

## CONVERGENCE OF HILL'S METHOD FOR NONSELFADJOINT OPERATORS\*

MATHEW A. JOHNSON<sup>†</sup> AND KEVIN ZUMBRUN<sup>‡</sup>

**Abstract.** By the introduction of a generalized Evans function defined by an appropriate 2-modified Fredholm determinant, we give a simple proof of convergence in location and multiplicity of Hill's method for numerical approximation of spectra of periodic-coefficient ordinary differential operators. Our results apply to operators of nondegenerate type under the condition that the principal coefficient matrix be symmetric positive definite (automatically satisfied in the scalar case). Notably, this includes a large class of non-self-adjoint operators which previously had not been treated in a simple way. The case of general coefficients depends on an interesting operator-theoretic question regarding properties of Toeplitz matrices.

**Key words.** Hill's method, periodic-coefficient operators, Floquet–Bloch decomposition, Fredholm determinant, Evans function

**AMS subject classifications.** 46N20, 46N40

**DOI.** 10.1137/100809349

**1. Introduction.** The study of stability of spatially periodic traveling wave solutions to various classes of partial differential equations motivates the study of  $L^2(\mathbb{R}; \mathbb{C}^n)$  (essential) spectra of periodic-coefficient differential operators

$$(1.1) \quad L = (\partial_x)^m a_m(x) + \cdots + \partial_x a_1(x) + a_0(x)$$

on the line, where coefficients  $a_j \in \mathbb{C}^{n \times n}$  are periodic with period  $X$ . By the Floquet theory, it is equivalent to study the  $L^2([0, X]; \mathbb{C}^n)$  point spectra of the family of Bloch operators

$$L_\sigma = (\partial_x + i\sigma)^m a_m(x) + \cdots + (\partial_x + i\sigma) a_1(x) + a_0(x),$$

where  $X$  is the common period of the coefficients and  $\sigma \in [0, 2\pi)$  acts as a parameter. Indeed, using this decomposition we have<sup>1</sup>

$$\operatorname{spec}_{L^2(\mathbb{R})}(L) = \bigcup_{\sigma \in [0, 2\pi)} \operatorname{spec}_{L^2_{\text{per}}([0, X])}(L_\sigma);$$

see, for example, [G] for more details.

Due to the mathematical difficulties involved in analytically computing the  $L^2(\mathbb{R})$  spectrum of such an—in general, variable-coefficient and vector-valued—operator or, equivalently, computing the periodic spectra of the full family of associated Bloch

---

\*Received by the editors September 21, 2010; accepted for publication (in revised form) October 17, 2011; published electronically January 19, 2012.

<http://www.siam.org/journals/sinum/50-1/80934.html>

<sup>†</sup>Department of Mathematics, University of Kansas, Lawrence, KS 66045 (matjohn@math.ku.edu). This author's research was partially supported by a postdoctoral fellowship under NSF grant DMS-0902192.

<sup>‡</sup>Department of Mathematics, Indiana University, Bloomington, IN 47405 (kzumbrun@indiana.edu). This author's research was partially supported by NSF grants DMS-0300487 and DMS-0801745.

<sup>1</sup>Unless otherwise stated, throughout this paper all functions are assumed to be complex valued and we adopt the notation  $L^2(\mathbb{R}) = L^2(\mathbb{R}; \mathbb{C})$  and similarly for  $L^2_{\text{per}}([0, X])$ .

operators, the determination of spectrum of periodic-coefficient operators is typically carried out numerically. This may be accomplished in a number of ways: for example, shooting, discretization, or various spectral and Galerkin methods. See [JZN, Appendix B] for further discussion.

A particularly natural and direct approach is Hill's method [DK],<sup>2</sup> a spectral Galerkin method carried out in a periodic Fourier basis, which is exact in the constant-coefficient case. In this method, to approximate the spectra of  $L_\sigma$  for a fixed  $\sigma \in [0, 2\pi)$ , one considers the eigenvalue problem

$$(1.2) \quad L_\sigma v = \lambda v$$

by expressing the coefficients  $a_j$  of  $L_\sigma$  and the function  $v$  as Fourier series in  $L^2_{\text{per}}([0, X])$ , as an infinite-dimensional matrix equation in  $\ell^2$ . Truncating the Fourier modes to frequencies  $|k| \leq J$  for each  $J \in \mathbb{N}$ , one then obtains a sequence of finite-dimensional matrix eigenvalue problems whose eigenvalues approximate true eigenvalues of the operator  $L_\sigma$  on  $L^2_{\text{per}}([0, X])$ . See section 3.2 for further details.

This method is fast and easy to use and in practice appears to give excellent results under quite general circumstances [DK, BJNRZ1, BJNRZ3, BJNRZ4]. However, up to now, an equivalently simple and direct convergence theory had been provided only in certain commonly occurring but restricted cases [CuD]. By convergence, we mean roughly that not only is Hill's method accurate, meaning that the numerically computed eigenvalues are always close to the actual eigenvalues of the associated Bloch operator (the "no-spurious-modes condition" of [CuD]), but also that the method is complete in the sense that it faithfully produces all of  $\sigma(L_\sigma)$  for a fixed  $\sigma$ . See [CuD] for a more precise discussion of convergence from this point of view. Here, we make the simpler operational definition of convergence as the property that on any bounded domain  $B = \{\lambda : |\lambda| \leq R\}$  whose boundary contains no eigenvalue of  $L_\sigma$ , the set of approximate eigenvalues lying in  $B$  converges to the set of exact eigenvalues of  $L$  in both location and number; see Corollary 3.9.<sup>3</sup>

Specifically, up to now a simple, direct proof of the convergence of Hill's method had been established, to our knowledge, only for self-adjoint operators with principal coefficient  $a_m = I$  [CuD]. In particular, though the accuracy of Hill's method was shown in [CuD] under quite general assumptions, completeness of the method in the non-self-adjoint case, which arises naturally, for example, in the applications in [BJNRZ1, BJNRZ2], does not seem to have been fully addressed. As noted in [CuD], there does exist an abstract framework established by Vainikko [V] by which convergence can be shown for a much more general class of approximation problems; we give a brief description of this framework and its application to Hill's method in Appendix A, including the important consequence of convergence to all orders for  $C^\infty$  coefficient operators, established in [CuD] for extremal eigenvalues.

In this short paper, we give a brief and simple proof of the convergence of Hill's method applying to the general class of operators (1.1) such that  $a_m$  is symmetric positive definite. In the scalar case, this condition on the principal coefficient  $a_m$  amounts to the mild requirement that the operator be nondegenerate type. In the system case, it is a genuine restriction, and it is an interesting and apparently non-trivial question, related to certain properties of Toeplitz matrices, to what extent the condition can be relaxed. Notably, our analysis applies to the important case where the operator  $L_\sigma$  is non-self-adjoint.

<sup>2</sup>A convenient implementation may be found in the numerical package SpectrUW [CDKK].

<sup>3</sup>This includes and slightly strengthens the definition of [CuD].

The main ingredient of our proof is the introduction of a generalized periodic Evans function, of interest in its own right, consisting of a 2-modified Fredholm determinant  $D_\sigma$  of an associated Birman–Schwinger type operator, whose roots we show to agree in location and multiplicity with the eigenvalues of  $L_\sigma$ . For related analysis in the solitary wave case, see [GLZ]. Once these properties are established, the desired convergence follows immediately by the observation that the corresponding 2-modified characteristic polynomial of the  $J$ th Galerkin-truncation of  $(L_\sigma - \lambda)v = 0$  are a subclass of the approximants used to define the aforementioned 2-modified Fredholm determinant in the limit as  $J \rightarrow \infty$ , and furthermore that these approximants are a sequence of analytic functions converging locally uniformly to the generalized periodic Evans function.

