ECONOMETRIC MODELING AND EVALUATION OF FISCAL-MONETARY POLICY INTERACTIONS

Fei Tan

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Doctoral Committee

Eric M. Leeper, Ph.D., Co-Chair

Todd B. Walker, Ph.D., Co-Chair

Joon Y. Park, Ph.D.

Juan Carlos Escanciano, Ph.D.

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How do fiscal and monetary policies interact to determine inflation? The conventional view rests on the Taylor principle, that central banks can control inflation by raising nominal interest rate more than one-for-one with inflation. This principle embeds an implicit assumption that the government always adjusts taxes or spending to assure fiscal solvency. But when the required fiscal adjustments are not assured, as may occur during periods of fiscal stress, monetary policy may no longer be able to determine inflation. Under this alternative view, policy roles are reversed, with fiscal policy determining the price level and monetary policy acting to stabilize debt. Because these two policy regimes imply starkly different policy advice, identifying the prevailing regime is a prerequisite to understanding the macro economy and to making good policy choices.

This dissertation employs econometric modeling and evaluation techniques to examine the empirical implications of the dynamic interactions between post-war U.S. fiscal and monetary policies. Chapter One compares two econometric interpretations of a dynamic macro model designed to study U.S. policy interactions. Two main findings emerge. First, the data overwhelmingly support the conventional view of inflation determination under the prevailing, "strong" econometric interpretation that takes literally *all* of the model's implications for the data. But this result is susceptible to any potential model misspecification. Second, according to the alternative, "minimal" econometric interpretation that is immune to the difficulties with the strong interpretation, the two views of inflation determination can explain the data about equally well. These findings imply that the apparent statistical support in favor of the conventional view over the alternative in the literature stems largely

from the strong interpretation rather than from compelling empirical evidence. Therefore, a prudent policymaker should broaden her perspective beyond any single view on the inflation process.

Chapter Two, joint with Todd B. Walker, develops an analytic function approach to solving generalized multivariate linear rational expectations models. This solution method is shown to provide important insights into equilibrium dynamics of well-known models. Chapter Three further demonstrates the usefulness of this method via a conventional new Keynesian model.

Eric M. Leeper, Ph.D., Co-Chair

Todd B. Walker, Ph.D., Co-Chair

Joon Y. Park, Ph.D.

Juan Carlos Escanciano, Ph.D.

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Chapter 1

Two Econometric Interpretations of U.S. Fiscal and Monetary Policy Interactions

1.1 Introduction

How do fiscal and monetary policies interact to determine inflation? Two distinct views on this fundamental economic question circulate in the profession. The conventional wisdom is the Taylor principle, that central banks can control inflation by systematically raising nominal interest rate more than one-for-one with inflation. This principle embeds an implicit assumption that the government always adjusts taxes or spending to assure fiscal solvency. But when the required fiscal adjustments are not assured, as may occur during periods of fiscal stress, then monetary policy may no longer be able to determine inflation. In this alternative regime, policy roles are reversed, with fiscal policy determining the price level and monetary policy acting to stabilize debt. Because these two policy regimes imply completely different mechanisms for determining inflation, different effects of fiscal and monetary disturbances and, therefore, starkly different policy advice, identifying the prevailing regime is a prerequisite to understanding the macro economy and to making good policy choices.

Efforts to test interactions between U.S. fiscal and monetary policies typically find that the alternative regime fails to explain post-war U.S. time series, leading to nearly uniform acceptance of the conventional view of inflation determination. This consensus emerged despite theoretical demonstrations that the two regimes can provide equally plausible interpretations of the data [Cochrane (2001, 2011), Sims (2011), Leeper and Walker (2011)]. To address the gap between theory and empirical consensus, this article compares two econometric procedures for interpreting a dynamic stochastic general equilibrium (DSGE) model designed to study U.S. policy interactions.

Recent developments in macroeconometric techniques have brought DSGE models to the forefront for guiding quantitative policymaking. Geweke (2010) points out that the relation between DSGE models and their measured economic behavior can be given two alternative econometric interpretations. [i.] The strong econometric interpretation is that the model provides a predictive distribution for all the observables. DSGE models do not fare well under this interpretation because it requires an explicit accounting of many aspects of the data that are poorly explained by the model, making inferences susceptible to any possible model misspecification.[1] [ii.] The minimal econometric interpretation, initiated by DeJong et al. (1996) and formalized by Geweke (2010), is that the model provides a predictive distribution for selected population moments. Since the model has no direct implications for the observable sequence, it is exempt from the difficulties with DSGEs interpreted in the strong sense.[2] This article compares the strong and minimal econometric interpretations of the post-war U.S. fiscal and monetary policy interactions. The comparison is presented with

[1]Rational expectations models in the 1970s and the early 1980s that failed to pass the likelihood-based specification tests can be understood to be interpreted in this strong sense. Recent notable exceptions are Smets and Wouters (2003, 2007), who find that large-scale DSGE models can attain competitive model fit relative to more profligately parameterized statistical models, such as VARs. Nevertheless, Del Negro et al. (2007) have shown that model fit comparisons between DSGE models and VAR models may not be robust since slight changes in the sample period can alter the fit ranking.

[2]Geweke (2010) actually includes a third, weak econometric interpretation under which the model provides a predictive distribution for selected sample moments. But other than making a dimensional reduction, its assumptions are in essence no weaker than the strong one. DSGE models estimated with the generalized method of moments (GMM) fall under this category, as do calibrated DSGE models if calibration is considered as a just-identified GMM. This article focuses on just two interpretations, leaving the exploration of weakly interpreted DSGE models of policy interactions for future research.

reference to a small-scale new Keynesian DSGE model with long-term nominal bonds—a key feature missing from many existing models—and two determinacy regions. Each region is indexed by a policy regime that postulates an institutional arrangement for controlling inflation and stabilizing government debt.

Economic theory had long recognized that fiscal and monetary policies jointly determine the price level [Sargent and Wallace (1981), Wallace (1981), Aiyagari and Gertler (1985), Sims (1988), Leeper (1991)]. Recent expansions of central bank balance sheets and soaring levels of sovereign debt, as well as the ensuing fiscal stress facing major advanced economies, have also alerted central bankers that the ability of monetary policy to control inflation and influence real activity rests fundamentally on the conduct of fiscal policy and on people's expectations of fiscal behavior [Leeper (2011)]. Our theory and evidence cast serious doubt on the belief that conventional macro models, which are built on the premise that inflation is determined solely by monetary policy, offer an adequate policy framework with which to confront theory with data.[3] The fiscal theory of the price level (FTPL), on the other hand, recognizes that fiscal policy can be a determinant of the inflation process and, therefore, also provides important insights into current policy discussions [Sims (2013)].

The distinction between conventional macro models and FTPL-type models is rooted in two fundamentally different channels through which debt valuation effects force linkages between equilibrium fiscal and monetary policies. Following Leeper (1991)'s terminology, the policy mix capturing the channel in conventional macro models arises from the active monetary-passive fiscal regime (regime M, for short)—the monetary authority systematically raises the nominal interest rate more than one-for-one with inflation, while the fiscal authority adjusts taxes or spending to assure fiscal solvency. The policy mix that underlies the alternative channel in FTPL models is the passive monetary-active fiscal regime (regime

[3]Good examples of the conventional macro models are those of the old monetarists and of the New Keynesians.

F)—the monetary authority forgoes inflation targeting and primary fiscal surpluses, for political or economic reasons, are insensitive to the state of government debt. The regime M or regime F DSGE models arise from different regions of the model's parameter space that deliver a determinate equilibrium.[4]

Two main findings emerge. First, although including a maturity structure for government debt plays a pronounced role in improving the model fit of regime F, the data still overwhelmingly support regime M under the strong econometric interpretation. But this likelihood-based result is susceptible to any possible model misspecification: it stems from taking literally *all* of the model's implications for the data, even though the model in each regime may be quite misspecified along some dimensions of the data that are not directly of interest. Second, according to the minimal econometric interpretation under which the DSGE model is used to elicit prior information, regimes M and F fit the data about equally well. Because the minimal econometric interpretation does not take the model's implications for the data literally, it greatly reduces the susceptibility of this result to potential model misspecification.

Taken together, these findings imply that the apparent statistical support in favor of regime M over regime F stems largely from the strong econometric interpretation rather than from compelling empirical evidence. Moreover, conventional DSGE models suggest that the two regimes are "nearly" observationally equivalent under the minimal econometric interpretation.[5] These results cast serious doubt on empirical tests of policy interactions that are based on simple correlations in the data [Bohn (1998), Canzoneri et al. (2001)].[6]

[4]Examining the indeterminacy—when fiscal and monetary policies both remain passive—may be of interest, but we do not pursue it here [Lubik and Schorfheide (2004), Bhattarai et al. (2012)].

[5]Economic models that reflect distinct behavioral hypotheses but are observationally equivalent for the set of targets they claim to explain are not rare. A prominent example is presented in Tobin (1970) who shows that the correlations and timing patterns observed by the monetarists can arise in an ultra-Keynesian model.

[6]Leeper and Walker (2011) also provides analytical results showing that both policy regimes can be consistent with a wide range of correlation patterns in the data.

Cautious macroeconometric modeling for monetary policy analyses should incorporate explicit and realistic treatment of the interactions between fiscal and monetary policies. This also echoes the general policy advice of Leeper and Walker (2011) and Sims (2013) that policymakers may wish to broaden their perspectives on inflation determination beyond the single, conventional view that dominates policy research and discussions.

The preceding paragraphs argue for the model space of central bankers to be expanded to encompass models operating under alternative regime and endowed with alternative econometric interpretation. Yet it is worthwhile to point out that many of the well-known new Keyesian DSGE models, such as Christiano et al. (2005) and Smets and Wouters (2003, 2007), have been estimated with fiscal behavior left unspecified. This amounts to implicitly assuming that the required fiscal adjustment is always forthcoming in response to any government budget imbalance, forcing fiscal and monetary policies to conform to regime M. One notable exception is Traum and Yang (2011) who use various post-war U.S. samples to estimate a new Keynesian DSGE model that allows for the possibility of either regime M or F. But their likelihood-based model evaluation exercises, which uniformly reject the regime F DSGE model across all samples, are tantamount to yielding a strong econometric interpretation of U.S. policy interactions. The minimal econometric interpretation, on the other hand, suggests that a source of regime F's forceful rejection under the strong interpretation is the false model implications along some dimensions of the data that are not of direct interest.

Existing regime-comparison results also presume that the underlying model space is *complete* in Geweke's (2010) sense, meaning that it has been adequately specified to include the "true" data generating process. But another explanation is possible: perhaps neither the regime M nor the regime F DSGE models even stay "close" to the true data generating process. This article also questions the completeness of the model space by presenting statistical evidence that the cross-equation restrictions implied by the two policy regime models are both misspecified, possibly to a similar degree.

Because this article takes a Bayesian stand on the identifiability of regime index, we briefly review a couple of key concepts drawn from Kadane's (1974) seminal work, which are essential for understanding the notion of identification in a Bayesian framework. We also illustrate how the observational equivalence between the minimally interpreted regime M and F DSGE models can be articulated in terms of these concepts. Finally, while it can be difficult in general to provide an explicit accounting for the DSGE model implications, interested readers are directed to Tan (2014) who uses a simple analytical example to characterize a rational expectations model's cross-equation restrictions under each econometric interpretation.

1.2 Regime Index as (Un)identified Hyperparameter

Not surprisingly, the expanded model space alluded to in Section 1.1 raises several issues on the statistical inference about the competing models, among which identification figures most pressingly. It then follows naturally that the object of identifying interest falls not on the parameters of each model but rather the regime index as an additional parameter of the model space. Put differently, we will study the identifiability of regime index as a hyperparameter that is otherwise treated as given in many related work.[7] This section is intended to review a few important concepts concerning Bayesian identification, perhaps not widely known among macroeconomists, which allow us to define precisely the sense in which the notions of identification and observational equivalence used subsequently become meaningful. In doing so, we closely follow the seminal work of Kadane (1974).

[7] In a Bayesian context, this amounts to specifying a hierarchical prior consisting of a marginal distribution for the regime index and a distribution for the model parameters conditional on the regime index.

Definition 1. *[Parameter Identification] Let* $(\mathbb{S}, \mathcal{S})$ *be a measurable state space and* $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ *a family of probability distributions on* $(\mathbb{S}, \mathcal{S})$ *indexed by the parameter* θ*. Two parameters* θ_1 *and* θ_2 *are said to be observationally equivalent (denoted* $\theta_1 \sim \theta_2$*) if and only if*

$$P_{\theta_1}(A) = P_{\theta_2}(A) \quad \textit{for all } A \in \mathcal{S} \tag{1.2.1}$$

Moreover, the parameter space Θ *is said to be identified if and only if*

$$\forall\ \theta_1, \theta_2 \in \Theta, \quad \theta_1 \sim \theta_2 \ \Rightarrow\ \theta_1 = \theta_2 \tag{1.2.2}$$

This is the notion of parameter identification most frequently used in the existing work. It makes clear that identification is a property of the likelihood function alone—a parametric model is identified if and only if distinct parameter values produce observationally distinct behavior patterns characterized by the likelihood functions. It is not difficult to recognize that Definition 1 is subject to at least two sources of limitation. First, attempts to achieve full identifiability of the parameter space would inevitably demand unnecessary *a priori* restrictions to rule out the parameter values that are unidentified but only in ways harmless to the modeler's purpose. As argued forcefully by Hurwicz (1962), both the degree of and the need for identification are not absolute, but *relative* to the purpose for which the model is designed.[8] Second, in the presence of more than one modeling options available, the identification of parameters associated with each model is only of secondary importance. This is easiest to understand when regime M and F DSGE models are applied to study an economy with high inflation rates and high levels of government debt, as well as a fiscal

[8] A standard textbook paradigm, in light of Hurwicz (1962)'s argument, is that the failure of "invertibility" condition in a linear regression model should not disturb the modeler when her purpose for the model is not to give a partial effect interpretation but to make prediction. This may well leave the parameter space unidentified but only in a way irrelevant to her predictive intention.

authority not committed to budgetary solvency—an environment to which regime F fits well. It then makes little sense to identify the parameters in a monetary policy rule that satisfies the "Taylor principle" without discerning in the first place which policy regime is at work. After all, economic theory does not generally support such a policy configuration. Indeed, one goal of this article is to investigate whether two different understandings of the inflation process can be equally consistent with the data, so the object of identifying priority ought to be the regime index.[9] It turns out that both sources of limitation can be avoided by the notion of function identification, which allows one to focus on the identification of parameters only along dimensions of interest and of first-order importance.

Definition 2. *[Function Identification] Let $f : (\Theta, \mathcal{A}) \to (\Lambda, \mathcal{B})$ be a measurable function. f is said to be identified if and only if*

$$\forall\ \theta_1, \theta_2 \in \Theta, \quad \theta_1 \sim \theta_2 \ \Rightarrow\ f(\theta_1) = f(\theta_2) \tag{1.2.3}$$

The concept of function identification generalizes that of parameter identification because in the case $\Lambda = \Theta$ and f is given by an identity function, the identification of function f reduces to that of parameter θ. It maps all those parameterizations that imply observationally indistinguishable behavioral patterns to one single point in a transformed parameter space of possibly lower dimension. In what follows, let θ_M and θ_F be the parameter vectors by which regime M and F models are parameterized. Also, let a typical point in the joint parameter space be given by (D, θ_M, θ_F), where $D \in \{M, F\}$ is a hyperparameter referring to the regime index and represents the assumptions underlying regime D DSGE model. It

[9] This point is far from new but has not received enough attention in empirical work. For example, Qu and Tkachenko (2012) show that the Taylor rule coefficient of a small-scale determinant DSGE model is not locally identified at a particular value. In addition, Qu and Tkachenko (2013) and Lubik and Schorfheide (2004) examine the identification of key policy parameters for a similar DSGE model but under indeterminacy. An implicit assumption on the fiscal authority underlying these models is that fiscal policy is responsible for providing a real anchor—the real value of government debt. It still remains unclear whether their conclusions can survive if such assumption does not hold.

follows naturally that the function of our identifying interest can be written as

$$f(D, \theta_M, \theta_F) = D, \quad D \in \{M, F\} \tag{1.2.4}$$

To see how Definition 2 facilitates our discussion of identification and observational equivalence in a Bayesian context, we establish the following notations. Let the underlying experiment be characterized by the likelihood function $p(y|\theta)$, which is the conditional probability density function (pdf) of the observable y with respect to a generic measure $\mu(\cdot)$. Moreover, let the initial belief about θ be summarized by a prior distribution $\mathbb{P}(\cdot)$ on Θ. Observe that any specification of f induces a corresponding prior distribution $P_f(\cdot)$ on Λ given by $P_f(B) = \mathbb{P}(f^{-1}(B))$ for all $B \in \mathcal{B}$. Then the posterior distribution on Λ can be obtained by invoking the Bayes' theorem

$$P_f(B|C) = \frac{\int_C \int_{f^{-1}(B)} p(y|\theta)p(\theta)d\nu(\theta)d\mu(y)}{\int_C \int_\Theta p(y|\theta)p(\theta)d\nu(\theta)d\mu(y)}, \quad \forall\ B \in \mathcal{B},\ \forall\ C \in \mathcal{S}$$

where $p(\theta)$ is the pdf of θ with respect to a generic measure $\nu(\cdot)$. Using the posterior distribution on the transformed parameter space, we can define the informativeness of an experiment.

Definition 3. *[Informative Experiment] An experiment is informative about f with respect to the prior $\mathbb{P}$ if and only if there exist $B \in \mathcal{B}$ and $C \in \mathcal{S}$ with $0 < \mu(C) < 1$ such that*

$$P_f(B|C) \neq P_f(B|C^c) \tag{1.2.5}$$

An easier way to digest the content of Definition 3 is from its contrary—the design of an experiment provides no new information about the transformed parameter space relative to the prior belief if and only if the occurrence of any event of the experiment does not alter the posterior inference about that space. As noted by Kadane, the dependence of the informativeness of an experiment on the prior $\mathbb{P}$ and function f indicates that the notions of identification and informativeness are quite different. Were we to define observational equivalence as the unidentifiability of f in a Bayesian context, asserting that the regime index is unidentified would amount to verifying the uninformativeness of the experiment for f with respect to every distinct two-point prior [see Kadane (1974), Theorem 2, p. 181]. From a practical perspective, this is difficult to implement, thereby forcing us to introduce a strictly weaker sense of observational equivalence that is of more practical relevance.

Definition 4. *[Observational Equivalence] Let $p(\theta_M)$ and $p(\theta_F)$ be the priors of regime M and F model parameters, respectively. Regime M and F models are said to be observationally equivalent if and only if*

$$p(y|M) = p(y|F) \quad \textit{for all } y \in \mathbb{S} \tag{1.2.6}$$

where the marginal likelihood functions are given by

$$p(y|D) = \int p(y|D, \theta_D)p(\theta_D)d\theta_D, \quad D \in \{M, F\}$$

This is also the notion of unidentifiability used by Zellner (1971). Weaker as it is, the violation of Definition 4 in many model comparison exercises is oftentimes interpreted as if the model index is identified. Before detailing the statistical evidence, it is worthwhile to point out precisely in what sense one should interpret the empirical findings of this article. That there do exist minimally interpreted regime M and F DSGE models whose implied marginal data densities make criterion (1.2.6) approximately true for the data merely asserts that the joint experiment, characterized by the expanded likelihood function $p(y|D, \theta_M, \theta_F)$, is *uninformative* for the function f in (1.2.4), provided that we believe *a priori* the two policy regimes are equally likely, i.e. $p(M) = p(F) = \frac{1}{2}$. To verify this assertion, observe that for any $C \in \mathcal{S}$ with $0 < \mu(C) < 1$, one has

$$P_f(\{M\}|C) = \frac{\int_C p(y|M)p(M)d\mu(y)}{\int_C p(y|M)p(M)d\mu(y) + \int_C p(y|F)p(F)d\mu(y)} = \frac{1}{2}$$

where we have used the fact that $p(y|M) = p(y|F)$ for all $y \in \mathbb{S}$. By a similar token, one can show that $P_f(\{M\}|C^c) = P_f(\{F\}|C) = P_f(\{F\}|C^c) = \frac{1}{2}$. Now the proof is completed by invoking Definition 3. Therefore, the emergence of observationally equivalent policy regimes serves only as a necessary condition for claiming the unidentifiability of f. A much stronger observational equivalence result in the latter sense can be found, for example, in Leeper and Walker (2011) and Leeper et al. (2014). These work establish the observational equivalence of regime M and F only in very simple models that rely on the symmetry between fiscal and monetary policies. The results of this article suggests that it is likely that the difficulty in identifying the policy regime in simple models also generalizes to more realistic models of policy interactions.

1.3 Complete Model Space: I

This section explores the strong econometric interpretation of the post-war U.S. fiscal and monetary policy interactions. Regime M and F DSGE models, each featuring long-term nominal bonds, are considered *a priori* as two trusted, complete, econometric models for providing predictive distributions of the observable sequence. A tacit assumption is that the model space is complete—either regime M or F DSGE model produces what we observe in the data.[10] A similar exploration using DSGE models that feature only short-term nominal bonds can be found in Traum and Yang (2011). We show that the strong econometric in-

[10] Assuming the completeness of a model space is of course naive. A more defensible statement may be that of Box (1980): all models are wrong, but some are useful.

terpretation always remains unambiguous about regime selection; albeit the important role of long-term nominal bonds in improving model fit, there is no decisive statistical evidence supporting even the near observational equivalence, defined in the sense of Definition 4, of the strongly interpreted regime M and F DSGE models.

1.3.1 A Tale of Two Regimes

To fix ideas, we consider the state space representation of a prototypical New Keynesian DSGE model augmented with a fiscal rule, which can be estimated with Bayesian methods. The fiscal and monetary authorities are assigned with two primary policy objectives: controlling inflation and stabilizing government liabilities. To keep the model specification simple, we abstract from real money balances, wage rigidities, and capital accumulation but otherwise the model captures most of the important features about price level determination [Leeper (1991, 1993), Sims (1988, 1994), Woodford (1994, 1995, 1999, 2001), Cochrane (1998, 2001, 2005)].

Household's Problem

The representative household derives utility from consumption C_t relative to a habit stock, and disutility from hours worked H_t. We assume that the habit stock is given by the level of labor-augmenting technology A_t. This assumption assures that the economy evolves along a balanced growth path. The household has an infinite planning horizon and her preference is characterized as

$$E_t\left[\sum_{s=0}^{\infty}\beta^s\left(\frac{(C_{t+s}/A_{t+s})^{1-\tau}}{1-\tau}-\frac{H_{t+s}^{1+\varphi}}{1+\varphi}\right)\right] \tag{1.3.1}$$

where β is the discount factor, τ is the risk aversion parameter, and $1/\varphi$ is the Frisch elasticity of labor supply. The household supplies perfectly elastic labor services to the firms, taking the real wage W_t as given. She also has access to a complete financial market where nominal government bonds B_t, including state-contingent claims of many sorts, are traded. Following Woodford (2001), this general bond portfolio consists of perpetuities with coupons that decay exponentially. Specifically, suppose that a bond issued in period t pays ρ^k dollars $k+1$ periods later for each $k \geq 0$ with some decay factor $0 \leq \rho \leq \beta^{-1}$. This assumption allows us to mimic the behavior of arbitrary bond maturity structure with a single parameter ρ.[11] Meanwhile, we need only consider the equilibrium price of one type of bond in each period because a bond of this type that has been issued k periods ago is equivalent to ρ^k new bonds. Lastly, the household receives aggregate residual real profits D_t from the firms and pays lump-sum taxes T_t to the government. Thus the household's budget constraint takes the form

$$P_t C_t + P_{B,t} B_t = (1 + \rho P_{B,t}) B_{t-1} + [P_t W_t H_t + P_t D_t - T_t] \tag{1.3.2}$$

where $P_{B,t}$ is the price in period t of one unit of bond portfolio. The bond pricing relation implied by the household's optimal choice of bond portfolio is given by

$$P_{B,t} = \mathbb{E}_t[Q_{t+1|t}(1 + \rho P_{B,t+1})] \tag{1.3.3}$$

where the random variable $Q_{t+1|t}$ is a stochastic discount factor for pricing arbitrary nominal contingent claims. The household, as a price taker in financial markets, takes the evolution of the stochastic discount factor as being independent of her portfolio decisions. The (gross)

[11] The *average maturity* of such a bond portfolio is $(1 - \beta\rho)^{-1}$. In reality, the average maturities of outstanding government debt in many OECD countries are at above 20 quarters. In the U.S., it is at 20 quarters, corresponding to $\beta\rho = 0.95$ in the model.

nominal interest rate R_t on one-period riskless claim purchased in period t must satisfy

$$\frac{1}{R_t} = \mathbb{E}_t[Q_{t+1|t}] \tag{1.3.4}$$

Combining (1.3.3) and (1.3.4) gives the yield curve $R_t = \mathbb{E}_t[R_{B,t}] + R_t \text{Cov}[Q_{t+1|t}, R_{B,t}]$, where $R_{B,t} = \frac{1+\rho P_{B,t+1}}{P_{B,t}}$ is the (gross) nominal rate of return on general bond portfolio. Necessary and sufficient conditions for household optimization are given by the following intratemporal and intertemporal Euler equations

$$\frac{H_{t+s}^{\varphi}}{(C_{t+s}/A_{t+s})^{-\tau}} = \frac{W_{t+s}}{A_{t+s}}, \quad s \geq 0 \tag{1.3.5}$$

$$Q_{t+s|t} = \frac{P_t}{P_{t+s}} m_{t+s|t}, \quad s \geq 0 \tag{1.3.6}$$

where $m_{t+s|t}$ is a stochastic discount factor for pricing arbitrary real contingent claims

$$m_{t+s|t} = \beta^s \left(\frac{C_{t+s}/A_{t+s}}{C_t/A_t} \right)^{-\tau} \frac{A_t}{A_{t+s}} \tag{1.3.7}$$

Substitution of (1.3.6) and (1.3.7) into (1.3.4) then yields

$$1 = \beta \mathbb{E}_t \left[\frac{R_t}{P_{t+1}/P_t} \left(\frac{C_{t+1}/A_{t+1}}{C_t/A_t} \right)^{-\tau} \frac{A_t}{A_{t+1}} \right] \tag{1.3.8}$$

This optimality condition, when imposed by market clearing conditions, is a sort of "Fisher equation", linking short-term nominal interest rate and expected inflation, as well as endogenous real factors that determine the equilibrium real interest rate.

Firms' Problem

The perfectly competitive, representative, final goods producing firm combines a continuum of intermediate goods indexed by $j \in [0, 1]$ using the technology

$$Y_t = \left(\int_0^1 Y_t(j)^{1-v} dj \right)^{\frac{1}{1-v}} \tag{1.3.9}$$

where $1/v$ represents the elasticity of demand for each intermediate good. The firm takes input prices $P_t(j)$ and output prices P_t as given. Profit maximization implies that the demand for intermediate goods is

$$Y_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-1/v} Y_t \tag{1.3.10}$$

and the aggregate price index relating intermediate goods prices and final good price is

$$P_t = \left(\int_0^1 P_t(j)^{\frac{v-1}{v}} dj \right)^{\frac{v}{v-1}} \tag{1.3.11}$$

Though each intermediate firm produces a differentiated good, they all use an identical technology represented by the production function

$$Y_t(j) = z_t A_t N_t(j) \quad \text{with} \quad A_t = \gamma^t A_0 \tag{1.3.12}$$

where z_t is an exogenous productivity process, $N_t(j)$ is the labor input of firm j, and $z_t A_t$ is common to all firms. It implies that the economy grows at the rate γ in steady state. Labor is hired in a perfectly competitive factor market at the real wage W_t. Nominal rigidity is introduced by assuming that intermediate firms face quadratic price adjustment costs. When a firm changes its price away from π^*, the steady-state inflation rate associated with the final good, it incurs menu costs in the form of lost output

$$AC_t(j) = \frac{\phi}{2}\left(\frac{P_t(j)}{P_{t-1}(j)} - \pi^*\right)^2 Y_t(j) \tag{1.3.13}$$

where ϕ governs the degree of price stickiness. Firm j chooses its price $P_t(j)$ to maximize the present value of all its future profits

$$E_t\left[\sum_{s=0}^{\infty} m_{t+s|t}\left(\frac{P_{t+s}(j)}{P_{t+s}}Y_{t+s}(j) - W_{t+s}N_{t+s}(j) - AC_{t+s}(j)\right)\right] \tag{1.3.14}$$

where $m_{t+s|t}$ is treated as exogenous by the firm.

Fiscal and Monetary Policies

The fiscal authority consumes a fraction η_t of aggregate output Y_t, where $\eta_t \in [0, 1]$ follows an exogenous process with steady state $g^{(Q)}$. Moreover, it collects the lump-sum net taxes (taxes net of transfers) to finance any shortfalls in government revenues. The nominal flow government budget constraint is given by

$$P_{B,t}B_t^s + T_t = P_tG_t + (1 + \rho P_{B,t})B_{t-1}^s \tag{1.3.15}$$

where $G_t = \eta_t Y_t$. It is convenient to denote real debt-to-GDP ratio and real lump-sum tax-to-GDP ratio by $b_t = \frac{P_{B,t}B_t^s}{P_tY_t}$ and $\tau_t = \frac{T_t}{P_tY_t}$.[12] The fiscal authority follows a taxation feedback rule that responses to deviations of real debt-to-GDP ratio in previous period from

[12] The real flow government budget constraint corresponding to long-term and short-term debt cases can be written as

$$b_t + \tau_t = \eta_t + \frac{1 + \rho P_{B,t}}{\pi_t \gamma_t}\frac{b_{t-1}}{P_{B,t-1}} \quad \text{(long-term debt)}$$
$$b_t + \tau_t = \eta_t + \frac{R_{t-1}}{\pi_t \gamma_t}b_{t-1} \quad \text{(short-term debt)}$$

where $\gamma_t = Y_t/Y_{t-1}$ is the gross output growth rate.

its steady state

$$\tau_t = \tau_t^{*1-\rho^F} \tau_{t-1}^{\rho^F} e^{\psi_t^F} \quad \text{with} \quad \tau_t^* = \tau^* \left(\frac{b_{t-1}}{b^*}\right)^{\delta^b} \tag{1.3.16}$$

where ψ_t^F is a fiscal policy shock, τ^* is the steady state real tax-to-GDP ratio, and b^* is the steady-state real debt-to-GDP ratio. The parameter $0 \leq \rho^F < 1$ determines the degree of tax ratio smoothing. The monetary authority follows an interest rate feedback rule that reacts to deviations of inflation and output from their respective target levels

$$R_t = R_t^{*1-\rho^M} R_{t-1}^{\rho^M} e^{\theta_t^M} \quad \text{with} \quad R_t^* = r^* \pi^* \left(\frac{\pi_t}{\pi^*}\right)^{\psi^\pi} \left(\frac{Y_t}{Y_t^*}\right)^{\psi^y} \tag{1.3.17}$$

where θ_t^M is a monetary policy shock, r^* is the steady-state real interest rate, π_t is the gross inflation rate defined as $\pi_t = \frac{P_t}{P_{t-1}}$, and π^* is the target inflation rate, which in equilibrium coincides with the steady-state inflation rate. The parameter $0 \leq \rho^M < 1$ determines the degree of interest rate smoothing. Finally, Y_t^* is the level of potential output that would prevail in the absence of nominal rigidities.

Structural Disturbances

The model economy is perturbed by four exogenous shocks—technology shock $\ln z_t$, fiscal shock ψ_t^F, monetary shock θ_t^M, and government spending shock $g_t = \frac{1}{1-\eta_t}$—that are assumed to follow stationary AR (1) processes

$$\hat{z}_t = \rho_z \hat{z}_{t-1} + \epsilon_t^Z, \quad \psi_t^F = \rho_\psi \psi_{t-1}^F + \epsilon_t^F, \quad \theta_t^M = \rho_\theta \theta_{t-1}^M + \epsilon_t^M, \quad \hat{g}_t = \rho_g \hat{g}_{t-1} + \epsilon_t^G \tag{1.3.18}$$

where $\hat{x}_t = \ln(x_t/x^*)$ denotes the percentage deviation of x_t from its steady-state x^*, $g^* = \frac{1}{1-g^{(Q)}}$, and $0 \leq \rho_z, \rho_\psi, \rho_\theta, \rho_g < 1$.[13] The four innovations $\{\epsilon_t^Z, \epsilon_t^F, \epsilon_t^M, \epsilon_t^G\}$ are assumed to be serially uncorrelated by themselves, independent of each other at all leads and lags, and normally distributed with means zero and standard deviations $\{\sigma_Z, \sigma_F, \sigma_M, \sigma_G\}$.

Equilibrium

We consider the symmetric equilibrium in which all intermediate firms make identical choices. The market clearing conditions are given by

$$Y_t = C_t + G_t + AC_t, \qquad H_t = N_t, \qquad B_t = B_t^s \tag{1.3.19}$$

To shed light on all possible interactions between fiscal and monetary policies that are consistent with a uniquely determined price level, it is useful to focus on the *intertemporal equilibrium condition* linking the real debt-to-GDP ratio at the beginning of each period to the present value of current and expected future primary surpluses

$$\underbrace{\left(1 + \rho \sum_{k=0}^{\infty} \rho^k \mathbb{E}_t \left[\frac{1}{R_t R_{t+1} \cdots R_{t+k}}\right]\right)}_{\text{current and future MP}} \frac{1}{\pi_t \gamma_t} \frac{b_{t-1}}{P_{B,t-1}} = \underbrace{\sum_{k=0}^{\infty} \mathbb{E}_t[\bar{m}_{t+k|t} s_{t+k}]}_{\text{current and future FP}} \tag{1.3.20}$$

where $s_t = \tau_t - \eta_t$ is the real primary-surplus-to-GDP ratio, $\gamma_t = Y_t/Y_{t-1}$ is the gross output growth rate, and $\bar{m}_{t+k|t} = m_{t+k|t}\gamma_{t+k}$ is the stochastic discount factor adjusted by output growth rate.[14] The appearance of ρ in (1.3.20) highlights the key role of long-term nominal

[13]The AR (1) structures of technology and government spending shocks are commonly assumed in the literature and need no further explanation. There are plausible reasons to specify AR (1) processes for fiscal and monetary shocks. For instance, business cycles usually take years to complete, making it likely for ψ_t^F and θ_t^M to be persistent and positively autocorrelated for an extended period of time. Some political factors, e.g. election cycles, will even pronounce the positive autocorrelation of fiscal shock. Indeed, Canzoneri et al. (2001) use the estimated positive autocorrelation of primary surplus process to argue in favor of the empirical plausibility of Ricardian regime (regime M) against non-Ricardian regime (regime F).

[14]See Appendix A for derivation details. Sims (2013) and Leeper and Zhou (2013) have also demonstrated conditions similar in essence to (1.3.20).

debt in providing a sizeable source of fiscal financing—surprise changes in current inflation, or current and future nominal interest rates (through changes in the market value of debt), or any combination all serve as a cushion against fiscal shocks. This effectively alleviates the complete reliance of fiscal financing on surprise inflation or deflation in a FTPL-type model with only short-term nominal bonds. As hinted by Leeper and Zhou (2013), condition (1.3.20) reflects a fundamental symmetry between fiscal policy (FP) and monetary policy (MP)—any current and expected future policy mix that uniquely determines the inflation today must satisfy (1.3.20), irrespective of the policy regime and its welfare properties.

Formally, there are two decoupled regions spanning the parameter subspace of (δ^b, ψ^π) that deliver a unique DSGE model solution. The first set of solutions is obtained under regime M when $\delta^b > \frac{\gamma}{\beta} - 1$ and $\psi^\pi > 1$. This is the working assumption underlying basic New Keynesian models—(1.3.20) works as a constraint for the fiscal behavior and output and inflation can be determined without reference to the government budget constraint and fiscal policy. The second set of solutions is obtained under regime F when $|\delta^b| < \frac{\gamma}{\beta} - 1$ and $0 \leq \psi^\pi < 1$. By symmetry, this is the working assumption underlying models in which FTPL is valid—(1.3.20) works as a constraint for the monetary behavior and output and inflation must be determined by the whole system of equations characterizing the equilibrium.

State Space Representation

We work with the log-linearized version of the model economy because it leads to a state-space representation that can be evaluated with the Kalman filter. Stacking the endogenous variables and exogenous shocks in the vector s_t and the shock innovations in the vector ϵ_t, the linearized regime D DSGE model solution takes the form $s_t = \Phi(s_{t-1}, \epsilon_t; \theta_D)$, where Φ is a linear function in s_{t-1} and ϵ_t, θ_D is the vector containing all regime D DSGE model parameters

$$\theta_D = [\tau, \varphi, \kappa, \rho, \delta^b, \psi^\pi, \psi^y, \rho^F, \rho^M, r^{(A)}, \pi^{(A)}, \gamma^{(Q)}, b^{(Q)}, \theta_s]'$$

and the subvector $\theta_s = [\rho_z, \rho_\psi, \rho_\theta, \rho_g, \sigma_Z, \sigma_F, \sigma_M, \sigma_G]'$ collects all parameters in the AR (1) representations of the shock processes. The model description is completed by defining a set of measurement equations that relate the model variables s_t to a set of observables—quarter-to-quarter per capita GDP growth rate (YGR), annualized quarter-to-quarter inflation rate (INF), annualized nominal interest rate (INT), and quarterly real debt-to-GDP ratio (DTY)—to which the model is fitted. All models are estimated for three post-war U.S. samples with quarterly frequency. The first sample, ranging from 1955:Q1 to 1979:Q2, begins with a period shown to be consistent with regime F [Davig and Leeper (2006), Davig and Leeper (2011)], followed by the "Great Inflation" period, and ends after the appointment of Paul Volcker as Chairman of the Federal Reserve Board in August 1979. The second sample, ranging from 1982:Q4 to 2007:Q4, roughly corresponds to the "Great Moderation" period as recognized in the literature and ends with the burst of recent recession. The third sample is quite short-lived, ranging from 2008:Q1 to 2014:Q2, and spans the worst period of the "Great Recession". For convenience, we also term the first two samples as "pre-Volcker" and "post-Volcker" samples.

We follow the conventional notation throughout: The $n \times 1$ vector y_t stacks the quarter t observation; The sample ranges from $t = 1$ to T and its observations are collected in the matrix Y with rows y_t'; We denote the likelihood function by $p(Y|\theta_D)$ and the posterior density by $p(\theta_D|Y)$. See Appendix A for a detailed analysis of the state space representation of linearized regime M and F DSGE models and Appendix B for a description of the data set.

