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Gravity, Lorentz violation, and the standard model

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The role of the gravitational sector in the Lorentz- and *CPT*-violating standard-model extension (SME) is studied. A framework is developed for addressing this topic in the context of Riemann-Cartan spacetimes, which include as limiting cases the usual Riemann and Minkowski geometries. The methodology is first illustrated in the context of the QED extension in a Riemann-Cartan background. The full SME in this background is then considered, and the leading-order terms in the SME action involving operators of mass dimension three and four are constructed. The incorporation of arbitrary Lorentz and *CPT* violation into general relativity and other theories of gravity based on Riemann-Cartan geometries is discussed. The dominant terms in the effective low-energy action for the gravitational sector are provided, thereby completing the formulation of the leading-order terms in the SME with gravity. Explicit Lorentz symmetry breaking is found to be incompatible with generic Riemann-Cartan geometries, but spontaneous Lorentz breaking evades this difficulty.

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I. INTRODUCTION

The combination of Einstein's general relativity and the standard model (SM) of particle physics provides a remarkably successful description of nature. The former theory describes gravitation at the classical level, while the latter encompasses all other phenomena involving the basic particles and forces down to the quantum level. These two field theories are expected to merge at the Planck scale, $m_P \simeq 10^{19}$ GeV, into a single unified and quantum-consistent description of nature.

Uncovering experimental confirmation of this idea is challenging because direct experiments at the Planck scale are impractical. However, suppressed effects emerging from the underlying unified quantum gravity theory might be observable in sensitive experiments performed at our presently attainable low-energy scales. One candidate set of Planck-scale signals is relativity violations, which are associated with the breaking of Lorentz symmetry [1].

Any observable signals of Lorentz violation can be described using effective field theory [2]. To ensure that known physics is reproduced, a realistic theory of this type must contain both general relativity and the SM, perhaps together with suppressed higher-order terms in the gravitational and SM sectors. Incorporating in addition terms describing arbitrary coordinate-independent Lorentz violation yields an effective field theory called the standard-model extension (SME). At the classical level, the dominant terms in the SME action include the pure-gravity and minimally coupled SM actions, together with all leading-order terms introducing violations of Lorentz symmetry that can be constructed from gravitational and SM fields.

The SME has been extensively studied in the Minkowski-spacetime limit, where all terms expected to dominate at low energies are known [3]. A primary goal of the present work is to construct explicitly the modifications appearing in non-Minkowski spacetimes, including both those in the pure-gravity sector and those involving gravitational couplings in the matter and gauge sectors. Some previous work along these lines has been performed, and in fact the Lorentz-violating gravitational sector was among the first pieces of the SME to be studied [4]. However, an explicit construction of all dominant gravitational couplings in the SME action has been lacking to date.

The investigation of local Lorentz violation in non-Minkowski spacetimes requires a geometrical framework allowing for nonzero vacuum quantities that violate local Lorentz invariance but preserve general coordinate invariance. The Riemann-Cartan geometry is well suited to this task, and it also naturally handles minimal gravitational couplings of spinors [5,6]. The present work studies the SME in a general Riemann-Cartan spacetime, allowing for dynamical curvature and torsion modes. The general-relativistic version of this theory is readily recovered in the limit of zero torsion.

The Lorentz-violating terms in the SME take the form of Lorentz-violating operators coupled to coefficients with Lorentz indices. Nonzero coefficients of this type could emerge in various ways. One attractive and generic mechanism is spontaneous Lorentz violation, studied in string theory and field theories with gravity [4,7]. Noncommutative field theories also contain Lorentz violation, with realistic models involving a subset of SME operators of higher mass dimension [8]. Other suggestions for sources of Lorentz violation include, for example, various nonstring approaches to quantum gravity [9], random dynamics models [10], multiverses [11], brane-world scenarios [12], and cosmologically varying fields [13,14].

In the Minkowski-spacetime limit of the SME, the Lorentz-violating terms can be classified according to their properties under *CPT*. Indeed, since *CPT* violation implies Lorentz violation in this limit [15], the SME also incorporates general *CPT* breaking. To determine the *CPT* properties of a given operator in Minkowski spacetime, it suffices in practice to count the number of indices on the corresponding coefficient for Lorentz violation. A Lorentz-violating term breaks *CPT* when this number is odd. However, in non-Minkowski spacetimes, establishing a satisfactory definition of *CPT* and its properties is challenging, and a complete

understanding is lacking at present. In this work, a practical definition is adopted: *CPT*-odd operators are taken to be those with an odd total number of spacetime and local Lorentz indices. This suffices for present purposes and ensures a smooth match to the Minkowski-spacetime limit. With this understanding, the SME serves as a realistic general basis for studies of Lorentz violation in Riemann-Cartan spacetimes, with or without *CPT* breaking.

Since no compelling experimental evidence for Lorentz violation has been uncovered as yet, it is plausible to assume that the SME coefficients for Lorentz violation are small in any concordant frame [16]. Indeed, sensitivity to the SME coefficients has attained Planck-suppressed levels in a number of experiments, including ones with mesons [2,17–19], baryons [20–22], electrons [23–25], photons [13,26–29], and muons [30], and discovery potential exists in experiments with neutrinos [3,31,32]. Only a comparatively small part of the coefficient space has been explored to date, and the present work is expected eventually to provide further directions in which to pursue experimental searches for Lorentz violation.

The organization of this paper is as follows. The framework for local Lorentz violations is discussed in Sec. II A, while the structure of the action and the derivation of covariant conservation laws in the presence of Lorentz violation are provided in Sec. II B. Section III considers the QED extension with gravitational couplings, and contains separate subsections devoted to the fermion and photon actions. The SME in a Riemann-Cartan background is presented in Sec. IV. The leading-order terms in the pure-gravity sector are constructed in Sec. V A, while the limiting Riemann-spacetime case is considered in Sec. V B. Section V C addresses the issue of the compatibility of explicit Lorentz violation with the underlying Riemann-Cartan geometry. The body of the paper concludes with a summary in Sec. VI. Appendix A lists conventions adopted in this work and some key results for Riemann-Cartan geometry. Appendix B presents a class of models for Lorentz violation used to illustrate various concepts throughout this work.

II. FRAMEWORK

A. Local Lorentz violation

The classic description of gravity in a Riemann spacetime invokes a metric and a covariant derivative that acts on vector or tensor representations of $\mathrm{Gl}(4,R)$. However, $\mathrm{Gl}(4,R)$ has no spinor representations, whereas the fundamental constituents of ordinary matter, leptons and quarks, are known to be spinors. One framework that incorporates spinors and distinguishes naturally between local Lorentz and general coordinate transformations is the vierbein formalism [5], which is adopted in the present work.

In the vierbein formalism, the basic gravitational fields can be taken as the vierbein $e_\mu{}^a$ and the spin connection $\omega_\mu{}^{ab}$. The corresponding Riemann-Cartan spacetimes are determined by the curvature tensor $R^\kappa{}_{\lambda\mu\nu}$ and the torsion tensor $T^\lambda{}_{\mu\nu}$. The usual Riemann spacetime of Einstein's general relativity can be recovered in the zero-torsion limit, while Minkowski spacetime is a special case with zero curvature and torsion. One well-known gravitation theory based on Riemann-Cartan geometry is the Einstein-Cartan theory, which has gravitational action of the Einstein-Hilbert form. The torsion in this theory is static, and in the absence of matter the solutions of the theory are equivalent to those of general relativity. However, more general gravitation theories in Riemann-Cartan spacetime contain propagating vierbein and spin-connection fields, describing dynamical torsion and curvature [6].

The vierbein formalism has a close parallel to the description of local symmetry in gauge theory. A key feature is the separation of local Lorentz transformations and general coordinate transformations. At each spacetime point, the action of the local Lorentz group allows three rotations and three boosts, independent of general coordinate transformations. This situation is ideal for studies of local Lorentz violation in which it is desired to maintain the usual freedom of choice of coordinates without affecting the physics. Within this framework, local Lorentz violation is analogous to the violation of local gauge invariance.

The presence of Lorentz violation in a local Lorentz frame is signaled by a nonzero vacuum value for one or more quantities carrying local Lorentz indices, called coefficients for Lorentz violation. As a simple example, consider a toy theory in which a nonzero timelike vacuum value $b_a=(b,0,0,0)$ exists in a certain local Lorentz frame at some point P. One explicit theory of this type is the bumblebee model described in Appendix B. The presence of the coefficient b_a for Lorentz violation implies that a preferred direction is selected at P within the local Lorentz frame, leading to equivalence-principle violations. Physical Lorentz breaking occurs at P whenever particles or fields have observable interactions with b_a.

