\theta_h^*}, & \mu_h^{**} = 0, & g_h^{**} = p^{**} y_h^*, & p^{**} = \frac{\gamma^*}{\gamma^{C*}} &
\end{array}
$$

■

Lemma 10 $\{\xi^{**}\} = G^{-1}(0)$.

Proof. Suppose there exists $\widehat{\xi} \equiv \left(\widehat{y}, \widehat{y}^g, \widehat{\alpha}, \widehat{x}, \widehat{g}, \widehat{\mu}, \widehat{\lambda}, \widehat{p}^{\setminus}\right) \in G^{-1}(0)$, with $\widehat{\xi} \neq \xi^{**}$. Observe that $(\widehat{x}, \widehat{y}) = (x^*, y^*)$. Otherwise,

$$
(x'', y^{g\prime\prime}, y'') \equiv \frac{1}{2}\left(\widehat{x}, \widehat{y}^g, \widehat{y}\right) + \frac{1}{2}\left(x^*, y^{g*}, y^*\right) \tag{10}
$$

would be feasible and Pareto superior to (x^*, y^{g*}, y^*), contradicting Lemma 7. The proof of that results uses the strict quasiconcavity of u_h and the strict concavity of f. Then, from $(f.h.1)$, $(f.h.2)$ and $(h.5)$in system (9), and the definition of y^{g*}, we have also that $\widehat{y^g} = y^{g*}$. The other equalities follow easily from system (9). ■

Lemma 11 *$D_\xi G(\xi^{**})$ has full rank.*

Proof. $D_\xi G(\xi^{**})$ is displayed below.

$$\begin{bmatrix}
D^2_h f & -p^T & & & & & & & \begin{matrix}-\alpha_h I\\ 0\end{matrix} \\
-p & & & & 1 & & & & -y_h^{\backslash T} \\
& & D^2_{x_h x_h} & D^2_{x_h y^g_h} & & ;p^T & & & \begin{matrix}-\lambda_h I\\ 0\end{matrix} \\
& & D^2_{x_h y^g_h} & D^2_{y^g_h y^g_h} & & & -1 & & \\
& \eta_h & & & & -1 & \alpha_h & 1 & \\
& & -p & & -1 & & & & y_h^{\backslash T} \\
& & & -1 & \alpha_h & & & & -\alpha_h y_h^{\backslash} \\
& & & & & & & 1 & \\
-I0 & & -I0 & & & & & &
\end{bmatrix}$$

In fact, it has full rank if the following matrix M does.

$$\begin{bmatrix}
D^2_h f & -p^T & & & & & & \begin{matrix}-\alpha_h I\\ 0\end{matrix} \\
-p & & & & 1 & & & \\
& & D^2_{x_h x_h} & D^2_{x_h y^g_h} & & -p^T & & \begin{matrix}-\lambda_h I\\ 0\end{matrix} \\
& & D^2_{x_h y^g_h} & D^2_{y^g_h y^g_h} & & & -1 & \\
& \eta_h & & & & -1 & \alpha_h & \\
& & -p & & -1 & & & \\
& & & -1 & \alpha_h & & & \\
-I0 & & -I0 & & & & &
\end{bmatrix}$$

We are going to show that $\ker M = \{0\}$. Defining $M\Delta = 0$ with $\Delta \equiv \left((\Delta y_h, \Delta\alpha_h, \Delta x_h, \Delta y^g_h, \Delta g_h, \Delta\lambda_h, \Delta\eta_h)_{h=1}^H, \Delta p^{\backslash} \right)$, $M\Delta = 0$ is displayed below.

$$\begin{cases}
D^2_h f \Delta y_h - p^T \Delta\alpha_h + \begin{bmatrix}-\alpha_h I\\ 0\end{bmatrix} \Delta p^{\backslash} & = 0 \\
-p\Delta y_h + \Delta g_h & = 0 \\
D^2_{x_h x_h}\Delta x_h + D^2_{x_h y^g_h}\Delta y^g_h - p^T\Delta\lambda_h \begin{bmatrix}-\lambda_h I\\ 0\end{bmatrix} \Delta p^{\backslash} & = 0 \\
D^2_{x_h y^g_h}\Delta x_h + D^2_{y^g_h y^g_h}\Delta y^g_h - \Delta\eta_h & = 0 \\
\eta_h \Delta\alpha_h - \Delta\lambda_h + \alpha_h\Delta\eta_h & = 0 \\
-p\Delta x_h - \Delta g_h & = 0 \\
-\Delta y^g_h + \alpha_h \Delta g_h & = 0 \\
[-I0] \sum_h (\Delta y_h + \Delta x_h) & = 0
\end{cases}$$

The proof proceeds through the following steps.

a. $\exists h$ such that $(\Delta x_h, \Delta y_h^g, \Delta y_h) \neq 0$; b. $Du_h(x_h, y_h^g)\begin{pmatrix}\Delta x_h \\ \Delta y_h^g\end{pmatrix} = 0$; c. $\sum_h \Delta y_h \frac{D^2 f(y_h)}{\alpha_h} \Delta y_h + \sum_h (\Delta x_h, \Delta y_h^g) \frac{D^2 u_h(x_h, y_h^g)}{\lambda_h} (\Delta x_h, \Delta y_h^g) = 0$.