1.3.2 Strong Econometric Interpretation

A typical finding under the strong econometric interpretation of the U.S. policy interactions is that regime F, due to its implied excessive volatility of inflation, always gets rejected against regime M [Traum and Yang (2011)]. This section examines the robustness of such finding with respect to the maturity structure by assessing the quantitative importance of long-term government debt. It is found that, relative to the "null" policy regime, there is decisive statistical evidence in favor of regime M for the pre- and post-Volcker samples, reaffirming the empirical finding of Traum and Yang (2011), and that in favor of regime F for the Great Recession sample.

To provide a predictive distribution of the observable sequence Y, we must combine our *a priori* belief about the vector θ_D that parameterizes regime D DSGE model with its likelihood function. Doing so yields the first, complete, model space of policy interactions

$$\mathcal{M}_1 = \{(p(\theta_M), p(Y|M, \theta_M)), (p(\theta_F), p(Y|F, \theta_F))\} \tag{1.3.21}$$

where the pair of prior distribution and likelihood function, $(p(\theta_D), p(Y|D, \theta_D))$, refers to a specific point in the model space $\mathcal{M}_1$. Translated into the terminology of Section 1.2, we apply Bayesian methods to estimate each element of $\mathcal{M}_1$ and evaluate the informativeness of the joint experiment, $p(Y|D, \theta_M, \theta_F) = p(Y|D, \theta_D)$, about function $f(D, \theta_D) = D$ with respect to the prior distribution over the joint parameter space of (D, θ_D).[15] Interested readers are directed to An and Schorfheide (2007) for a thorough review on Bayesian methods that have been developed in recent years to estimate and evaluate DSGE models.

[15] Since regime M and F DSGE models are identically parameterized, (D, θ_M, θ_F) boils down to (D, θ_D).

Priors and Posteriors

Table 1.2 in Appendix C reports the marginal prior distributions of regime M and F DSGE model parameters employed in this article. We assume *a priori* that all parameters are independent and consider only the parameter subspace that implies a unique DSGE model solution. The two policy parameters (δ^b, ψ^π) and the decay factor ρ play a central role in driving the major model dynamics. We place relatively tight priors on (δ^b, ψ^π) that put little probability mass outside the determinacy region of each policy regime. The first prior is concentrated within the realm of Regime M. The prior for the responsive coefficient of fiscal instrument to debt deviation δ^b has a mean of 0.15 that stays far above the mean benchmark value of $\frac{\gamma}{\beta} - 1$; fiscal authority passively adjusts net tax revenues to stabilize debt.[16] The prior for the responsive coefficient of monetary instrument to inflation deviation ψ^π is centered at Taylor (1999)'s value 1.5; monetary authority actively adjusts policy rate to stabilize price level. The second prior focuses on the realm of regime F. The prior for δ^b has a zero mean and its absolute value tightly stays within the mean benchmark value of $\frac{\gamma}{\beta} - 1$; fiscal authority makes no systematic adjustment in net tax revenues in response to debt deviation. The prior for ψ^π is centered at 0.5; monetary authority forgoes the conventional policy prescription of controlling inflation. Because the decay factor ρ becomes very sensitive as it approaches to unity—a tiny change in its value alters the average maturity of government debt significantly—a tight prior is placed on ρ with mean 0.9. Given the mean value of $r^{(A)}$, it implies an average maturity of about ten quarters.[17]

Appendix C details the numerical implementation of Random-Walk Metropolis (RWM)

[16] Table 1.2 implies the prior means $\gamma = 1.004$ and $\beta = 0.995$, leading to a benchmark value of $\frac{\gamma}{\beta} - 1 = 0.009$ for δ^b. Relaxing the fiscal rule to a less stringent specification can blur the borderline between regime M and F. For example, Canzoneri et al. (2001) argues that what is necessary to result in regime M is that the private sector expects there will sooner or later be a fiscal retrenchment. Davig et al. (2011) shows that a nontrivial probability of fiscal limit in the future can influence the inflation today and poses a serious challenge to a central bank pursuing an inflation target.

[17] See Appendix C for a detailed description about the priors of other DSGE model parameters.

algorithm for generating the posterior draws of DSGE model parameters [Schorfeide (2000)]. Cross-regime comparison of the priors and posteriors in Tables 1.3–1.5 of Appendix C indicates that overall the likelihood is informative and contains quite different information for the two policy regimes. Relative to the priors, the pre-Volcker steady state annual inflation rate $\pi^{(A)}$ is significantly revised upward under regime M with posterior mean 7.49, whereas it is shifted downward by more than half under regime F with posterior mean 1.47. Given the distinct implications about inflation volatility from the two policy regimes, this result is not surprising—the posteriors are supposed to drive a wedge between regime M and F estimates of $\pi^{(A)}$ so as to account for the same high levels of inflation in the data.[18] The wedge is preserved for the post-Volcker and Great Recession samples but with a reversed direction for the latter, perhaps revealing a hidden inflationary pressure brought about by the great expansion of the Fed balance sheet in the last few years. Moreover, both policy regimes tend to attribute a substantial portion of uncertainty to one single shock, e.g. fiscal shock under regime M for all samples and technology shock under regime F for pre-Volcker sample. This signifies the inability of the four shocks to accommodate in a credible way for a large number of random disturbances in reality, because the strong econometric interpretation sets out to account for many more dimensions of variation in the data than those that can be accounted for in the model. While incorporating additional shocks may resolve this issue, Section 1.4 shows that such phenomenon becomes much less obvious once regime M and F DSGE models are interpreted in the minimal sense.

For comparison ease, Figures 1.4–1.6 in Appendix C plot the prior and posterior pdf's of $(\delta^b, \psi^\pi, \rho)$. The pre- and post-Volcker posteriors for δ^b are slightly revised downward under regime M, whereas those for ψ^π are moderately shifted upward. In addition, the higher

[18] A common misperception of the FTPL-type model is that it implies high levels of average inflation. The drastically different estimates of $\pi^{(A)}$ suggest that it is possible for regime M and F DSGE models to generate similar sample means of the inflation series. See, for instance, the prior predictive analysis conducted in Leeper et al. (2014).

post-Volcker posteriors for (δ^b, ψ^π) indicate that the fiscal and monetary authorities took a stronger stand on jointly controlling inflation and stabilizing debt from pre- to post-Volcker periods. Because the Great Recession sample is quite short-lived, its posteriors for (δ^b, ψ^π) closely mimic the priors. There seems to be no substantial difference between the short-term and long-term debt posteriors for (δ^b, ψ^π) under regime M.[19] In sharp contrast, the long-term debt posteriors for ψ^π under regime F are tightly centered at values much above the short-term debt posterior means across all samples, and nearly hit the upper bound for post-Volcker and Great Recession samples. This result reaffirms the role of long-term nominal debt in reducing the reliance of fiscal financing on surprise inflation or deflation—it allows the central bank to engineer a strong, though less than one-for-one, change in current policy rate in the same direction as current inflation deviation so as to absorb a sizeable portion of any fiscal shock. With the issuance of only one-period nominal debt, however, such "strong" response will result in an increased issuance of nominal debt as the only fiscal outcome of any deliberate increase in policy rate. Central banks, understanding this, will then pursue a much weaker response. Finally, all samples remain uninformative about ρ under regime M as its posteriors roughly mimic the prior, verifying a type of Ricardian equivalence result that holds for changes in the average maturity of government debt. See Appendix A for a further discussion on this point. Nevertheless, the posteriors for ρ under regime F are significantly revised upward across all samples and tightly center at values very close to unity.[20] It highlights the important role of long-term nominal bonds in smoothing the inflation series and policy rates under regime F—longer maturity (higher ρ) can greatly reduce the required adjustment in inflation and policy rates in response to a given fiscal shock.

[19]The sole exception is the pre-Volcker ψ^π, whose long-term debt posterior centers at a value much beyond its short-term debt posterior mean, denoted by superscript * in Figure 1.3.

[20]The post-Volcker posterior for ρ has a so small dispersion that it becomes an almost vertical line.

Impulse Response Functions

To gain further insights into the distinct model dynamics under regime M and F, it is useful to highlight the various financing schemes of government debt from the linearized government budget constraint

$$\hat{b}_t = -\underbrace{\left[\left(\frac{g^{(Q)}}{b^*} + \frac{1}{\beta} - 1\right)\hat{\tau}_t - \frac{1-g^{(Q)}}{b^*}\hat{g}_t\right]}_{\text{primary-surplus-to-GDP ratio}} - \underbrace{\frac{1}{\beta}\hat{\pi}_t}_{\text{surprise inflation}} + \underbrace{\frac{\rho}{\gamma\pi^*}\hat{P}_{B,t}}_{\text{bond price}} - \underbrace{\frac{1}{\beta}(\hat{y}_t - \hat{y}_{t-1})}_{\text{output growth}}$$

$$+ \underbrace{\frac{1}{\beta}(\hat{b}_{t-1} - \hat{P}_{B,t-1})}_{\text{remainder term}} \qquad \text{(see (1.5.11) in Appendix A)}$$

The right-hand side (RHS) of (1.5.11) makes it clear that a fiscal consolidation can be accomplished through several channels—higher primary-surplus-to-GDP ratio, surprise inflation, lower bond price, and higher output growth—or any of their combinations, irrespective of which policy regime is in place. One fundamental difference between the two policy regimes lies in their fiscal financing schemes; regime M relies primarily on direct taxation, whereas regime F hinges crucially on the debt revaluation effect of surprise inflation. Moreover, a pairwise comparison of the terms in (1.5.11) and those in the linearized government budget constraint with one-period debt (1.5.16) indicates that, while a higher nominal interest rate, through higher interest payments on outstanding debt, tends to raise debt ratio in a short-term debt model, it can lower debt ratio in a long-term debt model by reducing the bond price and hence the market value of outstanding debt. Figures 1.10–1.12 in Appendix C depict the dynamic responses of output growth, inflation, nominal interest rate, and real debt-to-GDP ratio to one percent unanticipated changes in the fiscal and monetary instruments under regime M (left two columns) and F (right two columns). The qualitative features of these impulse responses largely conform to those of the responses evaluated at the prior means, detailed interpretation of which can be found in Appendix A. We shed light on two key findings, which hold across all samples, about the contemporaneous correlations between the debt ratio and the two policy instruments.

First, analogous to the typical story of Ricardian equivalence, a fiscal contraction (first and third columns) has practically none equilibrium effects on non-fiscal variables under regime M. With unchanged output growth, inflation, and nominal interest rate but higher real primary surplus ratio, the real debt ratio must fall by (1.5.11), giving rise to a negative contemporaneous correlation between debt ratio and fiscal instrument. This is in sharp contrast to the scenario under regime F. By (1.3.20) a fiscal contraction forces the market value of government liabilities to be backed up more than sufficiently by the primary surpluses, making the household feel less wealthier and hence try to substitute consumption for government bonds. While a surprise deflation in the current period must occur under regime F to eliminate the negative wealth effect so that (1.3.20) can be restored, long-term nominal bonds also make possible the role of surprise changes in current and future nominal interest rates in revaluing government liabilities. In response to the falling inflation and output, central bank can engineer a monetary expansion by moderately reducing the nominal interest rate. This bids up the bond price and hence the market value of government liabilities, effectively reducing the complete reliance on current deflation. With falling inflation, output, and nominal interest rate, the real debt ratio increases by (1.5.11), giving rise to a positive contemporaneous correlation between debt ratio and fiscal instrument. It is worthwhile to point out that the argument of Canzoneri et al. (2001) that both regime M and F can produce such positive contemporaneous correlation indeed rests on their version of regime M fiscal policy rule based on Bohn (1998)'s empirical finding: primary surpluses have responded positively to *contemporaneous* government liabilities. This is quite different from the fiscal policy rule (1.3.16).

Second, due to the costly price adjustment, a monetary contraction (second and fourth columns) preserves its usual contractionary effects on the economy under regime M. Higher policy rate also bids down the bond price but the contractionary effect of lower bond price on debt ratio is fully offset by those of lower output growth and inflation. As a result, the real debt ratio increases by (1.5.11), giving rise to a positive contemporaneous correlation between debt ratio and monetary instrument. This is again in sharp contrast to the scenario under regime F. By (1.3.20) a monetary contraction lowers the bond price and hence forces the market value of government liabilities to be backed up more than sufficiently by the primary surpluses. It follows almost tautologically, as in the fiscal contraction under regime F, that a surprise deflation in the current period must occur, mimicking the initial impact of a monetary contraction on inflation under regime M. But as the nominal interest rate falls back to its steady-state, bond price starts increasing and (1.3.20) requires a surprise inflation in the future periods so as to reestablish (1.3.20), thereby producing the "inflation reversal" phenomenon documented by Kim (2003). Put differently, long-term nominal bonds allow central bankers aiming at inflation stabilization to make a tradeoff between current and future inflation or deflation in response to various shocks in the economy. Because the contractionary effect of lower bond price on debt ratio dominates those of the lower output growth and inflation, the real debt ratio decreases by (1.5.11), giving rise to a negative contemporaneous correlation between debt ratio and monetary instrument.[21]

But that regime M and F DSGE models can produce opposite implications for simple correlations among selected observables ought not to be regarded as powerful identifying restrictions to discern which policy regime generated the data. For example, the sign of contemporaneous correlation between debt ratio and monetary instrument relies heavily on the magnitude of the effect of changes in bond price relative to those of changes in output growth and inflation taken together, making such correlation a fragile identifying restriction.

[21] Such negative correlation agrees, at least qualitatively, with those calculated from the pre- and post-Volcker samples.

Posterior Odds

Since the model space $\mathcal{M}_1$ contains more than one element, it is natural to make cross-regime comparison by evaluating the relative fit of each model, measured by the posterior odds. With equal prior model probabilities, $p(M) = p(F) = \frac{1}{2}$, this amounts to comparing the marginal data densities of regime M and F DSGE models. Conditional on the prior means, we simulate the long-term debt regime M and F DSGE models to obtain two data sets with 80 observations each. The upper panel reports the log marginal data densities of long-term debt models for the two simulated samples, whereas the lower panel compares those of short-term and long-term debt models for the three actual samples. Throughout this article we treat regime F as the "null regime" and follow the guidance of Jeffreys (1961) for interpreting the magnitude of Bayes factor—statistical evidence in favor of regime M as opposed to regime F. Factors in $[1, 3.2]$ signify "very slight evidence"; Factors in $[3.2, 10]$ signify "slight evidence"; Factors in $[10, 100]$ signify "strong to very strong evidence"; Factors exceeding 100, or 4.6 in logarithm, signify "decisive evidence".[22] We summarize the results in Table 1.1 below.

[22] A log Bayes factor of 4.6 plays a role similar to a significance level in the frequentist approach.

Table 1.1: Model Fit of Regime M and F DSGE Models

	$\ln p(Y\|M)$		$\ln p(Y\|F)$		Log Bayes Factor	
Regime	Long Debt		Long Debt		Long Debt	
M [L]	394.9		226.4		168.6*	
F [L]	166.1		315.0		−148.9*	
Sample	Short	Long	Short	Long	Short	Long
55 – 79	−286.3	−278.2	−371.8	−313.5	85.6*	35.3*
82 – 07	−114.7	−107.7	−354.9	−156.9	240.2*	49.1*
08 – 14	−153.5	−151.4	−240.4	−145.5	86.9*	−5.8*

NOTES: All log marginal data densities are approximated using Geweke (1999)'s modified harmonic mean estimator. Debt maturities are indicated in brackets. The log Bayes factors indicate the extent to which data supports regime M against regime F DSGE models. Decisive evidence in favor of the regime with superior fit is denoted by superscript *, corresponding to a Bayes factor greater than 100, or 4.6 in logarithm.

First, the enormous magnitudes of Bayes factors for the simulated samples indicate that the true policy regime is always decisively favored within a complete model space, making it unlikely to select the misspecified one by Jeffreys' criterion. For example, when the true policy regime is regime M, the log marginal data density of regime F is 226.4, which translates into a Bayes factor of about e^{169}. This is significantly larger than 100, constituting decisive evidence in favor of regime M. Similarly, when the data is generated by regime F, the inverse of the Bayes factor is about e^{149}, again significantly larger than the 100 benchmark ratio for decisive evidence in favor of regime F. Given the starkly different model dynamics under regime M and F, the immediate rejection of the misspecified policy regime is not surprising at all. Recall that a fiscal contraction generates practically none equilibrium effects on non-fiscal variables under regime M, whereas it preserves the contractionary effects on the economy under regime F. Also, regime M and F can produce opposite implications for the contemporaneous correlation between debt ratio and policy instruments. These reverse model dynamics could easily emerge as obvious forms of misspecification. It follows naturally that the identification of policy regime, in the sense of data informativeness about function f, from the equilibrium time series will not pose a serious challenge, provided that the model space $\mathcal{M}_1$ is complete. This sanguine view, however, were it brought to guide our regime selection for the actual samples, would turn out to be inconclusive once the econometric interpretation for regime M and F DSGE models takes its weakest form.

Second, the strong econometric interpretation always remains unambiguous about regime selection for the long-term debt models; there are decisive statistical evidence in favor of regime M for the pre- and post-Volcker samples, extending the empirical finding of Traum and Yang (2011) to the long-term debt case, and that in favor of regime F for the Great Recession sample. The significant increase in Bayes factors from pre-Volcker (e^{35}) to post-Volcker (e^{49}) periods turns out to be expected and is by and large attributable to the distinct mechanisms of fiscal financing underlying the two policy regimes. For example, to the extent of accounting for the much smoother inflation rates in the post-Volcker sample, that the fiscal backing of government debt is always forthcoming under regime M allows the Fed to actively pursue its inflation target, greatly reducing the variability of inflation series. The fiscal backing, however, does not take place under regime F in which government liabilities are primarily financed through surprise changes in inflation and policy rates. To confront the data with regime F DSGE model, a sizeable extension of the average maturity of government debt is needed to reduce the required adjustments in inflation and policy rates—by (1.3.20) the longer the average duration of government debt (higher ρ), the less the required adjustments. This explains why the post-Volcker posterior distribution of ρ tightly centers at about 0.997 that is significantly above its prior mean. Furthermore, because of the great fiscal uncertainty and its resulting unanchored fiscal expectations during the Great Recession—a period to which regime F fits well—the inverse of the Bayes factor for this latest sample is approximately 330, a value constituting decisive evidence in favor of regime F. Lastly, building on the previous result, the much milder magnitudes of Bayes factors for the actual samples than those for the simulated samples tend to forge some statistical evidence against the completeness assumption of model space.

Third, a cross-regime comparison of the Bayes factors in favor of long-term debt models as opposed to short-term debt ones suggests that extending the average maturity of government debt always plays a role in improving model fit, and the improvement gets much more pronounced under regime F. This confirms a conjecture of Traum and Yang (2011) that introducing nominal bonds of longer maturity may bring closer the two policy regimes in terms of model fit. For example, these factors (in logarithm) are given by $[8.1, 7.0, 2.1]$ for regime M, and $[58.3, 198.0, 94.9]$ for regime F. Not surprisingly, the most significant improvement happens to regime F with the post-Volcker sample. The short-term debt model receives a big penalty in model fit under regime F because it relies entirely on surprise inflation or deflation as a fiscal shock absorber, and the implied excessive volatility of inflation is apparently at odds with the data. But more importantly, to the extent that regime selection based on relative model fit is integral to interpretations of data, this last result makes it clear that abstraction from government debt of longer maturity can fundamentally alter our understanding of the interactions between fiscal and monetary policies. The short-term debt Bayes factors, which are $[85.6, 240.2, 86.9]$ in logarithm, all constitute decisive evidence against regime F, leading to a uniform acceptance of the conventional perspective on inflation determination that dominates policy thinking.

As a final remark, we emphasize that the decisive rejection of either policy regime rests more fundamentally on the fact that their DSGE models are forced to provide a fully trusted marginal likelihood function—a complete probabilistic characterization of the observables

Y. Although the components of Y are not collectively confined to a degenerate space, such provision by regime M and F DSGE models with a set of selected shocks can still be ambitious. By a usual transformation argument, one can always use these primitive predictive densities to determine the derived ones of arbitrary functions of Y, most of which are likely to be at odds with the data. This explains why many rational expectations models in their early stage failed to pass the likelihood-based specification tests. As Thomas J. Sargent recalled in Evans and Honkapohja (2005):

> My recollection is that Bob Lucas and Ed Prescott were initially very enthusiastic about rational expectations econometrics...But after about five years of doing likelihood ratio tests on rational expectations models, I recall Bob Lucas and Ed Prescott both telling me that those tests were rejecting too many good models.

After all, regime M and F DSGE models are mainly intended to study various aspects of policy interactions and there are likely to be other dimensions of the reality along which they are highly misspecified. Therefore, it is not the behavioral implications of regime M and F to be blamed, but rather the difficulty of developing their DSGE models to the point of accounting for what they claim to characterize in a credible way. Because the strong econometric interpretation requires an explicit characterization of too many features in the data that are poorly accounted for in the DSGE models, policy regime selection under this category is subject to serious suspicion. On the positive side, however, the suspicion can be avoided by considering a more modest claim for regime M and F DSGE models.

1.4 Complete Model Space: II

To accommodate the limited scope of DSGE models and avoid the logical pitfall of strong econometric interpretation, this section treats regime M and F DSGE models as two partially trusted econometric models for providing predictive distributions of the selected population moments of Y. Following the terminology of Geweke (2010), we investigate the minimal econometric interpretation of the post-war U.S. fiscal and monetary policy interactions. It leads to a major departure from the conventional likelihood-based econometrics underlying the strongly interpreted DSGE models, and greatly alters the nature of the empirical findings about regime selection as established in Section 1.3. The use of regime M and F DSGE models as two sources of *a priori* information about the selected population moments renders these models *incomplete* since they do not yield immediately testable implications for the observables. Therefore, it is necessary to posit a separate link between the population moments and the observables by integrating regime M and F DSGE models with an auxiliary econometric model that captures the empirical regularities reasonably well. While maintaining the completeness assumption of model space, this step creates a set of hybrid models that enlarges the model space of policy interactions upon $\mathcal{M}_1$. We show that there is barely decisive statistical evidence supporting either of the two policy regimes in the context of minimal econometric interpretation, rendering regime M and F DSGE models nearly observationally equivalent in the sense of Definition 4.

1.4.1 Methodological Issues

We establish some notations. Let the collection of selected population moments be denoted by $m = \mathbb{E}_{\theta_D}[g(Y)]$ for $D \in \{M, F\}$, where $g(Y)$ represents a matrix of selected sample moments and expectation is taken with respect to the likelihood function of θ_D. Note that the prior $p(\theta_D)$ induces a corresponding prior $p(m|D)$, but otherwise regime D DSGE model has implications for neither Y nor $f(Y)$. To rebate regime D its empirical contents, it is necessary to merge both policy regimes into a third econometric model E, parameterized by θ_E, that specifies a likelihood function $p(Y|\theta_E, m, E)$ together with a conditional prior $p(\theta_E|m, E)$. Model E is also incomplete as it provides no prior $p(m|E)$. Instead, the prior for m is supplied by either regime M though $p(m|M)$ or regime F through $p(m|F)$. Under certain regularity conditions [see Geweke (2010), Condition 4.1, p. 111] that are satisfied in this article, policy regime selection can be based on the following posterior odds

$$\frac{p(M|Y,E)}{p(F|Y,E)} = \underbrace{\frac{p(M|E)}{p(F|E)}}_{\text{prior odds}} \underbrace{\frac{\int p(m|M)p(Y|m,E)dm}{\int p(m|F)p(Y|m,E)dm}}_{\text{Bayes factor}} \tag{1.4.1}$$

where we assume *a priori* that both policy regimes are equally likely given model E, i.e. $p(M|E) = p(F|E) = \frac{1}{2}$. In the example of DSGE models designed to study the U.S. equity premium, Geweke (2010) assesses the Bayes factor in (1.4.1) by means of kernel density approximation, and shows that the so-called equity premium puzzle disappears when m consists of the population means for risk free rate and equity premium and model E is given by a Gaussian vector autoregression (VAR). While this nonparametric approach has the advantage of allowing the researcher to specify at will the population moments, it is subject to the usual curse of dimensionality problem when m is of high dimension.

We adopt an alternative but closely related method advocated in Del Negro and Schorfheide (2004), called DSGE-VAR approach, in which the sole function of regime M and F DSGE models is to provide prior distributions for reduced-form VARs. Its earlier predecessors include DeJong and Whiteman (1993) and Ingram and Whiteman (1994), which are subsequently extended by, among others, Del Negro and Schorfheide (2006, 2009), Del Negro et al. (2007), and Park (2011). This approach turns out to be very efficient from a computational perspective. What is given up is the flexibility of selecting the set of population moments that DSGE models aim to characterize. Nevertheless, the contents of m in the DSGE-VAR approach should be rich enough to incorporate most types of population moments whose sample counterparts are frequently used to calibrate and evaluate similar DSGE models.

To implement the DSGE-VAR approach, let $y_t = [\mathrm{YGR}_t, \mathrm{INF}_t, \mathrm{INT}_t, \mathrm{DTY}_t]'$ be an $n \times 1$ vector ($n = 4$) of observables and consider the following p-th order ($p = 4$) VAR model

$$y_t = \Phi_0 + \Phi_1 y_{t-1} + \cdots + \Phi_p y_{t-p} + u_t \tag{1.4.2}$$

where u_t is a vector of one-step-ahead forecast errors that has a multivariate Gaussian distribution $\mathbb{N}(0, \Sigma_u)$ conditional on the past observations of y_t. Let Y be a $T \times n$ matrix with rows y_t'. Also, define $k = 1 + np$ and let X be a $T \times K$ matrix with rows $x_t' = [1, y_{t-1}', \ldots, y_{t-p}']$, U a $T \times n$ matrix with rows u_t', and $\Phi = [\Phi_0, \Phi_1, \ldots, \Phi_p]'$. Then model E can be compactly expressed as $Y = X\Phi + U$ with $(\theta_E, m) = (\Phi, \Sigma_u)$ since any choice of m can be recovered through (1.4.2) as a function of (Φ, Σ_u). To induce regime D DSGE model prior for the VAR parameters (Φ, Σ_u), we let the data (Y, X) be augmented with $T^* = \lambda_D T$ dummy observations $(Y^*(\theta_D), X^*(\theta_D))$ that are simulated from regime D DSGE model. The scaling parameter λ_D can be thought of as a tightness measure that controls the weight of regime D DSGE model prior relative to VAR likelihood. Note that we may interpret the following artificial likelihood function

$$\begin{aligned} & p_{\lambda_D}(Y^*(\theta_D)|\Phi, \Sigma_u) \\ & \propto |\Sigma_u|^{-\lambda_D T/2} \exp\left\{-\frac{1}{2}\mathrm{trace}[\Sigma_u^{-1}(Y^{*'}Y^* - \Phi' X^{*'} Y^* - Y^{*'} X^* \Phi + \Phi' X^{*'} X^* \Phi)]\right\} \end{aligned} \tag{1.4.3}$$

as a prior kernel density for (Φ, Σ_u) that summarizes the information about VAR parameters contained in the simulated sample. To elicit the minimal econometric interpretation, we replace the collection of nonstandardized artificial sample moments $[Y^{*'}Y^*, Y^{*'}X^*, X^{*'}X^*]$

in (1.4.3) by their expected values, and denote the resulting regime D DSGE model prior by $p_{\lambda_D}(\Phi, \Sigma_u|\theta_D)$. This step delivers an explicit specification for m

$$m = \Big[\mathbb{E}_{\theta_D}[Y^{*'}Y^*], \mathbb{E}_{\theta_D}[Y^{*'}X^*], \mathbb{E}_{\theta_D}[X^{*'}X^*]\Big] \tag{1.4.4}$$

It is straightforward to see that m includes most population moments of our interest, ranging from the population means of observables to up to their p-th order population autocovariance matrices. For example, m retains the volatility implications for inflation and nominal interest rate, two of the most distinctive features of regime M and F DSGE models. Combining the Regime D DSGE model prior with the VAR likelihood $p(Y|\Phi, \Sigma_u, E)$, we call the resulting hybrid model Regime D DSGE-VAR model. Observe that $p_{\lambda_D}(\Phi, \Sigma_u|\theta_D)$ depends on θ_D only through m. This property allows us to compute the Bayes factor in favor of Regime M as opposed to Regime F DSGE-VAR models as

$$\begin{aligned}
\frac{p_{\lambda_M}(Y|M,E)}{p_{\lambda_F}(Y|F,E)} &= \frac{\int\int p(Y|\Phi,\Sigma_u,E)p_{\lambda_M}(\Phi,\Sigma_u|\theta_M)p(\theta_M)d(\Phi,\Sigma_u)d\theta_M}{\int\int p(Y|\Phi,\Sigma_u,E)p_{\lambda_F}(\Phi,\Sigma_u|\theta_F)p(\theta_F)d(\Phi,\Sigma_u)d\theta_F} \\
&= \frac{\int\int p(Y|\Phi,\Sigma_u,E)p_{\lambda_M}(\Phi,\Sigma_u|m)p(m|M)d(\Phi,\Sigma_u)dm}{\int\int p(Y|\Phi,\Sigma_u,E)p_{\lambda_F}(\Phi,\Sigma_u|m)p(m|F)d(\Phi,\Sigma_u)dm} \\
&= \frac{\int p(m|M)p_{\lambda_M}(Y|m,E)dm}{\int p(m|F)p_{\lambda_F}(Y|m,E)dm}
\end{aligned} \tag{1.4.5}$$

echoing the Bayes factor in (1.4.1); it conforms to the regime selection criterion under minimal econometric interpretation.

1.4.2 Minimal Econometric Interpretation

To fix idea, we let the scaling parameter λ_D take values in a finite set Λ of grid points and index the strongly interpreted DSGE model by $\lambda_D = \infty$.[23] To endow Regime D with

[23]The notation differs from Del Negro and Schorfheide (2004), where $\lambda = \infty$ refers to a VAR approximation of the DSGE model subject to no misspecification.

refutable implications, we must combine the prior for regime D DSGE-VAR model parameters $(\theta_D, \Phi, \Sigma_u)$ with its VAR likelihood function. Doing so gives the second, complete, model space of policy interactions

$$\mathcal{M}_2 = \{(p_{\lambda_D}(\theta_D, \Phi, \Sigma_u), p(Y|\Phi, \Sigma_u, E)) \;:\; D \in \{M, F\},\; \lambda_D \in \{\Lambda, \infty\}\} \tag{1.4.6}$$

where the pair of prior distribution and likelihood function, $(p_{\lambda_D}(\theta_D, \Phi, \Sigma_u), p(Y|\Phi, \Sigma_u, E))$, represents a specific point in the model space $\mathcal{M}_2$. Translated into the terminology of Section 1.2, we apply Bayesian methods to estimate each element of $\mathcal{M}_2$ and evaluate the informativeness of the joint experiment, $p(Y|D, \lambda_D, \theta_D, \Phi, \Sigma_u, E) = p(Y|\Phi, \Sigma_u, E)$, about function

$$f(D, \lambda_D, \theta_D, \Phi, \Sigma_u) = D, \quad D \in \{M, F\} \tag{1.4.7}$$

with respect to the prior distribution over the joint parameter space of $(D, \lambda_D, \theta_D, \Phi, \Sigma_u)$.

The road ahead. Two sets of useful exercises can shed light on the empirical implications of regime M and F DSGE-VAR models. First, we use a data-driven procedure to determine the value of λ_D that attains the highest model fit, measured by the marginal data density. This value, denoted as $\hat{\lambda}_D$, measures the overall degree of misspecification associated with Regime -D DSGE model, thereby quantifying how consistent regime D is with the data. The posterior draws of $(\theta_D, \Phi, \Sigma_u)$ corresponding to $\hat{\lambda}_D$ are then used to compute the impulse response functions of regime D DSGE and DSGE-VAR models. Second, we make cross-regime comparison for each $\lambda_M = \lambda_F$ and examine whether there exist regime M and F DSGE models in $\mathcal{M}_2$, either interpreted strongly or minimally, that are nearly observationally indistinguishable.

Posteriors of DSGE Parameters

One major improvement of the DSGE-VAR approach over earlier procedures is that it enables posterior inference with respect to the DSGE model parameters. In particular, the posterior for regime D DSGE model parameters $p_{\lambda_D}(\theta_D|Y, E)$ can be obtained by combining its prior $p(\theta_D)$ with

$$p_{\lambda_D}(Y|\theta_D, E) = \int p(Y|\Phi, \Sigma_u, E) p_{\lambda_D}(\Phi, \Sigma_u|\theta_D) d(\Phi, \Sigma_u) \tag{1.4.8}$$

See Appendix A of Del Negro and Schorfheide (2004) for the expression of $p_{\lambda_D}(Y|\theta_D, E)$. The posterior estimates of regime M and F DSGE model parameters are reported in Tables 1.6–1.8 of Appendix C. Relative to the strong econometric interpretation, the likelihood leads to a more modest updating on the prior and contains less different information for the two policy regimes. Except for the post-Volcker $\pi^{(A)}$ under regime M, the estimates of $\pi^{(A)}$ lie much closer to the prior under both policy regimes, and the cross-regime wedge needed to reconcile the distinct implications for inflation volatility from regime M and F turns out to be much smaller. Similar to Del Negro et al. (2007), most of the scaled standard deviation parameters are estimated to be much lower than those under strong econometric interpretation, with the sole exception being $100\sigma_F$ under regime F. Moreover, only regime M tends to attribute a major portion of uncertainty to one single shock—fiscal shock—for all samples. This is because by claiming to account for the selected population moments m, the minimally interpreted regime M and F DSGE models void a direct characterization of the variation in the data that can never be fully accounted for in the model with a set of selected shocks.

For comparison ease, Figures 1.7–1.9 in Appendix C plot the prior and posterior pdf's of $(\delta^b, \psi^\pi, \rho)$. The posteriors roughly mimic those under strong econometric interpretation.

The only exception is the pre-Volcker posterior for ψ^{π} under regime F, whose DSGE-VAR estimate is tightly centered at a value close to unity, much above its DSGE posterior mean. In other words, all the actual samples turn out to be quite informative about the role of long-term nominal debt in smoothing the inflation series and policy rates under the minimally interpreted regime F DSGE model.

Impulse Response Functions

Another major improvement of the DSGE-VAR approach is that it allows for a detailed assessment of DSGE model misspecification by comparing the impulse responses from the DSGE model and a benchmark DSGE-VAR model identified by the DSGE cross-equation restrictions. Following Del Negro and Schorfheide (2004), this section uses regime M and F DSGE cross-equation restrictions to identify their DSGE-VAR models and assesses the DSGE model misspecification in terms of predicting the policy effects under each regime. It can be shown that the initial impacts of the structural shocks ϵ_t on the observables y_t in any exactly identified VAR (1.4.2) and the state space representation of regime D DSGE model are given by

$$\left(\frac{\partial y_t}{\partial \epsilon_t'}\right)_{\mathrm{VAR}} = \Sigma_{\mathrm{tr}}\Omega \quad \text{and} \quad \left(\frac{\partial y_t}{\partial \epsilon_t'}\right)_{\mathrm{DSGE}} = \Sigma_{\mathrm{tr}}^{D}(\theta_D)\Omega^{D}(\theta_D) \tag{1.4.9}$$

where Σ_{tr} is the Cholesky decomposition of Σ_u, Ω is an arbitrary orthonormal matrix, $\Sigma_{\mathrm{tr}}^{D}(\theta_D)$ is lower triangular, $\Omega^{D}(\theta_D)$ is orthonormal, and ϵ_t is normalized to have unit variance. To identify regime D DSGE-VAR model, we maintain Σ_{tr} but replace Ω with $\Omega^{D}(\theta_D)$ in (1.4.9). Figures 1.13–1.15 in Appendix C depict the dynamic responses of output growth, inflation, nominal interest rate, and real debt-to-GDP ratio to one percent unanticipated changes in the fiscal and monetary instruments under regime M (left two columns) and F (right two columns). The VAR impulse responses, at least qualitatively, conform to those of the DSGE models and in many dimensions place most probability mass in regions that encompass the DSGE impulse responses. This suggests that Σ_{tr} and $\Sigma^{D}_{\mathrm{tr}}(\theta_D)$ are quite similar.[24] In what follows, we document some evidence for DSGE model misspecification by focusing on the dimensions along which the DSGE impulse responses escape from the VAR 90% confidence bands.

First, where the DSGE model predicts practically none equilibrium effects on non-fiscal variables in response to a fiscal contraction under regime M (first column), the pre-Volcker VAR impulse responses suggest that it does raise inflation and policy rates in the subsequent periods. This is because the data predicts a much more persistent dampening effect of higher net taxes on debt ratio than the DSGE model. The household, while having a firm belief about fiscal authority's mandate on debt stabilization, will anticipate lower future net taxes for an extended period of time. By (1.3.20) the market value of government liabilities is forced to be backed up less than sufficiently by the primary surpluses, making the household feel wealthier and try to substitute government bonds for consumption. In response to the rising aggregate demand and inflation, central bank reacts actively by raising the nominal interest rate. In other words, the Ricardian equivalence story breaks down during the pre-Volcker periods, which indicates that the assumption of non-distortionary taxation seems to be a misspecified structure under regime M. However, no obvious evidence against such assumption is found for the post-Volcker and Great Recession samples as the 90% confidence band of any non-fiscal variable centers around zero. Another notable discrepancy between the VAR and DSGE impulse responses is that regime M and F DSGE models tend to underestimate the persistent impacts of a fiscal contraction (first and third columns) on debt ratio for the two pre-recession samples, perhaps suggesting that a more flexible exogenous process of fiscal shock is preferable.

[24] Such similarity becomes most apparent during the Great Recession as the VAR and DSGE posterior mean responses nearly coincide. This is because the Great Recession sample is quite short-lived and the DSGE model priors play a dominating role in shaping the DSGE-VAR posteriors.

Second, where regime F DSGE model predicts an "inflation reversal" phenomenon to take place in about one year following a monetary contraction (fourth column) for the two pre-recession samples, the VAR impulse responses suggest that such phenomenon happens much sooner for the pre-Volcker sample but nearly disappears for the post-Volcker sample. This seems to be evidence favoring a more flexible exogenous process of monetary shock. In addition, the signs of the estimated impacts of a monetary contraction on debt ratio from regime M and F DSGE-VAR models remain quite ambiguous across all samples. In other words, the data turns out to be uninformative about the magnitude of the effect of changes in bond price relative to those of changes in output growth and inflation lumped together. This makes the sign of contemporaneous correlation between debt ratio and monetary instrument a fragile identifying restriction, a point made earlier in Section 1.3.2.