A novel feature of the present analysis is that our argument yields convergence of the spectrum in both location and multiplicity, whereas the results of [CuD] concerned only location. A second novelty of our work is to make the connection to the Evans function, putting the work in a broader context. As observed in [BJNRZ1, BJNRZ2, BJNRZ3, BJNRZ4, BJZ2], the Evans function can serve as a useful complement to direct methods as exemplified by those described in Appendix A, in particular, giving a posteriori and/or winding number estimates guaranteeing stability or instability of the total spectrum of  $L$  that are in general difficult to obtain by direct methods. We note that the beyond-all-orders rate of convergence established in [CuD] for the eigenvalue of minimum modulus and in [V] for general eigenvalues are *qualitative results*, for which coefficients are not known: important theoretical justification of the general power of Hill’s method, but not giving guaranteed accuracy in any specific case. Moreover, the general Evans function approach appears useful in wider contexts, as, for example, in the “dual” study of finite difference approximations carried out in [BJZ1].

*Note.* Since submission of this paper, the conjecture of Remark 3.5 has been verified in [Z], showing that the generalized Evans function defined here as a spectral determinant in fact agrees with the standard “Jost-function type” periodic Evans function of Gardner, defined as a Wronskian of solutions of the eigenvalue ODE.

## 2. Hilbert–Schmidt operators and 2-modified Fredholm determinants.

We begin by recalling the basic properties of 2-modified Fredholm determinants, defined for Hilbert–Schmidt perturbations of the identity; see [GGK1, GGK2], [GGK3, Ch. XIII], [GK, section IV.2], [Si1], [Si2, Ch. 3], [GLZ, section 2] for more details.

For a given Hilbert space  $\mathcal{H}$ ,<sup>4</sup> the Hilbert–Schmidt class  $\mathcal{B}_2(\mathcal{H})$  is defined as the set of all bounded linear operators  $A$  on  $\mathcal{H}$  for which the norm

$$\|A\|_{\mathcal{B}_2(\mathcal{H})}^2 := \sum_{j,k} |\langle Ae_j, e_k \rangle|^2 = \operatorname{tr}_{\mathcal{H}}(A^*A)$$

is finite, where  $\{e_j\}$  is any orthonormal basis. Evidently,  $\|\cdot\|_{\mathcal{B}_2(\mathcal{H})}$  is independent of the basis chosen. Moreover, every operator in  $\mathcal{B}_2(\mathcal{H})$  is compact (Fredholm).

On a finite-dimensional space  $\mathcal{H}$ , we define the 2-modified Fredholm determinant as

$$(2.1) \quad \det_{2,\mathcal{H}}(I_{\mathcal{H}} - A) := \det_{\mathcal{H}}((I_{\mathcal{H}} - A)e^A) = \det_{\mathcal{H}}(I_{\mathcal{H}} - A) e^{\operatorname{tr}_{\mathcal{H}}(A)},$$

where  $\det_{\mathcal{H}}$  and  $\operatorname{tr}_{\mathcal{H}}$  denotes the usual determinant and trace, respectively. From this

---

<sup>4</sup>Throughout this paper, we will always assume that our Hilbert spaces are separable.

definition, we have the useful estimates

$$(2.2) \quad \left| \det_{2, \mathcal{H}}(I_{\mathcal{H}} - A) \right| \leq e^{C \|A\|_{\mathcal{B}_2(\mathcal{H})}^2}$$

and

$$(2.3) \quad \left| \det_{2, \mathcal{H}}(I_{\mathcal{H}} - A) - \det_{2, \mathcal{H}}(I_{\mathcal{H}} - B) \right| \leq \|A - B\|_{\mathcal{B}_2(\mathcal{H})} e^{C[\|A\|_{\mathcal{B}_2(\mathcal{H})} + \|B\|_{\mathcal{B}_2(\mathcal{H})} + 1]^2},$$

where  $C > 0$  is a constant independent of the dimension of  $\mathcal{H}$ .

To extend this notion of a determinant to an infinite-dimensional Hilbert space  $\mathcal{H}$ , we note that for any  $A \in \mathcal{B}_2(\mathcal{H})$  the estimate (2.3) allows us to define the 2-modified Fredholm determinant unambiguously as the limit

$$(2.4) \quad \det_{2, \mathcal{H}}(I_{\mathcal{H}} - A) := \lim_{J \rightarrow \infty} \det_{2, \mathcal{H}_J}(I_{\mathcal{H}_J} - A_J),$$

where  $\mathcal{H}_J$  is any increasing sequence of finite-dimensional subspaces filling up  $\mathcal{H}$ , and  $A_J$  denotes the Galerkin approximation  $P_{\mathcal{H}_J} A|_{\mathcal{H}_J}$ , where  $P_J : \mathcal{H} \rightarrow \mathcal{H}_J$  is the orthogonal projection onto  $\mathcal{H}_J$ . That is, thinking of the infinite-dimensional matrix representation of  $A$ , the 2-modified Fredholm determinant is defined as the limit of such determinants on finite,  $J$ -dimensional, minors as  $J \rightarrow \infty$ .

Alternatively, denoting the (countably many, since  $A$  is Fredholm) eigenvalues of  $A$  as  $\{\alpha_j\}_{j=1}^{\infty}$  and taking  $\mathcal{H}_J$  to be the (total) eigenspace associated with the eigenvalues  $\{\alpha_j\}_{j=1}^J$  we find that

$$(2.5) \quad \det_{2, \mathcal{H}}(I_{\mathcal{H}} - A) = \lim_{J \rightarrow \infty} \prod_{k=1}^J (1 - \alpha_k) e^{\alpha_k},$$

which, by  $\prod_k (1 - \alpha_k) e^{\alpha_k} \lesssim \prod_k (1 + \alpha_k^2) \sim e^{\sum_k \alpha_k^2} \leq e^{\|A\|_{\mathcal{B}_2(\mathcal{H})}^2}$ , is readily seen to converge for all  $A \in \mathcal{B}_2(\mathcal{H})$  by Weyl's inequality  $\sum |\alpha_j|^r \leq \sum |s_j|^r$  for  $r \geq 0$ , where  $s_j$  denote the eigenvalues of  $|A| := (A^* A)^{1/2}$  [Si1, W]. This shows how the renormalization of the standard determinant  $\det(I_{\mathcal{H}} - A) := \prod_j (1 - \alpha_j)$  by factor  $e^{\text{tr}_{\mathcal{H}}(A)}$  cancels the possibly divergent first-order terms in  $\prod_k (1 - \alpha_k) \sim e^{\sum_k \alpha_k}$ , allowing the treatment of operators  $A$  that are not in trace class  $\mathcal{B}_1 := \{A : \| |A|^{1/2} \|_{\mathcal{B}_2(\mathcal{H})} < +\infty\}$ .<sup>5</sup>

**PROPOSITION 2.1.** *For  $A \in \mathcal{B}_2(\mathcal{H})$ , the operator  $(I_{\mathcal{H}} - A)$  is invertible if and only if  $\det_{2, \mathcal{H}}(I_{\mathcal{H}} - A)$  is nonzero.*

*Proof.* By standard Fredholm theory, this is equivalent to the statement that 0 is an eigenvalue of  $(I_{\mathcal{H}} - A)$  if and only if  $\det_{2, \mathcal{H}}(I_{\mathcal{H}} - A) = 0$ . Note that since  $A$  is Fredholm, it possesses a countable number of isolated eigenvalues  $\{\alpha_j\}$  of finite multiplicity, except possibly at zero. Choosing  $J \in \mathbb{N}$  sufficiently large, then, we may factor the product formula (2.5) as

$$\det_{2, \mathcal{H}}(I_{\mathcal{H}} - K) = \left( \prod_{j=1}^J (1 - \alpha_j) e^{\alpha_j} \right) \left( \prod_{j=J+1}^{\infty} (1 - \alpha_j) e^{\alpha_j} \right),$$