Rotations or boosts of particles or localized field distributions in a given local Lorentz frame at P can be performed that leave b_a unaffected. Lorentz transformations of this kind are called local *particle* Lorentz transformations, and under them b_a behaves as a set of four scalars. However, the choice of the local Lorentz frame itself remains arbitrary up to spacetime rotations and boosts. Rotations or boosts changing the local Lorentz frame are called local *observer* Lorentz transformations, and under them b_a behaves covariantly as a four-vector. The theory thus maintains local observer Lorentz covariance, despite the presence of local particle Lorentz violation.

The conversion from the local Lorentz frame to spacetime coordinates is implemented via the vierbein, $b_\mu=e_\mu{}^a b_a$. A change of the observer's spacetime coordinates x^μ induces a conventional general coordinate transformation on b_μ. The description of the physics is therefore invariant under general coordinate transformations, as is to be expected for coordinate-independent behavior.

Different local observer Lorentz frames can be reached using different vierbeins, related by local observer Lorentz transformations. In a local neighborhood containing P, b_μ is typically a function $b_\mu(x)$ of position. Assuming for definiteness that b_μ has constant magnitude $b^\mu b_\mu$, the local observer Lorentz freedom in the vierbein $e_\mu{}^a(x)$ can be used to choose $b_a=(b,0,0,0)$ everywhere in the neighborhood. This

defines a preferred set of frames over the neighborhood.

Note that the existence of preferred frames is a special feature of this simple model. Extending the model to one with a second nonzero coefficient for Lorentz violation c_a typically destroys the existence of preferred frames at P and in the neighborhood. Observer Lorentz transformations have only six degrees of freedom, which are used in selecting the preferred frame for b_a at P. In this preferred frame, c_a generically has the arbitrary form $c_a=(c_1,c_2,c_3,c_4)$. Moreover, once $e_\mu{}^a(x)$ has been selected to maintain the preferred position-independent form of b_a over a neighborhood of P, c_a can vary with position. Although another frame at P can be found in which c_a does have a preferred (timelike, spacelike, or lightlike) form, then b_a no longer has the preferred form $b_a=(b,0,0,0)$. The notion of preferred frame therefore loses meaning in the generic case.

It is natural and convenient, although not necessary, to assume $b_\mu(x)$ is a smooth vector field over the neighborhood of P and over most of the spacetime, except perhaps for singularities. Since most applications involve second-order differential equations, $\mathcal{C}^2$ smoothness suffices. However, a smooth extension of $b_\mu(x)$ over the *entire* spacetime may be precluded by topological conditions analogous to the Hopf theorem, which states that smooth vector fields can exist on a compact manifold if and only if its Euler characteristic χ vanishes. Note that, if indeed singularities of b_μ occur, their location can differ from those of singularities in the curvature and torsion. Note also that some standard topological constraints on the spacetime itself are implied by the general framework adopted here. For example, the presence of spinor fields requires a spinor structure on the spacetime, so the corresponding manifold must be a spin manifold and have trivial second Steifel-Whitney class.

Studies of Lorentz violation in the Minkowski-spacetime limit commonly assume that the coefficients for Lorentz violation are constants over the spacetime, which ensures the useful simplifying physical consequence that energy and momentum remain conserved. Various physical arguments can be used to justify this assumption. For example, some mechanisms for Lorentz violation may attribute higher overall energy to coefficients with nontrivial spacetime dependence, so that constant coefficients are naturally preferred. More generally, if the Lorentz breaking originates at the Planck scale and there is an inflationary period in cosmology, then a present-day configuration with constant coefficients over the Hubble radius is a plausible consequence. Also, for sufficiently slow spacetime variation of the coefficients, the assumption of constancy can be viewed as the leading approximation in a series expansion. However, all arguments of this type are ultimately physical choices. From the formal perspective, any vector or tensor field with smooth integral curves is also an acceptable candidate. The choice of constant coefficients for Lorentz violation can therefore be viewed as a kind of boundary condition for the theory.

For the simple toy model in the present example, the condition of constant coefficients in Minkowski spacetime can be written $\partial_\mu b_\nu=0$. In a more general Riemann-Cartan spacetime, it might seem natural to impose the covariant generalization of this,

$$D_\mu b_a \equiv \partial_\mu b_a - \omega_\mu{}^b{}_a b_b = 0. \tag{1}$$

However, the integrability conditions for this equation can be satisfied globally only for special spacetimes, in particular, for parallelizable manifolds. Such manifolds have zero curvature, are comparatively rare in four or more dimensions, and appear of lesser interest for theories of gravity. It is therefore reasonable to suppose that $D_\mu b_a \neq 0$ at least in some region of spacetime. This in turn implies nontrivial consequences for the energy-momentum tensor. Section II B discusses these consequences and obtains the covariant conservation law in the presence of Lorentz violation. In any case, an arbitrary *a priori* specification of $b_\mu(x)$ in a given spacetime can be expected to be inconsistent with the simple condition (1).

A consistent prescription for determining $b_\mu(x)$ and hence $D_\mu b_a$ exists in some cases. For example, this is true when $b_\mu(x)$ arises through a dynamical procedure, such as the development of a vacuum expectation value in the context of spontaneous Lorentz breaking. The dynamical equations for the spacetime curvature and torsion can then be solved simultaneously with the dynamical equations for b_μ, yielding a self-consistent solution. As usual, appropriate boundary conditions are needed for all variables to fix the solution. In the case of asymptotically Minkowski spacetimes, which are relevant for many experimental purposes, it may be physically reasonable to adopt as part of the boundary conditions the criterion (1) in the asymptotic limit where the curvature and torsion vanish. Solutions of this form then merge with those of the SME in Minkowski spacetime. More complicated solutions involving asymptotic coefficients varying with spacetime position could also be considered. The corresponding potential experimental signals would include violations of energy-momentum conservation. In most of what follows, no particular special assumptions about the global structure of the spacetime or about asymptotic properties of the coefficients are made, and in particular Eq. (1) is *not* assumed.

For illustrative purposes, the above discussion uses a simple toy model with a single coefficient $b_a(x)$ that behaves like a vector under local observer Lorentz transformations. More generally, there can be a (finite or infinite) number of coefficients for Lorentz violation, each transforming as a specific representation of the local observer Lorentz group. In what follows, a generic coefficient with compound local Lorentz index x transforming in the representation $(X_{[ab]})^x{}_y$ is denoted k_x. The considerations presented above for b_a apply to the more general k_x. In any case, the introduction of coefficients for Lorentz violation suffices to encompass the description of Lorentz violation from any source that maintains coordinate independence of physics.

B. Action and covariant conservation laws

From the perspective of physics at our present comparatively low energies, the underlying fundamental theory of nature appears as a four-dimensional effective field theory. The action of this theory is expected to incorporate the standard model (SM) of particle physics, including gravitational

couplings and a purely gravitational sector. Assuming that gravity can be described using the vierbein and spin connection, it is reasonable to suppose that the action of the effective theory also contains the usual minimal gravitational couplings and the Einstein-Hilbert action among its terms.

Whatever the underlying structure, the physics of the effective field theory is also expected to be coordinate independent. This corresponds to covariance under general coordinate and local observer Lorentz transformations. Then, assuming the fundamental theory indeed incorporates a mechanism for Lorentz violation, it follows that the action contains terms involving operators with nontrivial local Lorentz transformations contracted with coefficients for Lorentz violation. The resulting effective field theory is the SME, as already mentioned in the Introduction.

The present work considers the structure and some implications of the SME in Riemann-Cartan spacetime. As an effective field theory, the SME action contains an infinite number of terms, but typically the physics is dominated by operators of low mass dimension. In addition to the usual SM and Einstein-Hilbert terms, possible higher-order terms involving SM fields, and possible higher-order curvature and torsion couplings, the terms of comparatively low mass dimension include ones violating local Lorentz symmetry. Later sections of this work explicitly display the dominant Lorentz-violating terms involving the vierbein, spin connection, and SM fields. It is straightforward to extend the analysis to include Lorentz-violating couplings of other hypothesized fields.

The Lorentz-violating piece $S_{\rm LV}$ of the SME effective action $S_{\rm SME}$ consists of a series of terms, each of which can be expressed as the observer-covariant integral of the product of a coefficient k_x for Lorentz violation with an operator J^x:

$$S_{\rm LV} \supset \int d^4x e k_x J^x. \tag{2}$$

The coefficient k_x transforms in the covariant x representation of the observer Lorentz group, while the operator J^x transforms in the corresponding contravariant representation. In the present context, J^x is understood to be formed from the vierbein, spin connection, and SM fields and is invariant under general coordinate transformations. This structure of the effective action is independent of the origin of the Lorentz violation, and in particular it is independent of whether the violation in the underlying theory is spontaneous or explicit. In practice, for many (but not all) calculations, the coefficient k_x can be treated as if it represents explicit violation even when its origin lies in the development of a vacuum value.