Posterior Odds

It is not surprising that the minimally interpreted regime M and F DSGE models can also produce starkly different dynamic implications simply because model E is identified by two sets of fundamentally different cross-equation restrictions. Nevertheless, we show that the distinct model dynamics do not translate into appreciable differences in the model fit as under strong econometric interpretation. We also document the overall degree of model misspecification for each policy regime by finding the values of $(\hat{\lambda}_M, \hat{\lambda}_F)$ that attain the highest marginal data densities. The larger the value of $\hat{\lambda}_D$, the less by which regime D DSGE cross-equation restrictions must be relaxed so as to maintain a balance between model fit and model complexity. Due to computational concerns, Λ is taken to be a finite set of grid points that are sufficient to trace out the typical "inverted-U" shape of the marginal data density as a function of λ_D [Del Negro and Schorfheide (2006), Del Negro et al. (2007)]. Figure 1.1 below gives a graphical illustration of the log marginal data densities (left three panels) for regime M and F DSGE-VAR models and the associated log Bayes factors (right three panels). We report the numerical results in Table 1.9 of Appendix C.

First, overall the log marginal likelihood functions, as depicted in Panels (a,c,e) of Figure 1.1, display the expected inverted-U shape, constituting strong statistical evidence of DSGE model misspecification—as regime M and F DSGE model priors shrink the estimates of VAR parameters toward the neighborhoods of their cross-equation restrictions, the DSGE-VAR model fits deteriorate substantially.[25] In addition, the overall degrees of regime M and F DSGE model misspecification turn out to be quite similar as the blue and red curves peak at roughly the same value. For example, $\hat{\lambda}_M \approx \hat{\lambda}_F \approx 0.5$ for the pre-Volcker sample, whereas $\hat{\lambda}_M \approx \hat{\lambda}_F \approx 1$ for the post-Volcker sample. The sole exception happens to be regime F DSGE-VAR model for the Great Recession sample. Its log marginal likelihood function persistently stays near the peak level for $\lambda_F > \hat{\lambda}_F \approx 2$ and displays no tendency to drop even when the artificial sample size is nine times as the actual one. In contrast, the log marginal likelihood function of regime M DSGE-VAR model eventually drops though it peaks at a similar value to regime F. On one hand, this may suggest that the likelihood remains uninformative about the overall degree of regime F DSGE model misspecification for the short-lived Great Recession sample. But on the other, considering Lawrence Christiano's comment on the DSGE-VAR approach—it is rare for the log marginal likelihood function of a correctly specified DSGE model to be steeply sloped for $\lambda > \hat{\lambda}$—it seems to hint that regime F conforms quite well to the macroeconomic features in recent years, at least better than its cousin. After all, the fiscal development during the Great Recession and its accompanying worldwide financial crisis has made it clear that we are in a period of remarkable shifts in fiscal policy and people's expectations about future fiscal behavior are likely to be uncertain. Therefore, a macro model of empirical relevance should be one that

[25] An and Schorfheide (2007) shows that the marginal likelihood function becomes monotonically increasing when the DSGE model is correctly specified.

treats fiscal policy and its interactions with monetary policy in a serious and realistic way.

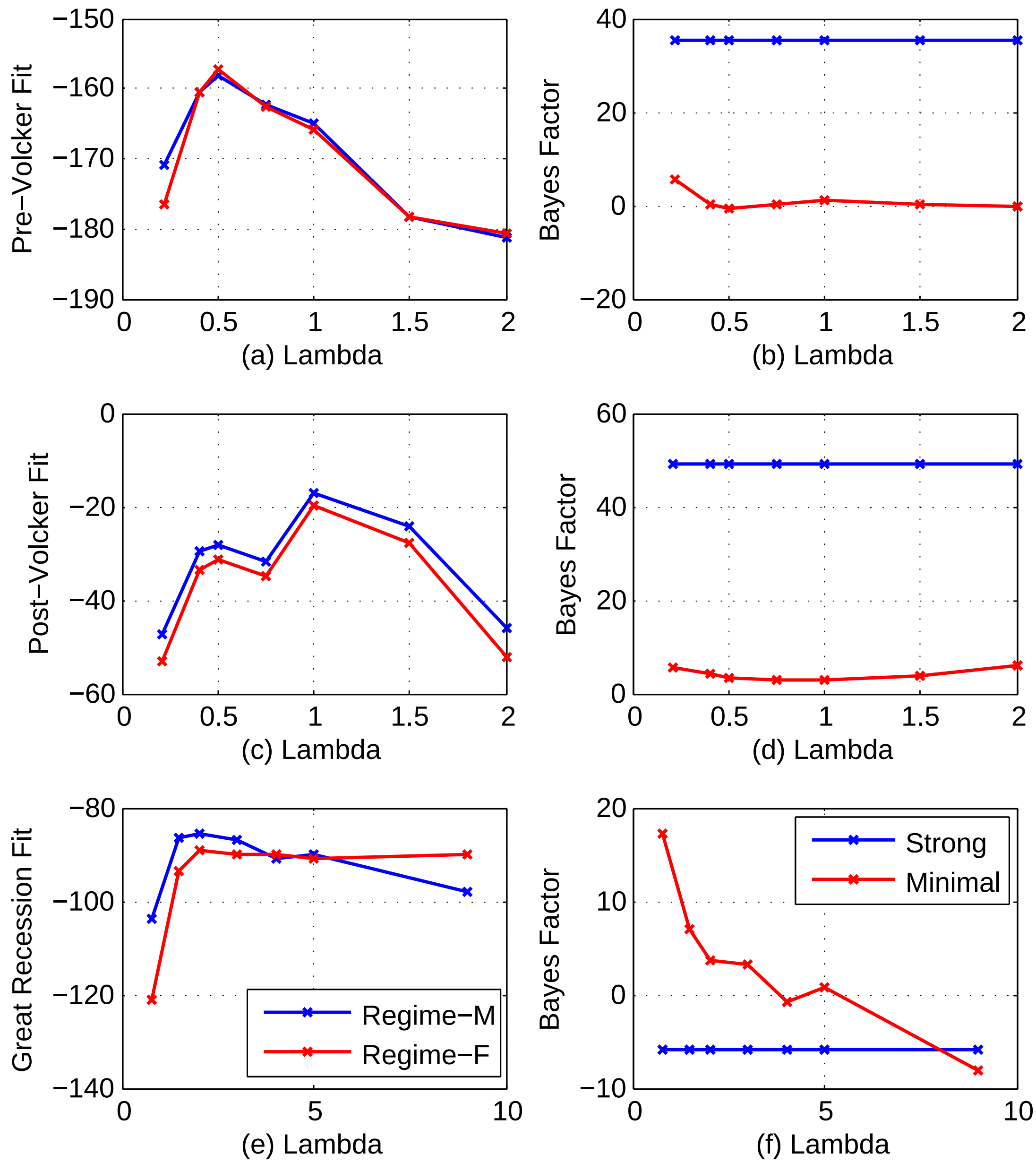

Figure 1.1: Model Fit and Bayes Factor. Notes: Panels (a,c,e) display the log marginal data densities (y-axis) of regime M (blue) and F (red) DSGE-VAR models as functions of DSGE prior weight λ (x-axis) for all samples. Panels (b,d,f) display the log Bayes factors associated with strong and minimal econometric interpretations. for all samples

Second, except for a few extreme cases marked by superscript * in Table 1.9, the Bayes factors in favor of the policy regime with superior fit, as depicted in Panels (b,d,f) of Figure 1.1, uniformly fall below those under strong econometric interpretation, and consistently stay within a "small" neighborhood of zero that signifies no decisive statistical evidence.[26] It underscores the essence of minimal econometric interpretation of the U.S. policy interactions—the minimally interpreted regime M and F DSGE models are nearly observationally equivalent. This greatly alters the nature of the findings regarding policy regime evaluation established under strong econometric interpretation in Section 1.3. To the extent that the long-term debt regime M and F DSGE models also share some strong similarity in model dynamics, this near observational equivalence result may not be too surprising. Recall that because a contemporaneous surprise deflation is required to reestablish the equilibrium, a monetary contraction under regime F, despite operating through a different mechanism, preserves its contractionary effects, mimicking the initial impact of a monetary contraction on inflation under regime M. Going forward, this subtle observational equivalence result makes the practice of selecting the policy regime with highest posterior probability unreliable, though it has been shown, e.g. Fernandez-Villaverde and Francisco Rubio-Ramirez (2004), that posterior odds asymptotically favor the DSGE model closest to the true model in the Kullback-Leibler sense.

The minimal econometric interpretation of the U.S. policy interactions also sheds new light on the existing understandings of the macroeconomic dynamics during pre- and post-Volcker periods. First, it is often discussed in the literature as if the high level and volatility of inflation of the 1970s could have been prevented had the monetary policy followed the "right" track. Fiscal policy, however, also underwent dramatic changes at that time and economic theory on policy interactions makes it clear that monetary policy will lose its

[26] Because regime F DSGE model fares better in model fit under strong econometric interpretation, all the Bayes factors in Panel (f) should be interpreted with negative signs.

power of controlling inflation in an environment where expectations about fiscal behavior are unanchored.[27] In other words, regime F seems to fit well into what was happening in the U.S. in the 1970s though it is found in Traum and Yang (2011) that the strongly estimated regime F DSGE model receives a flat rejection. As can be seen from Panel (a) that the blue and red curves closely track each other for most values of (λ_M, λ_F), there is barely decisive statistical evidence supporting either of the two minimally interpreted policy regimes. Second, the "Good Policy" explanation, as opposed to "Good Luck", that has been attributed to the increasing macroeconomic stability during post-Volcker periods argues that monetary policy has been active, characterized by Taylor-type rule, since the appointment of Paul Volcker as Chairman of the Federal Reserve Board. This argument also renders monetary policy omnipotent and ignores the alternative transmission mechanism through which fiscal policy can play a key role in determining the price level. Although the blue and red curves do not closely track each other as in Panel (a), Panel (c) suggests again that there is virtually no decisive statistical evidence supporting either of the two minimally interpreted policy regimes.

As a final remark, that regime M and F DSGE models are both detected to be misspecified and to a similar degree reinforces the statistical evidence against the completeness assumption of model space found in Section 1.3. This makes the identification of policy regime in practice even more challenging. We conjecture that such identification problem also generalizes to medium- or large-scale regime M and F DSGE models with richer model dynamics and more observables, and leave its exploration for future research.

[27] For example, the Ford tax cut and tax rebate had spurred primary deficits beginning in 1975 that sustained at an annual rate of 20% of the market value of outstanding debt [see Figure 1, Sims (2011)].

1.5 Concluding Remarks

This article compares two econometric interpretations of the post-war U.S. fiscal and monetary policy interactions. The main findings were summarized in the introduction and I repeat them more briefly here: [i.] the strong, likelihood-based econometric interpretation tends to powerfully favor the conventional view on inflation determination; [ii.] this result breaks down under the alternative, minimal econometric interpretation. Taken together, these findings imply that the apparent statistical support in favor of the conventional view over the fiscal theory stems largely from the strong econometric interpretation rather than from compelling empirical evidence. Conventional DSGE models suggest that the two views on inflation determination are nearly observationally equivalent under the minimal econometric interpretation.

Because statistical evidence suggests that both policy regime models are misspecified, this article also questions the completeness assumption of the model space underlying existing regime-comparison exercises. Perhaps neither the regime M nor the regime F DSGE models even stay close to the true data generating process. Therefore, a prudent policymaker, while contemplating the potential impacts of a deliberate policy intervention, is unlikely to rest her policy thinking solely upon any single regime. From a modeling perspective, this requires combining the dynamic implications from both policy regimes. For example, one route is to allow for the spillover effects of alternative policy interaction in a regime-switching framework [Davig and Leeper (2006, 2011), Bianchi and Melosi (2013)]. Another promising route is to model the decision makers as those who regard regime M and F as two approximations to the "true" policy behavior, and want decision rules to be robust with respect to a set of policy regimes nearby their approximating ones [Hansen and Sargent (2007)]. Following Geweke (2010) and Negro et al. (2014), we also suggest a third route that explores the idea of linear opinion pool models to overcome the identification issue with policy regime, and formally address it in Leeper et al. (2014). These approaches all point toward modeling the policy interactions within an *incomplete* model space where it is not assumed that the true data generating process has been included.

It is worthwhile to equip the foregoing small-scale DSGE model with more structural features and observables so that even the minimal econometric interpretation might become informative about the prevailing regime. But two points made here should stand the test of these extensions. First, including a maturity structure for government debt always plays a role in improving the model fit, and the improvement gets much more pronounced under regime F. Second, Bayes factors will typically be deflated in general once the underlying econometric interpretation is relaxed from being strong to minimal, making the strongly interpreted policy regime with inferior model fit at best nominally rejected.

Online Appendix

Appendix A: DSGE Model Analysis

First, substituting the bond pricing relation (1.3.3) into household's budget constraint (1.3.2) and rearranging terms yield

$$P_tC_t + \mathbb{E}_t[Q_{t+1|t}(1+\rho P_{B,t+1})B_t] = (1+\rho P_{B,t})B_{t-1} + [P_tW_tH_t + P_tD_t - T_t] \qquad (1.5.1)$$

where we also assume a borrowing limit each period, according to which the household's portfolio must satisfy

$$(1+\rho P_{B,t+1})B_t \geq -\sum_{s=t+1}^{\infty} \mathbb{E}_{t+1}[Q_{s|t+1}(P_tW_tH_t + P_tD_t - T_t)]$$

for each possible state in period $t+1$; it says that the household must never accumulate debts greater than the present value of all future after-tax income, which rules out the Ponzi schemes. The usual transversality condition on asset accumulation applies, i.e.

$$\lim_{T\to\infty} \mathbb{E}_t[Q_{T|t}(1+\rho P_{B,T})B_{T-1}] = 0 \tag{1.5.2}$$

which says that it is not optimal for household to overaccumulate assets. The sequence of flow budget constraints (1.5.1) combined with the transversality condition (1.5.2) is then equivalent to an intertemporal budget constraint

$$\sum_{s=t}^{\infty} \mathbb{E}_t[Q_{s|t}P_sC_s] = (1+\rho P_{B,t})B_{t-1} + \sum_{s=t}^{\infty} \mathbb{E}_t[Q_{s|t}(P_tW_tH_t + P_tD_t - T_t)] < \infty \tag{1.5.3}$$

where $Q_{s|t}$ for discounting income in period s back to period t is defined as the product of factors $Q_{i+1|i}$ for i running from t through $s-1$ with $Q_{t|t} = 1$. Thus we can state the household's problem, looking forward from any date t, as the choice of a sequence of planned consumption and working hours to maximize (1.3.1) subject to (1.5.3), given financial wealth $(1+\rho P_{B,t})B_{t-1}$. This gives the necessary and sufficient conditions (1.3.5) and (1.3.6) for household optimization.

To derive the intertemporal equilibrium condition (1.3.20), iterate on the bond pricing relation (1.3.3) forward and impose the terminal condition $\lim_{T\to\infty} \rho^{T-2}\mathbb{E}_t[Q_{T|t}P_{B,T}] = 0$ to obtain

$$P_{B,t} = \sum_{j=0}^{\infty} \rho^j \mathbb{E}_t[Q_{t+1+j|t}] = \sum_{j=0}^{\infty} \rho^j \mathbb{E}_t\left[\left(\prod_{i=0}^{j} \frac{1}{\pi_{t+1+i}}\right) m_{t+1+j|t}\right] \tag{1.5.4}$$

which implies that $P_{B,t+1}$ is determined by the expected, discounted, future stochastic discount factors from period $t+1$ onwards. So we may assume without loss of generality that $\mathrm{Cov}[Q_{t+1|t}, P_{B,t+1}] = 0$. Then the price of bond portfolio can be written in terms of current and expected future nominal interest rates

$$P_{B,t} = \sum_{k=0}^{\infty} \rho^k \mathbb{E}_t \left[\frac{1}{R_t R_{t+1} \cdots R_{t+k}} \right] \tag{1.5.5}$$

Also, substituting the bond pricing relation (1.3.3) into the government budget constraint (1.3.15) and dividing through by the nominal income $P_t Y_t$ give

$$\frac{(1+\rho P_{B,t}) B_{t-1}^s}{P_t Y_t} = s_t + \mathbb{E}_t \left[\bar{m}_{t+1|t} \frac{(1+\rho P_{B,t+1}) B_t^s}{P_{t+1} Y_{t+1}} \right] \tag{1.5.6}$$

where s_t, γ_t, and $\bar{m}_{t+1|t}$ are defined as in the text. Iterating on (1.5.6) forward, imposing the terminal condition $\lim_{T\to\infty} \mathbb{E}_t \left[\bar{m}_{T|t} \frac{(1+\rho P_{B,T}) B_{T-1}^s}{P_T Y_T} \right] = 0$, and rearranging give

$$\frac{1+\rho P_{B,t}}{\pi_t \gamma_t} \frac{b_{t-1}^s}{P_{B,t-1}} = \sum_{k=0}^{\infty} \mathbb{E}_t [\bar{m}_{t+k|t} s_{t+k}] \tag{1.5.7}$$

Substituting (1.5.5) into (1.5.7) and imposing the bond market clearing condition yield (1.3.20).

Next, it can be shown that consumption, labor, output, nominal interest rates, inflation, bond prices, and real wage have to satisfy the following optimality conditions derived from consumer's utility maximization and firms' optimal price-setting problems

$$\begin{aligned}
\frac{N_t^{\varphi}}{(C_t/A_t)^{-\tau}} &= \frac{W_t}{A_t} \\
1 &= \beta E_t \left[\left(\frac{C_{t+1}/A_{t+1}}{C_t/A_t} \right)^{-\tau} \frac{A_t}{A_{t+1}} \frac{R_t}{\pi_{t+1}} \right] \\
P_{B,t} &= \beta E_t \left[\left(\frac{C_{t+1}/A_{t+1}}{C_t/A_t} \right)^{-\tau} \frac{A_t}{A_{t+1}} \frac{1+\rho P_{B,t+1}}{\pi_{t+1}} \right] \\
1 &= \frac{1}{v} \left(1 - \frac{W_t}{z_t A_t} \right) + \phi(\pi_t - \pi^*) \left[\left(1 - \frac{1}{2v} \right) \pi_t + \frac{\pi^*}{2v} \right] \\
&\quad -\phi \beta E_t \left[\left(\frac{C_{t+1}/A_{t+1}}{C_t/A_t} \right)^{-\tau} \frac{Y_{t+1}/A_{t+1}}{Y_t/A_t} (\pi_{t+1} - \pi^*) \pi_{t+1} \right]
\end{aligned}$$

In the absence of nominal rigidities ($\phi = 0$), the target level of output in the monetary policy rule is given by

$$Y_t^* = \left[(1-v)^{\frac{1}{\tau}} z_t^{\frac{1+\varphi}{\tau}} g_t\right]^{\frac{\tau}{\tau+\varphi}} A_t$$

Since the non-stationary technology process $z_t A_t$ induces stochastic growth in consumption and output, it is convenient to express the model in terms of detrended and stationary variables $c_t = C_t/A_t$ and $y_t = Y_t/A_t$. The model economy has a unique steady-state associated with the detrended variables that is attained when the innovations $\{\epsilon_t^z, \epsilon_t^F, \epsilon_t^M, \epsilon_t^G\}$ are zero at all times. The steady-state values are given by

$$r^* = \frac{\gamma}{\beta}, \qquad R^* = r^*\pi^* = \frac{1+\rho P_B^*}{P_B^*}, \qquad \frac{\tau^*}{b^*} = \frac{g^{(Q)}}{b^*} + \frac{1}{\beta} - 1,$$
$$\text{and} \qquad y^* = c^* g^* = \left[(1-v)^{1/\tau} g^*\right]^{\frac{\tau}{\tau+\varphi}}$$

Linearization of the optimality conditions, government budget constraint, aggregate resource constraint, aggregate production relationship, as well as fiscal and monetary policy rules yields

$$\hat{c}_t = \mathbb{E}_t[\hat{c}_{t+1}] - \frac{1}{\tau}(\hat{R}_t - \mathbb{E}_t[\hat{\pi}_{t+1}]) \tag{1.5.8}$$

$$\hat{P}_{B,t} = \frac{\beta\rho}{\gamma\pi^*}\mathbb{E}_t[\hat{P}_{B,t+1}] - \tau(\mathbb{E}_t[\hat{c}_{t+1}] - \hat{c}_t) - \mathbb{E}_t[\hat{\pi}_{t+1}] \tag{1.5.9}$$

$$\hat{\pi}_t = \beta\mathbb{E}_t[\hat{\pi}_{t+1}] + \kappa\hat{c}_t + \frac{\kappa\varphi}{\tau}\hat{N}_t - \frac{\kappa}{\tau}\hat{z}_t \tag{1.5.10}$$

$$\begin{aligned}\hat{b}_t = &- \left[\left(\frac{g^{(Q)}}{b^*} + \frac{1}{\beta} - 1\right)\hat{\tau}_t - \frac{1-g^{(Q)}}{b^*}\hat{g}_t\right] - \frac{1}{\beta}\hat{\pi}_t + \frac{\rho}{\gamma\pi^*}\hat{P}_{B,t} - \frac{1}{\beta}(\hat{y}_t - \hat{y}_{t-1}) \\ &+ \frac{1}{\beta}(\hat{b}_{t-1} - \hat{P}_{B,t-1})\end{aligned} \tag{1.5.11}$$

$$\hat{c}_t = \hat{y}_t - \hat{g}_t \tag{1.5.12}$$

$$\hat{N}_t = \hat{y}_t - \hat{z}_t \tag{1.5.13}$$

$$\hat{\tau}_t = \rho^F\hat{\tau}_{t-1} + (1-\rho^F)\delta^b\hat{b}_{t-1} + \psi_t^F \tag{1.5.14}$$

$$\hat{R}_t = \rho^M\hat{R}_{t-1} + (1-\rho^M)\psi^\pi\hat{\pi}_t + (1-\rho^M)\psi^y\hat{y}_t + \theta_t^M \tag{1.5.15}$$

where $\kappa = \frac{\tau(1-v)}{v\phi\pi^2}$. These equations, when combined with the four exogenous shock processes, form a linear rational expectations system in the vector of variables

$$s_t = [\hat{y}_t, \hat{N}_t, \hat{\pi}_t, \hat{R}_t, \hat{P}_{B,t}, \hat{b}_t, \hat{\tau}_t, \hat{z}_t, \psi_t^F, \theta_t^M, \hat{g}_t]'$$

that is driven by the vector of innovations $\epsilon_t = [\epsilon_t^Z, \epsilon_t^F, \epsilon_t^M, \epsilon_t^G]'$. With the issuance of only one-period nominal bonds, (1.5.9) disappears from the system and (1.5.11) is replaced by

$$\hat{b}_t = -\left[\left(\frac{g^{(Q)}}{b^*} + \frac{1}{\beta} - 1\right)\hat{\tau}_t - \frac{1-g^{(Q)}}{b^*}\hat{g}_t\right] - \frac{1}{\beta}\hat{\pi}_t - \frac{1}{\beta}(\hat{y}_t - \hat{y}_{t-1}) + \frac{1}{\beta}(\hat{b}_{t-1} + \hat{R}_{t-1}) \tag{1.5.16}$$

Note that the model with only one-period nominal bonds is not exactly equivalent to the one in the text by setting $\rho = 0$, though Table 1.1 shows that the two models have "close" fit under regime M.

We assume that the time period t in the model corresponds to one quarter and the relationship between the four observable series and the model variables is given by the following measurement equations

$$\text{YGR}_t = \gamma^{(Q)} + 100(\hat{y}_t - \hat{y}_{t-1}) \tag{1.5.17}$$

$$\text{INF}_t = \pi^{(A)} + 400\hat{\pi}_t \tag{1.5.18}$$

$$\text{INT}_t = \pi^{(A)} + r^{(A)} + 4\gamma^{(Q)} + 400\hat{R}_t \tag{1.5.19}$$

$$\text{DTY}_t = b^{(Q)} + b^{(Q)}\hat{b}_t \tag{1.5.20}$$

where the parameters $\gamma^{(Q)}$, $\pi^{(A)}$, $r^{(A)}$, and $b^{(Q)}$ are related to the steady-states of the model economy by

$$\gamma^{(Q)} = 100(\gamma - 1), \quad r^{(A)} = 400\left(\frac{1}{\beta} - 1\right), \quad \pi^{(A)} = 400(\pi^* - 1), \quad b^{(Q)} = b^*$$

The system of transition equations (1.5.8)–(1.5.15) and the set of measurement equations (1.5.17)–(1.5.20) together form a state space representation available for estimation.

Lastly, we display the impulse response functions of DSGE models. Figure 1.2 documents the effects of four structural shocks on four observables under regime M.

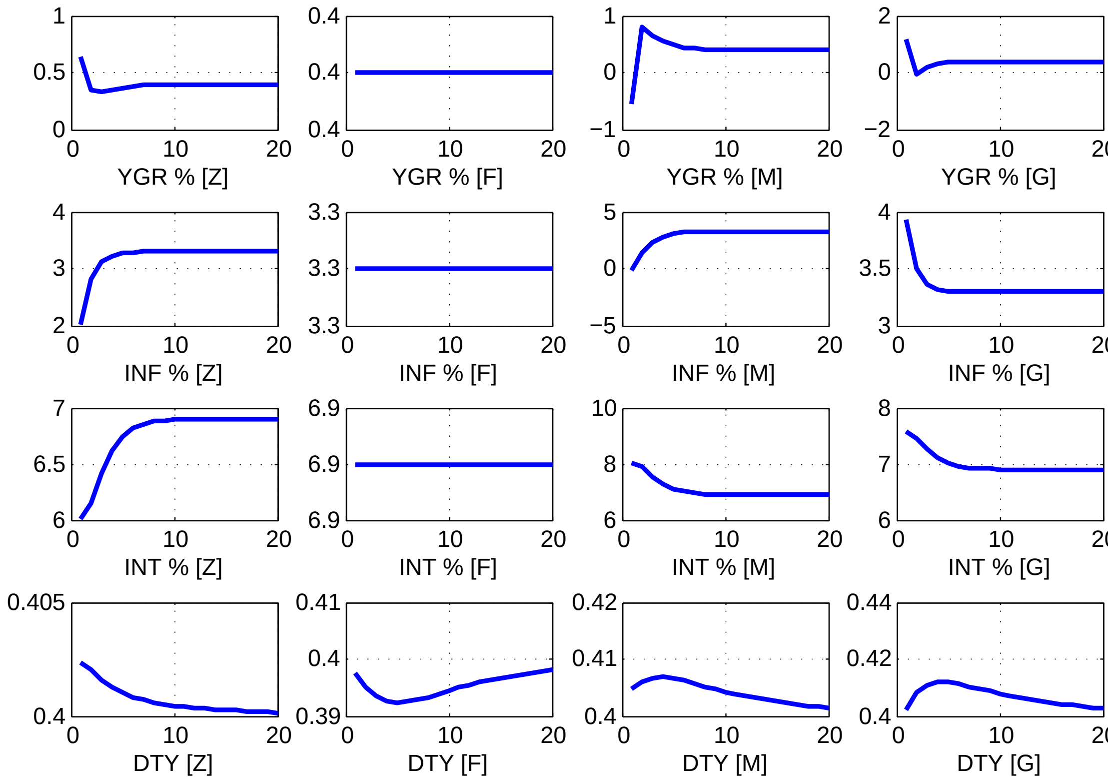

Figure 1.2: Impulse Response Functions of regime M DSGE Model. Notes: The solid lines display the effects on the four observables (YGR, INF, INT, DTY) of the four structural shocks (Z, F, M, G). Parameters are set according to the last column (DGP) of Table 1.2.

First column: A technological progress works as a favorable aggregate supply shock that increases output growth and decreases inflation. Since the central bank responses to inflation deviation much more strongly than output under regime M, the net effect of rising output and falling inflation is to reduce the nominal interest rate, which raises the bond price. Because the effect of higher output growth on debt ratio is fully offset by those of lower inflation and higher bond price, by (1.5.11) the real debt ratio rises. In response to the rising debt ratio, fiscal authority reacts passively by raising sufficient net tax revenues now and in the future so as to make the government budget constraint satisfied.

Second column: A fiscal contraction in the form of higher net taxes has no real effect on household's consumption streams because the household, while having a firm belief about fiscal authority's mandate on debt stabilization, simply decreases her holding of government bonds so as to maintain the original consumption profile, expecting lower net taxes in the future due to lower debt ratio. The anticipation of lower future net taxes eliminates the wealth effect of higher current net taxes. Analogous to the typical story of Ricardian equivalence, this leaves the present value of current and expected future primary surpluses unchanged, and hence the output growth, inflation, and nominal interest rate. However, the real debt ratio decreases by (1.5.11), giving rise to a negative contemporaneous correlation between debt ratio and primary surplus ratio. In sum, a positive net tax shock does not have any equilibrium effect on non-fiscal variables and it only lowers the debt ratio under regime M. This type of Ricardian equivalence result also holds for changes in the average maturity of nominal bonds. For example, an extension of the average maturity (ρ increases) raises the bond price and hence the real debt ratio by (1.5.11), leading to an expectation of higher future net taxes. By (1.3.20) this makes it have no real impact.

Third column: Because prices are sticky, a monetary contraction in the form of higher nominal interest rate raises the real interest rate. By (1.5.8) consumption today becomes more costly relative to tomorrow, leading to a decrease in consumption and hence output growth and inflation. In addition, the effect of lower bond price and hence the market value of government liabilities due to higher nominal interest rate is fully offset by those of the lower output growth and inflation. By (1.5.11) the real debt ratio increases, giving rise to a positive contemporaneous correlation between debt ratio and nominal interest rate under regime M. In response to the rising debt ratio, fiscal authority reacts passively by raising net tax revenues now and in the future sufficiently to retire the additional government deficits so that the government budget remains solvent.

Fourth column: A fiscal expansion in the form of higher government spending spurs the aggregate demand and hence raises output growth and inflation. In response to the rising inflation and output, the central bank reacts aggressively by raising the nominal interest rate under regime M. Because prices are sticky, this raises the real interest rate and decreases consumption, which partly offsets the increase in output. The monetary contraction eventually brings output growth and inflation back to the steady states. In addition, since the rising government deficits brought about by the higher government expenditure are not fully absorbed by the inflation taxes, lower market value of government liabilities due to lower bond price, as well as higher output growth, by (1.5.11) the real debt ratio increases. In response to the rising debt ratio, fiscal authority reacts passively by raising sufficient net tax revenues now and in the future so as to make the government budget constraint satisfied.

Figure 1.3 documents the effects of four structural shocks on four observables under regime F.

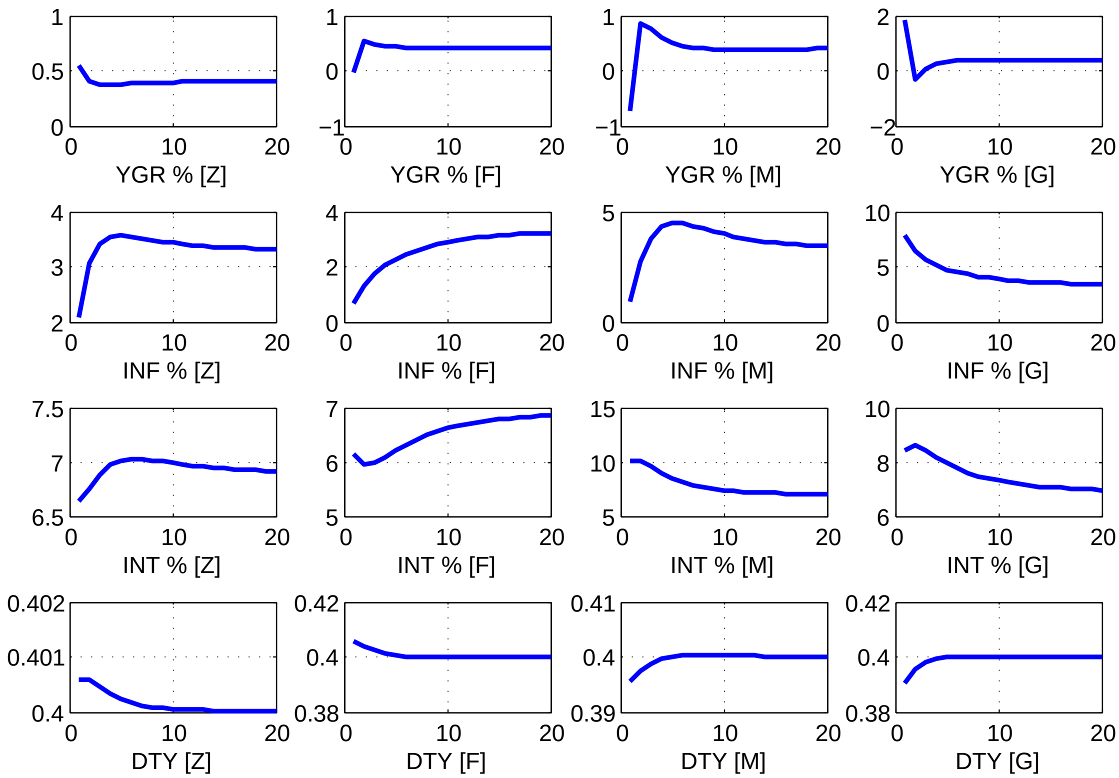

Figure 1.3: Impulse Response Functions of Regime F DSGE Model. Notes: See Figure 1.2.

First column: A technological progress increases output growth and decreases inflation. Since the central bank responses to inflation deviation more strongly than output under regime F, this leads to an initial decrease in the nominal interest rate and increase in the debt ratio. By (1.3.20) the real debt ratio at the beginning of the next period exceeds the present value of current and expected future primary surpluses, and a surprise inflation in the next period is required so as to restore (1.3.20). This produces an "inflation reversal" phenomenon documented in Kim (2003)—an initial decrease in inflation below the steady state due to a favorable aggregate supply shock is followed by a subsequent rise in inflation above the steady state. In addition, such inflation reversal also induces an associated "nominal interest rate reversal". This is in contrast to the effects of a technological progress under regime M in which the initial fall in inflation and nominal interest rate both smoothly die out over time.

Second column: By (1.3.20) a fiscal contraction, while the household having no anticipation of lower future net taxes, makes the real debt ratio at the beginning of the period fall below the present value of current and expected future primary surpluses. In the presence of only one-period nominal bonds, a surprise deflation in the current period must occur under regime F by just enough amount so as to restore (1.3.20). This is because the market value of government liabilities are backed up more than sufficiently by the primary surpluses, making the households feel less wealthier and hence try to substitute consumption for government bonds. As a result, output growth and inflation fall. Long-term nominal bonds, however, introduce the possibility of surprise changes in current and future nominal interest rates as an additional channel to revalue government liabilities. In response to the falling inflation and output, central bank can engineer a monetary expansion by moderately reducing the nominal interest rate. This bids up the bond price and hence the market value of government liabilities, effectively reducing the complete reliance on current deflation. By (1.5.11), the falling inflation, output growth, and nominal interest rate all tend to raise the real debt ratio, giving rise to a positive contemporaneous correlation between debt ratio and primary surplus ratio. In contrast to the scenario under regime M, a fiscal contraction does have real impacts on the economy. So do changes in the average maturity of nominal bonds. For example, an extension of the average maturity (ρ increases) raises the bond price and hence the debt ratio but it generates no expectation of higher future net taxes. As the household feels wealthier, her demand for consumption goods rises, which pushes up inflation and output.

Third column: A monetary contraction lowers the bond price, which by (1.3.20) makes the real debt ratio at the beginning of the period fall below the present value of current and expected future primary surpluses. As a result, a surprise deflation in the current period must occur under regime F by just enough amount so as to restore (1.3.20), mimicking the initial impact of a monetary contraction on inflation under regime M. This is because the lower bond price or higher bond yield makes the government bond more attractive and the household tries to substitute out consumption goods into government bonds, which lowers inflation and output in the current period. But as the nominal interest rate falls back to its steady-state, bond price starts increasing and (1.3.20) requires a surprise inflation in the future periods so as to reestablish (1.3.20). Again this is because the household tries to convert government bonds into consumption goods in response to the higher bond price, leading to an increase in inflation and output in the future periods. In contrast, a monetary contraction under regime F with only one-period nominal bonds decreases inflation initially, which then smoothly dies out over time. In addition, the effect of lower bond price and hence the market value of government liabilities due to higher nominal interest rate dominates those of the lower inflation and output growth. By (1.5.11) real debt ratio decreases, giving rise to a negative contemporaneous correlation between debt ratio and nominal interest rate.

Fourth column: By (1.3.20) higher government spending decreases the present value of current and expected future primary surpluses below the real debt ratio at the beginning of the period. In the presence of only one-period nominal bonds, a surprise inflation in the current period must occur under regime F by just enough amount so as to restore (1.3.20). This is because the market value of government liabilities are not backed up sufficiently by the primary surpluses, making the household feel wealthier and hence try to substitute government bonds for consumption. As a result, output growth and inflation rise. In response to the rising inflation and output, the central bank moderately raises the nominal interest rate. With long-term nominal bonds, this bids down the bond price and hence the market value of government liabilities, effectively reducing the complete reliance on current inflation to devalue government debt. The overall responses of output growth, inflation, and nominal interest rate are larger than those under regime M, which by (1.5.11) completely offset the effect of rising government deficits. As a result, the real debt ratio falls even though the rising deficits induces no fiscal adjustment under regime F.

Appendix B: U.S. Data Set

Unless otherwise stated, the following data are drawn from the National Income and Product Accounts Tables released by the Bureau of Economic Analysis. All data in levels are nominal values. The four observable sequences in the text are constructed as follows:

1. **Quarter-to-quarter per capita GDP growth rate**, YGR. Per capita (real) GDP is obtained by dividing the nominal GDP, defined as the sum of total personal consumption expenditures (Table 1.1.5, line 2) and federal government consumption expenditures and gross investment (Table 1.1.5, line 22), by population 16 years and older (Bureau of Labor Statistics, series ID: CLF16OV) and deflating using GDP deflator (Table 1.1.4, line 1).
2. **Annualized quarter-to-quarter inflation rate**, INF, is defined as the growth rate of GDP deflator (Table 1.1.4, line 1) and converted into annualized percentage.
3. **Annualized short-term nominal interest rate**, INT, corresponds to the effective federal funds rate (Board of Governors of the Federal Reserve System, series ID: FEDFUNDS) and is also in percentage.
4. **Quarterly real debt-to-GDP ratio**, DTY, is obtained by dividing the market value of privately held gross federal debt (Federal Reserve Bank of Dallas) by nominal GDP.