From a., we are left with considering two cases. Case 1.: $\exists h$ such that $(\Delta x_h, \Delta y_h^g) \neq 0$; and Case 2.: $\exists h$ such that $\Delta y_h \neq 0$. In both cases, given that u_h is (differentiably) strictly quasiconcave, and f is differentiably strictly concave, we have a. and b. contradict c. ■

Lemma 12 $H^{-1}(0)$ *is compact.*

Proof. Taken an arbitrary sequence $(\xi^v, \tau^v)_{v \in \mathbb{N}}$ in $H^{-1}(0)$, we want to show that, up to a subsequence, it does converge to $(\overline{\xi}, \overline{\tau}) \in H^{-1}(0)$.

Preliminary Remark: As verified below, $\forall h$, $\left\{(x_h^v, y_h^{gv})_h : v \in \mathbb{N}\right\}$ is contained in a closed (in the topology of $\mathbb{R}^C$) subset of $\mathbb{R}^{C+1}_{++}$.

Equations (1.1) $-$ (1.6) in system (9) are Kuhn-Tucker system associated with the problem

$$
\begin{array}{llll}
\max_{(x_1, g_1, y_1^g) \in \mathbb{R}^C_{++} \times \mathbb{R}} u_1(x_1, y_1^g) & s.t. & -px_1 + p\left[(1-\tau) e_1 + \tau(x_1^* + y_1^*)\right] - g_1 = 0 & \lambda_1 \\
& & g_1 \geq 0 & \mu_1 \\
& & -y_1^g + \widehat{f}\left(p, g_1 + (1-\tau) G_{\setminus 1}\right) \geq 0 & \eta_1
\end{array}
$$

Equations $(h.1)$ $-$ $(h.6)$ are the Kuhn-Tucker system of equations associated with the problem

$$
\begin{array}{llll}
\max_{(x_h, g_h, y_h^g) \in \mathbb{R}^C_{++} \times \mathbb{R} \times \mathbb{R}_{++}} u_h(x_h, y_h^g) & s.t. & \begin{array}{l} -px_h + p\left[(1-\tau) e_h + \tau(x_h^* + y_h^*)\right] + \\ -g_h - \tau(1-\tau) G_{\setminus h} = 0 \end{array} & \lambda_h \\
& & g_h \geq 0 & \mu_h \\
& & -y_h^g + \widehat{f}\left(p, g_h + (1-\tau) G_{\setminus h}\right) \geq 0 & \eta_h
\end{array}
$$

Observe that $\forall h$

$$
\begin{aligned}
\widetilde{x}_h &= \widetilde{x} \equiv \frac{1}{3H}\left(\min_{h \in \mathcal{H}} \left\{(1-\tau) e_h^c + \tau(x_h^{*c} + y_h^{*c})\right\}\right)_{c=1}^{C} > 0 \\
\widetilde{g}_h &= \widetilde{g} \equiv p\widetilde{x} > 0 \\
\widetilde{y}_h^g &= \widetilde{y}^g \equiv \widehat{f}\left(p, \widetilde{g}_h + (1-\tau)\sum_{h' \neq h} \widetilde{g}_{h'}\right) \geq \widehat{f}(p, \widetilde{g}_h) = f(\widetilde{x}) > 0
\end{aligned}
$$

belongs to the constraint set of household h for each value of $e_h \in \mathbb{R}^C_{++}$ and $\tau \in [0, 1]$. Therefore $\forall v$

$$
u_h(x_h^v, y_h^{gv}) \geq u_h(\widetilde{x}, f(\widetilde{x}))
$$

and, from Assumption 1, we get the desired result.

Since $\{\tau^v : v \in \mathbb{N}\} \subseteq [0, 1]$, up to a subsequence, $(\tau^n)_{n \in \mathbb{N}}$ converges, say, to $\overline{\tau}$. We distinguish two cases: Case 1. $\overline{\tau} = 0$, and Case 2. $\overline{\tau} \in (0, 1]$.