The covariant energy-momentum conservation law and the symmetry property of the energy-momentum tensor are modified in the presence of explicit Lorentz violation. To obtain these conditions, separate the action $S_{\rm SME}$ into a piece $S_{\rm gravity}$ involving only the vierbein and spin connection and a piece $S_{\rm matter}$ containing the remainder. The matter action $S_{\rm matter}$ in turn can be split into a Lorentz-invariant part $S_{\rm matter,0}$ and a Lorentz-violating part $S_{\rm matter,LV}$. In accordance with the above discussion, any term in the latter then has the general form

$$S_{\rm matter,LV} = \int d^4x e k_x J^x(f^y, e^\mu{}_a D_\mu f^y), \tag{3}$$

where the operator J^x can in this case be viewed as a current formed from matter fields f^y and their covariant derivatives, assuming minimal couplings for simplicity. The desired energy-momentum conditions follow from the properties of these terms under local Lorentz and general coordinate transformations when the vierbein and spin connection are treated as background couplings fixing the Riemann-Cartan spacetime.

Consider in particular a special variation of the action $S_{\rm matter}$ in which all fields and backgrounds are allowed to vary, including the coefficients for explicit Lorentz violation, but in which the equations of motion are obeyed for the dynamical fields f^x. The resulting change in the action takes the form

$$\delta S_{\rm matter} = \int d^4x e \left(T_e^{\mu\nu} e_{\nu a} \delta e_\mu{}^a + \frac{1}{2} S_\omega{}^\mu{}_{ab} \delta \omega_\mu{}^{ab} + e J^x \delta k_x \right). \tag{4}$$

This expression can be taken to define the energy-momentum tensor $T_e{}^{\mu\nu}$ associated with the vierbein and the spin-density tensor $S_\omega{}^\mu{}_{ab}$ associated with the spin connection. The reader is cautioned that in a Riemann-Cartan spacetime $T_e{}^{\mu\nu}$ typically differs from the (Belinfante) energy-momentum tensor $T_g{}^{\mu\nu}$ obtained by variation with respect to the metric, whether or not Lorentz violation is present. Similarly, the definition of $S_\omega{}^\lambda{}_{ab}$ differs from those of the spin-density tensors $S_T{}^\lambda{}_{\mu\nu}$ and $S_K{}^\lambda{}_{\mu\nu}$ obtained by varying with respect to the torsion and contortion, respectively. The tensors defined here are the most convenient for practical purposes because they are the sources in the equations of motion for the vierbein and the spin connection. The usual Einstein general relativity involving coupling to the symmetric energy-momentum tensor $T_g{}^{\mu\nu}$ is contained in this discussion as a special case with vanishing torsion.

When the special variation (4) is induced by infinitesimal local Lorentz transformations parametrized by ϵ^{ab}, the relevant infinitesimal changes in the vierbein, spin connection, and coefficients for Lorentz violation take the form

$$\begin{aligned} \delta e_\mu{}^a &= -\epsilon^a{}_b e_\mu{}^b, \\ \delta \omega_\mu{}^{ab} &= -\epsilon^a{}_c \omega_\mu{}^{cb} + \epsilon^{cb} \omega_\mu{}^a{}_c + \partial_\mu \epsilon^{ab}, \\ \delta k_x &= -\frac{1}{2} \epsilon^{ab} (X_{[ab]})^y{}_x k_y . \end{aligned} \tag{5}$$

A suitable substitution of these results into Eq. (4) followed by some manipulation then yields the desired condition on the symmetry of the energy-momentum tensor $T_e{}^{\mu\nu}$ in the presence of coefficients for explicit Lorentz violation:

$$T_e{}^{\mu\nu} - T_e{}^{\nu\mu} = (D_\alpha - T^\beta{}_{\beta\alpha}) S_\omega{}^{\alpha\mu\nu} + e^{\mu a} e^{\nu b} k_x (X_{[ab]})^x{}_y J^y. \tag{6}$$

In the Minkowski-spacetime limit, this equation becomes

$$\Theta_c{}^{\mu\nu} - \Theta_c{}^{\mu\nu} = \partial_\alpha S_c{}^{\alpha\mu\nu} + k_x (X^{[\mu\nu]})^x{}_y J^y, \tag{7}$$

where $\Theta_c{}^{\mu\nu}$ is the canonical energy-momentum tensor and $S_c{}^{\lambda\mu\nu}$ is the canonical spin-density tensor. With appropriate substitutions for the matter fields and coefficients for Lorentz violation, Eq. (7) correctly reproduces the results in Minkowski spacetime obtained in Ref. [3].

If instead the special variation (4) is induced by a general coordinate transformation with parameter ϵ^μ, the relevant field variations are the Lie derivatives

$$\begin{aligned} \delta e_\mu{}^a &= \mathcal{L}_\epsilon e_\mu{}^a = e_\nu{}^a \partial_\mu \epsilon^\nu + \partial_\nu e_\mu{}^a \epsilon^\nu, \\ \delta \omega_\mu{}^{ab} &= \mathcal{L}_\epsilon \omega_\mu{}^{ab} = \omega_\nu{}^{ab} \partial_\mu \epsilon^\nu + \partial_\nu \omega_\mu{}^{ab} \epsilon^\nu, \\ \delta k_x &= \mathcal{L}_\epsilon k_x = \epsilon^\mu \partial_\mu k_x . \end{aligned} \tag{8}$$

Substituting these expressions appropriately into Eq. (4), manipulating the result, and incorporating the condition (6) yields the covariant energy-momentum conservation law in the presence of coefficients for explicit Lorentz violation:

$$(D_\mu - T^\lambda{}_{\lambda\mu}) T_e{}^\mu{}_\nu + T^\lambda{}_{\mu\nu} T_e{}^\mu{}_\lambda + \frac{1}{2} R^{ab}{}_{\mu\nu} S_\omega{}^\mu{}_{ab} - J^x D_\nu k_x = 0. \tag{9}$$

In the limiting case of Minkowski spacetime, where the curvature and torsion vanish, this equation becomes a modified conservation law for the canonical energy-momentum tensor

$$\partial_\mu \Theta_c{}^{\mu\nu} = J^x \partial^\nu k_x . \tag{10}$$

Explicit substitution for the fields and currents shows that this result agrees with the Minkowski-spacetime results of Ref. [3], as expected. The interesting issue of the compatibility of the relations (6), (9) with the underlying geometrical assumptions of the Riemann-Cartan spacetime is discussed in Sec. V.

A similar chain of reasoning can be adopted to obtain the symmetry property of the energy-momentum tensor and the covariant energy-momentum conservation law relevant in the case of spontaneous Lorentz violation. Since spontaneous violation of a symmetry leaves unaffected the associated conserved currents, it is to be expected that in this case the terms involving k_x in Eqs. (6) and (9) are absent. This is indeed confirmed by calculation. The basic point is that coefficients originating from spontaneous breaking are vacuum values of fields, and so they must obey the corresponding equations of motion. Just as the variations δf^x of other dynamical fields f^x have vanishing coefficients in Eq. (4) and so provide no contributions to the covariant energy-momentum and spin-density conservation laws, no contributions arise from the variation δk_x when Lorentz symmetry is spontaneously broken.

III. QED EXTENSION

The basic nongravitational fields for the Lorentz- and CPT-violating QED extension in Riemann-Cartan spacetime are a Dirac fermion ψ and the photon A_μ. The action for the theory can be expressed as a sum of partial actions of the form

$$S = S_\psi + S_A + S_{\text{gravity}} + \cdots . \tag{11}$$

The fermion part S_ψ of the action S contains terms dominating at low energies that involve fermions and their minimal couplings to photons and gravity. The photon part S_A contains terms dominating at low energies that involve only photons and their minimal couplings to gravity, while the pure-gravity part S_{gravity} involves only the vierbein and the spin connection. The ellipsis represents higher-order terms, including ones involving fermions and photons that are nonrenormalizable in the Minkowski-spacetime limit, ones involving nonminimal and higher-order gravitational couplings, and ones involving field operators of dimension greater than 4 that couple curvature and torsion to the matter and photon fields. Other possible nonminimal operators formed from the fermion and photon fields, such as ones breaking U(1) gauge invariance, may also be of interest for certain considerations and can be included as appropriate.

This section presents the explicit form of the two partial actions S_ψ and S_A and some of their basic physical implications. Discussion of the gravity partial action is deferred to Sec. V.