<sup>5</sup>For  $A \in \mathcal{B}_1$ ,  $\text{tr}_{\mathcal{H}}(A) = \sum_j \alpha_j$  is absolutely convergent by Weyl's inequality with  $r = 1$ , and so the standard determinant  $\det_{\mathcal{H}}(I_{\mathcal{H}} - A) = \prod_j (1 - \alpha_j)$  converges. For  $A$  self-adjoint,  $\|A\|_{\mathcal{B}_1} := \| |A|^{1/2} \|_{\mathcal{B}_2(\mathcal{H})} = \sum_j |\alpha_j|$  and  $\|A\|_{\mathcal{B}_2(\mathcal{H})} = \sum_j |\alpha_j|^2$ .

where

$$\prod_{j=J+1}^{\infty} (1 - \alpha_j) e^{\alpha_j} \approx e^{\sum_{j=J+1}^{\infty} \alpha_j^2} \neq 0.$$

It follows then that  $\det_{2,\mathcal{H}}(I_{\mathcal{H}} - A)$  vanishes if and only if  $1 - \alpha_j = 0$  for some  $1 \leq j \leq J$  and hence, since  $J \in \mathbb{N}$  was arbitrary, if and only if 0 is an eigenvalue of  $(I_{\mathcal{H}} - A)$ .  $\square$

**3. Analysis of a simple case.** With the above preliminaries in hand, we now turn to our proof of convergence. As a first step in this analysis, we present a complete proof in the case of a second-order operator with identity principal part. In later sections, we will then describe the extension of this proof to more general cases, noting that most of the ideas can be found in this simpler context.

Consider a periodic-coefficient differential operator

$$L_{\sigma} = (\partial_x + i\sigma)^2 + (\partial_x + i\sigma)a_1(x) + a_0(x)$$

acting on vector-valued functions in  $L^2_{\text{per}}([0, X])$ ,  $\sigma \in [0, 2\pi)$  the Floquet parameter, and  $a_j \in L^2([0, X])$  matrix-valued and periodic on  $x \in [0, X]$ . We can rewrite this more generally as a family of operators in the simpler form

$$(3.1) \quad L_{\sigma} = \partial_x^2 + \partial_x A_1(\sigma, x) + A_0(\sigma, x),$$

where

$$A_1 = a_1 + 2i\sigma, \quad A_0 = a_0 - \sigma^2 + i\sigma a_1.$$

In order to analyze the (necessarily discrete) spectrum of the operator  $L_{\sigma}$ , we introduce a generalization of the periodic Evans function, a complex analytic function whose roots coincide in location and multiplicity with the eigenvalues of  $L_{\sigma}$  [G], expressed in terms of a 2-modified Fredholm determinant. To this end, notice that associated with the eigenvalue problem

$$(3.2) \quad (L_{\sigma} - \lambda)U = 0$$

is the equivalent problem

$$(3.3) \quad (I + K(\sigma, \lambda))U = 0,$$

where here  $I$  is the identity operator on  $L^2_{\text{per}}([0, X])$  and  $K = K_1 + K_0$  with

$$K_1 = \partial_x(\partial_x^2 - 1)^{-1}A_1, \quad K_0 = (\partial_x^2 - 1)^{-1}(A_0 + 1 - \lambda).$$

In particular, notice that  $\lambda$  is an eigenvalue of  $L_{\sigma}$  if and only if 0 is an eigenvalue of the operator  $(I + K(\sigma, \lambda))$ . Before we can define the appropriate generalization of the Evans function, we need the following fundamental lemma.

**LEMMA 3.1.** *For  $A_j \in L^2_{\text{per}}([0, X])$ , the operator  $K$  is Hilbert–Schmidt.*

*Proof.* Expressing  $K_m$  in matrix form  $\mathcal{K}_m$  with respect to the infinite-dimensional Fourier basis, we find that the corresponding matrix elements can be expressed as

$$[\mathcal{K}_1]_{j,k} = \frac{ij}{1+j^2} \hat{A}_1(j-k),$$

where  $\hat{A}_1(m)$  denotes the  $m$ th Fourier coefficient of  $A_1$ , and  $i := \sqrt{-1}$ . Computing explicitly, we find by Parseval's Theorem that<sup>6</sup>

$$\begin{aligned} \|K_1\|_{\mathcal{B}_2} &= \|\mathcal{K}_1\|_{\mathcal{B}_2} = \sum_j \frac{j^2}{(1+j^2)^2} \sum_k |\hat{A}_1(j-k)|^2 \\ &= \sum_j \frac{j^2}{(1+j^2)^2} \|A_1\|_{L^2_{\text{per}}([0,X])} < +\infty, \end{aligned}$$

hence  $K_1$  is a Hilbert–Schmidt operator. Similarly, we find that  $K_0$  is Hilbert–Schmidt with norm

$$\|K_0\|_{\mathcal{B}_2} = \sum_j \frac{1}{(1+j^2)^2} \sum_k \left| \hat{A}_0(j-k) + (1-\lambda)\delta_j^k \right|^2,$$

which implies that  $K = K_1 + K_0 \in \mathcal{B}_2$  as claimed.  $\square$

*Remark 3.2.* On the other hand,  $K$  is not trace class if  $\hat{A}_1(0) := \int_0^X A_1(x)dx \neq 0$ , since then  $\sum_j |\mathcal{K}_{jj}| = |\hat{A}_1(0)| \sum_j \frac{|j|}{1+|j|^2} + O(1) \sum_j \frac{1}{1+|j|^2} = +\infty$ . Thus, it is necessary to introduce the 2-modified Fredholm determinant in order to define a notion of determinant of  $(I + K)$ .

**3.1. Generalized periodic Evans function.** By Lemma 3.1 in conjunction with Proposition 2.1, it follows that the zero eigenvalues of  $(I_{L^2_{\text{per}}([0,X])} - K(\sigma, \lambda))$  can be identified through the use of a 2-modified Fredholm determinant. This leads us to the following definition.

**DEFINITION 3.3.** *For a fixed  $\sigma \in [0, 2\pi)$ , we define the generalized periodic Evans function  $D_\sigma : \mathbb{C} \rightarrow \mathbb{C}$  by*

$$(3.4) \quad D_\sigma(\lambda) := \det_{2, L^2_{\text{per}}([0,X])} (I_{L^2_{\text{per}}([0,X])} + K(\sigma, \lambda)).$$

For ease of notation, throughout the rest of our analysis we will drop the dependence on the Hilbert space  $L^2_{\text{per}}([0, X])$  on the identity operator and all 2-modified Fredholm determinants. In particular, we will write  $D_\sigma(\lambda) = \det_2(I + K(\sigma, \lambda))$  for the above generalized Evans function.

**THEOREM 3.4.** *For  $A_j \in L^2_{\text{per}}([0, X])$ , the function  $D_\sigma$  is complex-analytic in  $\lambda$  and continuous in the parameter  $\sigma$ . Furthermore, the roots of  $D_\sigma$  for a fixed  $\sigma \in [0, 2\pi)$  correspond in location and multiplicity with the eigenvalues of  $L_\sigma$ .*

*Proof.* Following the notation in Lemma 3.1, for each  $J \in \mathbb{N}$  we let  $\mathcal{K}_J := ([\mathcal{K}]_{j,k})_{|j|, |k| \leq J}$  be the finite-dimensional Galerkin matrix approximation of the bi-infinite-dimensional matrix representation of the operator  $K$  defined above. Clearly, then, for each fixed  $J \in \mathbb{N}$  the finite-dimensional approximation  $\Delta_J(\sigma, \lambda) := \det_2(I + \mathcal{K}_J(\sigma, \lambda))$  is complex-analytic in  $\lambda$  and continuous in  $\sigma \in [0, 2\pi)$ . Furthermore, as in the proof of Lemma 3.1 we have

$$(3.5) \quad \|\mathcal{K}_{1,J}(\sigma, \lambda) - \mathcal{K}_1(\sigma, \lambda)\|_{\mathcal{B}_2} \leq \|A_1\|_{L^2([0,X])} \sum_{|j| \geq J+1} \frac{j^2}{(1+j^2)^2},$$

where  $\mathcal{K}_{1,J}$  denotes the truncation of  $\mathcal{K}_1$ , and hence we find that  $\mathcal{K}_{1,J} \rightarrow \mathcal{K}_1$  in  $\mathcal{B}_2$  uniformly in both  $\sigma$  and  $\lambda$ . Similarly, we find that  $\mathcal{K}_{0,J}(\sigma, \lambda) \rightarrow \mathcal{K}_0(\sigma, \lambda)$  in  $\mathcal{B}_2$