Case 1. $\overline{\tau} = 0$.

x_1, y_1, y_1^g

$\{(x^v, y_1^v) : v \in \mathbb{N}\}$ is bounded below by zero and, by Walras law, it is bounded above by total resources. From the Preliminary Remark above, $\{\ (x_1^v, y_1^{gv}) : v \in \mathbb{N}\}$ is contained in a closed subset of $\mathbb{R}_{++}^{C+1}$. Therefore $(x_1^v, y_1^{gv})_v$ converges to an element $(\overline{x}_1, \overline{y}_1^g)$ belonging to $\mathbb{R}_{++}^{C+1}$. Moreover, from $(f.1.1), (f.1.2)$ and (1.5) in system (5),

$$y_1^{gv} = f(y_1^v)$$

and therefore $\{\ y_1^{gv} : v \in \mathbb{N}\}$ is bounded above, because $\{\ y_1^v : v \in \mathbb{N}\}$ is bounded above. Take $\overline{\overline{y}}_1^g = \min\{y_1^{gv} : n \in \mathbb{N}\} \cup \{\overline{y}_1^g\}$. Then $\forall v$, $y_1^{gv} \geq \overline{\overline{y}}_1^g$. On the other hand,

$$\{y_1^v : v \in \mathbb{N}\} \subseteq \left\{y_1' \in \mathbb{R}_{++}^C : f(y_h') \geq \overline{\overline{y}}_1^g\right\}$$

otherwise $\exists \overline{v}$ such that

$$f\left(y_1^{\overline{v}}\right) < \overline{\overline{y}}_1^g$$

but

$$\forall v, \ \ \overline{\overline{y}}_1^g \leq y_1^{gv} = f(y_1^v)$$

a contradiction.

Therefore, y_1^v is bounded above and below and it is contained in closed set of $\mathbb{R}_{++}^C$ and therefore converges.

$\lambda_1, p, \eta_1, \alpha_1, \mu_1, g_1$

$(\lambda_1^v, p^v), \eta_1^v, \alpha_1^v, \mu_1^v, g_1^v$ converge from (1.1), (1.2), $(f.1.1)$, (1.3), and (1.4) in system (5), respectively.

$(y_h^g)_{h\neq 1}$

From $(fh.1), (fh.2)$ and $(h.5)$, $f(y_h^v) = y_h^{gv} = y_1^{gv}$and therefore $(y_h^{gv})_{h\neq 1}$ converges.

$(x_h)_{h\neq 1}$

Using market clearing, i.e., (M) and Walras'law, for x_h and the above fact for $(y_h^g)_{h\neq 1}$,we have that (x_h^v, y_h^{gv}) is bounded above. It is bounded below by assumption, and from the Preliminary Remark it is contained in a closed subset of $\mathbb{R}_{++}^{(C+1)(H-1)}$.

$(g_h)_{h\neq 1}$

From $(h.4)$ for each h,we get

$$A^v \left(g_h^v\right)_{h=1}^H = \left(\gamma_h^v\right)_{h=1}^H \tag{11}$$

where

$$A \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ \delta & 1 & ... & \delta \\ ... & ... & ... & ... \\ \delta & \delta & ... & 1 \end{bmatrix}$$

and $\delta \equiv \tau(1-\tau)$.

Taking limits for $v \to \infty$ of both sides of (11), we have that the right hand side does converge say to γ because of previous steps, and we have

$$I\left(\lim_{v\to\infty} g_h^v\right)_{h=1}^{H} = \gamma$$

as desired.

$\boxed{(y_h)_{h\neq 1}}$

By assumption $\{ y_h^v : v \in \mathbb{N}\}$is bounded below. It is bounded above, otherwise $(f.h.2)$ would be violated. Following the same argument presented above for the convergence of y_1^v we can show that$\{y_h^v : n \in \mathbb{N}\}$ is contained in a closed subset of $\mathbb{R}^C_{++}$.

$\boxed{(\lambda_h, \eta_h, \alpha_h, \mu_h)_{h\neq 1}}$

The argument is the same as that one used to show convergence of the corresponding variables for household 1.

Case 2. $\overline{\tau} \in (0, 1]$.

$\boxed{(x_h, y_h, y_h^g)_h}$

Observe that $\{(x^v, y^v) : v \in \mathbb{N}\}$ is bounded below by zero and, since Walras law for the homotopy does hold, it is bounded above by total resources, as illustrated below. $\forall\ v$

$$y_1^v \leq \left[(1-\tau^v)\sum_{h=1}^{H} e_h^{\backslash} + \tau^v \sum_{h=1}^{H}\left(x_h^{*\backslash} + y_h^{*\backslash}\right)\right] - \sum_{h=1}^{H} x_h^{v\backslash} - \tau^v \sum_{h\neq 1} y_h^{v\backslash}$$

Since $\tau^v \to \overline{\tau} > 0$, for v sufficiently large $\tau^v > 0$. Take $\overline{\overline{\tau}} = \min\{\tau^v : v \in \mathbb{N}\} \cup \{\overline{\tau}\}$. For $h \neq 1$, $\forall v$

$$y_h^v \leq \tfrac{1}{\tau^v}\left(-\textstyle\sum_{h=1}^{H} x_h^{v\backslash} - y_1^{v\backslash} - \tau \sum_{h'\neq 1,h} y_{h'}^{v\backslash} + \left[(1-\tau^v)\sum_{h=1}^{H} e_h^{\backslash} + \tau^v \sum_{h=1}^{H}\left(x_h^{*\backslash} + y_h^{*\backslash}\right)\right]\right) \leq \tfrac{1}{\overline{\overline{\tau}}}\left[\textstyle\sum_{h=1}^{H} e_h^{\backslash} + \sum_{h=1}^{H}\left(x_h^{*\backslash} + y_h^{*\backslash}\right)\right]$$

Moreover, $\forall h, v$

$$x_h^v \leq (1-\tau^v)\sum_{h=1}^{H} e_h^{\backslash} + \tau^v \sum_{h=1}^{H}\left(x_h^{*\backslash} + y_h^{*\backslash}\right)$$

From $(f.h.1), (f.h.2)$ and $(h.5)$,

$$y_h^{gv} = f\left(y_h^v\right)$$

and therefore y_h^{gv} is bounded above, because y_h^v is bounded above.