All growth rates are computed using quarter-to-quarter log differences and converted into percentage by multiplying by 100. Quarterly data are the monthly data at the beginning of each quarter. The prior means of steady-state parameters are calibrated using data from 1955:Q1 to 2014:Q2.

Appendix C: Bayesian Implementation

First, Table 1.2 below lists the marginal prior distributions of regime M and F DSGE model parameters, truncated to the determinacy region of each policy regime. The priors for τ and φ are both loosely centered at 2 with the implied 90% credible sets encompassing the empirical ranges of the degree of risk aversion and the Frisch elasticity of labor supply based on microlevel studies.[28] The slope of Phillips curve centers at 0.2 and lies between 0.07 and 0.39, so chosen as to be consistent with values used in the usual calibration exercises. Based on the finding of almost no response of policy rate to output deviation for the pre- and post-Volcker samples in Boivin and Giannoni (2006), ψ^y has a tight prior centered at 0.125, about one fourth of the value in univariate Taylor rule regressions with annualized data. We are *a priori* uncertain about the smoothing parameters (ρ^F, ρ^M) in the fiscal and monetary policy rules and the autoregressive coefficients $(\rho_z, \rho_\psi, \rho_\theta, \rho_g)$ in the structural shocks, so their prior means are all centered at the midpoint of unit interval.[29] A presample of observations, ranging from 1955:Q1 to 2014:Q2, is used to provide guidance on choosing the priors for all steady state parameters. A tight prior with mean 0.4 is imposed on the steady state quarterly growth rate of technology $\gamma^{(Q)}$ (in percentage) to match the average quarterly growth rate of output per capita. The prior for the inverse of annual discount factor $r^{(A)}$ (in percentage) implies a growth-adjusted annual real interest rate of 3.6% on average. Due to the high volatility of inflation in the pre-Volcker sample, we put a fairly diffuse prior on the steady state annual inflation rate $\pi^{(A)}$ (in percentage) with mean 3.3 to match the average annual inflation rate. Because the real debt-to-GDP ratio exhibits a modest magnitude of variability, its steady state $b^{(Q)}$ (not in percentage) has a relatively tight prior with mean 0.4 to match the average quarterly real debt-to-GDP

[28]The prior on φ implies that the Frisch elasticity of labor supply, $1/\varphi$, lies between 0.35 and 0.80 with 90% probability.

[29]The priors are also relatively tight to prevent these parameters from hitting the boundary.

ratio. Since we do not include the time series for government spending in the data set, the steady state government-spending-to-GDP ratio $g^{(Q)}$ is calibrated to the mean value of the presample, which is 0.25. Finally, the choice of priors for the standard deviation parameters $(\sigma_Z, \sigma_F, \sigma_M, \sigma_G)$, all scaled by 100 to convert their units into percentages, is based on a prior predictive check so as to obtain realistic volatilities of the observables, and their means are all centered at 0.2.

Next, Tables 1.3–1.8 make cross-regime comparison for the posterior means and 90% credible sets of regime M and F DSGE model parameters. Figures 1.4–1.9 make further cross-regime comparison for the posterior density plots of the parameters $(\rho, \delta^b, \psi^\pi)$. For each policy regime, i.e. $D \in \{M, F\}$, the posterior draws can be obtained by the following RWM algorithm:

1. Use the Matlab programs of BFGS quasi-Newton algorithm written by Chris Sims to maximize the posterior density kernel $p(Y|\theta_D)p(\theta_D)$. If the value of θ_D implies either non-existence or non-uniqueness of a stable rational expectations solution, then set $p(Y|\theta_D)p(\theta_D)$ to be zero. Otherwise, use the Kalman filter to evaluate the linear state space system. Denote the posterior mode by $\hat{\theta}_D$ and the inverse of the negative Hessian evaluated at $\hat{\theta}_D$ by $\hat{\Sigma}_D$.
2. Initialize the Markov chain by setting $\theta_D^{(0)} = \hat{\theta}_D$.
3. For $s = 1, \ldots, N$, draw θ_D from the proposal distribution $\mathbb{N}(\theta_D^{(s-1)}, c^2\hat{\Sigma}_D)$ where c serves as a scaling parameter. The jump from $\theta_D^{(s-1)}$ is accepted $(\theta_D^{(s)} = \theta_D)$ with probability $\min\{1, r(\theta_D^{(s-1)}, \theta_D|Y)\}$ and rejected $(\theta_D^{(s)} = \theta_D^{(s-1)})$ otherwise. Here

$$r(\theta_D^{(s-1)}, \theta_D|Y) = \begin{cases} \frac{p(Y|\theta_D)p(\theta_D)}{p(Y|\theta_D^{(s-1)})p(\theta_D^{(s-1)})} & \text{if solution exists and is unique} \\ 0 & \text{otherwise} \end{cases}$$

To implement the DSGE-VAR approach, we generate draws from the joint posterior distribution of DSGE model parameters and VAR parameters for each policy regime, i.e. $D \in \{M, F\}$, by following the steps below:

1. For each $\lambda_D \in \Lambda$, use the RWM algorithm described above to generate draws from $p_{\lambda_D}(\theta_D|Y) \propto p_{\lambda_D}(Y|\theta_D)p(\theta_D)$, where the expression for the marginal likelihood function of θ_D, $p_{\lambda_D}(Y|\theta_D)$, can be found in Appendix A.1 of Del Negro and Schorfheide (2004). Note that conditional on θ_D, the population moments m can be computed analytically from the solution of linearized regime M and F DSGE models.
2. Based on these posterior draws, apply Geweke (1999)'s modified harmonic mean estimator to obtain numerical approximations of the marginal data densities $p_{\lambda_D}(Y|D, E)$.
3. Find the value of $\hat{\lambda}_D$ that achieves the highest marginal data density.
4. Select the posterior draws of $\{\theta_D^{(s)}\}_{s=1}^N$ that correspond to $\hat{\lambda}_D$ and use standard methods to generate draws of (Φ, Σ_u) from $p_{\hat{\lambda}_D}(\Phi, \Sigma_u|Y, \theta_D^{(s)})$ for each $\theta_D^{(s)}$.
5. For each posterior draw of $\{\theta_D^{(s)}, \Sigma_u^{(s)}\}_{s=1}^N$, compute the corresponding Cholesky decomposition of $\Sigma_u^{(s)}$ and regime D DSGE model rotation under $\theta_D^{(s)}$.

Lastly, the RWM algorithm is applied to sample 1.01 million draws from the posterior distribution with the initial $10,000$ draws discarded, leaving a final sample size of one million for each estimation. We choose the value of c that yields a rejection rate roughly between 45% and 50% across estimations. As for the diagnostic check on the convergence of Markov chain, we rely on the graphical method suggested by An and Schorfheide (2007) that examines the convergence of the recursive means of all DSGE model parameters from multiple chains. Upon convergence, these posterior draws are then used to compute the impulse response functions of regime M and F DSGE and DSGE-VAR models as depicted in Figures 1.10–1.15, and to approximate the marginal data densities based on Geweke

(1999)'s modified harmonic mean estimator

$$\hat{p}(Y|D) = \left[\frac{1}{N} \sum_{s=1}^{N} \frac{f(\theta_D^{(s)})}{p(Y|\theta_D^{(s)})p(\theta_D^{(s)})} \right]^{-1}, \quad D \in \{M, F\}$$

where $f(\cdot)$ is the density function of a truncated multivariate normal distribution.

Table 1.2: Marginal Prior Distributions for DSGE Model Parameters

	Regime M $p(\theta_M)$ / Regime F $p(\theta_F)$				
Name	Domain	Density	Para (1)	Para (2)	DGP
τ	$\mathbb{R}^+$	$\mathbb{G}$	2.00	0.50	2.00
φ	$\mathbb{R}^+$	$\mathbb{G}$	2.00	0.50	2.00
κ	$\mathbb{R}^+$	$\mathbb{G}$	0.20	0.10	0.20
ρ	$[0, 1)$	$\mathbb{B}$	0.90	0.05	0.90
δ^b	$\mathbb{R}^+/\mathbb{R}$	$\mathbb{N}$	0.15/0.00	0.03/0.002	0.15/0.00
ψ^π	$(1, \infty)/[0, 1)$	$\mathbb{G}/\mathbb{B}$	1.50/0.50	0.15	1.50/0.50
ψ^y	$\mathbb{R}^+$	$\mathbb{G}$	0.125	0.05	0.125
ρ^F	$[0, 1)$	$\mathbb{B}$	0.50	0.10	0.50
ρ^M	$[0, 1)$	$\mathbb{B}$	0.50	0.10	0.50
$r^{(A)}$	$\mathbb{R}^+$	$\mathbb{G}$	2.00	0.50	2.00
$\pi^{(A)}$	$\mathbb{R}^+$	$\mathbb{G}$	3.30	1.50	3.30
$\gamma^{(Q)}$	$\mathbb{R}$	$\mathbb{N}$	0.40	0.05	0.40
$b^{(Q)}$	$\mathbb{R}^+$	$\mathbb{G}$	0.40	0.15	0.40
ρ_z	$[0, 1)$	$\mathbb{B}$	0.50	0.10	0.50
ρ_ψ	$[0, 1)$	$\mathbb{B}$	0.50	0.10	0.50
ρ_θ	$[0, 1)$	$\mathbb{B}$	0.50	0.10	0.50
ρ_g	$[0, 1)$	$\mathbb{B}$	0.50	0.10	0.50
$(100\sigma_Z)^2$	$\mathbb{R}^+$	$\mathbb{IG}$	10.00	0.36	0.20^2
$(100\sigma_F)^2$	$\mathbb{R}^+$	$\mathbb{IG}$	10.00	0.36	0.20^2
$(100\sigma_M)^2$	$\mathbb{R}^+$	$\mathbb{IG}$	10.00	0.36	0.20^2
$(100\sigma_G)^2$	$\mathbb{R}^+$	$\mathbb{IG}$	10.00	0.36	0.20^2

NOTES: Para (1) and Para (2) list the means and standard deviations for Normal ($\mathbb{N}$), Gamma ($\mathbb{G}$), and Beta ($\mathbb{B}$) distributions; the shape and scale parameters (α, β) for Inverse Gamma ($\mathbb{IG}$) distribution with pdf $p(x) \propto x^{-\alpha-1} \exp(-\beta/x)$. The Inverse Gamma distribution has mean $\frac{\beta}{\alpha-1}$ $(\alpha > 1)$ and variance $\frac{\beta^2}{(\alpha-1)^2(\alpha-2)}$ $(\alpha > 2)$. The standard deviation parameters $(\sigma_Z, \sigma_F, \sigma_M, \sigma_G)$ are scaled by 100 to convert them into percentages. The effective priors for regime M and F DSGE model parameters are truncated at the boundary of their determinacy regions.

Table 1.3: Posterior Estimates of DSGE Model Parameters [1955:Q1–1979:Q2]

	Prior		Regime M DSGE		Regime F DSGE	
Name	Mean	$[5, 95]$	Mean	$[5, 95]$	Mean	$[5, 95]$
τ	2.00	$[1.26, 2.89]$	2.88	$[2.26, 3.62]$	3.02	$[2.33, 3.80]$
φ	2.00	$[1.26, 2.89]$	3.27	$[2.17, 4.32]$	2.30	$[1.52, 3.23]$
κ	0.20	$[0.07, 0.39]$	0.31	$[0.19, 0.46]$	0.01	$[0.01, 0.02]$
ρ	0.90	$[0.81, 0.97]$	0.88	$[0.82, 0.93]$	0.98	$[0.97, 0.99]$
δ^b [M]	0.15	$[0.10, 0.20]$	0.11	$[0.06, 0.17]$	0*	***
δ^b [F]	0.00	***	0.15*	$[0.11, 0.20]$	0	***
ψ^π [M]	1.50	$[1.26, 1.76]$	1.66	$[1.46, 1.87]$	0.25*	$[0.12, 0.39]$
ψ^π [F]	0.50	$[0.26, 0.75]$	1.01*	$[1.01, 1.03]$	0.48	$[0.36, 0.58]$
ψ^y	0.125	$[0.06, 0.22]$	0.04	$[0.02, 0.07]$	0.04	$[0.02, 0.05]$
ρ^F	0.50	$[0.34, 0.66]$	0.49	$[0.31, 0.73]$	0.33	$[0.21, 0.45]$
ρ^M	0.50	$[0.34, 0.66]$	0.39	$[0.29, 0.50]$	0.51	$[0.40, 0.61]$
$r^{(A)}$	2.00	$[1.26, 2.88]$	1.14	$[0.74, 1.61]$	1.65	$[1.13, 2.23]$
$\pi^{(A)}$	3.30	$[1.30, 6.20]$	7.49	$[5.29, 9.72]$	1.47	$[0.59, 2.98]$
$\gamma^{(Q)}$	0.40	$[0.32, 0.48]$	0.48	$[0.43, 0.53]$	0.34	$[0.29, 0.40]$
$b^{(Q)}$	0.40	$[0.19, 0.68]$	0.43	$[0.34, 0.52]$	0.42	$[0.36, 0.48]$
ρ_z	0.50	$[0.34, 0.66]$	0.98	$[0.97, 0.99]$	0.97	$[0.94, 0.99]$
ρ_ψ	0.50	$[0.34, 0.66]$	0.58	$[0.37, 0.77]$	0.98	$[0.97, 0.98]$
ρ_θ	0.50	$[0.34, 0.66]$	0.51	$[0.42, 0.60]$	0.46	$[0.34, 0.57]$
ρ_g	0.50	$[0.34, 0.66]$	0.83	$[0.77, 0.89]$	0.42	$[0.33, 0.50]$
$100\sigma_Z$	0.20	$[0.15, 0.26]$	1.04	$[0.86, 1.18]$	3.16	$[2.07, 4.44]$
$100\sigma_F$	0.20	$[0.15, 0.26]$	3.82	$[3.34, 4.41]$	0.16	$[0.13, 0.19]$
$100\sigma_M$	0.20	$[0.15, 0.26]$	0.31	$[0.26, 0.37]$	0.20	$[0.18, 0.22]$
$100\sigma_G$	0.20	$[0.15, 0.26]$	0.70	$[0.53, 0.94]$	0.83	$[0.73, 0.94]$

NOTES: Posterior means and 90% credible sets for regime M and F DSGE model parameters are computed based on the output of RWM algorithm described in Appendix C. Estimates of (δ^b, ψ^π) for short-term debt DSGE models are distinguished by superscript *. Their prior counterparts with regime index indicated in brackets are provided for comparison ease. Triple-asterisk denotes inapplicable items. The standard deviation parameters $(\sigma_Z, \sigma_F, \sigma_M, \sigma_G)$ are scaled by 100 to convert them into percentages.

Table 1.4: Posterior Estimates of DSGE Model Parameters [1982:Q4–2007:Q4]

	Prior		Regime M DSGE		Regime F DSGE	
Name	Mean	$[5, 95]$	Mean	$[5, 95]$	Mean	$[5, 95]$
τ	2.00	$[1.26, 2.89]$	3.76	$[2.94, 4.70]$	3.76	$[2.94, 4.70]$
φ	2.00	$[1.26, 2.89]$	1.64	$[1.15, 2.23]$	1.67	$[1.00, 2.52]$
κ	0.20	$[0.07, 0.39]$	0.19	$[0.12, 0.28]$	0.09	$[0.05, 0.14]$
ρ	0.90	$[0.81, 0.97]$	0.92	$[0.85, 0.97]$	0.997	***
δ^b [M]	0.15	$[0.10, 0.20]$	0.13	$[0.08, 0.18]$	0*	***
δ^b [F]	0.00	***	0.13*	$[0.08, 0.18]$	0	***
ψ^π [M]	1.50	$[1.26, 1.76]$	1.71	$[1.46, 1.97]$	0.39*	$[0.20, 0.58]$
ψ^π [F]	0.50	$[0.26, 0.75]$	1.67*	$[1.43, 1.93]$	0.97	$[0.95, 0.99]$
ψ^y	0.125	$[0.06, 0.22]$	0.10	$[0.05, 0.16]$	0.11	$[0.05, 0.18]$
ρ^F	0.50	$[0.34, 0.66]$	0.43	$[0.24, 0.64]$	0.81	$[0.43, 0.95]$
ρ^M	0.50	$[0.34, 0.66]$	0.59	$[0.49, 0.67]$	0.67	$[0.59, 0.74]$
$r^{(A)}$	2.00	$[1.26, 2.88]$	1.51	$[1.07, 2.00]$	1.32	$[0.92, 1.77]$
$\pi^{(A)}$	3.30	$[1.30, 6.20]$	2.70	$[2.10, 3.34]$	1.09	$[0.45, 1.86]$
$\gamma^{(Q)}$	0.40	$[0.32, 0.48]$	0.52	$[0.50, 0.53]$	0.52	$[0.50, 0.53]$
$b^{(Q)}$	0.40	$[0.19, 0.68]$	0.50	$[0.39, 0.64]$	0.40	$[0.37, 0.42]$
ρ_z	0.50	$[0.34, 0.66]$	0.87	$[0.82, 0.92]$	0.94	$[0.91, 0.96]$
ρ_ψ	0.50	$[0.34, 0.66]$	0.49	$[0.28, 0.68]$	0.68	$[0.44, 0.94]$
ρ_θ	0.50	$[0.34, 0.66]$	0.61	$[0.52, 0.69]$	0.59	$[0.49, 0.68]$
ρ_g	0.50	$[0.34, 0.66]$	0.89	$[0.85, 0.92]$	0.67	$[0.62, 0.72]$
$100\sigma_Z$	0.20	$[0.15, 0.26]$	0.56	$[0.44, 0.70]$	0.48	$[0.34, 0.65]$
$100\sigma_F$	0.20	$[0.15, 0.26]$	4.40	$[3.92, 4.92]$	0.21	$[0.16, 0.27]$
$100\sigma_M$	0.20	$[0.15, 0.26]$	0.19	$[0.16, 0.22]$	0.15	$[0.14, 0.17]$
$100\sigma_G$	0.20	$[0.15, 0.26]$	0.59	$[0.51, 0.67]$	0.67	$[0.58, 0.76]$

NOTES: See Table 1.3.

Table 1.5: Posterior Estimates of DSGE Model Parameters [2008:Q1–2014:Q2]

	Prior		Regime M DSGE		Regime F DSGE	
Name	Mean	$[5, 95]$	Mean	$[5, 95]$	Mean	$[5, 95]$
τ	2.00	$[1.26, 2.89]$	2.84	$[2.07, 3.75]$	2.75	$[1.98, 3.66]$
φ	2.00	$[1.26, 2.89]$	1.89	$[1.31, 2.61]$	1.89	$[1.21, 2.69]$
κ	0.20	$[0.07, 0.39]$	0.15	$[0.09, 0.25]$	0.13	$[0.06, 0.22]$
ρ	0.90	$[0.81, 0.97]$	0.91	$[0.82, 0.97]$	0.995	***
δ^b [M]	0.15	$[0.10, 0.20]$	0.15	$[0.10, 0.20]$	0*	***
δ^b [F]	0.00	***	0.15*	$[0.10, 0.20]$	0	***
ψ^π [M]	1.50	$[1.26, 1.76]$	1.52	$[1.28, 1.77]$	0.28*	$[0.15, 0.44]$
ψ^π [F]	0.50	$[0.26, 0.75]$	1.53*	$[1.30, 1.77]$	0.98	$[0.95, 0.99]$
ψ^y	0.125	$[0.06, 0.22]$	0.13	$[0.06, 0.23]$	0.16	$[0.08, 0.27]$
ρ^F	0.50	$[0.34, 0.66]$	0.41	$[0.27, 0.56]$	0.83	$[0.68, 0.92]$
ρ^M	0.50	$[0.34, 0.66]$	0.56	$[0.44, 0.66]$	0.55	$[0.42, 0.66]$
$r^{(A)}$	2.00	$[1.26, 2.88]$	0.98	$[0.62, 1.42]$	1.09	$[0.68, 1.58]$
$\pi^{(A)}$	3.30	$[1.30, 6.20]$	0.76	$[0.33, 1.31]$	1.44	$[0.56, 2.49]$
$\gamma^{(Q)}$	0.40	$[0.32, 0.48]$	0.24	$[0.20, 0.29]$	0.22	$[0.18, 0.27]$
$b^{(Q)}$	0.40	$[0.19, 0.68]$	0.61	$[0.44, 0.81]$	0.64	$[0.58, 0.71]$
ρ_z	0.50	$[0.34, 0.66]$	0.65	$[0.46, 0.81]$	0.79	$[0.63, 0.90]$
ρ_ψ	0.50	$[0.34, 0.66]$	0.42	$[0.28, 0.57]$	0.84	$[0.72, 0.92]$
ρ_θ	0.50	$[0.34, 0.66]$	0.58	$[0.47, 0.68]$	0.60	$[0.47, 0.71]$
ρ_g	0.50	$[0.34, 0.66]$	0.65	$[0.51, 0.79]$	0.57	$[0.45, 0.67]$
$100\sigma_Z$	0.20	$[0.15, 0.26]$	0.41	$[0.30, 0.54]$	0.39	$[0.27, 0.54]$
$100\sigma_F$	0.20	$[0.15, 0.26]$	8.13	$[6.67, 9.80]$	0.30	$[0.20, 0.52]$
$100\sigma_M$	0.20	$[0.15, 0.26]$	0.23	$[0.18, 0.28]$	0.22	$[0.17, 0.27]$
$100\sigma_G$	0.20	$[0.15, 0.26]$	0.69	$[0.54, 0.85]$	0.63	$[0.51, 0.76]$

NOTES: See Table 1.3.

Table 1.6: Posterior Estimates of DSGE-VAR Model Parameters [1955:Q1–1979:Q2]

	Prior		Regime M VAR		Regime F VAR	
Name	Mean	$[5, 95]$	Mean	$[5, 95]$	Mean	$[5, 95]$
τ	2.00	$[1.26, 2.89]$	2.28	$[1.61, 3.08]$	2.32	$[1.61, 3.14]$
φ	2.00	$[1.26, 2.89]$	1.79	$[1.13, 2.61]$	1.81	$[1.14, 2.62]$
κ	0.20	$[0.07, 0.39]$	0.26	$[0.15, 0.40]$	0.21	$[0.11, 0.33]$
ρ	0.90	$[0.81, 0.97]$	0.90	$[0.81, 0.97]$	0.99	$[0.98, 1.00]$
δ^b [M]	0.15	$[0.10, 0.20]$	0.14	$[0.09, 0.19]$	0^*	***
δ^b [F]	0.00	***	0.11^*	$[0.06, 0.17]$	0	***
ψ^π [M]	1.50	$[1.26, 1.76]$	1.50	$[1.26, 1.75]$	0.48^*	$[0.36, 0.58]$
ψ^π [F]	0.50	$[0.26, 0.75]$	1.66^*	$[1.46, 1.87]$	0.94	$[0.89, 0.98]$
ψ^y	0.125	$[0.06, 0.22]$	0.14	$[0.06, 0.24]$	0.17	$[0.08, 0.28]$
ρ^F	0.50	$[0.34, 0.66]$	0.40	$[0.26, 0.54]$	0.68	$[0.51, 0.82]$
ρ^M	0.50	$[0.34, 0.66]$	0.46	$[0.34, 0.56]$	0.45	$[0.34, 0.55]$
$r^{(A)}$	2.00	$[1.26, 2.88]$	1.58	$[0.98, 2.34]$	1.50	$[0.93, 2.21]$
$\pi^{(A)}$	3.30	$[1.30, 6.20]$	2.66	$[1.27, 4.27]$	2.10	$[0.82, 3.85]$
$\gamma^{(Q)}$	0.40	$[0.32, 0.48]$	0.39	$[0.30, 0.47]$	0.38	$[0.30, 0.46]$
$b^{(Q)}$	0.40	$[0.19, 0.68]$	0.33	$[0.22, 0.49]$	0.31	$[0.23, 0.45]$
ρ_z	0.50	$[0.34, 0.66]$	0.68	$[0.55, 0.80]$	0.66	$[0.51, 0.80]$
ρ_ψ	0.50	$[0.34, 0.66]$	0.41	$[0.27, 0.56]$	0.68	$[0.51, 0.81]$
ρ_θ	0.50	$[0.34, 0.66]$	0.46	$[0.34, 0.58]$	0.43	$[0.31, 0.55]$
ρ_g	0.50	$[0.34, 0.66]$	0.46	$[0.32, 0.60]$	0.31	$[0.21, 0.42]$
$100\sigma_Z$	0.20	$[0.15, 0.26]$	0.38	$[0.28, 0.49]$	0.39	$[0.28, 0.52]$
$100\sigma_F$	0.20	$[0.15, 0.26]$	1.47	$[1.07, 1.88]$	0.24	$[0.18, 0.33]$
$100\sigma_M$	0.20	$[0.15, 0.26]$	0.22	$[0.18, 0.26]$	0.19	$[0.16, 0.22]$
$100\sigma_G$	0.20	$[0.15, 0.26]$	0.39	$[0.30, 0.48]$	0.36	$[0.28, 0.45]$

NOTES: Posterior means and 90% credible sets for regime M and F DSGE-VAR model parameters are computed based on the output of RWM algorithm described in Appendix C. The highest regime M and F marginal data densities correspond to $\hat{\lambda}_M = 0.5$ and $\hat{\lambda}_F = 0.5$.

Table 1.7: Posterior Estimates of DSGE-VAR Model Parameters [1982:Q4–2007:Q4]

	Prior		Regime M VAR		Regime F VAR	
Name	Mean	$[5, 95]$	Mean	$[5, 95]$	Mean	$[5, 95]$
τ	2.00	$[1.26, 2.89]$	2.45	$[1.77, 3.23]$	2.68	$[1.94, 3.55]$
φ	2.00	$[1.26, 2.89]$	1.49	$[0.93, 2.19]$	1.65	$[0.99, 2.42]$
κ	0.20	$[0.07, 0.39]$	0.20	$[0.12, 0.31]$	0.15	$[0.08, 0.24]$
ρ	0.90	$[0.81, 0.97]$	0.91	$[0.82, 0.97]$	0.99	$[0.98, 1.00]$
δ^b [M]	0.15	$[0.10, 0.20]$	0.14	$[0.09, 0.19]$	0*	***
δ^b [F]	0.00	***	0.13*	$[0.08, 0.18]$	0	***
ψ^π [M]	1.50	$[1.26, 1.76]$	1.63	$[1.39, 1.88]$	0.97*	$[0.95, 0.99]$
ψ^π [F]	0.50	$[0.26, 0.75]$	1.71*	$[1.46, 1.97]$	0.95	$[0.91, 0.98]$
ψ^y	0.125	$[0.06, 0.22]$	0.14	$[0.07, 0.24]$	0.18	$[0.09, 0.30]$
ρ^F	0.50	$[0.34, 0.66]$	0.35	$[0.22, 0.49]$	0.73	$[0.55, 0.86]$
ρ^M	0.50	$[0.34, 0.66]$	0.53	$[0.43, 0.63]$	0.54	$[0.45, 0.63]$
$r^{(A)}$	2.00	$[1.26, 2.88]$	1.94	$[1.23, 2.81]$	1.79	$[1.13, 2.59]$
$\pi^{(A)}$	3.30	$[1.30, 6.20]$	2.53	$[1.80, 3.27]$	2.23	$[1.02, 3.74]$
$\gamma^{(Q)}$	0.40	$[0.32, 0.48]$	0.41	$[0.34, 0.49]$	0.40	$[0.33, 0.48]$
$b^{(Q)}$	0.40	$[0.19, 0.68]$	0.55	$[0.44, 0.71]$	0.54	$[0.45, 0.68]$
ρ_z	0.50	$[0.34, 0.66]$	0.75	$[0.64, 0.84]$	0.68	$[0.52, 0.81]$
ρ_ψ	0.50	$[0.34, 0.66]$	0.36	$[0.23, 0.52]$	0.72	$[0.53, 0.86]$
ρ_θ	0.50	$[0.34, 0.66]$	0.49	$[0.38, 0.60]$	0.49	$[0.38, 0.60]$
ρ_g	0.50	$[0.34, 0.66]$	0.59	$[0.43, 0.75]$	0.40	$[0.29, 0.51]$
$100\sigma_Z$	0.20	$[0.15, 0.26]$	0.33	$[0.25, 0.42]$	0.41	$[0.26, 0.70]$
$100\sigma_F$	0.20	$[0.15, 0.26]$	2.55	$[2.09, 3.03]$	0.25	$[0.18, 0.36]$
$100\sigma_M$	0.20	$[0.15, 0.26]$	0.16	$[0.14, 0.19]$	0.15	$[0.13, 0.17]$
$100\sigma_G$	0.20	$[0.15, 0.26]$	0.32	$[0.26, 0.39]$	0.30	$[0.25, 0.35]$

NOTES: See Table 1.6. The highest regime M and F marginal data densities correspond to $\hat{\lambda}_M = 1.0$ and $\hat{\lambda}_F = 1.0$.

Table 1.8: Posterior Estimates of DSGE-VAR Model Parameters [2008:Q1–2014:Q2]

	Prior		Regime M VAR		Regime F VAR	
Name	Mean	$[5, 95]$	Mean	$[5, 95]$	Mean	$[5, 95]$
τ	2.00	$[1.26, 2.89]$	1.92	$[1.26, 2.70]$	2.00	$[1.34, 2.81]$
φ	2.00	$[1.26, 2.89]$	1.17	$[0.66, 1.82]$	1.64	$[1.01, 2.40]$
κ	0.20	$[0.07, 0.39]$	0.09	$[0.05, 0.16]$	0.13	$[0.07, 0.22]$
ρ	0.90	$[0.81, 0.97]$	0.90	$[0.80, 0.97]$	0.99	$[0.98, 1.00]$
δ^b [M]	0.15	$[0.10, 0.20]$	0.16	$[0.11, 0.21]$	0*	***
δ^b [F]	0.00	***	0.15*	$[0.10, 0.20]$	0	***
ψ^π [M]	1.50	$[1.26, 1.76]$	1.59	$[1.36, 1.83]$	0.98*	$[0.95, 0.99]$
ψ^π [F]	0.50	$[0.26, 0.75]$	1.52*	$[1.28, 1.77]$	0.96	$[0.93, 0.99]$
ψ^y	0.125	$[0.06, 0.22]$	0.10	$[0.04, 0.20]$	0.16	$[0.07, 0.27]$
ρ^F	0.50	$[0.34, 0.66]$	0.39	$[0.25, 0.53]$	0.77	$[0.61, 0.88]$
ρ^M	0.50	$[0.34, 0.66]$	0.60	$[0.48, 0.71]$	0.54	$[0.41, 0.66]$
$r^{(A)}$	2.00	$[1.26, 2.88]$	1.83	$[1.14, 2.72]$	1.72	$[1.07, 2.50]$
$\pi^{(A)}$	3.30	$[1.30, 6.20]$	2.18	$[0.83, 4.01]$	1.99	$[0.81, 3.62]$
$\gamma^{(Q)}$	0.40	$[0.32, 0.48]$	0.39	$[0.31, 0.47]$	0.37	$[0.29, 0.45]$
$b^{(Q)}$	0.40	$[0.19, 0.68]$	0.36	$[0.15, 0.64]$	0.81	$[0.67, 1.01]$
ρ_z	0.50	$[0.34, 0.66]$	0.59	$[0.42, 0.75]$	0.62	$[0.46, 0.76]$
ρ_ψ	0.50	$[0.34, 0.66]$	0.41	$[0.26, 0.56]$	0.77	$[0.62, 0.88]$
ρ_θ	0.50	$[0.34, 0.66]$	0.75	$[0.65, 0.83]$	0.61	$[0.47, 0.72]$
ρ_g	0.50	$[0.34, 0.66]$	0.53	$[0.36, 0.72]$	0.45	$[0.32, 0.57]$
$100\sigma_Z$	0.20	$[0.15, 0.26]$	0.22	$[0.17, 0.28]$	0.28	$[0.20, 0.37]$
$100\sigma_F$	0.20	$[0.15, 0.26]$	3.55	$[2.58, 4.61]$	0.31	$[0.21, 0.43]$
$100\sigma_M$	0.20	$[0.15, 0.26]$	0.16	$[0.13, 0.20]$	0.16	$[0.13, 0.20]$
$100\sigma_G$	0.20	$[0.15, 0.26]$	0.29	$[0.23, 0.37]$	0.33	$[0.25, 0.42]$

NOTES: See Table 1.6. The highest regime M and F marginal data densities correspond to $\hat{\lambda}_M = 2.0$ and $\hat{\lambda}_F = 2.0$.

Table 1.9: Model Fit of Regime M and F DSGE-VAR Models

1955:Q1–1979:Q2		1982:Q4–2007:Q4		2008:Q1–2014:Q2	
λ_D	Regime M/F	λ_D	Regime M/F	λ_D	Regime M/F
0.22^*	$-170.8/-176.6$	0.21^*	$-47.2/-52.9$	0.81^*	$-103.9/-121.2$
0.40	$-160.4/-160.5$	0.40	$-29.5/-33.8$	1.50^*	$-86.5/-93.5$
0.50	$-158.0/-157.3$	0.50	$-28.0/-31.6$	2.00	$-85.6/-89.1$
0.75	$-162.3/-162.5$	0.75	$-31.9/-34.9$	3.00	$-86.8/-90.0$
1.00	$-165.0/-166.0$	1.00	$-17.1/-19.8$	4.00	$-90.7/-90.0$
1.50	$-178.2/-178.2$	1.50	$-24.3/-28.0$	5.00	$-90.2/-91.1$
2.00	$-181.2/-180.8$	2.00^*	$-46.2/-52.1$	9.00^*	$-98.1/-90.0$
∞^*	$-278.2/-313.5$	∞^*	$-108/-157$	∞^*	$-151.4/-145.5$

NOTES: All log marginal data densities are approximated using Geweke (1999)'s modified harmonic mean estimator. Decisive evidence in favor of the policy regime with superior fit is denoted by superscript *, corresponding to a Bayes factor greater than 100, or 4.6 in logarithm.

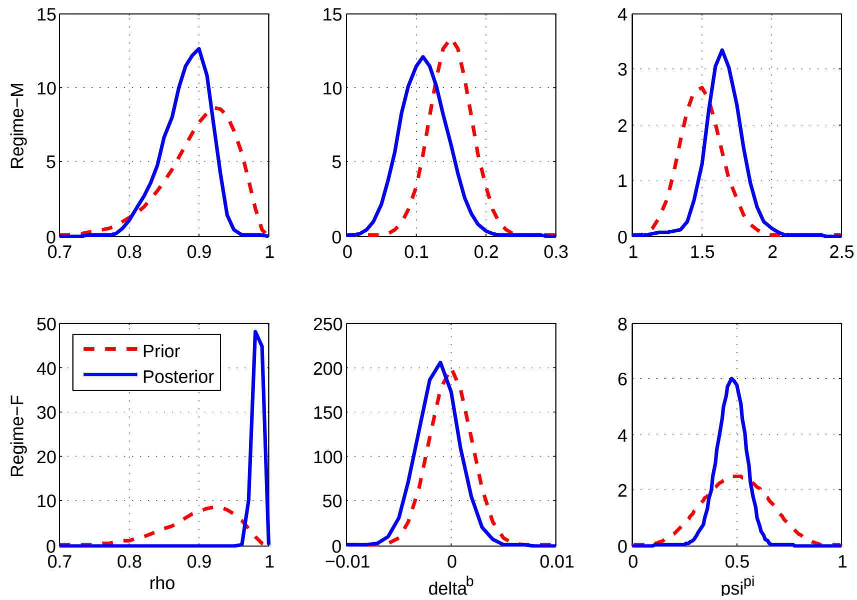

Figure 1.4: DSGE Prior and Posterior Density Functions [1955:Q1–1979:Q2]. Notes: The red-dashed and blue-solid lines display the prior and posterior density functions of the DSGE model parameters $(\rho, \delta^b, \psi^\pi)$. All posterior densities are estimated based on normal kernel function.

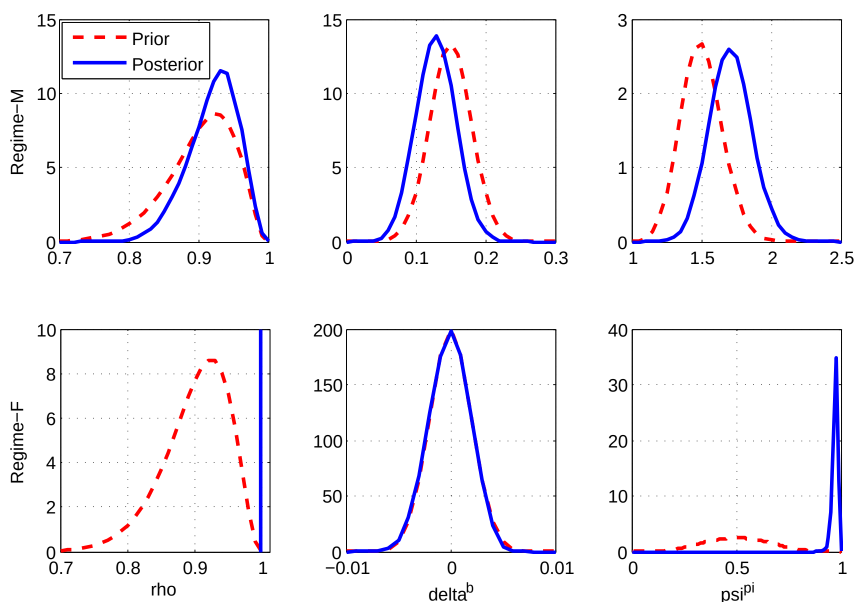

Figure 1.5: DSGE Prior and Posterior Density Functions [1982:Q4–2007:Q4]. Notes: See Figure 1.4.