---

<sup>6</sup>Henceforth, Hilbert–Schmidt spaces  $\mathcal{B}_2$  will always be considered on the Hilbert space  $L^2_{\text{per}}([0, X])$ . That is, we adopt the notation  $\mathcal{B}_2 := \mathcal{B}_2(L^2_{\text{per}}([0, X]))$ .

uniformly in  $\sigma$  and locally uniformly in  $\lambda$ , and hence the estimate (2.3) implies<sup>7</sup> that  $\Delta_J \rightarrow D_\sigma$  locally uniformly in  $\lambda \in \mathbb{C}$  and uniformly in  $\sigma \in [0, 2\pi)$ . It follows that the function  $(\sigma, \lambda) \mapsto D_\sigma(\lambda)$  inherits the same regularity properties in  $\lambda$  and  $\sigma$  as the limiting sequence  $\Delta_J$ , thus verifying the first claim of the theorem.

Next, by equivalence of problems (3.2) and (3.3) together with Proposition 2.1, we immediately obtain correspondence in location of the roots of  $D_\sigma$  and the eigenvalues of the operator  $L_\sigma$ . To obtain agreement in multiplicity, consider an eigenvalue  $\lambda_*$  of  $L_\sigma$  with corresponding eigenspace  $H_*$ . Recalling that by standard Fredholm theory the eigenvalues of  $L_\sigma$  are countable and isolated and have finite-multiplicity,<sup>8</sup> we find that there exists a closed ball  $B(\lambda_*, \varepsilon)$  of radius  $\varepsilon$ , centered at  $\lambda_*$ , containing no other eigenvalues of  $L_\sigma$ .

Consider now an increasing sequence of eigenspaces  $\{H_J\}_{J \in \mathbb{N}}$  of  $L_{\text{per}}^2([0, X])$  such that  $\lim_J H_J = L_{\text{per}}^2([0, X])$  and  $H_* \subset H_J$  for all  $J \in \mathbb{N}$ . For each  $J$ , let  $\{r_k\}_{k=1}^J$  be an orthonormal basis of  $H_J$  and let  $R_J = (r_1, \dots, r_J)$ . Then we can define the finite-dimensional approximants

$$(3.6) \quad \delta_J(\sigma, \lambda) := \det_2 \left( R_J^* (\partial_x^2 - 1)^{-1} (L_\sigma - \lambda I) R_J \right).$$

Since  $D_\sigma$  does not vanish on  $\partial B(\lambda_*, \varepsilon)$ , by the correspondence in location of roots and eigenvalues established above, and since  $\delta_J$  converges locally uniformly in  $\lambda$  to  $D_\sigma$  by (2.3), Rouché's theorem implies that there exists a  $J^* \in \mathbb{N}$  sufficiently large such that for  $J > J^*$  the winding number of  $D_\sigma$  around  $\partial B(\lambda_*, \varepsilon)$  is equal to the winding number of  $\delta_J$  around the same ball.

Finally, fixing  $J_0 > J^*$  and noticing that  $L_\sigma R_{J_0} = R_{J_0} M_{\sigma, J_0}$ , where  $M_{\sigma, J_0}$  is an  $J_0 \times J_0$  matrix representation of  $L_\sigma$  on the finite-dimensional invariant subspace  $H_{J_0}$ , we find from (3.6) that

$$(3.7) \quad \delta_{J_0}(\sigma, \lambda) = \det_2 \left( R_{J_0}^* (\partial_x^2 - 1)^{-1} R_{J_0} (M_{\sigma, J_0} - \lambda I) \right) = C \det_2 (M_{J_0} - \lambda I),$$

where  $C =: \det(R_{J_0}^* (\partial_x^2 - 1)^{-1} R_{J_0}) \neq 0$ ; hence  $\delta_{J_0}$  is a nonvanishing multiple of the characteristic polynomial of  $M_{\sigma, J_0}$ . Here, we are using the fact that  $R_{J_0}^* (\partial_x^2 - 1)^{-1} R_{J_0}$  is positive definite by positive symmetric definiteness of  $(\partial_x^2 - 1)^{-1}$ . It follows that  $\delta_{J_0}$  has a zero at  $\lambda_*$  of precisely the algebraic multiplicity of  $\lambda_*$  as an eigenvalue of  $L_\sigma$ . Thus, we conclude that the multiplicity of  $\lambda_*$  as a root of  $D_\sigma$  is equal to the winding number of  $\delta_{J_0}(\cdot, \sigma)$  about the ball  $\partial B(\lambda_*, \varepsilon)$ , which in turn is equal to the algebraic multiplicity of  $\lambda_*$  as an eigenvalue of  $L_\sigma$ , completing the proof.  $\square$

*Remark 3.5.* The truncated winding-number argument for agreement of multiplicity, to our knowledge, is new and seems of general use in similar situations. It would be interesting to prove this in a different way by establishing a direct correspondence between the Fredholm determinant and the standard periodic Evans function construction of Gardner [G], as done in the solitary-wave case in [GLM1, GLMZ2, GM] and in the periodic Schrödinger case in [GM, section 4], [GM2]. This would give at the same time an alternative proof of Gardner's fundamental result of agreement in location and multiplicity of roots of the standard periodic Evans function with eigenvalues of  $L_\sigma$  through the result of Theorem 3.4.

<sup>7</sup>To use the estimate (2.3) directly, one should consider the operator  $\mathcal{K}_J$ , which is technically defined on the finite-dimensional subspace  $H_J$ , as being defined on the larger space  $L_{\text{per}}^2([0, X])$ . Throughout the remainder of our analysis we will consider this extension without reserve.

<sup>8</sup>Note that in this standard theory, one inverts  $L_\sigma - \mu I$  rather than  $\mathcal{D}^2 - 1$ .

**3.2. Convergence of Hill's method.** Next, we use the machinery developed in the previous section to give a proof of the convergence of Hill's method. In order to precisely describe Hill's method, notice that by taking the Fourier transform, we may express (3.2) equivalently as the infinite-dimensional matrix system

$$(\mathcal{D}^2 + \mathcal{D}\mathcal{A}_1 + \mathcal{A}_0 - \lambda I)\mathcal{U} = 0,$$

where for each  $m = 0, 1$  and  $j, k \in \mathbb{Z}$ ,

$$(3.8) \quad \mathcal{D}_{jk} = \delta_j^k i j, \quad [\mathcal{A}_m]_{jk} = \widehat{A}_m(j - k), \quad \text{and } \mathcal{U}_j = \widehat{U}(j),$$

where  $\widehat{f}(k)$  denotes the discrete Fourier transform of  $f$  evaluated at Fourier frequency  $k$  and, as elsewhere,  $i = \sqrt{-1}$ . Hill's method then consists of fixing  $J \in \mathbb{N}$  and truncating the above infinite-dimensional matrix system at wave number  $J$ , that is, considering the  $(2J+1)$ -dimensional minor  $|(j, k)| \leq J$  and computing the eigenvalues of the finite-dimensional matrix

$$(3.9) \quad L_{\sigma, J} := \mathcal{D}_J^2 + \mathcal{D}_J \mathcal{A}_{1, J} + \mathcal{A}_{0, J},$$

where  $\mathcal{D}_J$  and  $\mathcal{A}_{m, J}$  denote the  $(2J+1)$ -dimensional matrices resulting from truncating the matrices  $\mathcal{D}$  and  $\mathcal{A}_m$  to frequencies  $|(j, k)| \leq J$  to obtain approximate eigenvalues for  $L_\sigma$ . Notice this can be done quite efficiently by applying modern numerical linear algebra techniques.