Summarizing, (x_h^v, y_h^{gv}) is bounded above and below, and, from the Preliminary Remark, it is contained in closed set contained in $\mathbb{R}^{C+1}_{++}$. Therefore it converges to an element $(\overline{x}_h, \overline{y}_h^g)$ belonging to $\mathbb{R}^{C+1}_{++}$. Convergence of y_h^v is shown using the same argument used for the convergence of y_1^v in Case 1 above.

Therefore, $\left\{ \left(x_h, y_h, y_h^g \right)^v \right\}$ is contained in a compact subset of $\mathbb{R}_{++}^{2C+1}$, and thus it converges.

$\boxed{\left(\lambda_h, \eta_h, \alpha_h, \mu_h \right)_h, p}$

The argument is the same as in Case 1.

$\boxed{\left(g_h \right)_h}$

From $(f.h.2)$ we have

$$\begin{bmatrix} 1 & (1-\tau^v) & ... & (1-\tau^v) \\ (1-\tau^v) & 1 & ... & (1-\tau^v) \\ ... & ... & ... & ... \\ (1-\tau^v) & (1-\tau^v) & ... & 1 \end{bmatrix} \begin{bmatrix} g_1^v \\ g_2^v \\ ... \\ g_H^v \end{bmatrix} = \begin{bmatrix} p^v y_1^v \\ p^v y_2^v \\ ... \\ p^v y_H^v \end{bmatrix}$$

Taking limits of both sides, it follows that it suffices to show that the following matrix has full rank.

$$\begin{bmatrix} 1 & (1-\overline{\tau}) & ... & (1-\overline{\tau}) \\ (1-\overline{\tau}) & 1 & ... & (1-\overline{\tau}) \\ ... & ... & ... & ... \\ (1-\overline{\tau}) & (1-\overline{\tau}) & ... & 1 \end{bmatrix}$$

That is the case iff $\overline{\tau} \neq 0$, which we assumed, and if $1 - (H-1)\overline{\tau} + H - 1 \neq 0$, which is certainly true. ■

The previous results lead to the main result of the section.

Theorem 13 *For every economy $\pi \in \Pi$, an equilibrium exists.*

4 Regularity

Lemma 14 *For every $e \in \mathbb{R}_{++}^{CH}$, ξ' is a solution to system (3) if and only if it is a solution to the following system*

$$\begin{array}{lll} (1) & Df(y) - \alpha p & = 0 \\ (2) & -py + \sum_h g_h & = 0 \\ (3) & y^g - f(y) & = 0 \\ (h.1) & D_{x_h} u_h (x_h, y^g) - \lambda_h p & = 0 \\ (h.3') & \alpha D_{y^g} u_h (x_h, y^g) - \lambda_h + \mu_h & = 0 \\ (h.4) & -p x_h + p e_h - g_h & = 0 \\ (h.6) & \min \{ g_h, \mu_h \} & = 0 \\ (M) & -\sum_{h=1}^H x_h^\setminus - y^\setminus + \sum_{h=1}^H e_h^\setminus & = 0 \\ (h.5') & -y_h^g + y^g & = 0 \\ (h.6') & \alpha \eta_h - \lambda_h + \mu_h & = 0 \end{array} \quad (12)$$

Proof. The proof follows from the comparison of systems (3) and (12), and from Remark 1. ■

Since η_h appears only in equation $(h.6')$ and it is uniquely determined by that equation, and y_h^g appears only in equation $(h.5')$ and it is uniquely determined by that equation,we can erase those variables and equations and get the following basically equivalent system.

$$
\begin{array}{lll}
(1) & Df\left(y\right)-\alpha p & =0 \\
(2) & -py+\sum_h g_h & =0 \\
(3) & y^g - f\left(y\right) & =0 \\
(h.1) & D_{x_h}u_h\left(x_h,y^g\right)-\lambda_h p & =0 \\
(h.2) & \alpha D_{y^g}u_h\left(x_h,y^g\right)-\lambda_h+\mu_h & =0 \\
(h.3) & -px_h+pe_h-g_h & =0 \\
(h.4) & \min\left\{g_h,\mu_h\right\} & =0 \\
(M) & -\sum_{h=1}^H x_h^{\setminus}-y^{\setminus}+\sum_{h=1}^H e_h^{\setminus} & =0
\end{array} \tag{13}
$$