A. Fermion sector

The fermion partial action for the QED extension can be written as

$$S_\psi = \int d^4x \left(\frac{1}{2} i e e^\mu{}_a \bar{\psi} \Gamma^a \overleftrightarrow{D}_\mu \psi - e \bar{\psi} M \psi \right). \tag{12}$$

In this equation, the symbols Γ^a and M are defined by

$$\begin{aligned} \Gamma^a \equiv{}& \gamma^a - c_{\mu\nu} e^{\nu a} e^\mu{}_b \gamma^b - d_{\mu\nu} e^{\nu a} e^\mu{}_b \gamma_5 \gamma^b - e_\mu e^{\mu a} \\ & - i f_\mu e^{\mu a} \gamma_5 - \frac{1}{2} g_{\lambda\mu\nu} e^{\nu a} e^\lambda{}_b e^\mu{}_c \sigma^{bc} \end{aligned} \tag{13}$$

and

$$M \equiv m + i m_5 \gamma_5 + a_\mu e^\mu{}_a \gamma^a + b_\mu e^\mu{}_a \gamma_5 \gamma^a + \frac{1}{2} H_{\mu\nu} e^\mu{}_a e^\nu{}_b \sigma^{ab}. \tag{14}$$

The first term of Eq. (13) leads to the usual Lorentz-invariant kinetic term for the Dirac field. Similarly, the first two terms of Eq. (14) lead to a Lorentz-invariant mass. In the absence of anomalies, the coefficient m_5 can be chirally rotated to zero in Minkowski spacetime without loss of generality. The same holds here provided suitable redefinitions of certain coefficients are made. The coefficients for Lorentz violation a_μ, b_μ, $c_{\mu\nu}$, $d_{\mu\nu}$, e_μ, f_μ, $g_{\lambda\mu\nu}$, $H_{\mu\nu}$ typically vary with position, in accordance with the discussion in Sec. II A. They

have no particular symmetry, except for the defining antisymmetry of $H_{\mu\nu}$ and of $g_{\lambda\mu\nu}$ on two indices. By assumption, the action (12) is hermitian, which constrains the coefficients for Lorentz violation to be real. Relaxing the latter constraint would permit the formalism to describe also nonhermitian Lorentz violation. Note the use of an uppercase letter for $H_{\mu\nu}$, which avoids conflicts with the metric fluctuation $h_{\mu\nu}$.

The action (12) is also locally U(1) invariant, by construction. The covariant derivative D_μ appearing in it is understood to be a combination of the spacetime covariant derivative, discussed in Appendix A, and the usual U(1) covariant derivative:

$$D_\mu\psi \equiv \partial_\mu\psi + \frac{1}{4} i\omega_\mu{}^{ab}\sigma_{ab}\psi - iqA_\mu\psi. \tag{15}$$

It is convenient to introduce the symbol $(\overline{\psi D_\mu})$ for the action of the covariant derivative on a Dirac-conjugate field $\overline{\psi}$:

$$(\overline{\psi D_\mu}) \equiv \partial_\mu\overline{\psi} - \frac{1}{4} i\omega_\mu{}^{ab}\overline{\psi}\sigma_{ab} + iqA_\mu\overline{\psi}. \tag{16}$$

In terms of these quantities, the covariant derivative appears in the action (12) in a combination defined by

$$\overline{\chi}\Gamma^a\vec{D}_\mu\psi \equiv \overline{\chi}\Gamma^a D_\mu\psi - (\overline{\chi D}_\mu)\Gamma^a\psi. \tag{17}$$

This definition is understood to hold even when Γ^a is spacetime-position dependent.

The generalized Dirac equation arising from the action S_ψ is

$$ie^\mu{}_a\Gamma^a D_\mu\psi - M\psi - \frac{1}{2} iT^\lambda{}_{\lambda\mu}e^\mu{}_a\Gamma^a\psi + \frac{1}{2} ie^\mu{}_a\omega_\mu{}^{bc}\left(\eta^a{}_b\Gamma_c + \frac{1}{4} i[\sigma_{bc},\Gamma^a]\right)\psi = 0. \tag{18}$$

As might be expected from nonderivative couplings, the Lorentz-violating terms involving M just add to the Dirac equation in a minimal way. However, those involving Γ^a appear both minimally and through commutation with the Lorentz generators in the covariant derivative. In particular, the Lorentz-invariant parts of the last two terms in Eq. (18) cancel, but the terms involving coefficients for Lorentz violation yield nonzero results.

Many physical features of this theory are expected to be similar to the QED extension in Minkowski spacetime introduced in Ref. [3]. Although beyond the scope of the present work, it would be of definite interest to investigate the corrections to established results [3,13,16,27,29,33] arising from the Riemann-Cartan couplings. A detailed study of quantum corrections and renormalization issues may be particularly challenging, since a satisfactory description of these is an open issue even for conventional Lorentz-invariant theories in curved spacetime [34]. Similar remarks apply to the causal and light-cone structure of the theory, which remains the subject of discussion even for Lorentz-invariant radiative corrections [35].

One difference between the QED extension in Minkowski and Riemann-Cartan spacetimes is that the presence of even weak gravitational couplings can change the effective properties of certain coefficients for Lorentz violation. Adopting the weak-field form of the vierbein and spin connection given in Eq. (A20) of Appendix A and extracting from the Lagrangian only terms that are linear in small quantities, one finds

$$\mathcal{L}_\psi \supset -i(c_{\text{eff}})_{\mu\nu}\overline{\psi}\gamma^\mu\partial^\nu\psi - (b_{\text{eff}})_\mu\overline{\psi}\gamma_5\gamma^\mu\psi, \tag{19}$$

where

$$(c_{\text{eff}})_{\mu\nu} \equiv c_{\mu\nu} - \frac{1}{2} h_{\mu\nu} + \chi_{\mu\nu},$$

$$(b_{\text{eff}})_\mu \equiv b_\mu - \frac{1}{4}\partial^\alpha\chi^{\beta\gamma}\epsilon_{\alpha\beta\gamma\mu} + \frac{1}{8} T^{\alpha\beta\gamma}\epsilon_{\alpha\beta\gamma\mu}. \tag{20}$$

In this expression, leading-order terms arising from the scaling of the vierbein determinant e are neglected because they are Lorentz invariant.

Equations (20) show that at leading order a weak background metric appears as a $c_{\mu\nu}$ term, while the dual of the antisymmetric part of the torsion behaves like a b_μ term, a result already noted elsewhere [36]. The latter is a *CPT*-violating term, so the presence of background torsion can mimic *CPT* violation. Experimental effects from these terms have been estimated for some situations, including hydrogen spectral line shifts in the solar gravitational field [37] and reinterpretations of various recent results [38]. Note, however, that these gravitational couplings are flavor independent, whereas the values of b_μ and $c_{\mu\nu}$ can depend on the fermion species. This implies caution is required in interpreting the existing experimental sensitivities to b_μ in terms of torsion, since some experiments are sensitive only to a nonzero difference in the value of b_μ for two fermion species. It further suggests that careful comparative experiments could distinguish background curvature and torsion effects from other sources of Lorentz and *CPT* violation. Note also that the inclusion of subleading terms in the derivation would yield additional Lorentz-violating effects. For example, at this level all dimension-one effective coefficients for Lorentz violation acquire a torsion dependence that can vary with flavor. Couplings of this type may play an important role in regions of possibly large torsion, such as spinning black holes or the early Universe.

Another issue worth mention is the observability of various types of Lorentz violation. A given coefficient k_x for Lorentz violation leads to observable effects only when the theory contains another conventional or Lorentz-violating coupling that precludes the elimination of k_x through field or coordinate redefinitions. In the Minkowski-spacetime limit of the QED extension, the comparatively small number of couplings leaves the freedom to eliminate some Lorentz-

violating terms [3,39,40]. As might be expected, the presence of the additional curvature and torsion couplings in the Riemann-Cartan spacetime reduces this freedom, but some options remain.

As a first example, consider a position-dependent redefinition of the phase of the spinor:

$$\psi(x) = \exp[if(x)]\chi(x). \tag{21}$$

This is *not* a gauge transformation, since A_μ remains unchanged. In the single-fermion Minkowski-spacetime limit with constant a_μ, the choice $f(x) = a_\mu x^\mu$ can be used to eliminate all four coefficients a_μ, so a_μ is unphysical. However, in Riemann-Cartan spacetime, the redefinition can typically be used to eliminate only one of the four coefficients a_μ. An exception to this occurs for special models in which a_μ arises as the four-derivative of a scalar, in which case a_μ is unphysical and can be removed.