*Remark 3.6.* In applications, one may of course encounter operators  $L$  that are not in divergence form (3.1). In this case, we point out that there is no effect in changing from nondivergence to divergence form except that we increase the regularity requirement on  $A_1$  from  $L^2$  to  $H^1$ . Indeed,, we may change from one form to the other using the Leibnitz rule  $\mathcal{A}_1 \mathcal{D} - \mathcal{D} \mathcal{A}_1 = (\mathcal{A}_1)'$ , where

$$(\mathcal{A}_1)'_{jk} = i(j - k)\mathcal{A}_1(j - k) = (\widehat{A_{1,x}})(j - k),$$

and noting that since  $\mathcal{D}$  is diagonal, this operation is respected by truncation. Thus, there is indeed no loss of generality in our representation of operators in divergence form, as it does not affect the result of Hill's method.

Following the construction of the generalized periodic Evans function (3.4), we may rewrite the truncated eigenvalue equation

$$(3.10) \quad (L_{\sigma, J} - \lambda I)\mathcal{U} = 0$$

as

$$(3.11) \quad (I + \mathcal{K}_J)\mathcal{U} = 0,$$

where  $\mathcal{K}_J = \mathcal{K}_{1, J} + \mathcal{K}_{2, J}$  is the truncation of the Fourier representation  $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$  of operator  $K$  to frequencies  $|(j, k)| \leq J$ , that is,

$$(3.12) \quad \mathcal{K}_{1, J} = \mathcal{D}_J(\mathcal{D}_J^2 - I)^{-1}\mathcal{A}_{1, J} \quad \text{and} \quad \mathcal{K}_{2, J} = (\mathcal{D}_J^2 - I)^{-1}(\mathcal{A}_{0, J} + 1 - \lambda).$$

Continuing to follow the above construction of  $D_\sigma$ , we now define the truncated periodic Evans function as

$$(3.13) \quad D_{\sigma, J}(\lambda) := \det_2(I + \mathcal{K}_J)$$

and notice that we have the following preliminary result.



LEMMA 3.7. *The zeros of  $D_{\sigma,J}$  correspond in location and multiplicity with those of  $L_{\sigma,J}$ .*

*Proof.* The proof is immediate by the nonsingularity of  $(\mathcal{D}_J^2 - I)^{-1}$  and properties of the (usual, finite-dimensional) characteristic polynomial, together with the observation that

$$\det_2(I + \mathcal{K}_J) = \det_2(\mathcal{D}_J^2 - I)^{-1} \det_2(\mathcal{D}_J^2 + \mathcal{D}_J \mathcal{A}_{1,J} + \mathcal{A}_{0,J} - \lambda I). \quad \square$$

With this construction in hand, we now state the main result of this section.

THEOREM 3.8. *For  $A_j \in L_{\text{per}}^2([0, X])$ , the sequence of determinants  $D_{\sigma,J}$  converges to  $D_\sigma$  as  $J \rightarrow \infty$  uniformly in  $\sigma$  and locally uniformly in  $\lambda$ .*

*Proof.* This convergence result follows from the proof of Theorem 3.4. Indeed, noting that  $D_{\sigma,J}$  is exactly such a sequence of approximate determinants, corresponding here to the ascending sequence of sinusoidal functions of integer wave number, by which the generalized periodic Evans function  $D_\sigma$  was defined in (3.4), we find by our definition of the 2-modified Fredholm determinant that  $D_{\sigma,J} \rightarrow D_\sigma$  pointwise in  $\lambda$  as  $J \rightarrow \infty$  for each fixed  $\sigma \in [0, 2\pi)$ . Moreover, recalling that the rate of convergence is determined by the difference between truncated operator  $\mathcal{K}_J$  and  $\mathcal{K}$  in  $\mathcal{B}_2$  norm, and noting that we have uniformly bounded  $\mathcal{B}_2$  estimates on each entry of  $\mathcal{K}_J$ , we find that this convergence is uniform in  $\sigma$  and locally uniform in  $\lambda$ .  $\square$

From Theorem 3.8 we immediately have convergence of Hill's method, as described in the introduction. For completeness, we state this result in the following corollary.

COROLLARY 3.9. *For  $A_j \in L_{\text{per}}^2([0, X])$ , the eigenvalues of  $L_{\sigma,J}$  defined in (3.9) approach the eigenvalues of  $L_\sigma$  in location and multiplicity as  $J \rightarrow \infty$ , uniformly on  $|\lambda| \leq R$ ,  $\sigma \in [0, 2\pi]$ , for any  $R$  such that  $\partial B(0, R)$  contains no eigenvalues of  $L_\sigma$ .*

*Proof.* The proof is immediate from Theorem 3.4, Lemma 3.7, and Theorem 3.8, along with basic properties of uniformly convergent analytic functions.  $\square$

**3.3. Rates of convergence.** Next, we address the issue of the rates of convergence of  $D_{\sigma,J}$  to  $D_\sigma$  and of the approximate spectra to the exact spectra. Assuming slightly more regularity on the function  $A_1$  in (3.1), we have the following easy convergence result.

THEOREM 3.10. *For  $A_j \in H_{\text{per}}^1([0, X])$  and each fixed  $R > 0$ , there exists a constant  $C = C(R) > 0$  such that for each fixed  $|\lambda| \leq R$*

$$|D_{\sigma,J}(\lambda) - D_\sigma(\lambda)| \leq C J^{-1/2}.$$

*In particular, this estimate is locally uniform in  $\lambda$  and uniform in  $\sigma$ .*

*Proof.* The rate of convergence is bounded by  $\|\mathcal{K}_J - \mathcal{K}\|_{\mathcal{B}_2}$  from which we readily obtain the result using the Cauchy–Schwarz estimate

$$\sum_{|j| \geq J} |\widehat{A^m}(j)|^2 \leq \sum_{|j| \geq J} |j|^{-2} \sum_{|j| \geq J} |j|^2 |\widehat{A^m}(j)|^2 \leq (C/J) \|A^m\|_{H^1([0, X])}^2$$

for each  $m \in \{0, 1\}$ . For details, see the very similar estimates in the proof of [GLZ, Theorem 4.9].  $\square$

Notice that Theorem 3.10 does not imply a rate of convergence of the roots of  $D_{\sigma,J}$  to the roots of  $D_\sigma$  or, equivalently, the eigenvalues of  $L_{\sigma,J}$  to the eigenvalues of  $L_\sigma$ . Indeed, the above convergence result is, with or without rate information, essentially an abstract one. For though we find convergence of analytic functions  $D_{\sigma,J}$  to  $D_\sigma$ , we don't obtain rates of convergence of their zeros without more structural

information about  $D_\sigma$  itself. In particular, we cannot conclude convergence rates of the approximate spectra to the true eigenvalues of  $L_\sigma$  using only the knowledge of the eigenvalues of  $L_{\sigma,J}$  computed in the course of Hill's method. This suggests the idea of computing the approximate Evans function  $D_{\sigma,J}$  directly, instead of using it as a purely analytical tool, an idea that would be interesting for future investigation. Though in principle slower due to the need for multiple evaluations of eigenvalues, this computation is better conditioned, so there might perhaps be some counterbalancing advantages to this approach, besides the possibility already mentioned of obtaining a posteriori estimates on the error bounds for eigenvalue approximations. We leave this as an interesting topic for further investigation, related to the larger question of relative advantages of standard periodic Evans function (as in [G]) versus Hill's computations.