Define

$$\widetilde{\Xi}\equiv\mathbb{R}_{++}^C\times\mathbb{R}_{++}\times\mathbb{R}_{++}\times\left(\mathbb{R}_{++}^C\times\mathbb{R}_{++}\times\mathbb{R}\right)^H\times\{g\in\mathbb{R}^H:\sum_{h=1}^H g_h>0\}\times\mathbb{R}_{++}^{C-1}$$

$$\widetilde{\xi}\equiv\left(y,\alpha,y^g,\left(x_h,\lambda_h,\mu_h\right)_{h=1}^H,g,p^{\setminus}\right)$$

and

$$F_2:\widetilde{\Xi}\times\mathbb{R}_{++}^{CH}\to\mathbb{R}^{\dim\widetilde{\Xi}},\quad F_2:\left(\widetilde{\xi},e\right)\mapsto\text{ left hand side of system (13)}$$

We can now prove that there is a large set of the endowments (the so-called regular economies) for which associated equilibria are finite in number, and that equilibria change smoothly with respect to endowments - see Theorem 17 below. To do this, we need to restrict the set of utility functions adding the following assumptions

Assumption 3. $\forall h$, u_h is differentiably strictly concave, i.e., $\forall\left(x_h,G\right)\in\mathbb{R}_{++}^{C+1}$, $D^2u_h\left(x_h\right)$ is negative definite.

Assumption 4. For all h and $\left(x_h,y^g\right)\in\mathbb{R}_{++}^{C+1}$

$$\det\begin{bmatrix} D_{x_hx_h}u_h\left(x_h,y^g\right) & \left[D_{x_h}u_h\left(x_h,y^g\right)\right]^T \\ D_{y^gx_h}u_h\left(x_h,y^g\right) & D_{y^g}u_h\left(x_h,y^g\right)\end{bmatrix}\neq 0$$

Remark 15 *In fact in the proof of regularity, we need that the following matrix has full rank*

$$\begin{bmatrix} D_{x_hx_h} & -p^T \\ \alpha D_{y^gx_h} & -1\end{bmatrix}$$

in the case of household h being a contributor (and therefore $\mu_h=0$) in equilibrium, which is in fact implied by Assumption 4.

The above assumption has an easy and appealing economic interpretation. Let $w_h \in \mathbb{R}_{++}$ denote the wealth of household h, and $g_h : \mathbb{R}^C_{++} \times \mathbb{R}_+ \to \mathbb{R}_+$, $\left(p^{\backslash}, w_h, G_{\backslash h}\right) \mapsto g_h\left(p^{\backslash}, w_h, G_{\backslash h}\right)$ denote the private contribution function of household h, i.e., part of the solution function to problem

$$\max_{\left(x_h, g_h, y_h^g\right) \in \mathbb{R}^C_{++} \times \mathbb{R} \times \mathbb{R}_{++}} u_h\left(x_h, y_h^g\right) \quad s.t. \quad \begin{array}{ll} -px_h + w_h - g_h \geq 0 & \lambda_h \\ g_h \geq 0 & \mu_h \\ -y_h^g + \widehat{f}\left(p, g_h + G_{\backslash h}\right) \geq 0 & \eta_h \end{array} \tag{14}$$

Lemma 16 *Assumption 4 is equivalent to*

for each $\left(p^{\backslash}, w_h, G_{\backslash h}\right) \in \mathbb{R}^C_{++} \times \mathbb{R}_+$ *such that* $g_h\left(p^{\backslash}, w_h, G_{\backslash h}\right) > 0,\ D_{w_h} g_h\left(p^{\backslash}, w_h, G_{\backslash h}\right) \neq 0$

Proof. The result follows from an application of the Implicit Function Theorem to the first order conditions of the household maximization problem (14).

To verify that the theorem can be applied we check that the Jacobian $\widetilde{M}$ of the left hand sides of the First Order Conditions of problem (14), with $g_h > 0$, has full rank. $\widetilde{M}$ is displayed below.

$$\begin{bmatrix} D_{x_h x_h} u_h & D_{x_h y_h^g} u_h & & -p & & \\ D_{y_h^g x_h} u_h & D_{y_h^g y_h^g} u_h & & & -1 & \\ & & \eta_h D_{GG}\widehat{f} & -1 & D_G \widehat{f} & \\ -p^T & & -1 & & & 1 \\ & -1 & D_G \widehat{f} & & & \end{bmatrix}$$

It is then enough to show that $\ker \widetilde{M} = \{0\}$. Define $\widetilde{\Delta} \equiv \left(\Delta x_h, \Delta y_h^g, \Delta g_h, \Delta\lambda_h, \Delta\eta_h\right)_{h=1}^H$. We first show that if $\left(\Delta x_h, \Delta y_h^g\right) = 0$, then $\Delta = 0$. If $\left(\Delta x_h, \Delta y_h^g\right) \neq 0$, then $\left(\Delta x_h, \Delta y_h^g\right) D^2 u_h\left(x_h, y_h^g\right)\left(\Delta x_h, \Delta y_h^g\right) + \left(\Delta g_h\right)^2 \eta_h D_{GG}\widehat{f} = 0$, contradicting the fact that $D^2 u_h$ is negative definite and $\left(\eta_h D_{GG}\widehat{f}\right) < 0$.