Another useful class of redefinitions consists of ones taking the general form

$$\psi(x) = [1 + v(x)\cdot\Gamma]\chi(x). \tag{22}$$

Here, $v(x)$ is a set of complex functions with appropriate local Lorentz indices and, for this equation only, Γ represents one of γ^a, $\gamma_5\gamma^a$, σ^{ab}. These redefinitions can be regarded as position-dependent mixings of components in spinor space. They can be used to show that, at leading order in coefficients for Lorentz violation, there are no physical effects from the coefficients e_μ, f_μ or from the antisymmetric parts of $c_{\mu\nu}$, $d_{\mu\nu}$. However, attempting to remove the antisymmetric and trace parts of $g_{\lambda\mu\nu}$ generically introduces spacetime-dependent mass terms proportional to the covariant derivative of v, a feature absent in the Minkowski-spacetime limit.

The freedom to redefine spacetime coordinates, perhaps accompanied by field and coupling rescalings, can also be viewed as a means of eliminating or interrelating certain coefficients for Lorentz violation. The symmetric piece of the coefficients $c_{\mu\nu}$ and the 9_s part of the photon-sector coefficient $(k_F)_{\kappa\lambda\mu\nu}$, which is introduced in the next subsection, appear in the action in a form similar to parts of the metric coupling. Appropriate coordinate choices can therefore appear to move the Lorentz violation from one sector to the other, or perhaps act to cancel effects between sectors. The coordinate frame used in reporting experimental results is often implicitly fixed by the experimental setup, for example, by the choice of a standard clock or rod. Particular care is therefore required in claiming or interpreting sensitivities to these types of coefficients. An explicit example of this type of redefinition is given for the case of Minkowski spacetime in Sec. II C of Ref. [29], where a constant coefficient of the type c_{00} is converted into the combination $(k_F)_{0j0j}$. When background curvature and torsion fields are present, the position dependence can complicate the analysis of these types of redefinitions and can introduce other effects such as spacetime-varying couplings.

To conclude this subsection, here are a few remarks about nonminimal gravitational couplings. For simplicity, attention is restricted here to operators of mass dimension four or less. In the QED extension there are comparatively few such nonminimal operators, and the only gauge-invariant ones are products of the torsion with fermion bilinears. The Lorentz-invariant possibilities are

$$\begin{aligned}\mathcal{L}_{\rm LI} = {} & ae{T^\lambda}_{\lambda\mu}\overline{\psi}\gamma^\mu\psi + be{T^\lambda}_{\lambda\mu}\overline{\psi}\gamma_5\gamma^\mu\psi \\ & + a_5 eT^{\alpha\beta\gamma}\epsilon_{\alpha\beta\gamma\mu}\overline{\psi}\gamma^\mu\psi + b_5 eT^{\alpha\beta\gamma}\epsilon_{\alpha\beta\gamma\mu}\overline{\psi}\gamma_5\gamma^\mu\psi.\end{aligned} \tag{23}$$

The last of these already occurs in the minimal couplings. The Lorentz-violating possibilities are

$$\begin{aligned}\mathcal{L}_{\rm LV} = {} & ek_{\alpha\beta\gamma}T^{\alpha\beta\gamma}\overline{\psi}\psi + ek_{5\alpha\beta\gamma}T^{\alpha\beta\gamma}\overline{\psi}\gamma_5\psi \\ & + ek_{\alpha\beta\gamma\delta}T^{\alpha\beta\gamma}\overline{\psi}\gamma^\delta\psi + ek_{5\alpha\beta\gamma\delta}T^{\alpha\beta\gamma}\overline{\psi}\gamma_5\gamma^\delta\psi \\ & + ek_{\alpha\beta\gamma\delta\epsilon}T^{\alpha\beta\gamma}\overline{\psi}\sigma^{\delta\epsilon}\psi.\end{aligned} \tag{24}$$

If Lorentz violation is suppressed as expected and the torsion is also small, then all five of the latter are subdominant. Also, if the torsion is constant or sufficiently slowly varying, only the last three are relevant. Nonetheless, all the above operators may be of interest in more exotic scenarios. Note that the presence of fundamental scalars, such as the Higgs doublet in the SME, permits other types of nonminimal gravitational couplings of dimension four or less, including ones involving both curvature and torsion. Note also that any operators of dimension greater than four must come with one or more inverse powers of mass, which may represent substantial Planck-scale suppression. However, some care is required in determining the relative dominance of operators. For example, a dimension-five Lorentz-invariant operator suppressed by the Planck mass m_P would produce effects comparable in magnitude to those of a dimension-four operator involving a coefficient for Lorentz violation suppressed by m_P.

B. Photon sector

The photon part of the action for the QED extension in Riemann-Cartan spacetime can be separated into two pieces,

$$S_A = \int d^4x(\mathcal{L}_F + \mathcal{L}_A), \tag{25}$$

where

$$\mathcal{L}_F = -\frac{1}{4}eF_{\mu\nu}F^{\mu\nu} - \frac{1}{4}e(k_F)_{\kappa\lambda\mu\nu}F^{\kappa\lambda}F^{\mu\nu}, \tag{26}$$

$$\mathcal{L}_A = \frac{1}{2}e(k_{AF})^\kappa\epsilon_{\kappa\lambda\mu\nu}A^\lambda F^{\mu\nu} - e(k_A)_\kappa A^\kappa. \tag{27}$$

The Lagrangian terms are hermitian provided the coefficients for Lorentz violation $(k_F)_{\kappa\lambda\mu\nu}$, $(k_{AF})_\mu$, and $(k_A)_\mu$ are real. The electromagnetic field strength $F_{\mu\nu}$ is defined by the locally U(1)-invariant form

$$F_{\mu\nu} \equiv D_\mu A_\nu - D_\nu A_\mu + {T^\lambda}_{\mu\nu}A_\lambda = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{28}$$

By definition, all curvature and torsion contributions cancel in the field strength. Gravitational effects in the photon-sector Lagrangian therefore are associated with the appearance of the metric in the index contractions and with the scaling by the vierbein determinant e.

The generalized Maxwell equations obtained from the action (25) are conveniently written using the standard Riemann-spacetime covariant derivative $\tilde{D}_\mu$, described in Appendix A. They consist of the homogeneous equations

$$\tilde{D}_\lambda F_{\mu\nu} + \tilde{D}_\mu F_{\nu\lambda} + \tilde{D}_\nu F_{\lambda\mu} = 0, \tag{29}$$

which follow from the definition (28) of the field strength, and the inhomogeneous equation obtained by varying the sum of the fermion action (12) and the photon action (25):

$$\tilde{D}_\alpha F_\mu{}^\alpha + \tilde{D}_\alpha[(k_F)_{\mu\alpha\beta\gamma} F^{\beta\gamma}] + (k_{AF})^\alpha \epsilon_{\mu\alpha\beta\gamma} F^{\beta\gamma} + (k_A)_\mu = j_\mu. \tag{30}$$

In this equation, the current j_μ is

$$j^\mu = q e^\mu{}_a \bar{\psi} \Gamma^a \psi. \tag{31}$$

These results correctly reduce to the usual QED extension in the Minkowski-spacetime limit.

Consider first the Lagrangian $\mathcal{L}_F$, which is invariant under local U(1) transformations by construction. The first term in $\mathcal{L}_F$ is the Lorentz-invariant action for photons in a Riemann-Cartan background, while the second term violates Lorentz invariance. Both terms are *CPT* even. The coefficient $(k_F)_{\kappa\lambda\mu\nu}$ for Lorentz violation is antisymmetric on the first two and on the last two indices, and it is symmetric under interchange of the first and last pair of indices. These symmetries reduce the number of independent components of $(k_F)_{\kappa\lambda\mu\nu}$ to 21. Decomposing into irreducible Lorentz multiplets gives $21 = 1_a + (1+9+10)_s$.

The antisymmetric singlet 1_a provides a Lorentz-invariant parity-odd coupling $k_1 \equiv \epsilon^{\kappa\lambda\mu\nu}(k_F)_{\kappa\lambda\mu\nu}$. Its coupling in the Lagrangian is therefore proportional to $e k_1 F_{\mu\nu} \tilde{F}^{\mu\nu}$, where $\tilde{F}$ is the dual field strength. Integrating by parts and discarding the surface term under the usual assumption of no monopoles converts this into an expression proportional to $e(D_\mu k_1) A_\nu \tilde{F}^{\mu\nu}$. In the Minkowski-spacetime limit with constant $(k_F)_{\kappa\lambda\mu\nu}$, no net effect results. In the present more general case with position-dependent $(k_F)_{\kappa\lambda\mu\nu}$, the expression can instead be absorbed into the term involving the coefficient $(k_{AF})_\mu$ in $\mathcal{L}_A$. This conversion of a scalar into a Lorentz-violating coefficient has features in common with the generation of a nonzero $(k_{AF})_\mu$ through the gradient of the axion in supergravity cosmology [13].