**4. Generalizations.** Here, we briefly discuss various generalizations of the theory developed in section 3.

**4.1. Operators with nontrivial principal coefficient.** Consider now a system of the more general form

$$(4.1) \quad L_\sigma = \partial_x^2 A_2 + \partial_x A_1(\sigma, x) + A_0(\sigma, x),$$

where  $A_2$  is symmetric positive definite, satisfying  $A_2(x) \geq C$  for some  $C > 0$ , uniformly on  $x \in [0, X]$ . Define as usual  $\mathcal{A}_2$  to be the infinite-dimensional matrix representation of  $A_2$  under Fourier transform; that is,  $\mathcal{A}_{2,jk} = \widehat{A}_2(j - k)$ . Then clearly  $\mathcal{A}_2$  is symmetric and, by Parseval's identity, satisfies  $\mathcal{A}_2 \geq C$  when considered as a quadratic form on  $\ell^2(\mathbb{N})$ . As a consequence, the  $J$ th truncation  $\mathcal{A}_{2,J}$ , as a principal minor of a positive definite symmetric matrix, must also be positive definite and satisfy the same bound  $\mathcal{A}_{2,J} \geq C$ .

In particular,  $\mathcal{A}_2$  is invertible with

$$\mathcal{A}_2^{-1} \geq 1/C, \quad \mathcal{A}_{2,J}^{-1} \geq 1/C.$$

LEMMA 4.1.  $\|AB\|_{\mathcal{B}_2} \leq \|A\|_{L^2} \|B\|_{\mathcal{B}_2}$ , where  $\|\cdot\|_{L^2}$  denotes  $L^2([0, X])$  operator norm.

*Proof.* The proof is straightforward from the definition of  $\|\cdot\|_{\mathcal{B}_2}$ .  $\square$

COROLLARY 4.2. For  $A_j \in L^2_{\text{per}}([0, X])$  and  $A_2$  symmetric positive definite with  $A_2(x) \geq C$ , the operator  $\mathcal{M} := \mathcal{A}_2^{-1} \mathcal{K}$  is Hilbert-Schmidt, where  $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$  is defined as in (3.12).

In this case, following the notation of Corollary 4.2, we define the generalized Evans function as  $D_\sigma(\lambda) := \det_2(I - \mathcal{M})$ , noting that the eigenvalue problem may be written equivalently as  $(I - \mathcal{M})\mathcal{U} = 0$ . The associated series of Fredholm approximants is  $D_{\sigma,J}(\lambda) := \det_2(I - \mathcal{M}_J)$  with  $D_{\sigma,J}(\lambda) \rightarrow D_\sigma(\lambda)$  uniformly as  $J \rightarrow \infty$ , just as before, and zeros of  $D_\sigma$  corresponding in location and multiplicity with eigenvalues of  $L_\sigma$ . However, the corresponding object obtained by Hill's method is not the truncated Fredholm determinant  $D_{\sigma,J}$  defined above, but rather the modified version

$$(4.2) \quad \check{D}_{\sigma,J}(\lambda) := \det_2(I + \mathcal{A}_{2,J}^{-1} \mathcal{K}_J),$$

and it is this function whose zeros correspond with the eigenvalues of the Hill approximant operator  $L_{\sigma,J}$ .

To verify convergence of Hill's method in this case, then, it is sufficient to show that

$$(4.3) \quad \|\mathcal{M}_J - \mathcal{A}_{2,J}^{-1} \mathcal{K}_J\|_{\mathcal{B}_2} = \|(\mathcal{A}_2^{-1} \mathcal{K})_J - \mathcal{A}_{2,J}^{-1} \mathcal{K}_J\|_{\mathcal{B}_2} \rightarrow 0$$

as  $J \rightarrow \infty$ . Indeed, with this convergence result in hand we may conclude by (2.3) that  $\lim_{J \rightarrow \infty} |\check{D}_{\sigma,J} - D_{\sigma,J}| = 0$ , and thus  $\check{D}_{\sigma,J} \rightarrow D_{\sigma}$  as  $J \rightarrow \infty$ , yielding the convergence result as before.

**THEOREM 4.3.** *For operators of the form (4.1), Hill's method converges in location and multiplicity provided that  $A_j \in L^2_{\text{per}}([0, X])$ .*

*Proof.* We sketch the proof of (4.3). By boundedness of  $\|A_2\|_{L^2([0, X])}$ , we may truncate  $\widehat{A_2}$  at wave number  $M$  to obtain an  $M$ -banded infinite-dimensional diagonal matrix centered around zero-frequency approximating  $\mathcal{A}_2$  to arbitrarily small order in the  $\ell^2(\mathbb{N})$  operator norm. Hence, for purposes of this argument, we may assume without loss of generality that  $\mathcal{A}_2$  is  $M$ -banded diagonal operator centered about zero-frequency. Furthermore, noting that since  $\widehat{\mathcal{A}_2^{-1}}$  is bounded in  $L^2(\mathbb{R})$ , for  $J \in \mathbb{N}$  sufficiently large the columns of  $\mathcal{A}_2^{-1}$  corresponding to frequencies  $|j| \leq J - M$  are small off the principal  $2J + 1 - M$  minor and hence a brief calculation reveals that

$$(\mathcal{A}_2^{-1})_J \mathcal{A}_{2,J} = \begin{pmatrix} E_M & 0 & 0 \\ 0 & I_{2J-2M} & 0 \\ 0 & 0 & F_M \end{pmatrix},$$

where  $E_M$  and  $F_M$  are  $M \times M$  matrices that are invertible by invertibility of  $(\mathcal{A}_2^{-1})_J \mathcal{A}_J$ , a property of principal minors of positive definite symmetric matrices. By a further left-multiplication by the block-diagonal matrix

$$\begin{pmatrix} E_M^{-1} & 0 & 0 \\ 0 & I_{2J-2M} & 0 \\ 0 & 0 & F_M^{-1} \end{pmatrix}$$

we obtain  $I_{2J+1}$ , demonstrating that  $(\mathcal{A}_{2,J})^{-1}$  agrees with  $(\mathcal{A}^{-1})_{2,J}$  on the central  $2J - 2M + 1$  dimensional minor. Recalling that  $\|\mathcal{K} - \mathcal{K}_J\|_{\mathcal{B}_2} \rightarrow 0$  as  $J \rightarrow 0$  by (3.5), we thus obtain by a straightforward calculation

$$\|(\mathcal{A}_2^{-1} \mathcal{K})_J - \mathcal{A}_{2,J}^{-1} \mathcal{K}_J\|_{\mathcal{B}_2} \sim \|(\mathcal{A}_2^{-1} \mathcal{K}_J)_J - \mathcal{A}_{2,J}^{-1} \mathcal{K}_J\|_{\mathcal{B}_2} \rightarrow 0,$$

completing the proof by (2.3).  $\square$

**4.2. Composite and higher-order operators.** The reader may easily verify that all the arguments in sections 3 and 4.1 carry over to the case when the operator (1.1) is replaced by a general periodic-coefficient operator

$$L = \partial_x^m a_m(x) + \partial_x^{m-1} a_{m-1}(x) + \cdots + a_0(x),$$

where  $a_j \in L^2_{\text{per}}([0, X])$  and where the principal coefficient  $a_m$  symmetric positive definite. Indeed, the analysis parallels that of previous sections except that one must substitute for  $(\partial_x^2 - 1)^{-1}$  everywhere the symmetric definite Fourier multiplier

$$(\partial_x^m + i^m)^{-1} = \mathcal{F}^{-1}((ij)^m + i^m)^{-1} \mathcal{F}$$

for  $m$  even, where  $j$  denotes the Fourier wave number,  $\mathcal{F}$  denotes Fourier transform, and the nonnormal but positive definite<sup>9</sup> Fourier multiplier

$$(\partial_x^m + 1)^{-1} = \mathcal{F}^{-1}((ij)^m + 1)^{-1} \mathcal{F}$$

<sup>9</sup>That is,  $\Re\langle v, (\partial_x^m + 1)^{-1} v \rangle > 0$ , which is sufficient to conclude as in (3.7) that  $\det R_{J_0}^* (\partial_x^m + 1)^{-1} R_{J_0} \neq 0$ .

for  $m$  odd. With these substitutions, our previous arguments immediately yield convergence of Hill's method in this case as well.