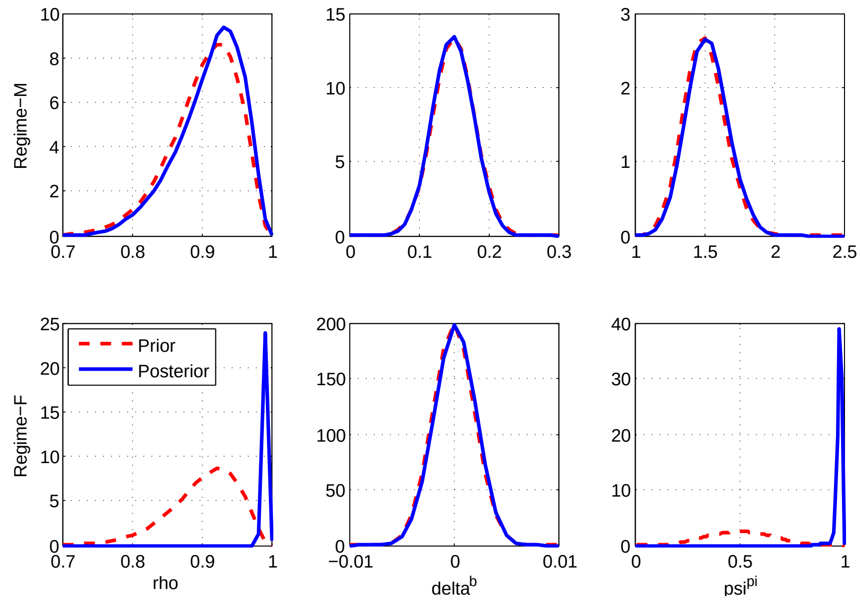

Figure 1.6: DSGE Prior and Posterior Density Functions [2008:Q1–2014:Q2]. Notes: See Figure 1.4.

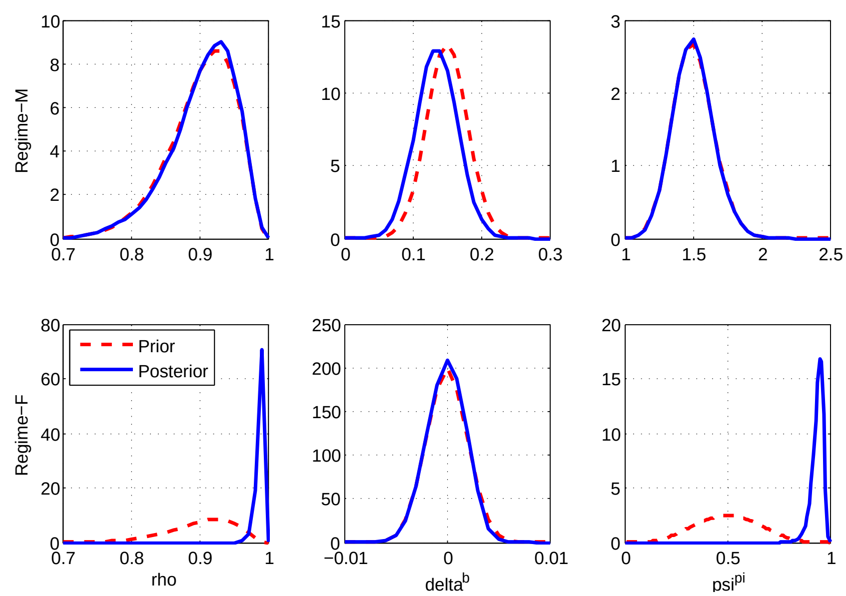

Figure 1.7: DSGE-VAR Prior and Posterior Density Functions [1955:Q1–1979:Q2]. Notes: The highest regime M and F marginal data densities correspond to $\hat{\lambda}_M = 0.5$ and $\hat{\lambda}_F = 0.5$.

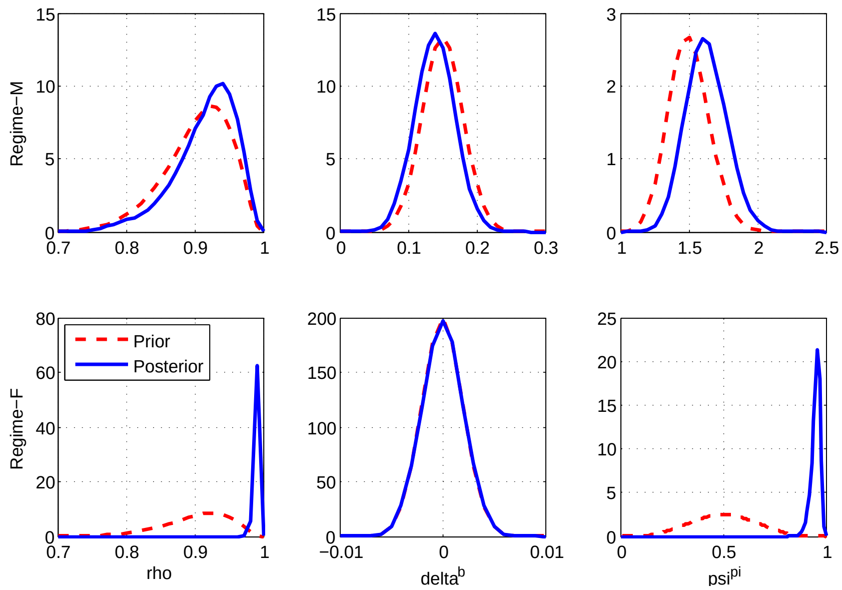

Figure 1.8: DSGE-VAR Prior and Posterior Density Functions [1982:Q4–2007:Q4]. Notes: See Figure 1.7. The highest regime M and F marginal data densities correspond to $\hat{\lambda}_M = 1.0$ and $\hat{\lambda}_F = 1.0$.

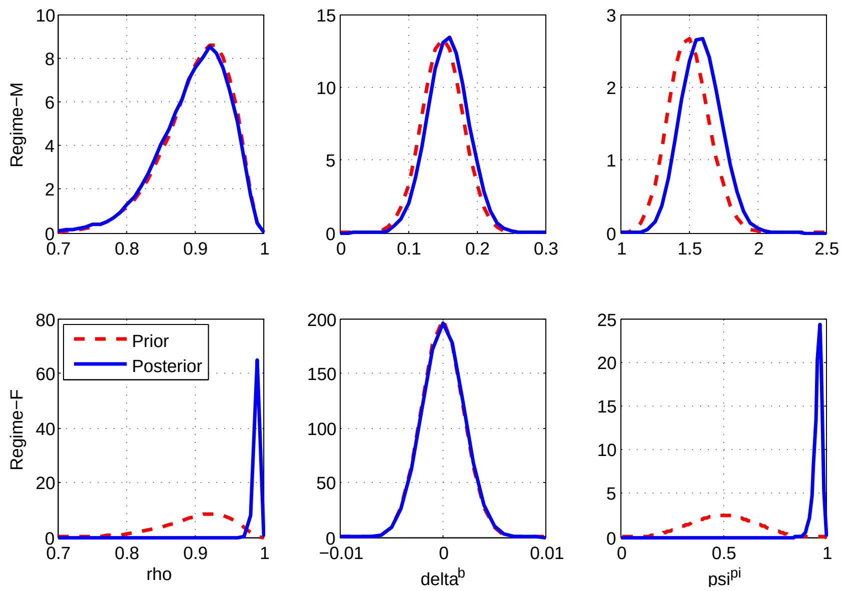

Figure 1.9: DSGE-VAR Prior and Posterior Density Functions [2008:Q1–2014:Q2]. Notes: See Figure 1.7. The highest regime M and F marginal data densities correspond to $\hat{\lambda}_M = 2.0$ and $\hat{\lambda}_F = 2.0$.

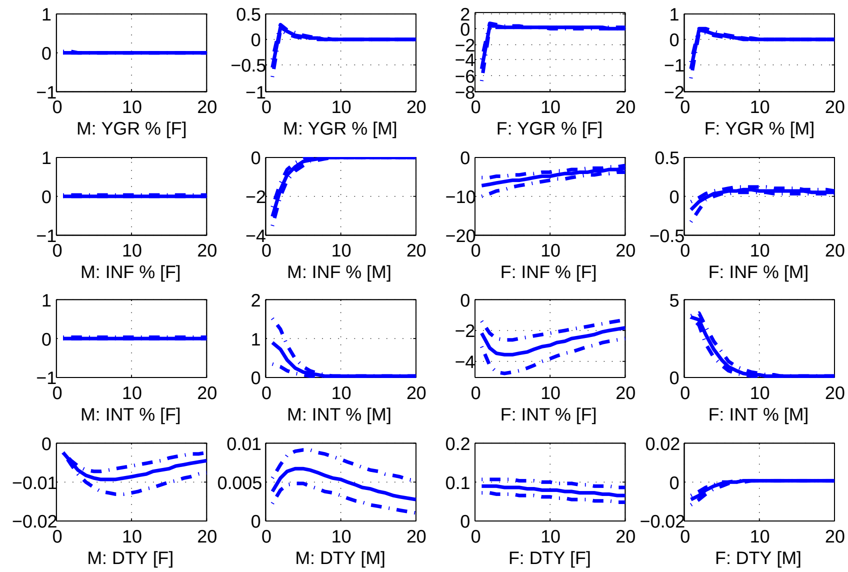

Figure 1.10: Impulse Response Functions of DSGE Models [1955:Q1–1979:Q2]. Notes: The solid lines display posterior mean effects on observables of fiscal and monetary shocks (F, M) under regime M and F. The dashed-dotted lines represent 90% confidence bands.

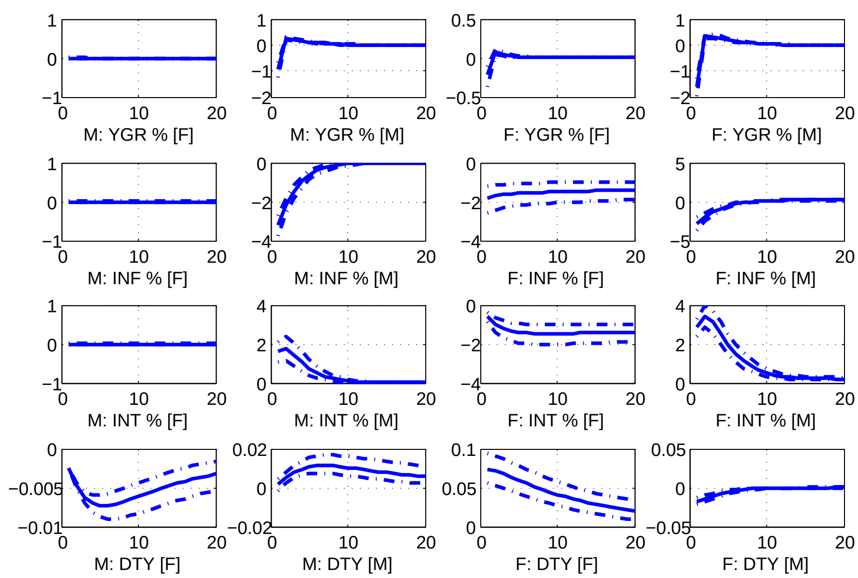

Figure 1.11: Impulse Response Functions of DSGE Models [1982:Q4–2007:Q4]. Notes: See Figure 1.10.

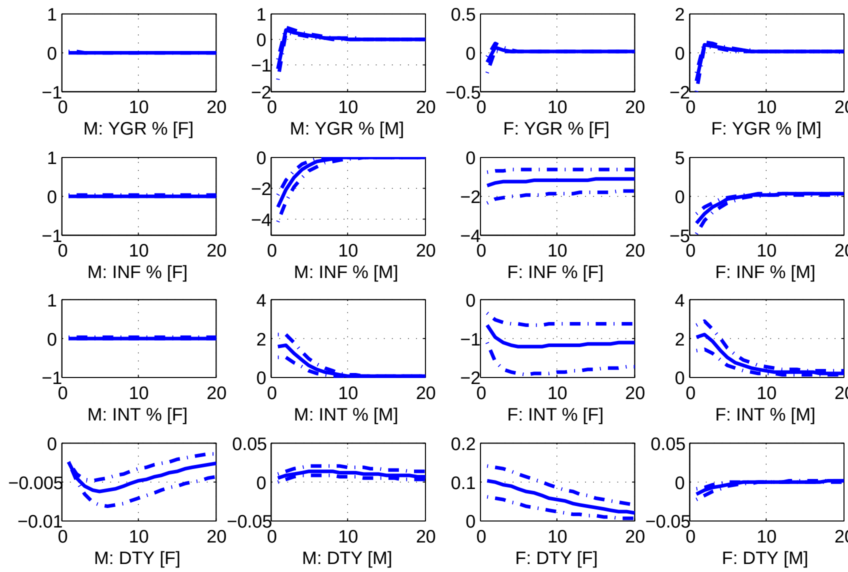

Figure 1.12: Impulse Response Functions of DSGE Models [2008:Q1–2014:Q2]. Notes: See Figure 1.10.

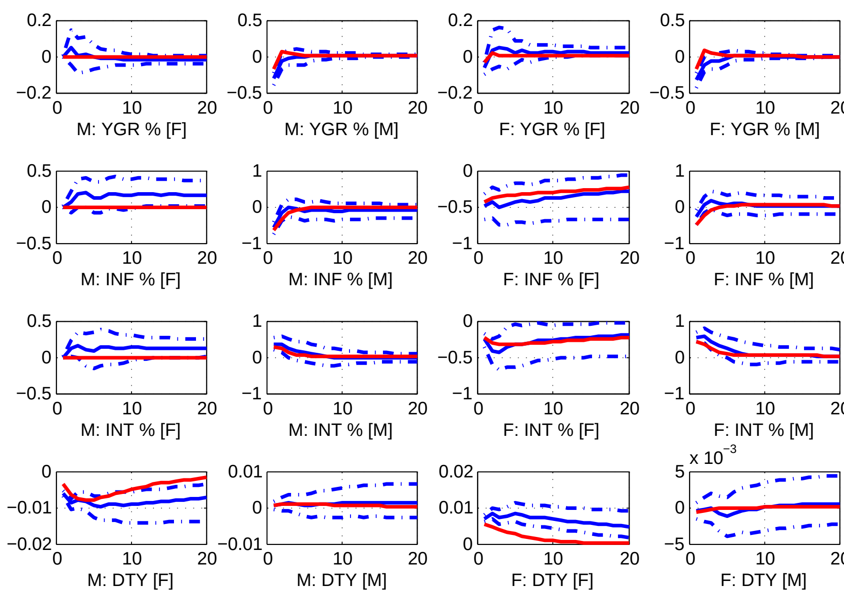

Figure 1.13: Impulse Response Functions of DSGE-VAR Models [1955:Q1–1979:Q2]. Notes: The blue-solid lines display posterior mean effects from VAR models. The blue-dashed-dotted lines represent 90% confidence bands. The red-solid lines are posterior mean responses from DSGE models.

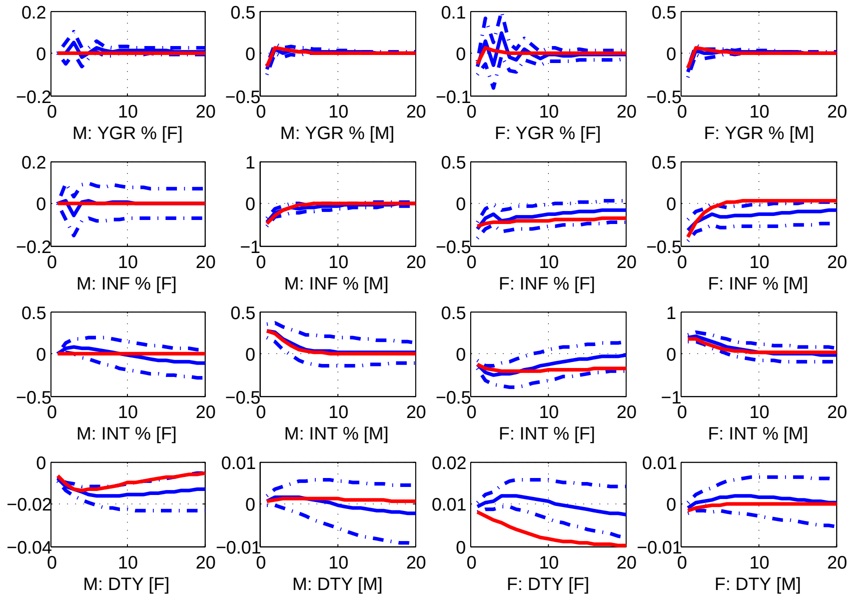

Figure 1.14: Impulse Response Functions of DSGE-VAR Models [1982:Q4–2007:Q4]. Notes: See Figure 1.13.

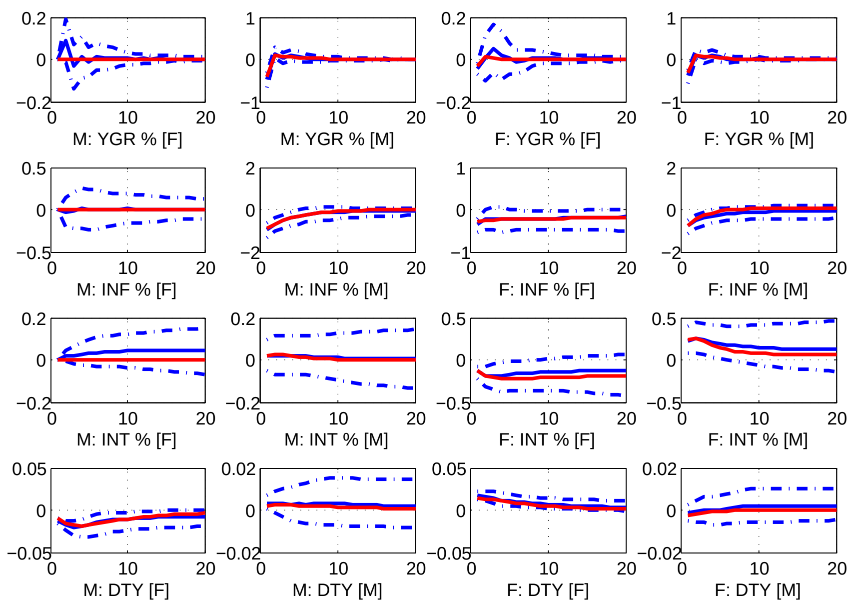

Figure 1.15: Impulse Response Functions of DSGE-VAR Models [2008:Q1–2014:Q2]. Notes: See Figure 1.13.

Chapter 2

Solving Generalized Multivariate Linear Rational Expectations Models

2.1 Introduction

Whiteman (1983) lays out a solution principle for solving stationary, linear rational expectations models. The four tenets of the solution principle are [i.] Exogenous driving processes are taken to be zero-mean linearly regular covariance stationary stochastic processes with known Wold representation; [ii.] Expectations are formed rationally and are computed using Wiener-Kolmogorov formulas; [iii.] Solutions are sought in the space spanned by time-independent square-summable linear combinations of the process fundamental for the driving process; [iv.] The rational expectations restrictions will be required to hold for all realizations of the driving process. The purpose of this paper is to extend the Whiteman solution principle to the multivariate setting.

The solution principle is general in the sense that the exogenous driving processes are assumed to only satisfy covariance stationarity. Solving for a rational expectations equilibrium is nontrivial under this assumption and Whiteman demonstrates how powerful z-transform techniques can be helpful in deriving the appropriate fixed point conditions.

The techniques advocated in Whiteman (1983) are not well known. This could be because the literature contains several well-vetted solution procedures for linearized rational expectations models (e.g., Sims (2002)) or because the solution procedure requires working knowledge of concepts unfamiliar to economists (e.g., z-transforms). We provide an introduction to these concepts and argue that there remain several advantages of the Whiteman approach on both theoretical and applied grounds. First, the approach only assumes that the exogenous driving processes possess a Wold representation, allowing for a relaxation of the standard assumption that exogenous driving processes follow an autoregressive process of order one, AR(1), specification. As recently emphasized in Curdia and Reis (2012), no justification is typically given for the AR(1) specification with little exploration into alternative stochastic processes despite obvious benefits to such deviations.[1] Second, models with incomplete information or heterogenous beliefs are easier to solve using the z-transform approach advocated by Whiteman. Kasa (2000) and Walker (2007) show how the analytic function approach can be used to solve models that were approximated in the time domain by Townsend (1983) and Singleton (1987).[2] Third, as shown in Kasa (2001) and Whiteman and Lewis (2008), the approach can easily be extended to allow for robustness as advocated by Hansen and Sargent (2011) or rational inattention as advocated by Sims (2001). Finally, there are potential insights into the econometrics of rational expectations models; two examples include Leeper et al. (2014), who show how observational equivalence problems can be clearly articulated in the space of analytic functions, and Qu and Tkachenko (2012), who demonstrate how working in the frequency domain can deliver simple identification conditions.

The contribution of the paper is to extend the approach of Whiteman (1983) to the multivariate setting and (re)introduce users of linear rational expectations models to the analytic function solution technique. We provide sufficient (though not exhaustive) background by introducing a few key theorems in Section 2.2.1 and walking readers through the

[1] This is true despite the fact that Kydland and Prescott (1982), the paper that arguably started the real business cycle literature, contains an interesting deviation from the AR(1) specification.

[2] Taub (1989), Kasa et al. (2013), Rondina (2009) and Rondina and Walker (2013) also use the space of analytic functions to characterize equilibrium in models with informational frictions. Seiler and Taub (2008), Bernhardt and Taub (2008), and Bernhardt et al. (2009) show how these methods can be used to accurately approximate asymmetric information equilibria in models with richer specifications of information.

univariate examples of Whiteman (Section 2.2.2). Section 2.3 establishes the main results of the paper. Section 2.4 provides a few examples that demonstrate the usefulness of solving linear rational expectations models in the frequency domain. An online Appendix C provides a user's guide to the MATLAB code that executes the solution procedure.

2.2 Preliminaries

Elementary results of the theory of stationary stochastic processes and the residue calculus are necessary for grasping the z-transform approach advocated here. This section introduces a few important theorems that are relatively well known but is by no means exhaustive. Interested readers are directed to Churchill and Brown (1990) and Whittle (1983) for good references on complex analysis and stochastic processes.[3]

2.2.1 A Few Useful Theorems

The first principle of Whiteman's solution procedure assumes that the exogenous driving process is a zero-mean linear covariance stationary stochastic processes with no other restrictions imposed. The Wold representation theorem allows for such a general specification.

Theorem 5. *[Wold Representation Theorem] Let $\{\mathbf{x}_t\}$ be any $(n \times 1)$ covariance stationary stochastic process with $\mathbb{E}(\mathbf{x}_t) = 0$. Then it can be uniquely represented in the form*

$$\mathbf{x}_t = \eta_t + \Psi(L)\epsilon_t \tag{2.2.1}$$

where $\Psi(L)$ is a matrix polynomial in the lag operator with $\Psi(0) = I_n$ and $\sum_{s=1}^{\infty} \Psi_s \Psi_s^{\intercal}$ is convergent. The process ϵ_t is n-variate white noise with $\mathbb{E}(\epsilon_t) = 0$, $\mathbb{E}(\epsilon_t \epsilon_t^{\intercal}) = \Sigma$ and $\mathbb{E}(\epsilon_t \epsilon_{t-m}^{\intercal}) = 0$ for $m \neq 0$. The process ϵ_t is the innovation in predicting $\mathbf{x}_t$ from its own

[3] Sargent (1987) provides a good introduction to these concepts and discusses economic applications.

past:

$$\epsilon_{t-s} = \mathbf{x}_{t-s} - P[\mathbf{x}_{t-s}|\mathbf{x}_{t-s-1}, \mathbf{x}_{t-s-2}, ...], \tag{2.2.2}$$

where $P[\cdot]$ denotes linear projection. The process η_t is linearly deterministic; there exists an n vector c_0 and $n \times n$ matrices C_s such that without error $\eta_t = c_0 + \sum_{s=1}^{\infty} C_s \eta_{t-s}$ and $\mathbb{E}[\epsilon_t \eta_{t-m}^{\intercal}] = 0$ for all m.

The Wold representation theorem states that *any* covariance stationary process can be written as a linear combination of a (possibly infinite) moving average representation where the innovations are the linear forecast errors for $\mathbf{x}_t$ and a process that can be predicted arbitrarily well by a linear function of past values of $\mathbf{x}_t$. The theorem is a *representation* determined by second moments of the stochastic process only and therefore may not fully capture the data generating process. For example, that the decomposition is linear suggests that a process could be deterministic in the strict sense and yet linearly non-deterministic; Whittle (1983) provides examples of such processes. The innovations in the Wold representation are generated by linear projections which need not be the same as the conditional expectation ($E[\mathbf{x}_{t-s}|\mathbf{x}_{t-s-1}, \mathbf{x}_{t-s-2}, ...]$). However, our focus here will be on linear Gaussian stochastic processes as is standard in the rational expectations literature. Under this assumption, the best conditional expectation coincides with linear projection.

The second principle advocated by Whiteman is that expectations are formed rationally and are computed using Wiener-Kolmogorov optimal prediction formulas. Consider minimizing the forecast error associated with the k-step ahead prediction of $x_t = A(L)\varepsilon_t =$

$\sum_{j=0}^{\infty} a_j \varepsilon_{t-j}$ by choosing $y_t = C(L)\varepsilon_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}$.

$$\begin{aligned}
\min_{y_t} \mathbb{E}(x_{t+k} - y_t)^2 &= \min_{c_j} \mathbb{E}\left(L^{-k} \sum_{j=0}^{\infty} a_j \varepsilon_{t-j} - \sum_{j=0}^{\infty} c_j \varepsilon_{t-j} \right)^2 \\
&= \min_{c_j} \mathbb{E}\left(\sum_{j=0}^{k-1} a_j \varepsilon_{t+k-j} + \sum_{j=0}^{\infty} (a_{j+k} - c_j) \varepsilon_{t-j} \right)^2 \\
&= \sigma_\varepsilon^2 \sum_{j=0}^{k-1} a_j^2 + \sigma_\varepsilon^2 \sum_{j=0}^{\infty} (a_{j+k} - c_j)^2 \qquad (2.2.3)
\end{aligned}$$

Obviously, (2.2.3) is minimized by setting $c_j = a_{j+k}$, which yields the mean-square forecast error of $\sigma^2 \varepsilon \sum_{j=0}^{\infty} c_j^2$.

Due to the Riesz-Fischer Theorem, this sequential problem has an equivalent representation as a functional problem.

Theorem 6. *[Riesz-Fischer Theorem] Let $D(\sqrt{r})$ denote a disk in the complex plane of radius $\sqrt{r}$ centered at the origin. There is an equivalence (i.e. an isometric isomorphism) between the space of r-summable sequences $\sum_j r^j |f_j|^2 < \infty$ and the Hardy space of analytic functions $f(z)$ in $D(\sqrt{r})$ satisfying the restriction*

$$\frac{1}{2\pi i} \oint f(z) f(rz^{-1}) \frac{dz}{z} < \infty$$

where $\oint$ denotes contour integration around $D(\sqrt{r})$. An analytic function satisfying the above condition is said to be r-integrable.[4]

The Riesz-Fischer theorem implies that the optimal forecasting rule (2.2.5) can be derived by finding the the analytic function $C(z)$ on the unit disk $|z| \leq 1$ corresponding to

[4]This theorem is usually proved for the case $r = 1$ and for functions defined on the boundary of a disk. For further exposition see Sargent (1987).

the z-transform of the $\{c_j\}$ sequence, $C(z) = \sum_{j=0}^{\infty} c_j z^j$ that solves

$$\min_{C(z)\in H^2} \frac{\sigma_\varepsilon^2}{2\pi i} \oint |z^{-k}A(z) - C(z)|^2 \frac{dz}{z} \tag{2.2.4}$$

where H^2 denotes the Hardy space of square-integrable analytic functions on the unit disk, and $\oint$ denotes integration about the unit circle. The restriction $C(z) \in H^2$ ensures that the forecast is casual (i.e., that the forecast contains no future values of ε's).

The sequential forecasting rule, $c_j = a_{j+k}$, has the functional equivalent

$$y_t = C(z) = \sum_{j=0}^{\infty} c_j z_j = \left[\frac{B(z)}{z^k}\right]_+ \tag{2.2.5}$$

where $B(z) = \sum_{j=0}^{\infty} b_j z_j$ and the operator $[\cdot]_+$ is defined, for a sum that contains both positive and negative powers of z, as the sum containing only the nonnegative powers of z.[5]

The beauty of the prediction formula (2.2.5) is its generality. It holds for any covariance stationary stochastic process. As an example, consider the AR(1), $x_t = \rho x_{t-1} + \varepsilon_t$. Here $B(z) = (1 - \rho z)^{-1}$ and (2.2.5) yields

$$\begin{aligned} C(z) &= [\frac{1}{(1-\rho z)z^k}]_+ = [z^{-k}(1 + \rho z + \rho^2 z^2 + \cdots)]_+ \\ &= \rho^k(1 + \rho z + \rho^2 z^2 + \cdots) = \frac{\rho^k}{1-\rho z} \end{aligned} \tag{2.2.6}$$

which delivers the well-known least squares predictor $\rho^k x_t$.[6]

The third principle assumes that solutions are sought in the space spanned by the time-independent square-summable linear combinations of the process fundamental for the driving process. Consider the moving average process $\mathbf{x}_t = \Gamma(L)u_t$; the innovations are said

[5]For a detailed derivation of (2.2.5) from (2.2.4) see Whiteman and Lewis (2008).

[6]It is often more convenient to express prediction formulas in terms of the x series as opposed to past forecast errors as in (2.2.5). If the process has an autoregressive representation, then one may write the prediction formula as $H(L)x_t$, where $H(z) = B(z)^{-1}[z^{-k}B(z)]_+$

to be fundamental for the $\mathbf{x}_t$ process if $u_t \in \overline{\text{span}}\{\mathbf{x}_{t-k}, k \geq 0\}$. That is, if the innovations span the same space as the current and past observables, then the innovations are fundamental for the $\mathbf{x}_t$ process. By construction, the innovations in the Wold representation theorem are fundamental. This implies that for any covariance stationary process there will always exist a *unique* representation that is fundamental for the exogenous driving process.[7]

2.2.2 Univariate Case

It is instructive to work through a univariate example of Whiteman (1983). There is nothing new here but it will set the stage for the generalizations in the next section. Consider the following generic rational expectations model

$$\mathbb{E}_t y_{t+1} - (\rho_1 + \rho_2) y_t + \rho_1 \rho_2 y_{t-1} = x_t, \tag{2.2.7}$$

$$x_t = A(L)\varepsilon_t, \qquad \varepsilon_t \overset{iid}{\sim} N(0,1) \tag{2.2.8}$$

where ε_t is assumed to be fundamental for x_t (i.e., $A(L)$ is assumed to have a one-sided inverse in non-negative powers of L). Following the solution principle, we will look for a solution in current and past ε, $y_t = C(L)\varepsilon_t$. If we invoke the optimal prediction formula (2.2.5), then $\mathbb{E}_t y_{t+1} = [C(z)/z]_+ = z^{-1}[C(z) - C_0]$. Together with the fourth tenet of the solution principle (i.e., that the rational expectation restrictions holds for all realization), this implies that (2.2.7) can be written in z-transform as

$$z^{-1}[C(z) - C_0] - (\rho_1 + \rho_2)C(z) + \rho_1 \rho_2 z C(z) = A(z)$$

[7]The spanning conditions prove extremely convenient for backing out the information content of exogenous and endogenous variables.

Multiplying by z and rearranging delivers

$$C(z) = \frac{zA(z) + C_0}{(1 - \rho_1 z)(1 - \rho_2 z)} \tag{2.2.9}$$

We seek a representation that is square summable in the Hilbert space generated by the fundamental shock, ε_t (tenet iii.). Appealing to the Riesz-Fischer Thereom, analyticity on the unit disk is tantamount to square-summability (stationarity) in the time domain.

As shown in Whiteman (1983), there are three cases one must consider. First, assume that $|\rho_1| < 1, |\rho_2| < 1$, then (2.2.9) is an analytic function on $|z| < 1$ and the representation is given by

$$y_t = \left(\frac{LA(L) + C_0}{(1 - \rho_1 L)(1 - \rho_2 L)} \right) \varepsilon_t \tag{2.2.10}$$

For any finite value of C_0, this is a solution that lies in the Hilbert space generated by $\{x_t\}$ and satisfies the tenets of the solution principle. Thus when $|\rho_1| < 1, |\rho_2| < 1$, no unique solution exists because C_0 can be set arbitrarily.

The second case to consider is $|\rho_1| < 1 < |\rho_2|$. In this case, the function $C(z)$ has an isolated singularity at $|\rho_2|$, implying that the $C(z)$ function is not analytic on the unit disk. In this case, the free parameter C_0 can be used to remove the singularity at $|\rho_2|$ by setting C_0 in such a way as to cause the residue of $C(\cdot)$ to be zero at $|\rho_2|$

$$\lim_{z \to \rho_2^{-1}} (1 - \rho_2) C(z) = \frac{\rho_2^{-1} A(\rho_2^{-1}) + C_0}{(1 - \rho_1 \rho_2^{-1})^{-1}} = 0 \tag{2.2.11}$$

Solving for C_0 delivers $C_0 = -\rho_2^{-1} A(\rho_2^{-1})$. Substituting this into (2.2.10) yields the following rational expectations equilibrium

$$y_t = \left(\frac{LA(L) - \rho_2^{-1}A(\rho_2^{-1})}{(1-\rho_2 L)(1-\rho_1 L)}\right)\varepsilon_t \tag{2.2.12}$$

This is the unique solution that lies in the Hilbert space generated by $\{x_t\}$. The solution is the ubiquitous Hansen-Sargent prediction formula that clearly captures the cross-equation restrictions that are the "hallmark of rational expectations models," [Hansen and Sargent (1980)].

The final case to consider is $1 < |\rho_1|$ and $1 < |\rho_2|$. In this case, (2.2.9) has two isolated singularities at ρ_1^{-1} and ρ_2^{-1} and C_0 cannot be set to remove both singularities.[8] Hence in this case, there is no solution that exists in the Hilbert space generated by $\{x_t\}$.

2.3 Generalization

This section extends the univariate solution method of Whiteman (1983) to the multivariate case. We also provide a connection to existing approaches.

2.3.1 Multivariate Case

The multivariate linear rational expectations models we are interested in can be cast in the form of

$$\mathbb{E}_t\left[\sum_{k=-n}^{m} \Gamma_k L^k y_t\right] = \mathbb{E}_t\left[\sum_{k=-n}^{l} \Psi_k L^k x_t\right] \tag{2.3.1}$$

where L is the lag operator: $L^k y_t = y_{t-k}$, y_t is a $(p \times 1)$ vector of endogenous variables, $\{\Gamma_k\}_{k=-n}^{m}$ and $\{\Psi_k\}_{k=-n}^{l}$ are $(p \times p)$ and $(p \times q)$ matrix coefficients, and $\mathbb{E}_t$ represents mathematical expectation given information available at time t including the model's structure

[8] As discussed by Whiteman (1983) the problem remains even if $\rho_1 = \rho_2$.

and all past and present realizations of the exogenous and endogenous processes. Moreover, x_t is a $(q \times 1)$ covariance stationary vector driving process with known Wold representation

$$x_t = \sum_{k=0}^{\infty} A_k \varepsilon_{t-k} \equiv A(L)\varepsilon_t \tag{2.3.2}$$

where $\varepsilon_t = x_t - \Pi[x_t | x_{t-1}, x_{t-2}, \ldots]$ and $\Pi[x_t | x_{t-1}, x_{t-2}, \ldots]$ is the optimal linear predictor for x_t conditional on observing $\{x_{t-j}\}_{j=1}^{\infty}$.[9] Also, each element of $\sum_{k=0}^{\infty} A_k A_k'$ is finite.

An illustrative example to show how we get a model into the form of (2.3.1) is given by the following simple RBC model. Consider the standard stochastic growth model with log preferences, inelastic labor supply, complete depreciation of capital, and Cobb-Douglas technology, $Y_t = A_t K_{t-1}^{\alpha}$. The Euler equation and aggregate resource constraint, after log linearizing, reduce to the following bivariate system in (c_t, k_t) which must hold for $t = 0, 1, 2, \ldots$, i.e.

$$E_t c_{t+1} = c_t + (\alpha - 1) k_t + E_t a_{t+1} \tag{2.3.3}$$

$$\frac{1 - \alpha\beta}{\alpha\beta} c_t + k_t = \frac{1}{\alpha\beta} a_t + \frac{1}{\beta} k_{t-1} \tag{2.3.4}$$

where we have used the steady state facts that $Y/K = 1/\alpha\beta$ and $C/K = (1 - \alpha\beta)/\alpha\beta$. Also, we assume that the technology shock is serially uncorrelated, which implies $E_t a_{t+1} = 0$.

[9] The inclusion of l periods of lags for exogenous driving process allows for the possibility that agents have foresight about some of the future endogenous variables.

Rewrite the above bivariate system into the form of (2.3.1)

$$\mathbb{E}_t\left[\left(\underbrace{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}}_{\Gamma_{-1}} L^{-1} + \underbrace{\begin{pmatrix}-1 & 1-\alpha\\ \frac{1-\alpha\beta}{\alpha\beta} & 1\end{pmatrix}}_{\Gamma_0} L^0 + \underbrace{\begin{pmatrix}0 & 0\\ 0 & -\frac{1}{\beta}\end{pmatrix}}_{\Gamma_1} L\right)\underbrace{\begin{pmatrix}c_t\\ k_t\end{pmatrix}}_{y_t}\right]$$

$$= \mathbb{E}_t\left[\left(\underbrace{\begin{pmatrix}1\\ 0\end{pmatrix}}_{\Psi_{-1}} L^{-1} + \underbrace{\begin{pmatrix}0\\ \frac{1}{\alpha\beta}\end{pmatrix}}_{\Psi_0} L^0\right)\underbrace{a_t}_{x_t}\right]$$

where $n = m = 1$, $l = 0$, $p = 2$, and $q = 1$.

Analogous to the univariate solution procedure outlined above, we exploit the properties of polynomial matrices to establish conditions for the existence and uniqueness of solutions to multivariate linear rational expectations models for general exogenous driving processes. Suppose a solution y_t to (2.3.1) is of the form

$$y_t = \sum_{k=0}^{\infty} C_k \varepsilon_{t-k} \equiv C(L)\varepsilon_t \tag{2.3.5}$$

where $\{y_t\}$ is taken to be covariance stationary. Let $\Gamma(z) = \sum_{k=-n}^{m} \Gamma_k z^k$. Further suppose that $z^n\Gamma(z)$ is nonsingular almost everywhere in the complex plane, and some of the roots of the polynomial in z, $\det[z^n\Gamma(z)] = \sum_{k=0}^{h} b_k z^k$, may have multiplicities greater than 1.[10] Note that such moving average representation of the solution is very convenient because it *is* the impulse response function. For example, the term $C_k(i,j)$ measures exactly the response of $y_{t+k}(i)$ to a shock $\varepsilon_t(j)$

$$(\mathbb{E}_t - \mathbb{E}_{t-1})y_{t+k}(i) = C_k(i,j)\varepsilon_t(j)$$

[10]Here $h = (m+n)p$.

where $C_k(i,j)$ denotes the (i,j)-th element of C_k, $y_{t+k}(i)$ denotes the i-th component of y_{t+k}, and $\varepsilon_t(j)$ denotes the j-th component of ε_t. In what follows, we describe the solution algorithm in detail and apply it to solve a few example models that demonstrate the usefulness of this frequency-domain solution method in Section 2.4.