Of the remaining 20 independent coefficients, the symmetric singlet 1_s is the irreducible double trace, which is Lorentz invariant. It can be regarded as renormalizing the Lorentz-invariant kinetic term. If $(k_F)_{\kappa\lambda\mu\nu}$ varies with position, this renormalization corresponds to a spacetime variation of the fine structure constant α. If instead $(k_F)_{\kappa\lambda\mu\nu}$ is constant, as is usually assumed in the Minkowski-spacetime limit, then the 1_s generates only an unobservable constant shift of α. The couplings of the remaining 9_s and 10_s Lorentz-violating terms are similar to those in Minkowski spacetime [3,29] but now typically vary with position. These 19 coefficients control the leading-order *CPT*-even Lorentz violation in the photon sector.

Next, consider the Lagrangian $\mathcal{L}_A$ in Eq. (27), which consists of *CPT*-odd terms. The corresponding partial action is U(1) gauge invariant only under special circumstances. Assuming no monopoles, as before, the coefficients for Lorentz violation must obey

$$\tilde{D}_\mu (k_{AF})_\nu - \tilde{D}_\nu (k_{AF})_\mu = 0,$$
$$\tilde{D}_\mu (k_A)^\mu = 0, \tag{32}$$

where the tilde again indicates the zero-torsion limit. These conditions must be satisfied in addition to any dynamical or other equations determining the form of $(k_{AF})_\mu$ and $(k_A)_\mu$. For $(k_{AF})_\mu$, an example of this is known: the mechanism for Lorentz violation in the supergravity cosmology of Ref. [13] enforces $(k_{AF})_\mu \equiv \partial_\mu \phi$ for an axion scalar ϕ, which satisfies the requirement (32). However, for the coefficient $(k_A)_\mu$, Eq. (32) implies $(k_A)_\mu = (k_0)_\mu / e$, where $(k_0)_\mu$ is a constant 4-vector. Generic manifolds do not admit such vectors, so $(k_A)_\mu$ must typically vanish. This is consistent with other requirements emerging in the Minkowski-spacetime limit [3].

As in the fermion sector, the presence of weak gravitational couplings can affect the interpretation of certain coefficients for Lorentz violation. The leading-order weak-field couplings can be extracted from the Lorentz-invariant part of the Lagrangian $\mathcal{L}_F$ using the expression (A20) of Appendix A. The result is a contribution that has the operator structure of the $(k_F)_{\kappa\lambda\mu\nu}$ term, with an effective coefficient $(k_{F,\mathrm{eff}})_{\kappa\lambda\mu\nu}$ given by

$$(k_{F,\mathrm{eff}})_{\kappa\lambda\mu\nu} = (k_F)_{\kappa\lambda\mu\nu} + \frac{1}{2}(\eta_{\kappa\mu} h_{\lambda\nu} + \eta_{\lambda\nu} h_{\kappa\mu} - \eta_{\kappa\nu} h_{\lambda\mu} - \eta_{\lambda\mu} h_{\kappa\nu}). \tag{33}$$

A weak-field background metric can therefore partially simulate the effect of the coefficient $(k_F)_{\kappa\lambda\mu\nu}$ for Lorentz violation. Some of the physical implications of this coupling can be appreciated by converting to the notation of Ref. [29]. Setting $(k_{AF})_\mu$ and $(k_F)_{\kappa\lambda\mu\nu}$ to zero for simplicity, only the coefficients $(\tilde{\kappa}_{e-})^{jk}$, $(\tilde{\kappa}_{o+})^{jk}$, $\tilde{\kappa}_{\mathrm{tr}}$ acquire nonzero contributions, given by

$$(\tilde{\kappa}_{e-})^{jk} = -h^{jk} + \frac{1}{3} h^{ll} \eta^{jk},$$
$$(\tilde{\kappa}_{o+})^{jk} = -\epsilon^{jkl} h^{0l},$$
$$\tilde{\kappa}_{\mathrm{tr}} = \frac{2}{3} h^{ll}. \tag{34}$$

One consequence is that both polarizations of light are affected in the same way, so no birefringence occurs. Experi-

ments with sensitivity to these coefficients could therefore be adapted to study the background-metric fluctuation $h_{\mu\nu}$, provided the signals in question involve no complete cancellation of the effects.

As a final remark, note that the combined action (12) and (25) for the leading-order QED extension can be used to obtain a general classical action for the Lorentz-violating behavior of point test particles and electrodynamic fields in a Riemann-Cartan background. Although its explicit form lies beyond the scope of the present work, the resulting theory would represent a useful test model for Lorentz-violating physics. For example, it could be used to provide insight into the interpretation of classical concepts such as mass, velocity, and geodesic trajectories, each of which typically is split by Lorentz violation into distinct notions that merge in the Lorentz-invariant limit [3]. It would also be of interest to obtain the connection between this theory and the $TH\epsilon\mu$ formalism [41,42], which is a widely used model involving a four-parameter action with modified classical test particles and electrodynamic fields in a conventional static and spherically symmetric Riemann background.

IV. STANDARD-MODEL EXTENSION

The action $S_{\rm SME}$ for the full SME in a Riemann-Cartan spacetime can conveniently be expressed as a sum of partial actions

$$S_{\rm SME}=S_{\rm SM}+S_{\rm LV}+S_{\rm gravity}+\cdots . \tag{35}$$

The term $S_{\rm SM}$ is the SM action, modified by the addition of gravitational couplings appropriate for a background Riemann-Cartan spacetime. The term $S_{\rm LV}$ contains all Lorentz- and *CPT*-violating terms that involve SM fields and dominate at low energies, including minimal gravitational couplings. The term $S_{\rm gravity}$ represents the pure-gravity sector, constructed from the vierbein and the spin connection and incorporating possible Lorentz and *CPT* violation. The ellipsis represents contributions to $S_{\rm SME}$ that are of higher order at low energies, some of which violate Lorentz symmetry. It includes terms nonrenormalizable in the Minkowski-spacetime limit, nonminimal and higher-order gravitational couplings, and operators of mass dimension greater than four coupling curvature and torsion to SM fields. Other possible nonminimal operators formed from SM fields, such as ones that break the $SU(3)\times SU(2)\times U(1)$ gauge invariance, can be included as needed. For example, these could play a significant role in the neutrino sector [32].

In this section, the explicit forms of $S_{\rm SM}$ and $S_{\rm LV}$ are presented, while discussion of the gravity action $S_{\rm gravity}$ is deferred to Sec. V. The notation adopted for the basic SM fields is as follows. First, consider the fermion sector. Introduce the generation index $A=1,2,3$, so that the three charged leptons are denoted $l_A\equiv(e,\mu,\tau)$, the three neutrinos are $\nu_A\equiv(\nu_e,\nu_\mu,\nu_\tau)$, and the six quark flavors are $u_A\equiv(u,c,t)$, $d_A\equiv(d,s,b)$. The color index on the quarks is suppressed for simplicity. Define as usual the left- and right-handed spinor components $\psi_L\equiv\frac{1}{2}(1-\gamma_5)\psi$, $\psi_R\equiv\frac{1}{2}(1+\gamma_5)\psi$. The right-handed leptons and quarks are SU(2) singlets, $R_A=(l_A)_R$, $U_A=(u_A)_R$, $D_A=(d_A)_R$. The left-handed leptons and quarks form SU(2) doublets, $L_A=[(\nu_A)_L,(l_A)_L]^T$, $Q_A=[(u_A)_L,(d_A)_L]^T$.

In the boson sector, the Higgs doublet ϕ is taken to have the form $\phi=(0,r_\phi)^T/\sqrt{2}$ in unitary gauge, and the conjugate doublet is denoted ϕ^c. The color gauge fields are denoted by the hermitian SU(3) adjoint matrix G_μ. The SU(2) gauge fields also form a hermitian adjoint matrix denoted W_μ, while the hermitian singlet hypercharge gauge field is B_μ. The associated field strengths are $G_{\mu\nu}$, $W_{\mu\nu}$, and $B_{\mu\nu}$. They are defined by expressions of the standard form in Minkowski spacetime, except that the Riemann-Cartan covariant derivative is used and a torsion term is added in analogy to Eq. (28). This ensures that all spacetime curvature and torsion contributions cancel in the field strengths, which therefore have conventional $SU(3)\times SU(2)\times U(1)$ properties.