Furthermore, it is straightforward to verify that all the analysis in sections 3 and 4.1 extends readily to the case of operators of "composite" type

$$L = \begin{pmatrix} \partial_x^{m_1} a_{m_1}^1 + \cdots \\ \vdots \\ \partial_x^{m_n} a_{m_n}^n + \cdots \end{pmatrix}$$

with  $a_k^j \in L^2_{\text{per}}([0, X])$  and  $a_{m_j}^j$  symmetric positive definite for each suitable choice of indices, that is, still assuming  $L$  is a nondegenerate ordinary differential operator in some sense.

*Remark 4.4.* The above observation applies to the numerics in [BJNRZ1, BJNRZ2], where the authors use Hill's method to numerically analyze the spectrum of the linearized St. Venant equations

$$\begin{aligned} \lambda\tau - c\tau' - u' &= 0, \\ \lambda u - cu' - (\bar{\tau}^{-3}(F^{-1} - 2\nu\bar{u}_x)\tau)' &= -(s+1)\bar{\tau}^s\bar{u}^r\tau - r\bar{\tau}^{s+1}\bar{u}^{r-1}u + \nu(\bar{\tau}^{-2}u)' \end{aligned}$$

about a given periodic or homoclinic orbit  $(\bar{u}, \bar{\tau})$ , where  $r$ ,  $s$ ,  $F$ , and  $\nu$  are physical parameters in the problem and  $\lambda$  is the corresponding spectral parameter.

**4.3. Operators with general coefficients.** Our results are completely general in the scalar case, applying to all nondegenerate operators. However, they are restricted in the system case by the condition that the principal coefficient(s) be symmetric positive definite. Whether this condition may be relaxed is an interesting operator-theoretic question regarding properties of Toeplitz matrices.

Specifically, the property that we need to carry out Hill's method (and indeed to complete our entire convergence analysis) is that the minor  $\mathcal{A}_{2,J}$  of a Toeplitz matrix  $[\mathcal{A}_2]_{mn} = \widehat{A}_2(k-n)$  be invertible for  $J$  sufficiently large. The question is what properties of  $A_2(x)$  are sufficient to guarantee this: in particular, is uniform invertibility enough? Alternatively, what are sufficient conditions on  $\widehat{A}_2$ ? This seems an interesting problem for further investigation.

**Appendix A. The results of Vainikko and convergence to all orders.** In this appendix, finally, we briefly recall the classical results of Vainikko [V] and show how they can be applied to Hill's method to yield convergence to all orders.

**A.1. Banach theory.** For  $T$  compact linear operators on a Banach space  $E$  with norm  $\|\cdot\|$ , let  $T_J$  be a sequence of compact linear operators on closed subspaces  $E_J$  and let  $P_J$  be a sequence of projections from  $E \rightarrow E_J$ . Denoting

$$(A.1) \quad R_J = T - TP_J, \quad S_J := T_J - P_JT|_{E_J},$$

assume that  $T_J$  approaches  $T$  in the sense that

$$(A.2) \quad |R_J|, |S_J| \rightarrow 0 \text{ as } J \rightarrow \infty.$$

Consider the eigenvalue problems

$$(A.3) \quad x = \mu Tx$$

and

$$(A.4) \quad x_J = \mu_J T_J x_J.$$

PROPOSITION A.1 (see [V, Theorems 1 and 3, case  $k = l$ ]). (i) *Every eigenvalue  $\mu$  of (A.3) is a limit of eigenvalues  $\mu_J$  of (A.4); conversely, eigenvalues  $\mu_J$  of (A.4) can only converge to eigenvalues  $\mu$  of (A.3) as  $J \rightarrow \infty$ .* (ii) *Let the eigenvalue  $\mu$  of  $T$  have eigenspace  $\Sigma$  of rank  $l$  and let  $\{\mu_J\}$  be a sequence of eigenvalues of  $T_J$  converging to  $\mu$  with associated eigenspaces  $\Sigma_J$ . Then, for some constant  $C > 0$ ,*

$$(A.5) \quad \begin{aligned} |\mu_J - \mu| &\leq C \left( \sup_{x_J \in \Sigma_J, \|x_J\|=1} \|S_J x_J\| + \sup_{x \in \Sigma, \|x\|=1} \|(I - P_J)x\| \right), \\ \sup_{x_J \in \Sigma_J, \|x_J\|=1} d(x_J, \Sigma) &\leq C \left( \sup_{x_J \in \Sigma_J, \|x_J\|=1} \|S_J x_J\| + \sup_{x \in \Sigma, \|x\|=1} \|(I - P_J)x\| \right). \end{aligned}$$

*Remark A.2.* The constant  $C > 0$  of the proposition depends on the lower bound for  $(I - \mu T)$  on a complementary subspace to  $\Sigma$  so is not a priori known or directly computable. For  $T$  self-adjoint, it is the inverse distance between  $\mu$  and the remaining spectrum of  $I - \mu T$ .

**A.2. Hilbert theory and Galerkin approximation.** For Galerkin approximation on a Hilbert Space  $\mathcal{H}$ , the operators  $P_J$  of the previous subsection reduce to orthogonal projection onto  $E_J$ ,  $|P_J| = 1$ , and the approximants  $T_J$  to

$$(A.6) \quad T_J = P_J T P_J.$$

In particular,  $S_J = T_J - P_J T P_J = 0$ , greatly simplifying the discussion.

Now, turning to the case of interest, taking

$$E = \mathcal{H} = L^2[0, 1]_{\text{per}}, \quad E_J = \mathcal{H}_J := \text{Span}\{e^{ijx} : |j| \leq J\},$$

and  $L$  as in (1.1), set  $T = (\lambda_0 - L)^{-1}$  with  $\lambda_0$  chosen so that

$$(A.7) \quad T \text{ and } T^* \text{ are bounded from } L^2[0, 1]_{\text{per}} \rightarrow H^m[0, m]_{\text{per}}, \quad m \geq 1 \text{ as in (1.1);}$$

in particular,  $T$  is compact.<sup>10</sup>

LEMMA A.3. *With  $T, E, P_J$  as just described,*

$$(A.8) \quad |(I - P_J)|_{H^s \rightarrow L^2} \leq J^{-s} \text{ and}$$

$$(A.9) \quad |R_J| \leq C J^{-2} \rightarrow 0 \text{ as } J \rightarrow \infty.$$

*Proof.* The first is standard, from  $\sum_{|j| \geq J} |\hat{f}_j|^2 \leq (1 + J^{2s})^{-1} \sum_{|j| \geq J} (1 + j^{2s}) |\hat{f}_j|^2$ , whence the second follows by  $|R_J| = |R_J^*| \leq |I - P_J|_{H^m \rightarrow L^2} |T^*|_{L^2 \rightarrow H^m}$ .  $\square$

COROLLARY A.4. *Under the above assumptions, the eigenvalues and eigenfunctions of  $L_J$  (resp.,  $T_J$ ) converge to those of  $L$  (resp.,  $T$ ) at rate  $CJ^{-m}$ . Moreover, if the coefficients of  $L$  are  $W^{k, \infty}$ , then convergence is at rate  $CJ^{-m-k}$ ; in particular, for  $C^\infty$  coefficients, convergence is at all (polynomial) orders.*

*Proof.* The first statement follows immediately from Lemma A.3 and Proposition A.1, together with the observation that  $\Sigma \subset H^m$  by  $(I - \mu T)^l \Sigma = 0$  and (A.7) so that  $\|(I - P_J)\Sigma\|$  decays by (A.8). The second follows by the observation that differentiating the eigenvalue ODE up to  $k$  times and bounding commutator terms by

<sup>10</sup>In the simplest case that  $a_m = I$ , it is readily seen that  $\lambda_0 = -Ci^m$  for  $C > 0$  sufficiently large will work; other cases can be treated by energy estimates.

the assumed regularity of coefficients, we find by induction that  $\Sigma \subset H^{m+k}$ , giving a faster rate of decay by factor  $J^{-k}$ .  $\square$

*Remark A.5.* Here, we have not tried to optimize rates of decay for finite regularity; for similar, somewhat more precise, results in the  $2r$ -elliptic setting, see [BO, O]. The results of this section generalize readily to multiple dimensions (PDE) with general domains and boundary conditions. Likewise, our results obtained in the main body of the paper using the generalized Evans function (Fredholm determinant) generalize immediately to PDE on multiply periodic domains ( $d$ -dimensional torii); that is, despite our description as an Evans function technique, this does not have the limitation to one dimension of the Jost-type formulations of Gardner and others.