2.3.2 Solution Procedure

If we define η_t (resp. ν_t) as a $(p \times 1)$ vector of endogenous (resp. exogenous) expectational errors, satisfying $\eta_{t+k} = y_{t+k} - \mathbb{E}_t y_{t+k}$ (resp. $\nu_{t+k} = x_{t+k} - \mathbb{E}_t x_{t+k}$) for all $k > 0$ and hence $\mathbb{E}_t \eta_{t+k} = 0$ (resp. $\mathbb{E}_t \nu_{t+k} = 0$), then we may write (2.3.1) as

$$\sum_{k=-n}^{m} \Gamma_k L^k y_t = \sum_{k=-n}^{l} \Psi_k L^k x_t + \sum_{k=1}^{n} \left(\Gamma_{-k}\eta_{t+k} - \Psi_{-k}\nu_{t+k}\right) \tag{2.3.6}$$

Similar to Sims (2002), it should be noted that the η terms are not given exogenously, but instead are treated as determined as part of the model solution.

First, rewrite model (2.3.6) as

$$\Gamma(L) y_t = \Psi(L) x_t + \sum_{k=1}^{n} \left(\Gamma_{-k}\eta_{t+k} - \Psi_{-k}\nu_{t+k}\right)$$

where $\Psi(L) = \sum_{k=-n}^{l} \Psi_k L^k$. Applying (2.3.5) and the Wiener-Kolmogorov optimal prediction formula gives

$$\eta_{t+k} = y_{t+k} - E_t y_{t+k} = L^{-k} \left(\sum_{i=0}^{k-1} C_i L^i \right) \varepsilon_t$$
$$\nu_{t+k} = x_{t+k} - E_t x_{t+k} = L^{-k} \left(\sum_{i=0}^{k-1} A_i L^i \right) \varepsilon_t$$

Substituting the above expressions, (2.3.2), and (2.3.5) into (2.3.6) gives

$$\Gamma(L)C(L)\varepsilon_t = \left\{\Psi(L)A(L) + \sum_{k=1}^{n}\left[\Gamma_{-k}L^{-k}\left(\sum_{i=0}^{k-1} C_i L^i\right) - \Psi_{-k}L^{-k}\left(\sum_{i=0}^{k-1} A_i L^i\right)\right]\right\}\varepsilon_t$$

which must hold for all realizations of $\{\varepsilon_t\}$. Thus, the coefficient matrices are related by the z-transform identities

$$z^n\Gamma(z)C(z) = z^n\Psi(z)A(z) + \sum_{t=1}^{n}\sum_{s=t}^{n}[\Gamma_{-s}C_{t-1} - \Psi_{-s}A_{t-1}]z^{n-s+t-1}$$

Next, applying the Smith canonical decomposition to the polynomial matrix $z^n\Gamma(z)$ gives

$$U(z)z^n\Gamma(z)V(z) = \begin{pmatrix} f_1(z) & & & \\ & f_2(z) & & \\ & & \ddots & \\ & & & f_p(z) \end{pmatrix} \tag{2.3.7}$$

where $f_1, \ldots, f_p$ are monic polynomials in z, $f_k | f_{k+1}$ for $1 \leq k \leq p-1$, $U(z)$ is a product of elementary row matrices, and $V(z)$ is a product of elementary column matrices. For $i = 1, \ldots, p$, let

$$f_i = \underbrace{\prod_{j=1}^{\underline{r}_i}(z - \underline{z}_{ij})^{\underline{m}_{ij}}}_{\underline{f}_i} \cdot \underbrace{\prod_{j=1}^{\overline{r}_i}(z - \overline{z}_{ij})^{\overline{m}_{ij}}}_{\overline{f}_i}$$

where $\underline{z}_{ij}$'s are complex-valued roots inside the unit circle with multiplicity $\underline{m}_{ij}$ and $\overline{z}_{ij}$'s are complex-valued roots on or outside the unit circle with multiplicity $\overline{m}_{ij}$. Then

$$z^n\Gamma(z) = \underbrace{U(z)^{-1}\begin{pmatrix} \underline{f}_1 & & & \\ & \underline{f}_2 & & \\ & & \ddots & \\ & & & \underline{f}_p \end{pmatrix}}_{S(z)} \underbrace{\begin{pmatrix} \overline{f}_1 & & & \\ & \overline{f}_2 & & \\ & & \ddots & \\ & & & \overline{f}_p \end{pmatrix} V(z)^{-1}}_{T(z)}$$

where $S(z)$ is a polynomial matrix such that all roots of $\det[S(z)]$ lie inside the unit circle while $T(z)$ is a polynomial matrix with all roots of $\det[T(z)]$ outside the unit circle. Therefore, we have

$$S(z)^{-1} = \begin{pmatrix} \frac{U_{1\cdot}(z)}{\prod_{k=1}^{\underline{r}_1}(z-\underline{z}_{1k})^{\underline{m}_{1k}}} \\ \frac{U_{2\cdot}(z)}{\prod_{k=1}^{\underline{r}_2}(z-\underline{z}_{2k})^{\underline{m}_{2k}}} \\ \vdots \\ \frac{U_{p\cdot}(z)}{\prod_{k=1}^{\underline{r}_p}(z-\underline{z}_{pk})^{\underline{m}_{pk}}} \end{pmatrix}$$

where $U_{j\cdot}(z)$ is the jth row of $U(z)$. Now the z-transform identities become

$$T_{j\cdot}(z)C(z) = \frac{U_{j\cdot}(z)}{\prod_{k=1}^{\underline{r}_j}(z-\underline{z}_{jk})^{\underline{m}_{jk}}} \left\{ z^n\Psi(z)A(z) + \sum_{t=1}^{n}\sum_{s=t}^{n}[\Gamma_{-s}C_{t-1} - \Psi_{-s}A_{t-1}]z^{n-s+t-1} \right\}$$

for $j = 1, \ldots, p$, which is valid for all z on the open unit disk except $z = \underline{z}_{jk}$ for $k = 1, \ldots, \underline{r}_j$. But since $C(z)$ is the z-transform of the moving average coefficients for y_t, it must exist for all $|z| < 1$. This condition places restrictions on the np^2 unknown parameters $C_0, C_1, \ldots, C_{n-1}$:

$$\frac{d^i}{dz^i}\left[\prod_{k=1}^{\underline{r}_j}(z-\underline{z}_{jk})^{\underline{m}_{jk}}T_{j\cdot}(z)C(z)\right]\Bigg|_{z=\underline{z}_{jk}} = 0, \quad i = 0, \ldots, \underline{m}_{jk}-1, \quad k = 1, \ldots, \underline{r}_j$$

Stacking the above expressions yields

$$\underbrace{\begin{pmatrix} \left[U_{j\cdot}(\underline{z}_{j1})(\underline{z}_{j1}^{n}\Psi(\underline{z}_{j1})A(\underline{z}_{j1}) - \sum_{t=1}^{n}\sum_{s=t}^{n}\Psi_{-s}A_{t-1}\underline{z}_{j1}^{n-s+t-1})\right]^{(0)} \\ \vdots \\ \left[U_{j\cdot}(\underline{z}_{j1})(\underline{z}_{j1}^{n}\Psi(\underline{z}_{j1})A(\underline{z}_{j1}) - \sum_{t=1}^{n}\sum_{s=t}^{n}\Psi_{-s}A_{t-1}\underline{z}_{j1}^{n-s+t-1})\right]^{(\underline{m}_{j1}-1)} \\ \vdots \\ \left[U_{j\cdot}(\underline{z}_{j\underline{r}_j})(\underline{z}_{j\underline{r}_j}^{n}\Psi(\underline{z}_{j\underline{r}_j})A(\underline{z}_{j\underline{r}_j}) - \sum_{t=1}^{n}\sum_{s=t}^{n}\Psi_{-s}A_{t-1}\underline{z}_{j\underline{r}_j}^{n-s+t-1})\right]^{(0)} \\ \vdots \\ \left[U_{j\cdot}(\underline{z}_{j\underline{r}_j})(\underline{z}_{j\underline{r}_j}^{n}\Psi(\underline{z}_{j\underline{r}_j})A(\underline{z}_{j\underline{r}_j}) - \sum_{t=1}^{n}\sum_{s=t}^{n}\Psi_{-s}A_{t-1}\underline{z}_{j\underline{r}_j}^{n-s+t-1})\right]^{(\underline{m}_{j\underline{r}_j}-1)} \end{pmatrix}}_{A_{j\cdot}} =$$

$$-\underbrace{\begin{pmatrix} \left[U_{j\cdot}(\underline{z}_{j1})\sum_{s=1}^{n}\Gamma_{-s}\underline{z}_{j1}^{n-s}\right]^{(0)} & \cdots & \left[U_{j\cdot}(\underline{z}_{j1})\Gamma_{-n}\underline{z}_{j1}^{n-1}\right]^{(0)} \\ \vdots & \ddots & \vdots \\ \left[U_{j\cdot}(\underline{z}_{j1})\sum_{s=1}^{n}\Gamma_{-s}\underline{z}_{j1}^{n-s}\right]^{(\underline{m}_{j1}-1)} & \cdots & \left[U_{j\cdot}(\underline{z}_{j1})\Gamma_{-n}\underline{z}_{j1}^{n-1}\right]^{(\underline{m}_{j1}-1)} \\ \vdots & \ddots & \vdots \\ \left[U_{j\cdot}(\underline{z}_{j\underline{r}_j})\sum_{s=1}^{n}\Gamma_{-s}\underline{z}_{j\underline{r}_j}^{n-s}\right]^{(0)} & \cdots & \left[U_{j\cdot}(\underline{z}_{j\underline{r}_j})\Gamma_{-n}\underline{z}_{j\underline{r}_j}^{n-1}\right]^{(0)} \\ \vdots & \ddots & \vdots \\ \left[U_{j\cdot}(\underline{z}_{j\underline{r}_j})\sum_{s=1}^{n}\Gamma_{-s}\underline{z}_{j\underline{r}_j}^{n-s}\right]^{(\underline{m}_{j\underline{r}_j}-1)} & \cdots & \left[U_{j\cdot}(\underline{z}_{j\underline{r}_j})\Gamma_{-n}\underline{z}_{j\underline{r}_j}^{n-1}\right]^{(\underline{m}_{j\underline{r}_j}-1)} \end{pmatrix}}_{R_{j\cdot}} \underbrace{\begin{pmatrix} C_0 \\ C_1 \\ \vdots \\ C_{n-1} \end{pmatrix}}_{C}$$

Further stacking over $j = 1, \ldots, p$ yields

$$\underset{[r\times q]}{A} = - \underset{[r\times np]}{R}\ \underset{[np\times q]}{C}$$

where $r = \sum_{j=1}^{p}\sum_{k=1}^{r_j}\underline{m}_{jk}$.

Lastly, we establish the existence and uniqueness conditions of the multivariate rational expectations model. As we show below, these conditions are more robust than the standard root counting analysis of Blanchard and Kahn (1980). Existence cannot be established if at least one column of A is outside the space spanned by the columns of R—the endogenous shocks or forecast errors η cannot adjust to offset the exogenous shocks x. Thus, the precise existence condition is that columns of R span the space spanned by the columns of A, i.e.

$$\operatorname{span}(A) \subseteq \operatorname{span}(R) \tag{2.3.8}$$

To check whether (2.3.8) is satisfied, we follow Sims (2002). Let the singular value decompositions of A and R be given by $A = U_A S_A V_A'$ and $R = U_R S_R V_R'$. Then R's column space spans A's if and only if $(I - U_R U_R')U_A = 0$, in which case one candidate of C can be computed as

$$C = -V_R S_R^{-1} U_R' A \tag{2.3.9}$$

When (2.3.8) is satisfied, we can obtain the analytical solution for y_t which is indexed by $C_0, C_1, \ldots, C_{n-1}$[11]

$$y_t = (L^n \Gamma(L))^{-1} \left\{ L^n \Psi(L) A(L) + \sum_{t=1}^{n} \sum_{s=t}^{n} [\Gamma_{-s} C_{t-1} - \Psi_{-s} A_{t-1}] L^{n-s+t-1} \right\}$$

The above solution captures all the multivariate cross-equation restrictions linking the Wold representation of the exogenous process to the endogenous variables of the model. This mapping is essentially a multivariate version of the celebrated Hansen-Sargent formula, and serves as a key ingredient in the analysis and econometric evaluation of dynamic rational expectations models.

In order for the solution to be unique, we must be able to determine $\{C_k\}_{k=0}^{\infty}$ from the parameter restrictions supplied by $A = -RC$. Since $V(\cdot)$ is unimodular, this is equivalent to determining the coefficients $\{D_k\}_{k=0}^{\infty}$ of $D(z) = V(z)^{-1} C(z)$, which can be computed

[11] We also need to impose a "consistency condition" when (2.3.1) is withholding—some relevant information is concealed from agents so that (2.3.1) contains terms like $\mathbb{E}_{t-i} y_{t+j}$ for some $i, j > 0$. See Whiteman (1983) for details.

using the inversion formula

$$\begin{aligned} D_k &= \frac{1}{2\pi i}\int_\Gamma D(z)z^{-k-1}dz \\ &= \text{sum of residues of } D(z^{-1})z^{k-1} \text{ at poles inside unit circle} \end{aligned}$$

Note that the jth row of $D(z^{-1})z^{k-1}$ is given by

$$\frac{U_{j\cdot}(z^{-1})z^{k-1}}{\prod_{k=1}^{\underline{r}_j}(z^{-1}-\underline{z}_{jk})^{\underline{m}_{jk}}\prod_{k=1}^{\overline{r}_j}(z^{-1}-\overline{z}_{jk})^{\overline{m}_{jk}}}\times \\ \left\{ z^{-n}\Psi(z^{-1})A(z^{-1}) + \sum_{t=1}^{n}\sum_{s=t}^{n}[\Gamma_{-s}C_{t-1} - \Psi_{-s}A_{t-1}]z^{-(n-s+t-1)} \right\}$$

which has poles inside unit circle at $\overline{z}_{jk}^{-1}$ with multiplicity $\overline{m}_{jk}$ for $k = 1, \ldots, \overline{r}_j$.[12] Some tedious algebra allows us to write the jth row of each D_k as a function of C that only shows up in the following common terms shared by all D_k's

$$\frac{d^i}{dz^i}\left[U_{j\cdot}(z^{-1})\sum_{t=1}^{n}\sum_{s=t}^{n}\Gamma_{-s}C_{t-1}z^{-(n-s+t-1)}\right]\Bigg|_{z=\overline{z}_{jk}^{-1}}, \quad i = 0, \ldots, \overline{m}_{jk}-1, \quad k = 1, \ldots, \overline{r}_j$$

Stacking the above expressions yields

$$\underbrace{\begin{pmatrix} \left[U_{j\cdot}(\overline{z}_{j1}^{-1})\sum_{s=1}^{n}\Gamma_{-s}\overline{z}_{j1}^{-(n-s)}\right]^{(0)} & \cdots & \left[U_{j\cdot}(\overline{z}_{j1}^{-1})\Gamma_{-n}\overline{z}_{j1}^{-(n-1)}\right]^{(0)} \\ \vdots & \ddots & \vdots \\ \left[U_{j\cdot}(\overline{z}_{j1}^{-1})\sum_{s=1}^{n}\Gamma_{-s}\overline{z}_{j1}^{-(n-s)}\right]^{(\overline{m}_{j1}-1)} & \cdots & \left[U_{j\cdot}(\overline{z}_{j1}^{-1})\Gamma_{-n}\overline{z}_{j1}^{-(n-1)}\right]^{(\overline{m}_{j1}-1)} \\ \vdots & \ddots & \vdots \\ \left[U_{j\cdot}(\overline{z}_{j\overline{r}_j}^{-1})\sum_{s=1}^{n}\Gamma_{-s}\overline{z}_{j\overline{r}_j}^{-(n-s)}\right]^{(0)} & \cdots & \left[U_{j\cdot}(\overline{z}_{j\overline{r}_j}^{-1})\Gamma_{-n}\overline{z}_{j\overline{r}_j}^{-(n-1)}\right]^{(0)} \\ \vdots & \ddots & \vdots \\ \left[U_{j\cdot}(\overline{z}_{j\overline{r}_j}^{-1})\sum_{s=1}^{n}\Gamma_{-s}\overline{z}_{j\overline{r}_j}^{-(n-s)}\right]^{(\overline{m}_{j\overline{r}_j}-1)} & \cdots & \left[U_{j\cdot}(\overline{z}_{j\overline{r}_j}^{-1})\Gamma_{-n}\overline{z}_{j\overline{r}_j}^{-(n-1)}\right]^{(\overline{m}_{j\overline{r}_j}-1)} \end{pmatrix}}_{Q_{j\cdot}} \underbrace{\begin{pmatrix} C_0 \\ C_1 \\ \vdots \\ C_{n-1} \end{pmatrix}}_{C}$$

[12] For $k = 0$, there is an additional pole inside unit circle at 0.

Further stacking over $j = 1, \ldots, p$ yields QC. Thus $A = -RC$ pins down all the error terms in the system that are influenced by the expectational error η. That is, we use RC to determine QC and the solution is unique if and only if

$$\text{span}(Q') \subseteq \text{span}(R') \tag{2.3.10}$$

In other words, determinacy of the solution requires that the columns of R' span the space spanned by the columns of Q', in which case we will have $QC = \Phi RC$ for some matrix Φ.[13] This completes the solution procedure.

2.4 Examples

We provide a few examples that demonstrate the usefulness of solving linear rational expectations models in the frequency domain. Some of the content is new to this paper but most of the examples are taken from the literature and are therefore not rigorous.

2.4.1 Incomplete Information

One of the more compelling reasons to solve models using the approach laid out above is the ease with which it handles incomplete information. The following example is a slightly modified version of Rondina and Walker (2013), which is based on Futia (1981).

Assume agents are risk neutral and discount the future at rate β. Agents trade an asset with price p_t and fundamentals given by s_t. Let there be a continuum of asymmetrically informed agents indexed by i. The model is given by

$$p_t = \beta \int_0^1 \mathbb{E}_t^i p_{t+1} di + s_t \tag{2.4.1}$$

[13] Similar to the space spanning condition for existence, (2.3.10) can be verified using the singular value decompositions of Q and R.

where $\mathbb{E}_t^i$ is the conditional expectation of agent i taken with respect to a filtration Ω_t^i.

The exogenous process (s_t) is driven by a Gaussian shock

$$s_t = A(L)\varepsilon_t, \qquad \varepsilon_t \overset{iid}{\sim} N(0, \sigma_\varepsilon^2) \tag{2.4.2}$$

where $A(L)$ is a square-summable polynomial in the lag operator L.

Solving models of the form of (2.4.1) is nontrivial because a rational expectations equilibrium will consist of a fixed point in *endogenous* information and the coefficients of the price process.

Definition 7. *A Rational Expectations Equilibrium is a stochastic process for* $\{p_t\}$ *and a stochastic process for the information sets* $\left\{\Omega_t^i, i \in [0,1]\right\}$ *such that: (i) each agent* i*, given the price and the information set, optimally forms expectations; (ii)* p_t *satisfies the equilibrium condition* (2.4.1)*.*

The solution procedure involves two steps: [i] guess a candidate solution that is minimal with respect to information and impose equilibrium conditions [ii] check the invertibility of the endogenous variables to ensure the informational fixed point condition holds. Through market interactions, the information conveyed by the candidate solution may be larger than the initial information set of step [i]. If this is the case, the new enlarged information set is used to generate a new candidate solution, and the process is repeated until convergence. Since the expansion of the information set is bounded above by the full information benchmark, the iteration is sure to converge.

A critical component of the solution procedure is initializing the recursion in information. Here one can follow Radner (1979), who advocated forming an "exogenous information equilibrium" as an initial guess for the IE. The exogenous information equilibrium assumes agents are only able to condition on exogenous information (e.g., private exogenous signal), which places a lower-bound restriction on the initial condition for information. Radner argued that such an equilibrium would persist only if every agent remained unsophisticated and ignored the information coming from the endogenous variables. A dynamic interpretation of Radner is to say that a "sophisticated" agent acting rationally will not generate forecast errors that are serially correlated with respect to their own information sets.

Following Rondina and Walker (2013), suppose there are two types of agents, *informed* and *uninformed*. The proportion of the informed agents is denoted by $\mu \in [0,1]$ and they are assumed to observe the entire history of the structural shock ε up to time t. The remaining $1-\mu$ agents are uninformed in the sense that they observe only equilibrium outcomes (i.e., the price sequence). The exogenous information is given by

$$\begin{aligned} U_t^i &= \mathbb{V}_t(\varepsilon) \quad \text{for} \quad i \in \mu \\ U_t^i &= \{0\} \quad \text{for} \quad i \in 1-\mu. \end{aligned}$$

The equilibrium is given by

$$p_t = \beta\big[\mu\mathbb{E}\big(p_{t+1}|V_t(\varepsilon) \vee \mathbb{M}_t(p)\big) + (1-\mu)\,\mathbb{E}\big(p_{t+1}|V_t(p) \vee \mathbb{M}_t(p)\big)\big] + s_t. \tag{2.4.3}$$

The following theorem is due to Rondina and Walker (2013).

Theorem 8. *Under the exogenous information assumption $U_t^i = \mathbb{V}_t(\varepsilon)$ for $i \in \mu$ and $U_t^i = \{0\}$ for $i \in 1-\mu$, a unique Information Equilibrium for (2.4.3) with $|\beta| < 1$ always exists and is determined as follows: If there exists a $|\lambda| < 1$ such that*

$$A(\lambda) - \frac{\mu\beta A(\beta)}{h(\beta)} = 0 \tag{2.4.4}$$

then the REE of (2.4.3) *is given by*

$$p_t = (L - \lambda)Q(L)\varepsilon_t = \frac{1}{L - \beta}\left\{LA(L) - \beta A(\beta)\frac{h(L)}{h(\beta)}\right\}\varepsilon_t \tag{2.4.5}$$

with

$$h(L) \equiv \mu\lambda - (1 - \mu)\mathcal{B}_\lambda(L), \qquad \mathcal{B}_\lambda(L) \equiv \frac{L - \lambda}{1 - \lambda L}$$

If restriction (2.4.4) *does not hold for* $|\lambda| < 1$*, the REE converges to a complete information equilibrium.*

The proof follows the solution procedure outlined above. The Radner "exogenous equilibrium" is a price sequence given by

$$p_t = Q(L)(L - \lambda)\varepsilon_t \tag{2.4.6}$$

where $|\lambda| < 1$ is assumed, and $Q(L)$ is assumed to contain no zeros inside the unit circle. Viewed as an analytic function, price process contains a zero inside the unit circle at $z = \lambda$. Thus, the right-hand side of (2.4.6) is not invertible. This implies that the price sequence p^t spans a smaller space than ε^t. For the uninformed agents, this space is characterized by a Blaschke factor [see Hansen and Sargent (1991), Lippi and Reichlin (1994)],

$$p_t = Q(L)(1 - \lambda L)\tilde{\varepsilon}_t \tag{2.4.7}$$

$$\tilde{\varepsilon}_t = \left[\frac{L - \lambda}{1 - \lambda L}\right]\varepsilon_t \tag{2.4.8}$$

Once we have an initial guess for our endogenous variables, we simply follow the solution procedure and take the conditional expectations for the informed and uninformed agents

$$E_t^I(p_{t+1}) = L^{-1}[(L-\lambda)Q(L) + \lambda Q_0]\varepsilon_t$$
$$E_t^U(p_{t+1}) = L^{-1}[(L-\lambda)Q(L) - Q_0\mathcal{B}_\lambda(L)]\varepsilon_t$$

Substituting the expectations into the equilibrium gives the z-transform in ε_t space as

$$(z-\lambda)Q(z) = \beta\mu z^{-1}[(z-\lambda)Q(z) + \lambda Q_0] + \beta(1-\mu)z^{-1}[(z-\lambda)Q(z) - Q_0\mathcal{B}_\lambda(z)] + A(z) \tag{2.4.9}$$

and re-arranging yields the following functional equation

$$(z-\lambda)(z-\beta)Q(z) = zA(z) + \beta Q_0[\mu\lambda - (1-\mu)\mathcal{B}_\lambda(z)]$$

The $Q(\cdot)$ process will not be analytic unless the process vanishes at the poles $z = \{\lambda, \beta\}$. Evaluating at $z = \lambda$ gives the restriction on $A(\cdot)$, $A(\lambda) = -\beta\mu Q_0$. Rearranging terms

$$\begin{aligned}(z-\beta)Q(z) &= \frac{1}{z-\lambda}\big\{zA(z) + \beta Q_0[\mu\lambda - (1-\mu)\mathcal{B}_\lambda(z)]\big\} \\ &= \frac{1}{z-\lambda}\big\{zA(z) + \beta Q_0 h(z)\big\} \end{aligned} \tag{2.4.10}$$

where $h(z) \equiv [\mu\lambda - (1-\mu)\mathcal{B}_\lambda(z)]$. Evaluating at $z = \beta$ gives Q_0 as $Q_0 = -\frac{A(\beta)}{h(\beta)}$. This implies the restriction on $A(\cdot)$ is

$$A(\lambda) = \frac{\beta\mu A(\beta)}{h(\beta)}$$

which is (2.4.4). Substituting this into (2.4.10) delivers (2.4.5).

Therefore, the only additional step to solving models with incomplete information is forming an initial guess for the endogenous variables. We advocate following the recursion described by Radner (1979). We can then follow the stand solution procedure of Whiteman (1983). Moreover, working with analytic functions makes keeping track of the information content of endogenous and exogenous variables straightforward.

2.4.2 Observational Equivalence

In this section, we first solve a cashless version of the model in Leeper (1991) with simple exogenous driving processes for policy shocks. Then we apply the solution methodology developed in this paper and show that the two disjoint determinacy regions in this model can generate observational equivalent equilibrium time series driven by carefully chosen exogenous driving process for each determinacy region, a result due to Leeper et al. (2014). In particular, an infinitely lived representative household is endowed each period with a constant quantity of nondurable goods, y. Government issues nominal one-period bonds, so the price level P can be defined as the rate at which bonds exchange for goods. The household chooses sequences of consumption and bonds, $\{c_t, B_t\}$, to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

where $0 < \beta < 1$, subject to the budget constraint

$$c_t + \frac{B_t}{P_t} + \tau_t = y + \frac{R_{t-1}B_{t-1}}{P_t}$$

taking prices and the initial principal and interest payments on debt, $R_{-1}B_{-1} > 0$, as given. The household pays lump sum taxes, τ_t, each period. Government spending is zero each period, so the government chooses sequences of taxes and debt to satisfy its flow constraint

$$\frac{B_t}{P_t} + \tau_t = \frac{R_{t-1}B_{t-1}}{P_t}$$

given $R_{-1}B_{-1} > 0$, while the monetary authority chooses a sequence for the nominal interest rate. After imposing goods market clearing, $c_t = y$ for $t \geq 0$, the household's consumption Euler equation reduces to the simple Fisher relation

$$\frac{1}{R_t} = \beta E_t \frac{P_t}{P_{t+1}}$$

For analytical convenience, we close the model laid out above by specifying the following monetary and fiscal policy rules

$$\begin{aligned} R_t &= R^*(\pi_t/\pi^*)^\alpha e^{\theta_t} \\ \tau_t &= \tau^*(b_{t-1}/b^*)^\gamma e^{\psi_t} \end{aligned}$$

where $\pi_t \equiv P_t/P_{t-1}$ and $b_t \equiv B_t/P_t$ and $*$ denotes steady state value for the corresponding variable. Log linearizing around the steady states and combining the above equations, the system can be reduced to a bivariate system in $(\widetilde{\pi}_t, \widetilde{b}_t)$ where tilde denotes log deviation from steady state value, which must hold for $t = 0, 1, 2, \ldots$, i.e.

$$E_t\widetilde{\pi}_{t+1} = \alpha\widetilde{\pi}_t + \theta_t \tag{2.4.11}$$

$$\widetilde{b}_t + \beta^{-1}\widetilde{\pi}_t = [\beta^{-1} - \gamma(\beta^{-1} - 1)]\widetilde{b}_{t-1} + \alpha\beta^{-1}\widetilde{\pi}_{t-1} - (\beta^{-1} - 1)\psi_t + \beta^{-1}\theta_{t-1} \tag{2.4.12}$$

Evidently, a unique bounded equilibrium can exist if either $|\alpha| > 1$ and $|\gamma| > 1$ or $|\alpha| < 1$ and $|\gamma| < 1$. This implies that the policy parameter space is divided into four disjoint regions according to whether monetary and fiscal policies are, in Leeper (1991) terminology, active or passive.

Case 1: $\alpha < 1$ and $\gamma > 1$. Then we have one root inside the unit circle, i.e. $z_1 = 0$, with the other two outside, i.e. $z_2 = \frac{1}{\alpha} > 1$ and $z_3 = \frac{1}{\frac{1}{\beta}-\gamma(\frac{1}{\beta}-1)} > 1$. Since the existence condition is satisfied but the uniqueness condition is violated, any candidate of C_0 that satisfies the existence condition may lead to a different solution for y_t and hence there are infinite solutions.

Case 2: $\alpha > 1$ and $\gamma > 1$. Then we have two roots inside the unit circle, i.e. $z_1 = 0$ and $z_2 = \frac{1}{\alpha} < 1$, with the other outside, $z_3 = \frac{1}{\frac{1}{\beta}-\gamma(\frac{1}{\beta}-1)} > 1$. Since both the existence and uniqueness conditions are satisfied, any candidate of C_0 that satisfies the existence condition leads to the same solution for y_t and hence the solution is unique. Finally, the z-transform of the coefficient matrices for y_t is given by

$$C(z) = (z\Gamma(z))^{-1}[z\Psi(z) + \Gamma_{-1}C_0] = \begin{pmatrix} -\frac{1}{\alpha} & 0 \\ \frac{-\frac{1}{\alpha}}{1-\gamma+\beta\gamma}\frac{1}{z-\frac{1}{\frac{1}{\beta}-\gamma(\frac{1}{\beta}-1)}} & \frac{1-\beta}{1-\gamma+\beta\gamma}\frac{1}{z-\frac{1}{\frac{1}{\beta}-\gamma(\frac{1}{\beta}-1)}} \end{pmatrix}$$

and thus the unique solution is given by

$$\begin{pmatrix} \widetilde{\pi}_t \\ \widetilde{b}_t \end{pmatrix} = C(L)\begin{pmatrix} \theta_t \\ \psi_t \end{pmatrix} = \underbrace{\begin{pmatrix} -\frac{1}{\alpha} & 0 \\ \frac{1}{\alpha\beta} & 1-\frac{1}{\beta} \end{pmatrix}}_{C_0}\begin{pmatrix} \theta_t \\ \psi_t \end{pmatrix} + \sum_{k=1}^{\infty}\underbrace{\begin{pmatrix} 0 & 0 \\ \frac{\rho^k}{\alpha\beta} & (1-\frac{1}{\beta})\rho^k \end{pmatrix}}_{C_k}\begin{pmatrix} \theta_{t-k} \\ \psi_{t-k} \end{pmatrix}$$

where $\rho = \frac{1}{\beta}-\gamma(\frac{1}{\beta}-1) < 1$ and C_0 not only satisfies the existence condition but is consistent as well. Also, observe that fiscal shock and its lags (ψ_{t-k}) do not enter the solution for $\widetilde{\pi}_t$. This consequence is consistent with Gensys because we have one unstable eigenvalue ($\alpha > 1$) in the Fisher relation containing expectational terms, which enables it to evolve separately from the real debt valuation equation and hence $\widetilde{\pi}_t$ is not affected by the fiscal shocks.

Case 3: $\alpha < 1$ and $\gamma < 1$. Then we have two roots inside the unit circle, i.e. $z_1 = 0$ and $z_3 = \frac{1}{\frac{1}{\beta}-\gamma(\frac{1}{\beta}-1)} < 1$, with the other outside, $z_2 = \frac{1}{\alpha} > 1$. Since both the existence and uniqueness conditions are satisfied, any candidate of C_0 that satisfies the existence condition leads to the same solution for y_t and hence the solution is unique. Finally, the z-transform of the coefficient matrices for y_t is given by

$$C(z) = (z\Gamma(z))^{-1}[z\Psi(z) + \Gamma_{-1}C_0] = \begin{pmatrix} -\frac{1}{\alpha}\frac{z}{z-\frac{1}{\alpha}} & \frac{1-\beta}{\alpha}\frac{1}{z-\frac{1}{\alpha}} \\ 0 & 0 \end{pmatrix}$$

and thus the unique solution is given by

$$\begin{pmatrix} \widetilde{\pi}_t \\ \widetilde{b}_t \end{pmatrix} = C(L)\begin{pmatrix} \theta_t \\ \psi_t \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \beta-1 \\ 0 & 0 \end{pmatrix}}_{C_0}\begin{pmatrix} \theta_t \\ \psi_t \end{pmatrix} + \sum_{k=1}^{\infty}\underbrace{\begin{pmatrix} \alpha^{k-1} & (\beta-1)\alpha^k \\ 0 & 0 \end{pmatrix}}_{C_k}\begin{pmatrix} \theta_{t-k} \\ \psi_{t-k} \end{pmatrix}$$

where C_0 not only satisfies the existence condition but is consistent as well. Also, observe that fiscal shock and its lags (ψ_{t-k}) enter the solution for $\widetilde{\pi}_t$. This consequence is also consistent with Gensys because the only unstable eigenvalue ($\frac{1}{\beta} - \gamma(\frac{1}{\beta} - 1) > 1$) lives in the real debt valuation equation not containing expectational terms. Determinacy of solution thus requires that such unstable eigenvalue be imported from the real debt valuation equation into the Fisher relation which entails bringing the fiscal shocks in the solution for $\widetilde{\pi}_t$.

Case 4: $\alpha > 1$ and $\gamma < 1$. Then all roots are inside the unit circle. Since the existence condition is violated in this case, there is no covariance stationary solution for y_t. See Appendix B for derivation details.

Further inspection of the preceding mechanism in pinning down C_0 provides a more intuitive and perhaps insightful interpretation on the consistency of solutions. A rational expectations equilibrium is a fixed point of the mapping between the perceived law of motion and the law of motion generated by those beliefs. Any candidate for C_0 designates a perceived law of motion, or agents' belief about how endogenous variables evolve, and a solution obtained by plugging in such C_0 represents a new law of motion with equilibrium conditions imposed that is generated by such belief. The consistency of solution simply requires that the two laws of motion be identical and hence the z-transform approach reduces to solving a fixed point problem. By applying this argument, it is easy to see that there are multiple fixed points in Case 1, unique fixed point in Case 2 & 3, and no fixed point in Case 4.

Given the differences in the equilibria described in the previous example, it seems straightforward to distinguish an equilibrium time series generated by active monetary/passive fiscal policies (Case 2) from a time series generated by passive monetary/active fiscal policy (Case 3). Unfortunately, subtle observational equivalence results may make it difficult to identify whether a regime is "active" or "passive". The theoretical framework proposed in this paper makes it possible to study such observational equivalence phenomenon and the implied identification challenges that potentially reside in well-known DSGE models. In this section, we highlight the point that simple models show that two decoupled determinacy regions may generate observational equivalent equilibrium time series driven by generic exogenous driving process. Consequently, identification of policy regimes may thus be achieved by imposing *ad hoc* identifying restrictions on the exogenous driving process. One iconoclastic or even depressing conclusion flows naturally from this result: empirically testing for the interactions between monetary and fiscal policies by examining simple correlations in the data may lead to spurious results and potentially false conclusions. This suggests that existing efforts to "test" for the fiscal theory may be more challenging than originally believed.

For simplicity, we assume that the Wold representations for the exogenous driving processes in Case 2 & 3 of the above example are given by the following

$$\begin{pmatrix}\theta_t\\ \psi_t\end{pmatrix} = \underbrace{\begin{pmatrix}A_{11}(L) & 0\\ 0 & A_{22}(L)\end{pmatrix}}_{A(L)}\begin{pmatrix}\varepsilon_{1t}\\ \varepsilon_{2t}\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\theta_t\\ \psi_t\end{pmatrix} = \underbrace{\begin{pmatrix}B_{11}(L) & 0\\ 0 & B_{22}(L)\end{pmatrix}}_{B(L)}\begin{pmatrix}\varepsilon_{1t}\\ \varepsilon_{2t}\end{pmatrix}$$

where the functional forms for $\{A_{11}(\cdot), A_{22}(\cdot), B_{11}(\cdot), B_{22}(\cdot)\}$ are not specified.[14] We proceed by solving the model for both cases. See Appendix B for derivation details.

Case 2: let $\alpha = \alpha_1 > 1$ and $\gamma = \gamma_1 > 1$. Then we have two roots inside the unit circle, i.e. 0 and $z_1^M = \frac{1}{\alpha_1} < 1$, with the other outside, $z_2^M = \frac{1}{\frac{1}{\beta}-\gamma_1(\frac{1}{\beta}-1)} > 1$. Therefore, the z-transform of the coefficient matrices for y_t is given by

$$C_1(z) = \begin{pmatrix} -z_1^M \frac{zA_{11}(z)-z_1^M A_{11}(z_1^M)}{z-z_1^M} & 0\\ -\frac{1}{\beta}\frac{z_1^M z_2^M A_{11}(z_1^M)}{z-z_2^M} & (\frac{1}{\beta}-1)z_2^M \frac{A_{22}(z)}{z-z_2^M}\end{pmatrix}$$

which gives the solution under active monetary/passive fiscal policy regime.

Case 3': let $\alpha = \alpha_2 < 1$ and $\gamma = \gamma_2 < 1$. Then we have two roots inside the unit circle, i.e. 0 and $z_2^F = \frac{1}{\frac{1}{\beta}-\gamma_2(\frac{1}{\beta}-1)} < 1$, with the other outside, $z_1^F = \frac{1}{\alpha_2} > 1$. Therefore, the z-transform of the coefficient matrices for y_t is given by

$$C_2(z) = \begin{pmatrix} -z_1^F \frac{zB_{11}(z)}{z-z_1^F} & (1-\beta)\frac{z_1^F B_{22}(z_2^F)}{z-z_1^F}\\ 0 & (\frac{1}{\beta}-1)z_2^F \frac{B_{22}(z)-B_{22}(z_2^F)}{z-z_2^F}\end{pmatrix}$$

which gives the solution under passive monetary/active fiscal policy regime.