The covariant derivative D_μ and its conjugate $\overline{D}_\mu$ are now understood to be both spacetime covariant and $SU(3)\times SU(2)\times U(1)$ covariant, in parallel with the electromagnetic-U(1) and spacetime covariant derivative (15) and its conjugate (16). The definition (17) is maintained. As usual, the coupling strengths for the three groups SU(3), SU(2), and U(1) are g_3, g, and g', respectively. Also, the charge q for the electromagnetic U(1) group and the angle θ_W are defined through $q=g\sin\theta_W=g'\cos\theta_W$.

Consider first the action $S_{\rm SM}$ for the SM in a Riemann-Cartan background. The corresponding Lagrangian $\mathcal{L}_{\rm SM}$ is $SU(3)\times SU(2)\times U(1)$ gauge invariant, and it is convenient to separate it into five parts:

$$\mathcal{L}_{\rm SM}=\mathcal{L}_{\rm lepton}+\mathcal{L}_{\rm quark}+\mathcal{L}_{\rm Yukawa}+\mathcal{L}_{\rm Higgs}+\mathcal{L}_{\rm gauge}. \tag{36}$$

The lepton sector has Lagrangian $\mathcal{L}_{\rm lepton}$ given by

$$\mathcal{L}_{\rm lepton}=\frac{1}{2}iee^\mu{}_a\overline{L}_A\gamma^a\overleftrightarrow{D}_\mu L_A+\frac{1}{2}iee^\mu{}_a\overline{R}_A\gamma^a\overleftrightarrow{D}_\mu R_A, \tag{37}$$

while the quark sector Lagrangian $\mathcal{L}_{\rm quark}$ is

$$\mathcal{L}_{\rm quark}=\frac{1}{2}iee^\mu{}_a\overline{Q}_A\gamma^a\overleftrightarrow{D}_\mu Q_A+\frac{1}{2}iee^\mu{}_a\overline{U}_A\gamma^a\overleftrightarrow{D}_\mu U_A+\frac{1}{2}iee^\mu{}_a\overline{D}_A\gamma^a\overleftrightarrow{D}_\mu D_A. \tag{38}$$

The Yukawa couplings are

$$\mathcal{L}_{\rm Yukawa}=-[(G_L)_{AB}e\overline{L}_A\phi R_B+(G_U)_{AB}e\overline{Q}_A\phi^c U_B+(G_D)_{AB}e\overline{Q}_A\phi D_B]+{\rm H.c.}, \tag{39}$$

where $(G_L)_{AB}$, $(G_U)_{AB}$, $(G_D)_{AB}$ are the Yukawa-coupling matrices. The Higgs sector has Lagrangian

$$\mathcal{L}_{\rm Higgs}=-e(D_\mu\phi)^\dagger D^\mu\phi+\mu^2e\phi^\dagger\phi-\frac{\lambda}{3!}e(\phi^\dagger\phi)^2, \tag{40}$$

while the gauge sector is

$$\mathcal{L}_{\rm gauge} = -\frac{1}{2}e\,{\rm Tr}(G_{\mu\nu}G^{\mu\nu}) - \frac{1}{2}e\,{\rm Tr}(W_{\mu\nu}W^{\mu\nu}) - \frac{1}{4}eB_{\mu\nu}B^{\mu\nu}. \tag{41}$$

Possible θ terms are omitted in the latter for simplicity.

Next, consider the partial action $S_{\rm LV}$ containing Lorentz- and *CPT*-violating operators constructed from SM fields of mass dimension four or less. In parallel with Eq. (36), the corresponding Lagrangian $\mathcal{L}_{\rm LV}$ can be decomposed as a sum of terms separating the contributions from the lepton, quark, Yukawa, Higgs, and gauge sectors. The Lagrangians for these five sectors can be further split into pieces that are *CPT* even and odd, except for the Yukawa-type couplings for which no *CPT*-odd terms arise:

$$\mathcal{L}_{\rm LV} = \mathcal{L}^{CPT+}_{\rm lepton} + \mathcal{L}^{CPT-}_{\rm lepton} + \mathcal{L}^{CPT+}_{\rm quark} + \mathcal{L}^{CPT-}_{\rm quark} + \mathcal{L}^{CPT+}_{\rm Yukawa} + \mathcal{L}^{CPT+}_{\rm Higgs} + \mathcal{L}^{CPT-}_{\rm Higgs} + \mathcal{L}^{CPT+}_{\rm gauge} + \mathcal{L}^{CPT-}_{\rm gauge}\,. \tag{42}$$

The Lagrangian for the *CPT*-even lepton sector is

$$\mathcal{L}^{CPT+}_{\rm lepton} = -\frac{1}{2}i(c_L)_{\mu\nu AB}ee^{\mu}{}_a\overline{L}_A\gamma^a\overleftrightarrow{D}^{\nu}L_B - \frac{1}{2}i(c_R)_{\mu\nu AB}ee^{\mu}{}_a\overline{R}_A\gamma^a\overleftrightarrow{D}^{\nu}R_B, \tag{43}$$

where the dimensionless coefficients $(c_L)_{\mu\nu AB}$ and $(c_R)_{\mu\nu AB}$ can be taken to be hermitian in generation space. The spacetime traces of these coefficients preserve Lorentz symmetry. In the Minkowski-spacetime limit with conserved energy and momentum, these traces act to renormalize the fermion fields and are unobservable, but in the present context the spacetime dependence can correspond to spacetime-varying couplings. The Lagrangian for the *CPT*-odd lepton sector is

$$\mathcal{L}^{CPT-}_{\rm lepton} = -(a_L)_{\mu AB}ee^{\mu}{}_a\overline{L}_A\gamma^aL_B - (a_R)_{\mu AB}ee^{\mu}{}_a\overline{R}_A\gamma^aR_B, \tag{44}$$

where the coefficients $(a_L)_{\mu AB}$ and $(a_R)_{\mu AB}$ are also hermitian in generation space but have dimensions of mass.

The quark-sector Lagrangians take a similar form:

$$\mathcal{L}^{CPT+}_{\rm quark} = -\frac{1}{2}i(c_Q)_{\mu\nu AB}ee^{\mu}{}_a\overline{Q}_A\gamma^a\overleftrightarrow{D}^{\nu}Q_B - \frac{1}{2}i(c_U)_{\mu\nu AB}ee^{\mu}{}_a\overline{U}_A\gamma^a\overleftrightarrow{D}^{\nu}U_B - \frac{1}{2}i(c_D)_{\mu\nu AB}ee^{\mu}{}_a\overline{D}_A\gamma^a\overleftrightarrow{D}^{\nu}D_B, \tag{45}$$

$$\mathcal{L}^{CPT-}_{\rm quark} = -(a_Q)_{\mu AB}ee^{\mu}{}_a\overline{Q}_A\gamma^aQ_B - (a_U)_{\mu AB}ee^{\mu}{}_a\overline{U}_A\gamma^aU_B - (a_D)_{\mu AB}ee^{\mu}{}_a\overline{D}_A\gamma^aD_B. \tag{46}$$

Remarks analogous to those for the lepton-sector coefficients for Lorentz violation also hold for the quark-sector coefficients in these equations.

The *CPT*-even Lorentz-violating Yukawa-type operators have the usual Yukawa gauge structure but involve different fermion bilinears. The Lagrangian for these terms is

$$\mathcal{L}^{CPT+}_{\rm Yukawa} = -\frac{1}{2}[(H_L)_{\mu\nu AB}ee^{\mu}{}_ae^{\nu}{}_b\overline{L}_A\phi\sigma^{ab}R_B + (H_U)_{\mu\nu AB}ee^{\mu}{}_ae^{\nu}{}_b\overline{Q}_A\phi^c\sigma^{ab}U_B + (H_D)_{\mu\nu AB}ee^{\mu}{}_ae^{\nu}{}_b\overline{Q}_A\phi\sigma^{ab}D_B] + {\rm H.c.} \tag{47}$$

The dimensionless coefficients $(H_{L,U,D})_{\mu\nu AB}$ are antisymmetric in the spacetime indices. Like the conventional Yukawa couplings $(G_{L,U,D})_{AB}$, they can violate hermiticity in generation space.