We then compute $D_{w_h} g_h$ as follows.

$$D_{w_h}\left(x_h, g_h, y_h^g, \lambda_h, \eta_h\right) = -\widetilde{M}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

We can now use the following facts. Take $A_{n \times n} \equiv \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ with A_{11}a square n_1 dimensional matrix, A_{22} a square n_2 dimensional matrix, $n_1 + n_2 = n$.

1. If both A_{11} and $C \equiv A_{22} - A_{21} A_{11}^{-1} A_{12}$ are invertible, then it can be easily checked that

$$A^{-1} \equiv \begin{bmatrix} A_{11}^{-1}\left(I + A_{12} C^{-1} A_{21} A_{11}^{-1}\right) & -A_{11}^{-1} A_{12} C^{-1} \\ -C^{-1} A_{21} A_{11}^{-1} & C^{-1} \end{bmatrix}$$

2.

$$\det A = \det A_{11} \det \left[A_{22} - A_{21}A_{11}^{-1}A_{12}\right] = \det A_{22} \det \left[A_{11} - A_{12}A_{22}^{-1}A_{21}\right]$$

if A_{11}^{-1} or A_{22}^{-1} exists, respectively - see 0.8.5, in Horn and Johnson (1985).

3. If $A_{n\times n}$ is negative definite and $rank\ C_{m\times n} = m$, then CAC^T is negative definite - see7.1.6, in Horn and Johnson (1985).

Using the above facts, we get

$$-D_w g = \left(D_{y^g y^g} u_h - D_{y^g x} u_h \left(D_{xx} u_h\right)^{-1} D_{y^g x} u_h\right)^{-1} \frac{\alpha}{\lambda_h} \left(-D_{y^g x} u_h \left(D_{xx} u_h\right)^{-1} D_x u_h + D_{y^g} u_h\right)$$

and finally

$$\det \begin{bmatrix} D_{xx}u_h & D_x u_h \\ D_{y^g x} u_h & D_{y^g} u_h \end{bmatrix} = \det D_{xx} u_h \cdot \det D_{y^g} u_h - D_{y^g x} u_h \left(D_{xx} u_h\right)^{-1} D_x u_h$$

as desired. ■

Call $\widetilde{\mathcal{U}}$ the subset of $\mathcal{U}$ whose elements satisfy Assumptions 3 and 4. Define

$$pr : F_2^{-1}(0) \to \mathbb{R}^{CH}_{++}, \quad pr : (\xi, e) \mapsto e$$

We can state the main theorem of this section.

Theorem 17 *For each $(u, f) \in \widetilde{\mathcal{U}} \times \mathcal{F}$, there exists an open and full measure subset $\mathcal{R}$ of $\mathbb{R}^{CH}_{++}$ such that*

1. *there exists $r \in \mathbb{N}$ such that $F_{2,e}^{-1}(0) = \left\{\widetilde{\xi}^i\right\}_{i=1}^{r}$;*
2. *there exist an open neighborhood Y of e in $\mathbb{R}^{CH}_{++}$, and for each i an open neighborhood U_i of $\left(\widetilde{\xi}^i, e\right)$ in $F_2^{-1}(0)$ such that $U_j \cap U_k = \emptyset$ if $j \neq k$, $pr^{-1}(Y) = \cup_{i=1}^r U_i$ and $pr_{|U_i} : U_i \to Y$ is a diffeomorphism.*

A key ingredient in the proof of the above theorem is to show that zero is a regular value for F_2. An obvious immediate problem is that the min function used in defining the equilibrium function F_2 is not even differentiable when both the constraint and the multiplier are equal to zero, which can be called a “border line” case. We therefore show that border line cases occur outside an open and full measure subset D^* of the economy space.

Lemma 18 *For each $(u, f) \in \widetilde{\mathcal{U}} \times \mathcal{F}$, there exists an open and full measure subset D^* of $\mathbb{R}^{CH}_{++}$ such that $\forall e \in D^*$ and $\forall \widetilde{\xi}$ such that $F_2\left(\widetilde{\xi}, e\right) = 0$, it is the case that*

$$\forall h \in \mathcal{H},\ \textit{either } g_h > 0 \textit{ or } \mu_h > 0$$

Proof. Define the set

$$C \equiv \left\{ \left(\widetilde{\xi}, e \right) \in F_2^{-1}(0) : \exists\, h \text{ such that } g_h = \mu_h = 0 \right\}$$

and observe that $D^* = \mathbb{R}_{++}^{CH} \setminus pr(C)$. Since C is a closed set, openness of D^* follows from properness of pr, which can be proven following the same arguments contained in Lemma 12. The proof of full measure proceeds in three steps.