Equating the polynomial matrix $C_1(z)$ with $C_2(z)$ element by element delivers the fol-

[14] Obviously, this modified model is not readily solvable by conventional approaches.

lowing system of restrictions on the exogenous driving processes in both cases

$$\begin{aligned}
&\frac{zA_{11}(z) - z_1^M A_{11}(z_1^M)}{z - z_1^M} = \mu \frac{zB_{11}(z)}{z - z_1^F} \\
&A_{11}(z_1^M) = 0 \\
&B_{22}(z_2^F) = 0 \\
&\frac{A_{22}(z)}{z - z_2^M} = \nu \frac{B_{22}(z) - B_{22}(z_2^F)}{z - z_2^F}
\end{aligned}$$

where $\mu = \frac{z_1^F}{z_1^M}$ and $\nu = \frac{z_2^F}{z_2^M}$. The above system of restrictions seems overly restrictive but the fact that there are sequences of infinite undetermined coefficients in the polynomial functions $\{A_{11}(z), A_{22}(z), B_{11}(z), B_{22}(z)\}$ buys one enough freedom of matching the coefficients. The following theorem is due to Leeper et al. (2014).

Theorem 9. *Let* $\{A_{11}(z), A_{22}(z), B_{11}(z), B_{22}(z)\}$ *be given by the following polynomials*

$$A_{11}(z) = a_0 + a_1 z \tag{2.4.13}$$

$$B_{11}(z) = b_0 + b_1 z \tag{2.4.14}$$

$$A_{22}(z) = c_0 + c_1 z \tag{2.4.15}$$

$$B_{22}(z) = d_0 + d_1 z \tag{2.4.16}$$

Then there exist an infinite sequence of solutions satisfying the above system of restrictions, *one of which is given by the following*[15]

$$a_0 = 1, \quad a_1 = -\frac{1}{z_1^M} \tag{2.4.17}$$

$$b_0 = 1, \quad b_1 = -\frac{1}{z_1^F} \tag{2.4.18}$$

$$c_0 = 1, \quad c_1 = -\frac{1}{z_2^M} \tag{2.4.19}$$

$$d_0 = 1, \quad d_1 = -\frac{1}{z_2^F} \tag{2.4.20}$$

Its proof is trivial and thus omitted. This simple monetary model shows that two disjoint determinacy regions can generate observational equivalent equilibrium time series driven by properly chosen exogenous driving processes for each determinacy region. However, further study is needed to examine whether such conclusion plagues in more complicated DSGE models that researchers and policy institutions employ to study monetary and fiscal policy interactions.

2.4.3 Decoupled System

We now give an example that is due to Sims (2007). Consider a linear rational expectations model that has no stable solution for which the Blanchard and Kahn (1980) regularity conditions do not hold and that is not "generic" in the terminology of Onatski (2006).

$$x_t = \alpha x_{t-1} + \varepsilon_t \tag{2.4.21}$$

$$E_t y_{t+1} = \beta y_t + v_t \tag{2.4.22}$$

where $|\alpha| > 1$, $|\beta| < 1$, and ε_t and v_t are exogenous, non-explosive stochastic processes.[16] The two equations are unrelated and there is no way to import the unstable root from the

[15]Under the specification given in Theorem 9, we have one free coefficient and hence there are infinite solutions.

[16]In Sims (2007), $\alpha = 1.1$ and $\beta = 0.9$.

first equation into the second equation. As Sims noted, "the Blanchard-Kahn rule, that the number of unstable roots must match the number of forward-looking variables, is satisfied, yet there is no stable solution. The reason is that the unstable root occurs in a part of the system that is decoupled from the expectational equation."

Now we apply the solution methodology proposed in this paper to solve the model. Note that $\det[z\Gamma(z)]$ has three distinct roots given by $z_1 = 0$, $z_2 = \frac{1}{\alpha}$, and $z_3 = \frac{1}{\beta}$. Here we have two roots inside the unit circle, i.e. $z_1 = 0$ and $z_2 = \frac{1}{\alpha} < 1$, with the other one outside, i.e. $z_3 = \frac{1}{\beta} > 1$. Imposing the restrictions $U_{2\cdot}(z)(z\Psi(z) + \Gamma_{-1}C_0)|_{z=0} = 0$ and $U_{2\cdot}(z)(z\Psi(z) + \Gamma_{-1}C_0)|_{z=\frac{1}{\alpha}} = 0$ gives the following system

$$-\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} C_0(1,1) & C_0(1,2) \\ C_0(2,1) & C_0(2,2) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{\beta-\alpha}{\alpha^3\beta} & 0 \end{pmatrix}$$

which requires that $\frac{\beta-\alpha}{\alpha^3\beta} = 0$, or $\alpha = \beta$, a contradiction.[17] Since the existence condition is violated in this case, there is no covariance stationary solution for (x_t, y_t). Though the winding number criterion in Onatski (2006) gives the wrong answer as pointed out by Sims (2007), our approach is immune to the trouble brought about by the decoupled system and hence gives the right answer. See Appendix B for derivation details.

2.5 Concluding Comments

There are many other solution methodology papers in the literature that, like this one, expand the range of models beyond that of Blanchard and Kahn (1980) [Anderson and Moore (1985), Broze et al. (1995), Klein (2000), Binder and Pesaran (1997), King and Wat-

[17] One may easily realize that one of the possibilities that such contradiction arises is due to the appearance of ε_t in the first equation. Removing ε_t from the first equation makes this contradiction dissipate but it also yields a "degenerate" covariance stationary solution for x_t, i.e. $x_t = 0$ for all t, which is consistent with the fact that existence condition holds when ε_t is dropped off. This shows that, unlike "root-counting" or "winding number" criteria, the approach in this paper is immune to the troubles posed by decoupling issue.

son (1998), McCallum (1998), Zadrozny (1998), Uhlig (1999), and Onatski (2006)]. There are compelling reasons why studying models with arbitrary number of lags of endogenous variables, or lagged expectations, or with expectations of more distant future values, and with generic exogenous driving processes may be interesting to economists. From a purely methodological perspective, analyzing more general models gives new insights about methods developed under more restrictive assumptions and allows their deeper interpretation. Moreover, as we argue here, new (or old) techniques could prove useful for solving complicated linear rational expectation models.

We show that the advantage of this frequency-domain approach over other popular time-domain approaches derives from its provision of new insights to solving several well-known challenging problems, e.g. forecasting the forecasts of others in Townsend (1983) and observational equivalence of monetary and fiscal policy interactions in Leeper et al. (2014), etc. In particular, we solve a simple cashless version of the model in Leeper (1991) and highlight the point that two decoupled determinacy regions may generate observational equivalent equilibrium time series driven by generic exogenous driving process. Therefore, our solution methodology proves to be an indispensable supplement to the existing approaches both from theoretical and applied perspectives.

One useful extension of the solution methodology proposed in this paper would be to accommodate continuous-time processes into our general framework as Sims (2002). On one hand, explicit extension to the continuous-time system enables one to tackle problems that can hardly be dealt with in the discrete-time system and thus brings new insights to the table. On the other hand, a continuous-time extension makes it possible to study various non-stationary or near non-stationary features commonly present in almost all important macroeconomic time series data. These non-stationarities usually cannot be fully removed by the simple detrending or transformations and very often, these detrending efforts may incur loss of important long-term information about the data that is potentially valuable to the researchers. Therefore, an explicit extension of our solution methodology to the continuous-time setting proves to be both non-trivial and useful. We leave this for future research.

Online Appendix

Appendix A: Proofs

The solution method derived in this paper is intimately related to many other approaches proposed in the literature. In this appendix, we will focus our attention only on its connection to one of the most popular solution methodologies, i.e. that of Sims (2002). We summarize the equivalence relation of our approach and Sims' approach in the following theorem.

Theorem 10. *Consider the following multivariate linear rational expectations model*[18]

$$(\Gamma_{-1}L^{-1} + \Gamma_0)y_t = \Psi_{-1}L^{-1}x_t + \Gamma_{-1}\eta_{t+1} \tag{2.5.1}$$

Assume that Γ_{-1} *is of full rank, and both the eigenvalues of* $-\Gamma_{-1}^{-1}\Gamma_0$ *and the roots of* $\det[\Gamma_{-1} + z\Gamma_0] = 0$ *are nonzero and distinct. Then*

1. *Factorization equivalence: the eigenvalues of* $-\Gamma_{-1}^{-1}\Gamma_0$ *are exactly the inverse of the corresponding roots of* $\det[\Gamma_{-1} + z\Gamma_0] = 0$*, or equivalently, those roots of the determinant of the Smith canonical form for* $\Gamma_{-1} + z\Gamma_0$*;*
2. *Existence equivalence: the restrictions imposed by the unstable eigenvalues in Sims (2002) are exactly those imposed by the roots inside unit circle in this paper.*

[18] Since all variables are taken to be zero-mean linearly regular covariance stationary stochastic processes in this paper, the vector of constants in Sims (2002) drops off from (2.5.1).

3. *Uniqueness equivalence: the conditions under which the solution to* (2.5.1) *is unique are equivalent between Sims (2002) and this paper.*

This theorem shows an equivalence relation between Sims (2002) and this paper, with Gensys as a special case. One of the major differences lies in the fact that the solution is expressed in ARMA form in Sims (2002) while this paper gives the solution in MA form. In what follows, we prove Theorem 10.

PROOF OF (1): first, the eigenvalue λ of $-\Gamma_{-1}^{-1}\Gamma_0$ can be computed as $|\Gamma_{-1}^{-1}\Gamma_0+\lambda I| = 0$. Also, since Γ_{-1} is assumed to be of full rank and $z \neq 0$, we have $|\Gamma_{-1}+z\Gamma_0| = |z\Gamma_{-1}||\Gamma_{-1}^{-1}\Gamma_0+\frac{1}{z}I| = 0$, or $|\Gamma_{-1}^{-1}\Gamma_0 + \frac{1}{z}I| = 0$. This establishes $\lambda = \frac{1}{z}$.

Second, let $\Gamma_{-1} + z\Gamma_0 = U(z)^{-1}P(z)V(z)^{-1}$ where $U(z)$ and $V(z)$ are unimodular matrices and $P(z)$ is the Smith canonical form for $\Gamma_{-1} + z\Gamma_0$. Since $|U(z)|$ and $|V(z)|$ are nonzero constants, the roots of $|\Gamma_{-1} + z\Gamma_0| = 0$ are exactly those of $|P(z)| = 0$.

Part (1) of Theorem 10 shows an equivalence relation between identifying the unstable eigenvalues in Sims (2002) and identifying the roots inside unit circle in this paper.

PROOF OF (2): first, we derive the restriction system in Sims (2002). Since all eigenvalues of $-\Gamma_{-1}^{-1}\Gamma_0$ are distinct, we know that $-\Gamma_{-1}^{-1}\Gamma_0$ is diagonalizable and can be factorized as

$$-\Gamma_{-1}^{-1}\Gamma_0 = P\Lambda P^{-1}$$

where P is the matrix of right-eigenvectors, P^{-1} is the matrix of left-eigenvectors, and Λ is a diagonal matrix with all eigenvalues of $-\Gamma_{-1}^{-1}\Gamma_0$ on its main diagonal. Stability conditions then require that for all t

$$P^{U\cdot}(\Gamma_{-1}^{-1}\Psi_{-1}x_{t+1} + \eta_{t+1}) = 0 \tag{2.5.2}$$

where $P^{U\cdot}$ collects all the rows of P^{-1} corresponding to unstable eigenvalues.

Second, we derive the restriction system in this paper. Note that the polynomial matrix $\Gamma_{-1} + z\Gamma_0$ can be factorized as

$$\Gamma_{-1} + z\Gamma_0 = U(z)^{-1}P(z)V(z)^{-1} = \underbrace{U(z)^{-1}P_1(z)}_{S(z)}\underbrace{P_2(z)V(z)^{-1}}_{T(z)}$$

where $U(z)$ and $V(z)$ are unimodular matrices and $S(z)$ is the Smith canonical form for $\Gamma_{-1} + z\Gamma_0$. Also, $S(z)$ is a polynomial matrix such that all the roots of $\det[S(z)]$ lie inside the unit circle while $T(z)$ is a polynomial matrix with all the roots of $\det[T(z)]$ outside the unit circle. Since all the roots of $\det[\Gamma_{-1} + z\Gamma_0]$ are distinct, the property that the (i,i) entry of Smith canonical form is divisible by its $(i-1,i-1)$ entry for $i = 2,\ldots,p$ implies that $P_1(z)$ is of the form

$$P_1(z) = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \prod_{j=1}^{r}(z - \underline{z}_j) \end{pmatrix}$$

and hence

$$S(z)^{-1} = \begin{pmatrix} U_{1\cdot}(z) \\ \vdots \\ U_{p-1\cdot}(z) \\ \frac{1}{\prod_{j=1}^{r}(z-\underline{z}_j)}U_{p\cdot}(z) \end{pmatrix}$$

This implies the following restriction system

$$\begin{pmatrix} U_{p\cdot}(\underline{z}_1) \\ \vdots \\ U_{p\cdot}(\underline{z}_{\underline{r}}) \end{pmatrix} \Gamma_{-1}(\Gamma_{-1}^{-1}\Psi_{-1} + C_0) = 0 \tag{2.5.3}$$

Observe that for $\forall \underline{z}_j$ with $j = 1, 2, \ldots, \underline{r}$, we have the following equation

$$U(\underline{z}_j)\Gamma_{-1}\left(\Gamma_{-1}^{-1}\Gamma_0 + \frac{1}{\underline{z}_j}I\right) = \frac{1}{\underline{z}_j}P(\underline{z}_j)V(\underline{z}_j)^{-1}$$

where the last row is given by

$$U_{p\cdot}(\underline{z}_j)\Gamma_{-1}\left(\Gamma_{-1}^{-1}\Gamma_0 + \frac{1}{\underline{z}_j}I\right) = (0 \cdots 0)$$

This implies that $U_{p\cdot}(\underline{z}_j)\Gamma_{-1}$ is exactly the left eigenvector corresponding to the unstable eigenvalue $\frac{1}{\underline{z}_j}$ of $-\Gamma_{-1}^{-1}\Gamma_0$. Stacking $U_{p\cdot}(\underline{z}_j)\Gamma_{-1} = P^{j\cdot}$ for $j = 1, 2, \ldots, \underline{r}$ then gives

$$\begin{pmatrix} U_{p\cdot}(\underline{z}_1) \\ \vdots \\ U_{p\cdot}(\underline{z}_{\underline{r}}) \end{pmatrix} \Gamma_{-1} = P^{U\cdot}$$

This implies that (2.5.3) is equivalent to

$$P^{U\cdot}(\Gamma_{-1}^{-1}\Psi_{-1} + C_0) = 0 \tag{2.5.4}$$

The proof is completed by noticing that both (2.5.2) and (2.5.4) hold if and only if the columns of $P^{U\cdot}$ span the space spanned by the columns of $P^{U\cdot}\Gamma_{-1}^{-1}\Psi_{-1}$, i.e.

$$\mathrm{span}(P^{U\cdot}\Gamma_{-1}^{-1}\Psi_{-1}) \subseteq \mathrm{span}(P^{U\cdot})$$

which shows that the algorithm in Sims (2002) and that discussed in this paper yield identical existence condition.

Proof of (3): first, the uniqueness condition in Sims (2002) requires that the knowledge of $P^{U\cdot}\eta$ can be used to determine $P^{S\cdot}\eta$, where $P^{S\cdot}$ is made up of all the rows of P^{-1} corresponding to stable eigenvalues.

Second, the uniqueness condition in this paper requires that the knowledge of

$$\begin{pmatrix} U_{p\cdot}(\underline{z}_1) \\ \vdots \\ U_{p\cdot}(\underline{z}_{\underline{r}}) \end{pmatrix} \Gamma_{-1} C_0$$

can be used to determine

$$\begin{pmatrix} U_{p\cdot}(\overline{z}_1^{-1}) \\ \vdots \\ U_{p\cdot}(\overline{z}_{\overline{r}}^{-1}) \end{pmatrix} \Gamma_{-1} C_0$$

where $\overline{z}_j$ for $j = 1, \ldots, \overline{r}$ are those roots outside unit circle for $\det[\Gamma_{-1} + z\Gamma_0] = 0$, and hence their inverses are exactly the stable eigenvalues of $-\Gamma_{-1}^{-1}\Gamma_0$ by part (1). Therefore, by part (2) the solution is unique when the knowledge of $P^{U\cdot}C_0$ can be used to determine $P^{S\cdot}C_0$.

The proof is completed by noticing that the uniqueness conditions in Sims (2002) and this paper both hold if and only if the columns of $(P^{U\cdot})'$ span the space spanned by the columns of $(P^{S\cdot})'$, i.e.

$$\text{span}((P^{S\cdot})') \subseteq \text{span}((P^{U\cdot})')$$

Appendix B: Solutions

First, we solve the example model in Section 2.4.2. Rewrite the bivariate system into the form of (2.3.6)

$$\left[\underbrace{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}}_{\Gamma_{-1}} L^{-1} + \underbrace{\begin{pmatrix}-\alpha & 0\\ \frac{1}{\beta} & 1\end{pmatrix}}_{\Gamma_0} L^{0} + \underbrace{\begin{pmatrix}0 & 0\\ -\frac{\alpha}{\beta} & -\left[\frac{1}{\beta}-\gamma(\frac{1}{\beta}-1)\right]\end{pmatrix}}_{\Gamma_1} L\right] \underbrace{\begin{pmatrix}\widetilde{\pi}_t\\ \widetilde{b}_t\end{pmatrix}}_{y_t}$$

$$= \left[\underbrace{\begin{pmatrix}1 & 0\\ 0 & -(\frac{1}{\beta}-1)\end{pmatrix}}_{\Psi_0} L^{0} + \underbrace{\begin{pmatrix}0 & 0\\ \frac{1}{\beta} & 0\end{pmatrix}}_{\Psi_1} L\right] \underbrace{\begin{pmatrix}\theta_t\\ \psi_t\end{pmatrix}}_{x_t} + \underbrace{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}}_{\Gamma_{-1}} \underbrace{\begin{pmatrix}\eta^{\pi}_{t+1}\\ \eta^{b}_{t+1}\end{pmatrix}}_{\eta_{t+1}}$$

where $n = m = l = 1$, $p = q = 2$, and $A(L)$ is taken to be a (2×2) identity matrix. Then we can obtain

$$z\Gamma(z) = U(z)^{-1} \begin{pmatrix}1 & 0\\ 0 & z\left(z-\frac{1}{\alpha}\right)\left(z-\frac{1}{\frac{1}{\beta}-\gamma(\frac{1}{\beta}-1)}\right)\end{pmatrix} V(z)^{-1}$$

where $U(z)$ and $V(z)$ are unimodular matrices. Obviously $\det[z\Gamma(z)]$ has three distinct roots, i.e. $z_1 = 0$, $z_2 = \frac{1}{\alpha}$, and $z_3 = \frac{1}{\frac{1}{\beta}-\gamma(\frac{1}{\beta}-1)}$. In this example, we have four cases.

Case 1: $\alpha < 1$ and $\gamma > 1$. Then we have one root inside the unit circle, i.e. $z_1 = 0$, with the other two outside, i.e. $z_2 = \frac{1}{\alpha} > 1$ and $z_3 = \frac{1}{\frac{1}{\beta}-\gamma(\frac{1}{\beta}-1)} > 1$. Therefore, the polynomial matrix $z\Gamma(z)$ can be decomposed as

$$z\Gamma(z) = \underbrace{U(z)^{-1} \begin{pmatrix}1 & 0\\ 0 & z\end{pmatrix}}_{S(z)} \underbrace{\begin{pmatrix}1 & 0\\ 0 & \left(z-\frac{1}{\alpha}\right)\left(z-\frac{1}{\frac{1}{\beta}-\gamma(\frac{1}{\beta}-1)}\right)\end{pmatrix} V(z)^{-1}}_{T(z)}$$

and hence

$$S(z)^{-1} = \begin{pmatrix} U_{1\cdot}(z) \\ \frac{1}{z}U_{2\cdot}(z) \end{pmatrix}$$

Imposing the restriction $U_{2\cdot}(z)[z\Psi(z) + \Gamma_{-1}C_0]|_{z=0} = 0$ gives the following system

$$-\begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} C_0(1,1) & C_0(1,2) \\ C_0(2,1) & C_0(2,2) \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

and hence we have no restrictions imposed on the unknown coefficient matrix C_0. Now we examine the uniqueness condition. Notice that

$$R = U_{2\cdot}(z_1)\Gamma_{-1} = \begin{pmatrix} 0 & 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} U_{2\cdot}(z_2^{-1})\Gamma_{-1} \\ U_{2\cdot}(z_3^{-1})\Gamma_{-1} \end{pmatrix} = \begin{pmatrix} \frac{\alpha(\alpha+1-\gamma+\beta\gamma)-(1+\beta)}{1-\gamma+\beta\gamma} & 0 \\ \frac{(\alpha+1-\gamma+\beta\gamma)(1-\gamma+\beta\gamma)-\beta(1+\beta)}{\alpha\beta^2} & 0 \end{pmatrix}$$

and since the columns of R' do not span the space spanned by the columns of Q', i.e. $\text{span}(Q') \nsubseteq \text{span}(R')$, any candidate of C_0 that satisfies the existence condition may lead to a different solution for y_t and hence there are infinite solutions.

Case 2: $\alpha > 1$ and $\gamma > 1$. Then we have two roots inside the unit circle, i.e. $z_1 = 0$ and $z_2 = \frac{1}{\alpha} < 1$, with the other outside, $z_3 = \frac{1}{\frac{1}{\beta}-\gamma(\frac{1}{\beta}-1)} > 1$. Therefore, the polynomial matrix $z\Gamma(z)$ can be decomposed as

$$z\Gamma(z) = \underbrace{U(z)^{-1}\begin{pmatrix} 1 & 0 \\ 0 & z\left(z-\frac{1}{\alpha}\right) \end{pmatrix}}_{S(z)} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & z - \frac{1}{\frac{1}{\beta}-\gamma(\frac{1}{\beta}-1)} \end{pmatrix} V(z)^{-1}}_{T(z)}$$

and hence

$$S(z)^{-1} = \begin{pmatrix} U_{1\cdot}(z) \\ \frac{1}{z(z-1/\alpha)}U_{2\cdot}(z) \end{pmatrix}$$

Imposing the restriction $U_{2\cdot}(z)[z\Psi(z)+\Gamma_{-1}C_0]|_{z=1/\alpha}=0$ gives the following system

$$-\begin{pmatrix}\frac{1-\gamma+\beta\gamma-\alpha\beta}{\alpha^3(1-\gamma+\beta\gamma)} & 0\end{pmatrix}\begin{pmatrix}C_0(1,1) & C_0(1,2)\\ C_0(2,1) & C_0(2,2)\end{pmatrix}=\begin{pmatrix}\frac{1-\gamma+\beta\gamma-\alpha\beta}{\alpha^4(1-\gamma+\beta\gamma)} & 0\end{pmatrix}$$

and hence $C_0(1,1)=-\frac{1}{\alpha}$ and $C_0(1,2)=0$.[19] Now we examine the uniqueness condition. Notice that

$$R=\begin{pmatrix}U_{2\cdot}(z_1)\Gamma_{-1}\\ U_{2\cdot}(z_2)\Gamma_{-1}\end{pmatrix}=\begin{pmatrix}0 & 0\\ \frac{1-\gamma+\beta\gamma-\alpha\beta}{\alpha^3(1-\gamma+\beta\gamma)} & 0\end{pmatrix},$$
$$Q=U_{2\cdot}(z_3^{-1})\Gamma_{-1}=\begin{pmatrix}\frac{(\alpha+1-\gamma+\beta\gamma)(1-\gamma+\beta\gamma)-\beta(1+\beta)}{\alpha\beta^2} & 0\end{pmatrix}$$

and the columns of R' span the space spanned by the columns of Q', i.e. $\text{span}(Q')\subseteq\text{span}(R')$ is satisfied. Therefore, for any candidate of C_0 that satisfies the existence condition, it leads to the same solution for y_t and thus the solution is unique. Finally, the z-transform of the coefficient matrices for y_t is given by

$$C(z)=(z\Gamma(z))^{-1}[z\Psi(z)+\Gamma_{-1}C_0]=\begin{pmatrix}-\frac{1}{\alpha} & 0\\ \frac{-\frac{1}{\alpha}}{1-\gamma+\beta\gamma}\frac{1}{z-\frac{1}{\frac{1}{\beta}-\gamma(\frac{1}{\beta}-1)}} & \frac{1-\beta}{1-\gamma+\beta\gamma}\frac{1}{z-\frac{1}{\frac{1}{\beta}-\gamma(\frac{1}{\beta}-1)}}\end{pmatrix}$$

and thus the unique solution is given by

$$\begin{pmatrix}\widetilde{\pi}_t\\ \widetilde{b}_t\end{pmatrix}=C(L)\begin{pmatrix}\theta_t\\ \psi_t\end{pmatrix}=\underbrace{\begin{pmatrix}-\frac{1}{\alpha} & 0\\ \frac{1}{\alpha\beta} & 1-\frac{1}{\beta}\end{pmatrix}}_{C_0}\begin{pmatrix}\theta_t\\ \psi_t\end{pmatrix}+\sum_{k=1}^{\infty}\underbrace{\begin{pmatrix}0 & 0\\ \frac{\rho^k}{\alpha\beta} & (1-\frac{1}{\beta})\rho^k\end{pmatrix}}_{C_k}\begin{pmatrix}\theta_{t-k}\\ \psi_{t-k}\end{pmatrix}$$

where $\rho=\frac{1}{\beta}-\gamma(\frac{1}{\beta}-1)<1$ and C_0 not only satisfies the existence condition but is consistent as well.

[19] Here we omit the restriction imposed by $z=0$ because it is unrestrictive.

CASE 3: $\alpha < 1$ and $\gamma < 1$. Then we have two roots inside the unit circle, i.e. $z_1 = 0$ and $z_3 = \frac{1}{\frac{1}{\beta}-\gamma(\frac{1}{\beta}-1)} < 1$, with the other outside, $z_2 = \frac{1}{\alpha} > 1$. Therefore, polynomial matrix $z\Gamma(z)$ can be decomposed as

$$z\Gamma(z) = \underbrace{U(z)^{-1}\begin{pmatrix} 1 & 0 \\ 0 & z\left(z - \frac{1}{\frac{1}{\beta}-\gamma(\frac{1}{\beta}-1)}\right)\end{pmatrix}}_{S(z)}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & z - \frac{1}{\alpha}\end{pmatrix} V(z)^{-1}}_{T(z)}$$

and hence

$$S(z)^{-1} = \begin{pmatrix} U_{1\cdot}(z) \\ \frac{1}{z\left(z - \frac{1}{\frac{1}{\beta}-\gamma(\frac{1}{\beta}-1)}\right)} U_{2\cdot}(z)\end{pmatrix}$$

Imposing the restriction $U_{2\cdot}(z)[z\Psi(z) + \Gamma_{-1}C_0]|_{z=\frac{1}{\frac{1}{\beta}-\gamma(\frac{1}{\beta}-1)}} = 0$ gives the following system

$$-\begin{pmatrix} -\frac{\beta(1-\gamma+\beta\gamma-\alpha\beta)}{\alpha(1-\gamma+\beta\gamma)^3} & 0\end{pmatrix}\begin{pmatrix} C_0(1,1) & C_0(1,2) \\ C_0(2,1) & C_0(2,2)\end{pmatrix} = \begin{pmatrix} 0 & -\frac{\beta(1-\beta)(1-\gamma+\beta\gamma-\alpha\beta)}{\alpha(1-\gamma+\beta\gamma)^3}\end{pmatrix}$$

and hence $C_0(1,1) = 0$ and $C_0(1,2) = \beta - 1$. Now we examine the uniqueness condition. Notice that

$$R = \begin{pmatrix} U_{2\cdot}(z_1)\Gamma_{-1} \\ U_{2\cdot}(z_3)\Gamma_{-1}\end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -\frac{\beta(1-\gamma+\beta\gamma-\alpha\beta)}{\alpha(1-\gamma+\beta\gamma)^3} & 0\end{pmatrix},$$

$$Q = U_{2\cdot}(z_2^{-1})\Gamma_{-1} = \begin{pmatrix} \frac{\alpha(\alpha+1-\gamma+\beta\gamma)-(1+\beta)}{1-\gamma+\beta\gamma} & 0\end{pmatrix}$$

and the columns of R' span the space spanned by the columns of Q', i.e. $\text{span}(Q') \subseteq \text{span}(R')$ is satisfied. Therefore, for any candidate of C_0 that satisfies the existence condition, it leads to the same solution for y_t and thus the solution is unique. Finally, the z-transform of the coefficient matrices for y_t is given by

$$C(z) = (z\Gamma(z))^{-1}[z\Psi(z) + \Gamma_{-1}C_0] = \begin{pmatrix} -\frac{1}{\alpha}\frac{z}{z-\frac{1}{\alpha}} & \frac{1-\beta}{\alpha}\frac{1}{z-\frac{1}{\alpha}} \\ 0 & 0 \end{pmatrix}$$

and thus the unique solution is given by

$$\begin{pmatrix} \widetilde{\pi}_t \\ \widetilde{b}_t \end{pmatrix} = C(L)\begin{pmatrix} \theta_t \\ \psi_t \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \beta-1 \\ 0 & 0 \end{pmatrix}}_{C_0}\begin{pmatrix} \theta_t \\ \psi_t \end{pmatrix} + \sum_{k=1}^{\infty}\underbrace{\begin{pmatrix} \alpha^{k-1} & (\beta-1)\alpha^k \\ 0 & 0 \end{pmatrix}}_{C_k}\begin{pmatrix} \theta_{t-k} \\ \psi_{t-k} \end{pmatrix}$$

where C_0 not only satisfies the existence condition but is consistent as well.

Case 4: $\alpha > 1$ and $\gamma < 1$. Then all roots are inside the unit circle. Therefore, the polynomial matrix $z\Gamma(z)$ can be decomposed as

$$z\Gamma(z) = \underbrace{U(z)^{-1}\begin{pmatrix} 1 & 0 \\ 0 & z\left(z-\frac{1}{\alpha}\right)\left(z-\frac{1}{\frac{1}{\beta}-\gamma(\frac{1}{\beta}-1)}\right) \end{pmatrix}}_{S(z)}\underbrace{V(z)^{-1}}_{T(z)}$$

and hence

$$S(z)^{-1} = \begin{pmatrix} U_{1\cdot}(z) \\ \frac{1}{z\left(z-\frac{1}{\alpha}\right)\left(z-\frac{1}{\frac{1}{\beta}-\gamma(\frac{1}{\beta}-1)}\right)}U_{2\cdot}(z) \end{pmatrix}$$

Imposing the restrictions $U_{2\cdot}(z)[z\Psi(z)+\Gamma_{-1}C_0]|_{z=\frac{1}{\alpha},\frac{1}{\frac{1}{\beta}-\gamma(\frac{1}{\beta}-1)}} = 0$ gives the following system

$$-\begin{pmatrix} \frac{1-\gamma+\beta\gamma-\alpha\beta}{\alpha^3(1-\gamma+\beta\gamma)} & 0 \\ -\frac{\beta(1-\gamma+\beta\gamma-\alpha\beta)}{\alpha(1-\gamma+\beta\gamma)^3} & 0 \end{pmatrix}\begin{pmatrix} C_0(1,1) & C_0(1,2) \\ C_0(2,1) & C_0(2,2) \end{pmatrix} = \begin{pmatrix} \frac{1-\gamma+\beta\gamma-\alpha\beta}{\alpha^4(1-\gamma+\beta\gamma)} & 0 \\ 0 & -\frac{\beta(1-\beta)(1-\gamma+\beta\gamma-\alpha\beta)}{\alpha(1-\gamma+\beta\gamma)^3} \end{pmatrix}$$

which does not have a solution. Since the existence condition is violated in this case, there is no covariance stationary solution for y_t.

Next, we solve for the exogenous driving processes under the two regimes so that we can perfectly match the equilibrium solutions across regimes. First, let $\alpha = \alpha_1 > 1$ and $\gamma = \gamma_1 > 1$. Then we have two roots inside the unit circle, i.e. 0 and $z_1^M = \frac{1}{\alpha_1} < 1$, with the other outside, $z_2^M = \frac{1}{\frac{1}{\beta} - \gamma_1(\frac{1}{\beta} - 1)} > 1$. Therefore, the polynomial matrix $z\Gamma(z)$ can be decomposed as

$$z\Gamma(z) = U(z)^{-1} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & z\left(z - z_1^M\right) \end{pmatrix}}_{S(z)} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & z - z_2^M \end{pmatrix} V(z)^{-1}}_{T(z)}$$

and hence

$$S(z)^{-1} = \begin{pmatrix} U_{1\cdot}(z) \\ \frac{1}{z\left(z - z_1^M\right)} U_{2\cdot}(z) \end{pmatrix}$$

Imposing the restriction $U_{2\cdot}(z)[z\Psi(z)A(z) + \Gamma_{-1}C_0]|_{z=z_1^M} = 0$ gives the following system

$$-\begin{pmatrix} \frac{1-\gamma_1+\beta\gamma_1-\alpha_1\beta}{\alpha_1^3(1-\gamma_1+\beta\gamma_2)} & 0 \end{pmatrix} \begin{pmatrix} C_0(1,1) & C_0(1,2) \\ C_0(2,1) & C_0(2,2) \end{pmatrix} = \begin{pmatrix} \frac{1-\gamma_1+\beta\gamma_1-\alpha_1\beta}{\alpha_1^4(1-\gamma_1+\beta\gamma_1)} A_{11}(z_1^M) & 0 \end{pmatrix}$$

and hence $C_0(1,1) = -z_1^M A_{11}(z_1^M)$ and $C_0(1,2) = 0$. Therefore, the z-transform of the coefficient matrices for y_t is given by

$$C_1(z) = \begin{pmatrix} -z_1^M \frac{zA_{11}(z) - z_1^M A_{11}(z_1^M)}{z - z_1^M} & 0 \\ -\frac{1}{\beta} \frac{z_1^M z_2^M A_{11}(z_1^M)}{z - z_2^M} & (\frac{1}{\beta} - 1) z_2^M \frac{A_{22}(z)}{z - z_2^M} \end{pmatrix}$$

which gives the solution under active monetary/passive fiscal policy regime.

Second, let $\alpha = \alpha_2 < 1$ and $\gamma = \gamma_2 < 1$. Then we have two roots inside the unit circle, i.e. 0 and $z_2^F = \frac{1}{\frac{1}{\beta} - \gamma_2(\frac{1}{\beta} - 1)} < 1$, with the other outside, $z_1^F = \frac{1}{\alpha_2} > 1$. Therefore, polynomial matrix $z\Gamma(z)$ can be decomposed as

$$z\Gamma(z) = \underbrace{U(z)^{-1}\begin{pmatrix}1 & 0 \\ 0 & z\left(z-z_2^F\right)\end{pmatrix}}_{S(z)}\underbrace{\begin{pmatrix}1 & 0 \\ 0 & z-z_1^F\end{pmatrix}V(z)^{-1}}_{T(z)}$$

and hence

$$S(z)^{-1} = \begin{pmatrix} U_{1\cdot}(z) \\ \frac{1}{z\left(z-z_2^F\right)}U_{2\cdot}(z)\end{pmatrix}$$

Imposing the restriction $U_{2\cdot}(z)[z\Psi(z)B(z)+\Gamma_{-1}C_0]|_{z=z_2^F}=0$ gives the following system

$$-\begin{pmatrix}-\frac{\beta(1-\gamma+\beta\gamma-\alpha\beta)}{\alpha(1-\gamma+\beta\gamma)^3} & 0\end{pmatrix}\begin{pmatrix}C_0(1,1) & C_0(1,2) \\ C_0(2,1) & C_0(2,2)\end{pmatrix} = \begin{pmatrix}0 & -\frac{\beta(1-\beta)(1-\gamma+\beta\gamma-\alpha\beta)}{\alpha(1-\gamma+\beta\gamma)^3}B_{22}(z_2^F)\end{pmatrix}$$

and hence $C_0(1,1)=0$ and $C_0(1,2)=(\beta-1)B_{22}(z_2^F)$. Therefore, the z-transform of the coefficient matrices for y_t is given by

$$C_2(z) = \begin{pmatrix}-z_1^F\frac{zB_{11}(z)}{z-z_1^F} & (1-\beta)\frac{z_1^F B_{22}(z_2^F)}{z-z_1^F} \\ 0 & (\frac{1}{\beta}-1)z_2^F\frac{B_{22}(z)-B_{22}(z_2^F)}{z-z_2^F}\end{pmatrix}$$

which gives the solution under passive monetary/active fiscal policy regime.

Lastly, we solve the example model in Section 2.4.3. Rewrite the bivariate system into the form of (2.3.6)

$$\left[\underbrace{\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}}_{\Gamma_{-1}} L^{-1} + \underbrace{\begin{pmatrix}1 & 0\\ 0 & -\beta\end{pmatrix}}_{\Gamma_0} L^{0} + \underbrace{\begin{pmatrix}-\alpha & 0\\ 0 & 0\end{pmatrix}}_{\Gamma_1} L\right]\begin{pmatrix}x_t\\ y_t\end{pmatrix}$$

$$= \underbrace{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}}_{\Psi_0}\begin{pmatrix}\varepsilon_t\\ v_t\end{pmatrix} + \underbrace{\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}}_{\Gamma_{-1}}\underbrace{\begin{pmatrix}\eta^x_{t+1}\\ \eta^y_{t+1}\end{pmatrix}}_{\eta_{t+1}}$$

where $n = m = 1$, $l = 0$, and $p = q = 2$.[20] Then we can obtain

$$z\Gamma(z) = \begin{pmatrix}z - \alpha z^2 & 0\\ 0 & 1 - \beta z\end{pmatrix} = U(z)^{-1}\begin{pmatrix}1 & 0\\ 0 & z(z - \frac{1}{\alpha})(z - \frac{1}{\beta})\end{pmatrix}$$

Obviously $\det[z\Gamma(z)]$ has three distinct roots, i.e. $z_1 = 0$, $z_2 = \frac{1}{\alpha}$, and $z_3 = \frac{1}{\beta}$. Here we have two roots inside the unit circle, i.e. $z_1 = 0$ and $z_2 = \frac{1}{\alpha} < 1$, with the other one outside, i.e. $z_3 = \frac{1}{\beta} > 1$. The polynomial matrix can be decomposed as

$$z\Gamma(z) = \underbrace{U(z)^{-1}\begin{pmatrix}1 & 0\\ 0 & z(z - \frac{1}{\alpha})\end{pmatrix}}_{S(z)}\underbrace{\begin{pmatrix}1 & 0\\ 0 & z - \frac{1}{\beta}\end{pmatrix} V(z)^{-1}}_{T(z)}$$

and hence

$$S(z)^{-1} = \begin{pmatrix}U_{1\cdot}(z)\\ \frac{1}{z(z - \frac{1}{\alpha})}U_{2\cdot}(z)\end{pmatrix}$$

Imposing the restrictions $U_{2\cdot}(z)(z\Psi(z)+\Gamma_{-1}C_0)|_{z=0} = 0$ and $U_{2\cdot}(z)(z\Psi(z)+\Gamma_{-1}C_0)|_{z=\frac{1}{\alpha}} = 0$

[20] Here $A(L)$ is taken to be a (2×2) identity matrix.

gives the following system

$$-\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} C_0(1,1) & C_0(1,2) \\ C_0(2,1) & C_0(2,2) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{\beta-\alpha}{\alpha^3\beta} & 0 \end{pmatrix}$$

which requires that $\frac{\beta-\alpha}{\alpha^3\beta} = 0$, or $\alpha = \beta$, a contradiction. Since the existence condition is violated in this case, there is no covariance stationary solution for (x_t, y_t).