### A.3. Concluding comments: Comparison with determinant methods.

As discussed in the introduction, the “direct” methods of [V, BO, O] give satisfying qualitative justification of the efficiency of Hill’s method, showing convergence beyond all polynomial orders for  $C^\infty$  coefficients. By contrast, convergence of the generalized Evans function is limited in rate even for arbitrarily smooth coefficients, due to large tails in the truncation of the product  $\prod(1 - \alpha_j)$ , coming from other, far-away, eigenvalues. For further discussion, and partial improvements by renormalization, see [BJZ2]. On the other hand, we may obtain from these determinant bounds guaranteed estimates on the winding number and stability or instability of the total spectrum of  $L$ , which are difficult to obtain by direct methods. Thus, the Evans function approach seems to be a useful complementary tool to add to existing methods of [V] and its later refinements.

**Acknowledgments.** Thanks to Bernard Deconink for pointing out references [CuD, CDKK, DK] and to the anonymous referees for pointing out references [BO, O] and the translated version of reference [V] (at present not listed on MathSciNet).

### REFERENCES

- [BJNRZ1] B. BARKER, M. JOHNSON, P. NOBLE, M. RODRIGUES, AND K. ZUMBRUN, *Witham averaged equations and modulational stability of periodic solutions of hyperbolic-parabolic balance laws*, in Proceedings and seminars, Centre de Mathématiques de l’École polytechnique, Port d’Albret, France, 2010; also available from [http://jedp.cedram.org/jedp-bin/fitem?id=JEDP\\_2010\\_—A3\\_0](http://jedp.cedram.org/jedp-bin/fitem?id=JEDP_2010_—A3_0).
- [BJNRZ2] B. BARKER, M. JOHNSON, P. NOBLE, M. RODRIGUES, AND K. ZUMBRUN, *Efficient numerical evaluation of the periodic Evans function of Gardner and spectral stability of periodic viscous roll waves*, in preparation.
- [BJNRZ3] B. BARKER, M. JOHNSON, P. NOBLE, M. RODRIGUES, AND K. ZUMBRUN, *Stability of periodic Kuramoto-Sivashinsky waves*, Appl. Math. Letters., (2011), doi:10.1016/j.aml.2011.10.026.
- [BJNRZ4] B. BARKER, M. JOHNSON, P. NOBLE, M. RODRIGUES, AND K. ZUMBRUN, *Nonlinear modulational stability of periodic traveling-wave solutions of the generalized Kuramoto-Sivashinsky equation*, in preparation.
- [BJZ1] B. BARKER, M. JOHNSON, AND K. ZUMBRUN, *Convergence and nonconvergence of approximation of spectra of periodic waves by discretization*, in preparation.
- [BJZ2] B. BARKER, M. JOHNSON, AND K. ZUMBRUN, *Evans-function based a posteriori error estimates for Hill’s method*, in preparation.
- [BO] J. H. BRAMBLE AND J. E. OSBORN, *Rate of convergence estimates for nonselfadjoint eigenvalue approximations*, Math. Comp., 27 (1973), pp. 525–549.
- [CuD] C. CURTIS AND B. DECONICK, *On the convergence of Hill’s method*, Math. Comp., 79 (2010), pp. 169–187.
- [CDKK] J. D. CARTER, B. DECONICK, F. KIYAK, AND J. N. KUTZ, *SpectrUW: a laboratory for the numerical exploration of spectra of linear operators*, Math. Comput. Simulation, 74 (2007), pp. 370–379.

- [DK] B. DECONINCK AND J. N. KUTZ, *Computing spectra of linear operators using Hill's method*, J. Comput. Physics, 219 (2006), pp. 296–321.
- [G] R. GARDNER, *On the structure of the spectra of periodic traveling waves*, J. Math. Pures Appl., 72 (1993), pp. 415–439.
- [GLM1] F. GESZTESY, Y. LATUSHKIN, AND K. A. MAKAROV, *Evans functions, Jost functions, and Fredholm determinants*, Arch. Ration. Mech. Anal., 186 (2007), pp. 361–421.
- [GLMZ2] F. GESZTESY, Y. LATUSHKIN, M. MITREA, AND M. ZINCHENKO, *Nonselfadjoint operators, infinite determinants, and some applications*, Russ. J. Math. Phys., 12 (2005), pp. 443–471.
- [GM] F. GESZTESY AND K. A. MAKAROV, *(Modified) Fredholm determinants for operators with matrix-valued semi-separable integral kernels revisited*, Integral Equations Oper. Theory, 47 (2003), pp. 457–497.
- [GM2] F. GESZTESY AND K. A. MAKAROV, *(Modified) Fredholm determinants for operators with matrix-valued semi-separable integral kernels revisited: Erratum*, Integral Equations Oper. Theory, 47 (2003), pp. 457–497; see also Integral Equations Oper. Theory, 48 (2004), pp. 561–602.
- [GLZ] F. GESZTESY, Y. LATUSHKIN, AND K. ZUMBRUN, *Derivatives of (modified) Fredholm determinants and stability of standing and traveling waves*, J. Math. Pures Appl., 90 (2008), pp. 160–200.
- [GGK1] I. GOHBERG, S. GOLDBERG, AND N. KRUPNIK, *Traces and determinants of linear operators*, Integral Equations Oper. Theory, 26 (1996), pp. 136–187.
- [GGK2] I. GOHBERG, S. GOLDBERG, AND N. KRUPNIK, *Hilbert–Carleman and regularized determinants for linear operators*, Integral Equations Oper. Theory, 27 (1997), pp. 10–47.
- [GGK3] I. GOHBERG, S. GOLDBERG, AND N. KRUPNIK, *Traces and Determinants for Linear Operators*, Oper. Theory Adv. Appl. 116, Birkhäuser, Basel, Switzerland, 2000.
- [GK] I. GOHBERG AND M. G. KREIN, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Transl. Math. Monogr. 18, American Mathematical Society, Providence, RI, 1969.
- [JZN] M. JOHNSON, K. ZUMBRUN, AND P. NOBLE, *Nonlinear stability of viscous roll waves*, SIAM J. Math. Anal., 43 (2011), pp. 577–611.
- [O] J. E. OSBORN, *Spectral approximation for compact operators*, Math. Comput., 29 (1975), pp. 712–725.
- [Si1] B. SIMON, *Notes on infinite determinants of Hilbert space operators*, Adv. Math., 24 (1977), pp. 244–273.
- [Si2] B. SIMON, *Trace Ideals and Their Applications*, 2nd ed., Math. Surveys Monogr. 120, American Mathematical Society, Providence, RI, 2005.
- [V] G. M. VAINIKKO, *A perturbed Galerkin method and the general theory of approximate methods for nonlinear equations*, Zh. Vychisl. Mat. Mat. Fiz., 7 (1967), pp. 723–751 (in Russian). U.S.S.R. Comput. Math. and Math. Phys., 7 (1967), pp. 18–32 (in English).
- [W] H. WEYL, *Inequalities between the two kinds of eigenvalues of a linear transformation*, Proc. Natl. Acad. Sci. USA, 35 (1949), pp. 408–411.
- [Z] K. ZUMBRUN, *2-modified Characteristic Fredholm Determinants, Hill's Method, and the Periodic Evans Function of Gardner*, Z. Anal. Anwend., to appear.