1. Let $\mathcal{P}$ be the family of all partitions of $\mathcal{H}$ into three subsets $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$. In the equilibrium system 13, in place of $\min\{g_h, \mu_h\} = 0$ substitute $\mu_h = 0$ for $h \in \mathcal{H}_1$, $g_h = 0$ for $h \in \mathcal{H}_2$, and $g_h = 0$ and $\mu_h = 0$ for $h \in \mathcal{H}_3$ (see the first column of Table below for demonstration of how this is done).

2. Define

$$F_{2,\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3} : \Xi \times \mathbb{R}_{++}^{CH} \to \mathbb{R}^{\dim \Xi + \#\mathcal{H}_3}$$

which associates the left hand side of the equilibrium system (13) modified as explained above to each $\left(\widetilde{\xi}, e \right)$ in the domain.

3. Define the set

$$B_{\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3} \equiv \{ e \in \mathbb{R}_{++}^{CH} : \exists \widetilde{\xi} \text{ such that } F_{2,\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3} \left(\widetilde{\xi}, e \right) = 0 \}$$

and observe that[7]

$$D^* \supseteq \mathbb{R}_{++}^{CH} \setminus \bigcup_{\{\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3\} \in \mathcal{P}} B_{\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3} \tag{15}$$

That $B_{\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3}$ is of measure zero follows from the parametric transversality theorem - see, for example, Hirsch (1976), Theorem 2.7, page 79 - and from the fact that zero is a regular value for $F_{2,\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3}$, which is shown below.

In the following Table, the components of $F_{2,\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3}$ are listed in the first column, the variables with respect to which derivatives are taken are listed in the first row, and in the remaining bottom right corner the corresponding partial Jacobian of $D_{\left(\widetilde{\xi}, e\right)} F_{2,\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3} \left(\widetilde{\xi}, e \right)$ is displayed. Call J that partial Jacobian. Observe that in the Table below, $h_1 \in \mathcal{H}_3$ and $h_3 \in \mathcal{H}_3$, and $z_h \equiv x_h - e_h$, $z \equiv \sum_h z_h$, $z^{\setminus} \equiv (z^c)_{c \neq C}$; $\forall h$, $D^h_{x_h x_h} \equiv D_{x_h x_h} u_h (x_h, y^g)$, $D^h_{x_h y^g} \equiv D_{x_h y^g} u_h (x_h, y^g)$, $D^h_{y^g x_h} \equiv D_{y^g x_h} u_h (x_h, y^g)$, $D^h_{y^g y^g} \equiv D_{y^g y^g} u_h (x_h, y^g)$. Note also that, to simplify the exposition of the proof, only one household from each

[7] We cannot use the equality sign in (15), because $B_{\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3}$ contains economies e such that $F_{\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3}(\xi, e) = 0$ and for which some components g_h and μ_h of ξ may be negative.

set is presented in the Table.

	y	α	x_1	g_1	λ_1	μ_1	x_2	g_2	λ_2	μ_2	y^g	$e_1^\backslash$	e_1^C	$e_2^\backslash$	e_2^C
$Df(y)$ $-\alpha p$ (1)	$D^2 f$	$-p^T$													
$-py+$ $\Sigma_h g_h$ (2)	$-p$			1				1							
$D_{x_1} u_1+$ $-\lambda_1 p$ (1.1)			$D^1_{x_1 x_1}$		$-p^T$						$D^1_{x_1 y^g}$				
$\alpha D_{y^g} u_1$ $-\lambda_1+\mu_1$ (1.2)		$D^1_{y^g}$	$\alpha D^1_{y^g x_1}$		-1	1					$\alpha D^1_{y^g y^g}$				
$-pz_1+$ $-g_1$ (1.3)			$-p$	-1								$p^\backslash$	1		
μ_1 (1.4)						1									
$D_{x_2} u+$ $-\lambda_2 p$ (2.1)							$D^2_{x_2 x_2}$		$-p^T$		$D^2_{x_2 y^g}$				
$\alpha D_{y^g} u_2$ $-\lambda_2+\mu_2$ (2.2)		$D^2_{y^g}$					$D^2_{y^g x_2}$		-1	1	$\alpha D^2_{y^g y^g}$				
$-pz_2+$ $-g_2$ (2.3)							$-p$	-1						$p^\backslash$	1
g_2 (2.4)								1							
$-x^\backslash - y^\backslash$ $+e$ (M)	$-I0$		$-I0$				$-I0$					I		I	
$-y^g+f(y)$ (3)	$Df(y)$										-1				
g_1 (G)				1											

To show that J has full rank, we can use elementary column operations. The basic idea is to use a full row rank submatrix with zero in its supercolumn to erase all the terms in its superrow. Below, we apply that method simply listing a. the name of the row, and b. the submatrix used to "clean up" that row. Of course, the order of those operations is crucial - even though there are different possibilities which work as well.