Appendix C: User's Guide

All of the routines required to implement this solution algorithm are written and compiled in MATLAB, which take the advantages of MATLAB *Symbolic Toolbox* and are executed with the following files:[21]

- **model.m** file serves as a template for inputting all of the matrix coefficients of a generalized multivariate linear rational expectations model of the form given by (2.3.1). It then calls the function **tranz(Gamma,Psi,A,n,T)** in **tranz.m**;
- **tranz.m** file serves as the main script that performs the z-transform algorithm for a given multivariate linear rational expectations model and computes its solution by invoking related functions in MATLAB *Symbolic Toolbox*. It also examines the model's existence and uniqueness conditions;
- **multroot.m** file finds all the distinct roots of a given polynomial with their corresponding multiplicities;
- **U.txt** file defines a MAPLE procedure that computes the (left) unimodular matrix $U(z)$ in the Smith canonical decomposition of a given polynomial matrix.

As an example, we use the model in Section 2.4.2 to outline how to implement the solution algorithm. There are a number of model-specific initializations that are specified by the user and break down into several easily implementable steps:

[21] This program is available upon request.

- Step 1 – define the symbolic variable z and the numerical values of the model's parameters. MATLAB code:

```
syms z             % symbolic z
beta = 0.9804;     % discount factor
alpha = 1.5;       % active monetary
gamma = 1.2;       % passive fiscal
```

- Step 2 – specify the indices for both endogenous and exogenous variables. MATLAB code:

```
npi = 1;           % inflation
nb = 2;            % real debt
ntheta = 1;        % monetary shock
npsi = 2;          % fiscal shock
```

- Step 3 – define the matrix coefficients and relevant parameters. MATLAB code:

```
p  = 2;                       % system dimension
n = 1;                        % number of leads
m = 1;                        % number of endo lags
l = 1;                        % number of exo lags
Gamma = zeros(p,p,n+m+1);     % endo matrix polynomial
Psi = zeros(p,p,n+l+1);       % exo matrix polynomial
A = [1 0;0 1];                % driving matrix polynomial
```

- Step 4 – enter the equilibrium equations one by one. MATLAB code:

```
% (1) Fisher equation
Gamma(1,npi,1) = 1;
Gamma(1,npi,2) = -alpha;
Psi(1,ntheta,2) = 1;

% (2) Government budget constraint
Gamma(2,npi,2) = 1/beta;
Gamma(2,nb,2) = 1;
Gamma(2,npi,3) = -alpha/beta;
Gamma(2,nb,3) = -(1/beta-gamma*(1/beta-1));
Psi(2,npsi,2) = -(1/beta-1);
Psi(2,ntheta,3) = 1/beta;
```

- Step 5 – construct the matrix polynomials and solve the model by calling the function **tranz(Gamma,Psi,A,n,T)** in **tranz.m**. The program returns two elements, i.e. **eu** (existence and uniqueness) and **sol** (first T moving average matrix coefficients of the solution). MATLAB code:

```
% construct matrix polynomials
Gamma = Gamma(:,:,1)/z+Gamma(:,:,2)+Gamma(:,:,3)*z;
Psi = Psi(:,:,1)/z+Psi(:,:,2)+Psi(:,:,3)*z;
% solve model
[eu,sol] = tranz(Gamma,Psi,A,n,T);
```

Chapter 3

An Analytical Approach to New Keynesian Models under the Fiscal Theory

3.1 Introduction

This article builds on the seminal work, most notably of Hansen and Sargent (1980) and Whiteman (1983), in providing analytical approaches for integrating dynamic economic theory with econometric methods for the purpose of formulating and interpreting economic time series. The paper is illustrative; we walk the reader through the frequency-domain methodology of our companion paper, Tan and Walker (2014), to solving linear rational expectations models, who generalizes its predecessors to the multivariate setting. This method is of wide applicability and facilitates the spectral estimation of these models.

Our approach is applied to a conventional new Keynesian model of the kind presented in Woodford (2003) and Galí (2008). This has the advantage of keeping the illustration simple and concrete, but it should be emphasized that the techniques we describe are widely applicable in more general settings, e.g. models with a maturity structure, which we leave for future research. We derive an *analytical* solution to a linearized version of the model under the assumption that primary surpluses evolve independently of government liabilities, a regime in which the fiscal theory of the price level is valid [Woodford (1998), Cochrane (2001), Kim (2003), Sims (2011)]. This solution is useful in characterizing the cross-equation restrictions and understanding the policy transmission mechanisms implied by the fiscal theory. An equivalent derivation using time-domain method and a more extensive study of the fiscal theory can be found in Leeper and Leith (2015).

3.2 A Conventional New Keynesian Model

The model's essential elements include: a representative household and a continuum of firms, each producing a differentiated good for which it sets the price; Calvo (1983) sticky prices in which only a fraction of firms can reset their prices each period; a cashless economy with one-period nominal bonds that pay a gross interest rate of R_t; lump-sum taxation and zero government spending so that consumption equals output, $c_t = y_t$; monetary policy follows an interest rate rule whereas fiscal policy sets primary surplus *exogenously*.

3.2.1 Linearized Model

Let $\hat{x} \equiv \ln(x_t) - \ln(x^*)$ denote the log-deviation of a variable x_t from its steady state x^*. First, the household's optimizing behavior, when imposed by the goods market clearing condition, implies

$$\hat{y}_t = \mathbb{E}_t \hat{y}_{t+1} - \sigma(\hat{R}_t - \mathbb{E}_t \hat{\pi}_{t+1}) \tag{3.2.1}$$

where $\sigma > 0$ is the intertemporal elasticity of substitution and $\hat{\pi}_t$ is the inflation between $t-1$ and t. (3.2.1) represents a "Fisher relation", linking the short-term nominal interest rate, expected inflation, and endogenous output that determines the equilibrium real interest rate. It is also referred to as the dynamic IS equation. The firm's optimal price-setting behavior reduces to

$$\hat{\pi}_t = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \kappa \hat{y}_t \tag{3.2.2}$$

where $0 < \beta < 1$ is the discount factor. (3.2.2) is referred to as the new Keynesian Phillips curve whose slope is given by $\kappa > 0$. (3.2.1) and (3.2.2) constitute the non-policy block of the basic new Keynesian model.

Next, the monetary authority follows an interest rate feedback rule that reacts to deviations of inflation from its steady state

$$\hat{R}_t = \alpha \hat{\pi}_t + \theta_t \tag{3.2.3}$$

where θ_t is an exogenous monetary policy shock.[1] The fiscal authority sets an *exogenous* primary surplus process, s_t, that evolves independently of government liabilities. This profligate fiscal policy requires that monetary policy adjust nominal interest rate only weakly to inflation so that $0 \leq \alpha < 1$. We assume that the joint $(\theta_t, \hat{s}_t)$ process is white noise, normally distributed with mean zero and diagonal covariance matrix Σ.

Lastly, policy choices must satisfy the flow government budget constraint

$$\hat{b}_t = \hat{R}_t + \beta^{-1}(\hat{b}_{t-1} - \hat{\pi}_t) - (\beta^{-1} - 1)\hat{s}_t \tag{3.2.4}$$

where $\hat{b}_t$ is the real value of government debt. (3.2.1)—(3.2.4) constitute a system of four expectational difference equations in the variables $\{\hat{y}_t, \hat{\pi}_t, \hat{R}_t, \hat{b}_t\}$, which fully characterizes the equilibrium conditions under the fiscal theory.

3.2.2 Analytical Solution

Substituting the monetary policy rule (3.2.3) into (3.2.1) and (3.2.4) and rewriting the resulting multivariate linear rational expectations model into the framework of Tan and

[1] For analytical clarity, we assume that monetary authority does not respond to output.

Walker (2014) yield

$$\left[\underbrace{\begin{pmatrix}1 & \sigma & 0\\ 0 & \beta & 0\\ 0 & 0 & 0\end{pmatrix}}_{\Gamma_{-1}} L^{-1} + \underbrace{\begin{pmatrix}-1 & -\alpha\sigma & 0\\ \kappa & -1 & 0\\ 0 & \beta^{-1}-\alpha & 1\end{pmatrix}}_{\Gamma_0} L^0 + \underbrace{\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -\beta^{-1}\end{pmatrix}}_{\Gamma_1} L\right] \underbrace{\begin{pmatrix}\hat{y}_t\\ \hat{\pi}_t\\ \hat{b}_t\end{pmatrix}}_{z_t}$$
$$= \underbrace{\begin{pmatrix}\sigma & 0\\ 0 & 0\\ 1 & 1-\beta^{-1}\end{pmatrix}}_{\Psi_0} L^0 \underbrace{\begin{pmatrix}\theta_t\\ \hat{s}_t\end{pmatrix}}_{\varepsilon_t} + \underbrace{\begin{pmatrix}1 & \sigma & 0\\ 0 & \beta & 0\\ 0 & 0 & 0\end{pmatrix}}_{\Gamma_{-1}} \underbrace{\begin{pmatrix}\eta^y_{t+1}\\ \eta^\pi_{t+1}\\ \eta^b_{t+1}\end{pmatrix}}_{\eta_{t+1}} \tag{3.2.5}$$

where L is the lag operator, i.e. $L^k x_t = x_{t-k}$, $\{\Gamma_{-1}, \Gamma_0, \Gamma_1, \Psi_0\}$ are matrix coefficients, and η^x_{t+1} is an endogenous forecasting error defined as $\eta^x_{t+1} = x_{t+1} - \mathbb{E}_t x_{t+1}$. Because $\mathbb{E}_t$ represents the conditional expectation given information available at time t that includes the model's structure and all past and current realizations of the exogenous and endogenous processes, we have $\mathbb{E}_t \eta^x_{t+1} = 0$.

Suppose a solution $z_t = [\hat{y}_t, \hat{\pi}_t, \hat{b}_t]'$ to (3.2.5) is of the form

$$z_t = \sum_{k=0}^{\infty} C_k \varepsilon_{t-k} \equiv C(L)\varepsilon_t \tag{3.2.6}$$

where $\varepsilon_t = [\theta_t, \hat{s}_t]'$ and z_t is taken to be covariance stationary. Note that such moving average representation of the solution is very useful because it is the impulse response function—the coefficient $C_k(i,j)$ measures exactly the response of $z_{t+k}(i)$ to a shock $\varepsilon_t(j)$. From (3.2.6), we may also easily obtain the spectrum of z_t

$$s(w|\alpha,\beta,\sigma,\kappa) = C(e^{-iw})\Sigma C(e^{iw})', \qquad w \in [0, 2\pi) \tag{3.2.7}$$

as a function of all the model parameters $[\alpha, \beta, \sigma, \kappa]'$, which can be estimated by the standard methods of spectral estimation. In what follows, we walk the reader through the solution algorithm of Tan and Walker (2014) in details.

First, evaluate the forecasting errors $\eta_{t+1} = [\eta^y_{t+1}, \eta^\pi_{t+1}, \eta^b_{t+1}]'$ using (3.2.6) and the Wiener-Kolmogorov optimal prediction formula

$$\eta_{t+1} = \left\{ C(L)L^{-1} - \left[\frac{C(L)}{L} \right]_+ \right\} \varepsilon_t = C_0 L^{-1} \varepsilon_t \tag{3.2.8}$$

where $[\cdot]_+$ is the annihilation operator that instructs us to ignore negative powers of L. Define $\Gamma(L) = \Gamma_{-1}L^{-1} + \Gamma_0 + \Gamma_1 L$ and substitute (3.2.6) and (3.2.8) into (3.2.5)

$$\Gamma(L)C(L)\varepsilon_t = (\Psi_0 + \Gamma_{-1}C_0 L^{-1})\varepsilon_t$$

which must hold for all realizations of ε_t. Therefore, the coefficient matrices are related by the z-transform identities

$$z\Gamma(z)C(z) = z\Psi_0 + \Gamma_{-1}C_0$$

Next, apply the Smith canonical factorization to the polynomial matrix $z\Gamma(z)$[2]

$$z\Gamma(z) = U(z)^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z(z-\beta)(z-\lambda_-)(z-\lambda_+) \end{pmatrix} V(z)^{-1}$$

where $U(z)$ and $V(z)$ are unimodular matrices and

$$\lambda_\pm = \frac{1 + \beta + \sigma\kappa \pm \sqrt{(1 + \beta + \sigma\kappa)^2 - 4\beta(1 + \alpha\sigma\kappa)}}{2(1 + \alpha\sigma\kappa)} \tag{3.2.9}$$

[2]The Smith decomposition is available in MAPLE or MATLAB's Symbolic Toolbox.

It is straightforward to show that $\frac{\partial\lambda_+}{\partial\alpha} < 0$ and $\frac{\partial\lambda_-}{\partial\alpha} > 0$.[3] Moreover, given the parameter restrictions underlying the fiscal theory, both roots are real, one inside the unit circle, $|\lambda_-| < 1$, and the other outside, $|\lambda_+| > 1$:

$$0 < \lambda_- < \frac{\beta}{1+\alpha\sigma\kappa} < \beta < 1, \qquad \lambda_+ > \frac{1+\sigma\kappa}{1+\alpha\sigma\kappa} > 1 \tag{3.2.10}$$

Separate the roots inside the unit circle, 0, β, and λ_-, from the one outside, λ_+, by decomposing $z\Gamma(z)$ as

$$z\Gamma(z) = \underbrace{U(z)^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & z(z-\beta)(z-\lambda_-)\end{pmatrix}}_{S(z)}\underbrace{\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & z-\lambda_+\end{pmatrix}V(z)^{-1}}_{T(z)}$$

Now the z-transform identities become

$$T(z)C(z) = \begin{pmatrix} U_{1\cdot}(z)\\ U_{2\cdot}(z)\\ \frac{1}{z(z-\beta)(z-\lambda_-)}U_{3\cdot}(z)\end{pmatrix}(z\Psi_0 + \Gamma_{-1}C_0)$$

where $U_{j\cdot}(z)$ is the jth row of $U(z)$. These identities are valid for all z on the open unit disk except for $z = 0, \beta, \lambda_-$. But since $C(z)$ is the z-transform of the moving average coefficients for z_t, it must exist for all $|z| < 1$. This condition places restrictions on the unknown matrix

[3] This is because

$$\frac{1}{\lambda_\pm} = \frac{1+\beta+\sigma\kappa \mp \sqrt{(1+\beta+\sigma\kappa)^2 - 4\beta(1+\alpha\sigma\kappa)}}{2\beta}$$

are increasing and decreasing in α, respectively. Since $0 \leq \alpha < 1$, we can also obtain the lower bound for the square root of the discriminant

$$\sqrt{(1+\beta+\sigma\kappa)^2 - 4\beta(1+\alpha\sigma\kappa)} > \sqrt{(1+\beta+\sigma\kappa)^2 - 4\beta(1+\sigma\kappa)} = 1 - \beta + \sigma\kappa > 0$$

coefficient C_0:

$$U_{3\cdot}(z)(z\Psi_0 + \Gamma_{-1}C_0)|_{z=0,\beta,\lambda_-} = 0 \tag{3.2.11}$$

Stacking the restrictions in (3.2.11) yields[4]

$$-\underbrace{\begin{pmatrix} \frac{\beta^2\kappa(\alpha\beta-1)}{1+\alpha\sigma\kappa} & \frac{\beta^2(\alpha\beta-1)(\beta-1+\sigma\kappa)}{1+\alpha\sigma\kappa} & 0 \\ \frac{\lambda_-^2\kappa(\alpha\beta-1)}{1+\alpha\sigma\kappa} & \frac{\lambda_-^2(\alpha\beta-1)(\sigma\kappa+\beta)}{1+\alpha\sigma\kappa} - \frac{\lambda_-\beta(\alpha\beta-1)}{1+\alpha\sigma\kappa} & 0 \end{pmatrix}}_{R} C_0 = \underbrace{\begin{pmatrix} 0 & \frac{\beta^2\sigma\kappa(1-\beta)(\alpha\beta-1)}{1+\alpha\sigma\kappa} \\ \frac{\sigma\kappa\lambda_-^3(\alpha\beta-1)}{1+\alpha\sigma\kappa} & 0 \end{pmatrix}}_{A}$$

Lastly, we establish the existence and uniqueness of solutions to this model.[5] Existence cannot be established if at least one column of A is outside the space spanned by the columns of R—the endogenous forecasting errors η cannot fully adjust to offset the exogenous shocks ε. Thus, the solution exists if and only if the column space of R spans the column space of A, i.e. $\text{span}(A) \subseteq \text{span}(R)$, which is satisfied here. Solving for C_0 gives

$$\begin{pmatrix} C_0(1,1) & C_0(1,2) \\ C_0(2,1) & C_0(2,2) \end{pmatrix} = \begin{pmatrix} \frac{\sigma\lambda_-^2(\beta-1+\sigma\kappa)}{\lambda_- - \beta} & -\frac{(1-\beta)\sigma[(\sigma\kappa+\beta)\lambda_- - \beta]}{\lambda_- - \beta} \\ -\frac{\sigma\kappa\lambda_-^2}{\lambda_- - \beta} & \frac{\sigma\kappa\lambda_-(1-\beta)}{\lambda_- - \beta} \end{pmatrix}$$

where $C_0(3,1)$ and $C_0(3,2)$ are left undetermined. In order for the solution to be unique, we must be able to determine $\{C_k\}_{k=0}^{\infty}$ from the parameter restrictions supplied by $-RC_0 = A$. This is tantamount to verifying whether the columns of R' span the space spanned by the rows of

$$Q = U_{3\cdot}(\lambda_+^{-1})\Gamma_{-1} = \begin{pmatrix} \frac{\kappa(\alpha\beta-1)}{\lambda_+^2(1+\alpha\sigma\kappa)} & \frac{(\alpha\beta-1)[\sigma\kappa+\beta(1-\lambda_+)]}{\lambda_+^2(1+\alpha\sigma\kappa)} & 0 \end{pmatrix}$$

[4] Here we omit the restriction imposed by $z = 0$ because it is unrestrictive. One key result proved in Tan and Walker (2014) is that, roots inside (outside) the unit circle in the frequency-domain are exactly the inverses of unstable (stable) eigenvalues in the time-domain approach to solving linear rational expectations models, e.g. Sims (2002).

[5] As pointed out by Sims (2007), the standard root-counting analysis of Blanchard and Kahn (1980) would fail if the unstable root occurs in a part of the system that is decoupled from the expectational equations.

i.e. $\mathrm{span}(Q') \subseteq \mathrm{span}(R')$, which is also satisfied here.[6] Therefore, any candidate of C_0 that satisfies the existence condition will lead to the same solution and hence the solution is unique, which can be computed as

$$\begin{pmatrix} \hat{y}_t \\ \hat{\pi}_t \\ \hat{b}_t \end{pmatrix} = \underbrace{[L\Gamma(L)]^{-1}(L\Psi_0 + \Gamma_{-1}C_0)}_{C(L)} \begin{pmatrix} \theta_t \\ \hat{s}_t \end{pmatrix}$$

$$= \begin{pmatrix} C_0(1,1)\frac{1-\frac{\beta-\lambda_-}{\beta\lambda_-(\beta-1+\sigma\kappa)}L}{1-\frac{1}{\lambda_+}L} & C_0(1,2)\frac{1}{1-\frac{1}{\lambda_+}L} \\ C_0(2,1)\frac{1-\frac{\lambda_--\beta}{\beta\lambda_-}L}{1-\frac{1}{\lambda_+}L} & C_0(2,2)\frac{1}{1-\frac{1}{\lambda_+}L} \\ C_0(3,1)\frac{1}{1-\frac{1}{\lambda_+}L} & C_0(3,2)\frac{1}{1-\frac{1}{\lambda_+}L} \end{pmatrix} \begin{pmatrix} \theta_t \\ \hat{s}_t \end{pmatrix} \tag{3.2.12}$$

where

$$C_0(3,1) = \frac{\beta+\sigma\kappa}{(1+\alpha\sigma\kappa)\lambda_+}, \qquad C_0(3,2) = \frac{\beta-1}{\lambda_+}$$

Evidently, $[\hat{y}_t, \hat{\pi}_t, \hat{b}_t]'$ follows a vector autoregressive moving average process of order $(1,1)$.

It is worth emphasizing that (3.2.12) clearly captures the cross-equation restrictions imposed by the hypothesis of rational expectations, which are the "hallmark of rational expectations models" [Hansen and Sargent (1980)]. For example, (3.2.12) implies that none of the entries in $C(L)$ is identically zero. But in the companion regime not examined in this paper, in which monetary authority adjusts nominal rate more than one-for-one to inflation and fiscal authority systematically raises primary surpluses to pay off government liabilities, the moving average representation of the joint $(\hat{\pi}_t, \hat{b}_t)$ process (and the $(\hat{y}_t, \hat{b}_t)$ process) will display a lower-triangular structure, indicating that $\hat{b}_t$ fails to Granger-cause $\hat{\pi}_t$ [Sims (1972), Theorem 1].[7] Therefore, an empirically plausible way to test for the underlying regime is

[6]Practically, the two space spanning conditions for existence and uniqueness and the computation of C_0 can be obtained by employing the singular value decompositions of $\{A, R, Q\}$.

[7]This result relies on the assumption of lump-sum taxation. Also, to apply Sims' theorem when $C(L)$ is not invertible, we may transform $C(L)$ into a fundamental representation using Blaschke

to regress $\hat{b}_t$ onto the entire $\hat{\pi}_t$ process, and test whether the coefficients of *future* $\hat{\pi}$'s are identically zero [Sims (1972), Theorem 2].

From the moving average representation of the solution (3.2.12), we can easily write output, inflation, and real debt as linear functions of all past and present policy shocks with unambiguously signed coefficients. In particular, output follows[8]

$$\begin{aligned}\hat{y}_t = \underbrace{C_0(1,1)}_{<0}\theta_t + \sum_{k=1}^{\infty}\underbrace{C_0(1,1)\left[\frac{1}{\lambda_+} - \frac{\beta-\lambda_-}{\beta\lambda_-(\beta-1+\sigma\kappa)}\right]\left(\frac{1}{\lambda_+}\right)^{k-1}}_{>0}\theta_{t-k} \\ + \underbrace{C_0(1,2)}_{<0}\hat{s}_t + \sum_{k=1}^{\infty}\underbrace{C_0(1,2)\left(\frac{1}{\lambda_+}\right)^{k}}_{<0}\hat{s}_{t-k} \qquad (3.2.13)\end{aligned}$$

inflation follows

$$\begin{aligned}\hat{\pi}_t = \underbrace{C_0(2,1)}_{>0}\theta_t + \sum_{k=1}^{\infty}\underbrace{C_0(2,1)\left[\frac{1}{\lambda_+} - \frac{\lambda_- -\beta}{\beta\lambda_-}\right]\left(\frac{1}{\lambda_+}\right)^{k-1}}_{>0}\theta_{t-k} \\ + \underbrace{C_0(2,2)}_{<0}\hat{s}_t + \sum_{k=1}^{\infty}\underbrace{C_0(2,2)\left(\frac{1}{\lambda_+}\right)^{k}}_{<0}\hat{s}_{t-k} \qquad (3.2.14)\end{aligned}$$

and real debt follows

$$\hat{b}_t = \underbrace{C_0(3,1)}_{>0}\theta_t + \sum_{k=1}^{\infty}\underbrace{C_0(3,1)\left(\frac{1}{\lambda_+}\right)^{k}}_{>0}\theta_{t-k} + \underbrace{C_0(3,2)}_{<0}\hat{s}_t + \sum_{k=1}^{\infty}\underbrace{C_0(3,2)\left(\frac{1}{\lambda_+}\right)^{k}}_{<0}\hat{s}_{t-k} \qquad (3.2.15)$$

where we have separated shocks in the current period from those in the past.

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factors. See Hansen and Sargent (1980) for a similar analysis of a labor demand model.

[8] $C_0(1,1) < 0$ holds by (3.2.10) and the fact that $\beta - 1 + \sigma\kappa > 0$ for most plausible values of $\{\beta, \sigma, \kappa\}$. $\frac{1}{\lambda_+} - \frac{\beta-\lambda_-}{\beta\lambda_-(\beta-1+\sigma\kappa)} < 0$ holds by (3.2.10) and the property that $\lambda_+\lambda_- = \frac{\beta}{1+\alpha\sigma\kappa}$. $C_0(1,2) < 0$ holds by (3.2.10) and the fact that $\frac{\partial\lambda_-}{\partial\alpha} > 0$. A similar argument can be used to sign the coefficients in $\hat{\pi}$ and $\hat{b}$.

3.2.3 Economic Interpretations

The closed-form solution (3.2.12), or (3.2.13)—(3.2.15), is useful in understanding how monetary and fiscal disturbances are transmitted to influence the endogenous variables under the fiscal theory. The economic interpretations hinge on a ubiquitous relation in any dynamic macro model, that the real value of government liabilities derives its value from the present value of current and expected future primary surpluses.[9] See also Kim (2003) for a more detailed, numerical analysis.

First, we examine the effects of a monetary contraction (an increase in the nominal interest rate). Since primary surpluses evolve exogenously and do not systematically respond to government liabilities, a higher nominal rate leads to more rapidly growing government debt services without changing the present value of current and expected future primary surpluses. This raises the real debt in (3.2.15) and makes government bonds more attractive, inducing households to convert consumption goods into government bonds in the current period. Thus, output falls initially in (3.2.13). However, because the real value of government liabilities is backed up less than sufficiently by the present value of primary surpluses in the next period, households will convert government bonds back into consumption goods. From (3.2.13) and (3.2.14), this aggregate demand increase pushes up both output and inflation in the next period. Inflation must also rise in the current period because the firm's price-setting behavior is forward looking. Therefore, a monetary tightening under the fiscal theory will lose its usual contractionary effects—(3.2.14) makes it clear that a more aggressive monetary policy stance (larger α) is not only inflationary but makes the effects of these shocks more persistent as well.[10]

Next, we examine the impacts of a fiscal contraction (an increase in the primary surplus).

[9] This relation can be obtained by combining the government budget constraint, a fiscal policy rule, and the Fisher equation.

[10] Observe that the decay factor, $1/\lambda_+$, is an increasing function of α.

Higher primary surplus allows the government to retire some of the outstanding liabilities, thereby reducing the real debt in (3.2.15). Moreover, since households have no anticipation of lower future taxation under an exogenous fiscal policy, the real value of government liabilities is backed up more than sufficiently by the present value of primary surpluses. As a result, households feel less wealthy and hence substitute consumption goods into government bonds. From (3.2.13) and (3.2.14), this aggregate demand decrease pushes down both output and inflation.

Lastly, we analyze the effects of policy shocks on the real interest rate. Let $u(c) = \frac{c^{1-1/\sigma}}{1-1/\sigma}$ denote the representative household's utility over consumption and define the *ex ante* real interest rate r_t such that

$$\frac{1}{r_t} = \mathbb{E}_t \left[\beta \frac{u'(y_{t+1})}{u'(y_t)} \right] = \mathbb{E}_t \left[\beta \left(\frac{y_{t+1}}{y_t} \right)^{-1/\sigma} \right]$$

Log-linearizing the above relation around the steady state and combining with (3.2.1) yield

$$\begin{aligned}
\hat{r}_t &= \hat{R}_t - \mathbb{E}_t \hat{\pi}_{t+1} \\
&= \left[1 + C_0(2,1) \left(\alpha - \frac{1}{\lambda_+} + \frac{\lambda_- - \beta}{\beta \lambda_-} \right) \right] \theta_t \\
&\quad + \sum_{k=1}^{\infty} C_0(2,1) \left(\frac{1}{\lambda_+} - \frac{\lambda_- - \beta}{\beta \lambda_-} \right) \left(\alpha - \frac{1}{\lambda_+} \right) \left(\frac{1}{\lambda_+} \right)^{k-1} \theta_{t-k} \\
&\quad + C_0(2,2) \left(\alpha - \frac{1}{\lambda_+} \right) \hat{s}_t + \sum_{k=1}^{\infty} C_0(2,2) \left(\alpha - \frac{1}{\lambda_+} \right) \left(\frac{1}{\lambda_+} \right)^{k} \hat{s}_{t-k}
\end{aligned} \tag{3.2.16}$$

where we have substituted (1.3.17) and (3.2.14) into (3.2.16). Thus, it remains to determine the sign of $\alpha - 1/\lambda_+$. Note that $1/\lambda_+$ is a strictly increasing, continuous function in $\alpha \in [0,1]$ and its two endpoint values stay within the open unit interval. This implies that the graph of $1/\lambda_+$ as a function of α intersects the 45-degree line of the unit square for an odd number of times. Therefore, there exists at least one fixed point at which $1/\lambda_+ = \alpha$ and the sign of $\alpha - 1/\lambda_+$ remains ambiguous, depending on the monetary policy behavior and all the model parameters.[11]

For example, (3.2.16) suggests that higher primary deficits need not necessarily lower the real rate, which is in contrast to what is commonly believed under the fiscal theory. This ambiguity of the effects of policy disturbances on real rate is not surprising. (3.2.13) makes it clear that higher primary deficits are expansionary, raising both current and expected future consumption. Furthermore, the consumption-Euler equation (3.2.1) suggests that the real rate is the price of current consumption *relative* to expected future consumption. Therefore, the change in real rate can be inferred from the relative increase in consumption from the current period to the next, which itself remains ambiguous.

3.3 Concluding Remarks

This article illustrates the frequency-domain solution methodology of Tan and Walker (2014) in the context of a conventional new Keynesian model under the fiscal theory of the price level. The analytical solution derived herein is useful in characterizing the cross-equation restrictions and understanding the policy transmission mechanisms implied by the fiscal theory. We conclude by pointing out that our approach can be easily extended to allow for robustness as advocated by Hansen and Sargent (2007) or rational inattention as advocated by Sims (2001), and defer these extensions to a sequel to this paper.

[11] For example, the sign of $\alpha - 1/\lambda_+$ can depend on the frequency of the data to which a model is calibrated or fit; κ depends on the degree of price stickiness which in turn depends on the data frequency, being much higher for higher frequency. Eric Leeper made this point to me.

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Neil Wallace. A modigliani-miller theorem for open-market operations. *American Economic Review*, 71(3):267–274, 1981.

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FEI TAN

Curriculum Vitae

Contact Information	Department of Economics	Coordinator Phone: +1(812)855-8453
	Wylie Hall 105	Cell Phone: +1(812)327-4084
	100 South Woodlawn	E-mail: tanf@indiana.edu
	Bloomington, IN 47405	Web: http://mypage.iu.edu/~tanf

EDUCATION

INSTITUTION	DEGREE	DATE
Indiana University	**Ph.D. Economics**	*Jun., 2015*
Doctoral Thesis Title:	Essays in Macroeconomics: Econometric Modeling and Evaluation of Fiscal-Monetary Policy Interactions	
Dissertation Committee:	Eric Leeper, Todd Walker, Joon Park, Juan Carlos Escanciano	
Indiana University	**M.A. Mathematics**	*Jan., 2015*
Indiana University	**M.A. Economics**	*Jun., 2011*
Zhejiang University	**B.A. Economics**	*Jun., 2009*

FIELDS OF SPECIALIZATION

PRIMARY: Macroeconomics, Monetary Economics, Macroeconometrics

SECONDARY: Financial Econometrics, Public Goods Game

RESEARCH

Refereed Publications

– (With Hang Ye, Mei Ding, Yongmin Jia and Yefeng Chen) Sympathy and Punishment: Evolution of Cooperation in Public Goods Game, *Journal of Artificial Societies and Social Simulation 14*(4), October 2011.

Working and Discussion Papers

– An Analytical Approach to New Keynesian Models under the Fiscal Theory, manuscript, April 2015.

– Two Econometric Interpretations of the U.S. Fiscal and Monetary Policy Interactions, Job Market Paper, manuscript, October 2014.

– (With Todd B. Walker) Solving Generalized Multivariate Linear Rational Expectations Models, Revision Requested by *Journal of Economic Dynamics and Control*, March 2014.

– Interpreting Rational Expectations Econometrics via Analytic Function Approach, manuscript, July 2014.

– (With Eric M. Leeper and Todd B. Walker) The Observational Equivalence of Monetary and Fiscal Policy Interactions, manuscript, December 2012.

– (With Hang Ye, Hong Zhang, Yefeng Chen, and Yongmin Jia) Punishing Just in Time: Public Cooperation and Economies of Scale, submitted, September 2012.

– Stability and Predictability with Linear Projections: A Tentative Evaluation of Policy Intervention, manuscript, April 2011.

Work in Progress

– (With Guang Zuo) Bayesian Analysis of High Frequency Regressions with Estimates of U.S. Interbank Uncertainty, in progress.

PROFESSIONAL EXPERIENCE

Presentations

14^{th} Annual Missouri Economics Conference, University of Missouri, *Mar. 21-22, 2014*

23^{rd} Annual Meeting of Midwest Econometrics Group, Indiana Univ., *Oct. 25-26, 2013*

Title: “Two Econometric Interpretations of U.S. Policy Interactions”

Fall 2012 Midwest Macroeconomics Conference, University of Colorado, *Nov. 10-11, 2012*

7^{th} Annual Washington University Grad Student Conference, St. Lous, *Oct. 13, 2012*

12^{th} Annual Missouri Economics Conference, University of Missouri, *Oct. 5-6, 2012*

8^{th} Annual Jordan River Economics Conference, Indiana University, *Apr. 27, 2012*

Title: "Solving Generalized Multivariate Linear RE Models"

11^{th} Annual Missouri Economics Conference, Washington University, *Oct. 14, 2011*

Title: "Stability and Predictability with Linear Projections"

Macro/Econometrics Brown Bag, Economics Department, Indiana Univ, *Apr., 2011–*

Various presentations of on-going research

Teaching Experience

Teaching Assistant: with partial teaching responsibility, Indiana Univ.

Macroeconomic Theory II, Professor Todd Walker, *Spring 2014*

Macroeconomic Theory I, Professor Amanda Michaud, *Fall 2013*

Financial Econometrics, Professor Joon Park, *Spring 13, 14, 15*

Optimization Theory in Econ, Professor Michael Kaganovich, *Fall 2012*

Associate Instructor, Summer Math Camp, Indiana Univ.

Preparatory math course for incoming Ph.D. students, *Summer 2012*

Associate Instructor: with full teaching responsibility, Indianan Univ.

Money and Banking, *Fall 14, Spring 15*

Intermediate Macroeconomic Theory, *Summer 2013, 2014*

Introduction to Microeconomics, *Spring 2012, 2013*

Introduction to Macroeconomics, *Fall 2011*

Other Activities and Services

Research Assistant for Professor Todd B. Walker, Indiana Univ., *Summer 2011*

Research Assistant for Professor Gerhard Glomm, Indiana Univ., *Fall 10, Spring 11*

Graduate Assistant, Statistical Analysis for Business and Economics, *Fall 09, Spring 10*

Professor Mary E. Camp, Indiana Univ.

Research Assistant, Interdisciplinary Center for Social Science, Zhejiang Univ., *2007.7–*

REFEREE EXPERIENCE

European Economic Review, Economics Letters

AWARDS AND HONORS

Henry M. Oliver Award for Excellence in Economic Theory, Indiana Univ., *Apr. 2014*

2012-2013 College of Arts & Sciences Travel Award, Indiana Univ., *Dec. 2012*

Department Conference Travel Funding, Indiana Univ., *Fall 2012*

Best Third Year Paper Award, Department of Economics, Indiana Univ., *Sep. 2012*

Best Graduation Thesis, College of Economics, Zhejiang Univ., *Nov. 2009*

Second Prize of Academic Research Scholarship, Zhejiang Univ., *Sep. 2008*

First Prize of *Ninth Challenge Cup Thesis Oral Defense*, Zhejiang Univ., *Apr. 2008*

Chu Kochen Honors College Students, Zhejiang Univ., *Sep. 2005*

PERSONAL

Nationality: China [F1 Visa]

Language: Chinese (native), English (fluent)

Computer Skills: MATLAB, C, C++, R

REFERENCES

Eric M. Leeper (Co-Chair)
Rudy Professor of Economics
Department of Economics, Indiana Univ.
100 S. Woodlawn, Bloomington, IN 47405
Phone: +1(812)855-9157
E-mail: eleeper@indiana.edu

Todd B. Walker (Co-Chair)
Associate Professor of Economics
Department of Economics, Indiana Univ.
100 S. Woodlawn, Bloomington, IN 47405
Phone: +1(812)856-2892
E-mail: walkertb@indiana.edu

Joon Y. Park
Wisnewsky Professor of Human Studies
Department of Economics, Indiana Univ.
100 S. Woodlawn, Bloomington, IN 47405
Phone: +1(812)855-3736
E-mail: joon@indiana.edu

Juan Carlos Escanciano
Associate Professor of Economics
Department of Economics, Indiana Univ.
100 S. Woodlawn, Bloomington, IN 47405
Phone: +1(812)855-7925
E-mail: jescanci@indiana.edu

Michael Kaganovich (Teaching)
Professor of Economics
Department of Economics, Indiana Univ.
100 S. Woodlawn, Bloomington, IN 47405
Phone: +1(812)855-6967
E-mail: mkaganov@indiana.edu

Paul Graf (Teaching)
Senior Lecturer
Department of Economics, Indiana Univ.
100 S. Woodlawn, Bloomington, IN 47405
Phone: +1(812)855-2543
E-mail: paulgraf@indiana.edu