$a.$	(1.3)	(2.3)	(M)	(1.1),(1.2)[8]	(1.4)	(2.1),(2.2)	(3)	(1),(2)	(2.4)	$(G$
$b.$	1	1	I	$\begin{bmatrix} D^1_{x_1x_1} & -p^T \\ \alpha D^1_{y^g x_1} & -1 \end{bmatrix}$	1	$\begin{bmatrix} D^2_{x_2x_2} & 0 \\ D^2_{y^g x_2} & 1 \end{bmatrix}$	-1	$\begin{bmatrix} D^2 f & -p^T \\ -p & 0 \end{bmatrix}$	1	1

■

To prove Theorem 17 it is enough to follow the steps of the proof of Theorem 18, apart from that one related to the perturbation of the equation (G).

[8] We are using Assumption 4 and athe fact that $\mu_1 = 0$.

5 Concluding Remarks

Subscription equilibrium mechanisms are perhaps the most decentralized and market oriented mechanisms a society can consider for provision of public goods. For the case where the public good is produced by a public (non-profit) firm, we show in this paper that the subscription equilibrium mechanism has some desirable consistency (i.e., existence of equilibria) and basic structural properties (typical finiteness and "nice" dependence of endogenous variables from exogenous variables). Those properties are a necessary first step in the direction of analyzing other (more) interesting questions like the effect of different government policies on the total amount of public good provided and on the welfare of households at a subscription equilibrium with public production. We address those questions in a companion paper (Villanacci and Zenginobuz, 2006).[9]

It is, of course, well known that in a large class of models, public good provision through voluntary subscriptions leads to levels of public good that are lower than those associated with Pareto optima.[10] The non-cooperative behavior of households in determining their subscription levels leads to the infamous free-rider problem. On the other hand, equilibrium notions that lead to Pareto optimal outcomes in the presence of public goods suffer from the fact that they are typically not implementable in the presence of asymmetric information about preferences of households. For example, the Lindahl equilibrium notion, which is an extension of the Walrasian equilibrium notion to public good economies, involve personalized prices for households and their determination involve full knowledge of household preferences by a central authority. The non-cooperative subscription equilibrium notion, on the other hand, does not involve a central authority. In future work we plan to allow for asymmetric information in a full-fledged general equilibrium model with public goods and study the impact of government interventions in such a setting. In that context, it will be appropriate to model the economic policy institution as being managed by private agents whose goals are not necessarily consistent with the benevolent purpose of Pareto improving upon the existing situation, which will in turn require modelling the behavior of the institution as part of a principal-agent model.

A model with public goods is clearly a particular case of large class of models that involve more general externalities.[11] Therefore, previous works on existence of competitive equilibrium with externalities in consumption and production relate to the model we study here, but, as we illustrate below, our existence result is novel.

[9] See Villanacci and Zenginobuz (forthcoming a, forthcoming b) for studies on the impact of goverment intervention on subscription equilibrium outcomes in the case where the public good is produced by competitive firms.

[10] See, for example, Cornes and Sandler (1985).

[11] A standard definition of externalities is the following one "An externality is present whenever some economic agent's welfare (utility or profit) includes real variables whose values are chosen by others without particular attention to the effect upon the welfare of the other agents they affect." (Myles (1995), page 313.) A model with private provision of public good is then a particular model with externalities: each household voluntary contribution influences the amount of public good produced and therefore other households utility levels.

McKenzie (1955) studies a model of externalities with production, but he considers a linear production technology. Arrow and Hahn (1971) allow for decreasing returns to scale in production, but they postulate the existence of a consumption vector whose utility is independent of other agents' choices, an assumption which fails to hold in our model. Shafer and Sonnenschein (1975) show the existence of competitive equilibria in a pure exchange framework, hence their results do not apply to our model which incorporates production. Moreover, the definition of equilibrium by Shafer and Sonnenschein (1975) is problematic due to the way they treat the (artificial) distinction between an household's actual consumption and her intended consumption (see pages 84 and 85)[12].

Florenzano (2003) presents a proof of existence for a general equilibrium model with externalities and production under quite general assumptions[13]. The main difference between Florenzano's set-up and the one in this paper is that in our case firms do not behave competitively. In fact, in our model, there is only one firm which maximizes the level of production under a balanced budget constraint. Moreover, not only consumer preferences depend upon other households' decisions and production, but, by very definition of the firm problem, production choices depend upon consumers' choices[14], a feature of the model which is not encompassed by Florenzano's framework.[15]

[12] Arrow and Hahn (1971) themselves are aware of that problem and discuss it - see page 132.

[13] To the best of our knowledge, Florenzano's book presents the most general results on the topic.

[14] The constraint set of the firm - see problem 1 - is determined by households' subscriptions.

[15] The present model could be reinterpreted as one in which there is a particular kind of consumer (the public firm) which maximizes her utility under a budget constraint. But in that case another modelling assumption in Florenzano's framework, namely the fact that the consumers' budget sets do not depend upon other consumers' choices, will be violated.

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