The *CPT*-even Lagrangian in the Higgs sector is

$$\mathcal{L}^{CPT+}_{\rm Higgs} = \frac{1}{2}(k_{\phi\phi})^{\mu\nu}e(D_\mu\phi)^\dagger D_\nu\phi + {\rm H.c.} - \frac{1}{2}(k_{\phi W})^{\mu\nu}e\phi^\dagger W_{\mu\nu}\phi - \frac{1}{2}(k_{\phi B})^{\mu\nu}e\phi^\dagger\phi B_{\mu\nu}. \tag{48}$$

All the coefficients for Lorentz violation in this equation are dimensionless. The coefficient $(k_{\phi\phi})^{\mu\nu}$ can be taken to have symmetric real and antisymmetric imaginary parts, while $(k_{\phi W})^{\mu\nu}$ and $(k_{\phi B})^{\mu\nu}$ are real antisymmetric. The last two terms directly couple the Higgs scalar to the SU(2)×U(1) field strengths. They have no analogue in the usual SM. The *CPT*-odd Higgs Lagrangian is

$$\mathcal{L}^{CPT-}_{\rm Higgs} = i(k_\phi)^\mu e\phi^\dagger D_\mu\phi + {\rm H.c.} \tag{49}$$

The coefficient $(k_\phi)^\mu$ is complex valued and has dimensions of mass.

The Lagrangian for the *CPT*-even gauge sector is

$$\mathcal{L}^{CPT+}_{\rm gauge} = -\frac{1}{2}(k_G)_{\kappa\lambda\mu\nu}e\,{\rm Tr}(G^{\kappa\lambda}G^{\mu\nu}) - \frac{1}{2}(k_W)_{\kappa\lambda\mu\nu}e\,{\rm Tr}(W^{\kappa\lambda}W^{\mu\nu}) - \frac{1}{4}(k_B)_{\kappa\lambda\mu\nu}eB^{\kappa\lambda}B^{\mu\nu}. \tag{50}$$

All the coefficients for Lorentz violation in this equation are real. Each is antisymmetric on the first two and on the last two indices, and each is symmetric under interchange of the first and last pair of indices. Their spacetime properties are similar to those of the coefficient $(k_F)_{\kappa\lambda\mu\nu}$ in the photon sector of the QED extension, discussed in Sec. III B, which is itself a combination of $(k_W)_{\kappa\lambda\mu\nu}$ and $(k_B)_{\kappa\lambda\mu\nu}$. Note that possible total-derivative terms analogous to the usual θ terms in the SM are neglected in Eq. (50) for simplicity.

It is also possible to construct some CPT-odd Lagrangian terms that under special circumstances are invariant under infinitesimal $SU(3)\times SU(2)\times U(1)$ transformations. They have the Chern-Simons form

$$\begin{aligned}\mathcal{L}_{\text{gauge}}^{CPT-} &= (k_3)_\kappa \epsilon^{\kappa\lambda\mu\nu} e\,\mathrm{Tr}(G_\lambda G_{\mu\nu} + \tfrac{2}{3} i g_3 G_\lambda G_\mu G_\nu) \\ &+ (k_2)_\kappa \epsilon^{\kappa\lambda\mu\nu} e\,\mathrm{Tr}(W_\lambda W_{\mu\nu} + \tfrac{2}{3} i g W_\lambda W_\mu W_\nu) \\ &+ (k_1)_\kappa \epsilon^{\kappa\lambda\mu\nu} e B_\lambda B_{\mu\nu} + (k_0)_\kappa e B^\kappa . \end{aligned} \tag{51}$$

All the coefficients for Lorentz violation in these equations can be taken real. The coefficients $(k_{1,2,3})_\kappa$ have dimensions of mass, while $(k_0)_\kappa$ has dimensions of mass cubed. These terms are the SM analogues of those in Eq. (27) of the QED extension, which they contain as a limiting case. Their invariance requires that subsidiary conditions generalizing those in Eq. (32) be satisfied, so Eq. (51) is relevant only in special circumstances.

The above equations describe the actions S_{SM} and S_{LV} prior to the breaking of the electroweak $SU(2)\times U(1)$ symmetry to the electromagnetic U(1) subgroup. In the minimal SM in Minkowski spacetime, arguments based on energetics make this breaking plausible, at least for some range of the couplings μ and λ in Eq. (40). However, it is an open issue whether the Higgs potential in Eq. (40) suffices to drive electroweak symmetry breaking to the charge subgroup in the SM in a curved spacetime background [43]. Suppose this is indeed the case for at least some types of background, perhaps such as weak gravitational fields. Then, the presence in S_{LV} of small Lorentz-violating terms involving the Higgs and charge-neutral fields changes the pattern of expectation values that break the $SU(2)\times U(1)$ symmetry. A small Lorentz-violating expectation value emerges for the neutral Z_μ^0 field, and the expectation value of the Higgs is shifted slightly. It has been shown that this breaking pattern preserves the electromagnetic U(1) in the Minkowski-spacetime limit [3]. A careful study of this issue in Riemann-Cartan spacetime would be of interest. Note also that the standard procedure of expanding the terms in S_{SM} and S_{LV} about the vacuum expectation values generates additional effective contributions to some of the coefficients for Lorentz violation.

The presence of weak curvature and torsion couplings in the actions S_{SM} and S_{LV} can modify the interpretation of certain coefficients for Lorentz violation. The contributions of this type from S_{LV} are proportional to the product of weak fields and coefficients for Lorentz violation, so they are suppressed relative to those from S_{SM}. The expansions (A20) of Appendix A can be used to extract from S_{SM} the dominant effects. The analysis follows a pattern similar to that in the QED extension leading to Eqs. (20) and (33), with the symmetric part of the metric generating effective contributions to certain *CPT*-even Lorentz-violating terms and the torsion generating contributions to *CPT*-odd ones. The effects of the vierbein and the torsion are independent of flavor at leading order, but the sign of the torsion contribution depends on the handedness of the fermion. This is reflected in Eq. (20) for the fermion sector of the QED extension, where the coefficient $b_\mu \sim (a_L)_{\mu AB} - (a_R)_{\mu AB}$ is affected but $a_\mu \sim (a_L)_{\mu AB} + (a_R)_{\mu AB}$ is unchanged.

As in the case of the QED extension, care is required in determining the observability of a given coefficient for Lorentz violation in S_{LV} because there is freedom to eliminate certain coefficients by appropriate field and coordinate redefinitions. For example, for each fermion field there is a phase degree of freedom of the form (21) and possible reinterpretations of the spinor-space components of the form (22). There is also freedom in the Higgs sector, including the phase redefinition

$$\phi(x) = \exp[-ig(x)]\rho(x). \tag{52}$$

For instance, the choice $g(x) = (k_\phi)_\mu x^\mu$ can be used to absorb part of the effects from the coefficient $(k_\phi)_\mu$. Also, suitable coordinate redefinitions can interrelate some of the fermion coefficients $c_{\mu\nu}$, Higgs coefficients $(k_{\phi\phi})_{\mu\nu}$, and 9_s Lorentz-irreducible pieces of the gauge coefficients $(k_G)_{\kappa\lambda\mu\nu}$, $(k_W)_{\kappa\lambda\mu\nu}$, $(k_B)_{\kappa\lambda\mu\nu}$. However, the presence of cross couplings between generations means that some types of coefficient unobservable in the QED extension are now physical under suitable experimental circumstances. For example, the presence of flavor-changing weak interactions in the SME quark sector means that differences between constant coefficients of the a_μ type become observable in interferometric experiments with neutral-meson oscillations, a feature absent in the QED extension [19].

V. GRAVITATIONAL SECTOR

A. Action

It is convenient to write the pure-gravity action as

$$S_{\text{gravity}} = \frac{1}{2\kappa}\int d^4x \mathcal{L}_{\text{gravity}}, \tag{53}$$

where the usual gravitational coupling constant $1/2\kappa \equiv 1/16\pi G_N \simeq 3\times 10^{36}\ \text{GeV}^2$ has been factored outside the integral for convenience. The Lagrangian $\mathcal{L}_{\text{gravity}}$ can then be separated as

$$\mathcal{L}_{\text{gravity}} = \mathcal{L}_{e,\omega}^{\text{LI}} + \mathcal{L}_{e,\omega}^{\text{LV}} + \cdots, \tag{54}$$

where the Lorentz-invariant piece $\mathcal{L}_{e,\omega}^{\text{LI}}$ and the Lorentz-violating piece $\mathcal{L}_{e,\omega}^{\text{LV}}$ are constructed using the vierbein $e_\mu{}^a$ and the spin connection $\omega_\mu{}^{ab}$. Following Sec. II A, the latter are viewed as basic dynamical objects for the gravitational field. The ellipsis represents possible dependence on other dynamical gravitational fields, which could be fundamental or composite and could have both Lorentz-invariant and Lorentz-violating parts. The Lagrangian (54) is assumed to combine with the matter and gauge sectors of the SME, perhaps along with other modes as yet unobserved, to yield a smooth connection to the underlying theory at the Planck scale.

The Lorentz-invariant Lagrangian $\mathcal{L}_{e,\omega}^{\text{LI}}$ can be written as a series in powers of the curvature, torsion, and covariant derivatives:

$$\mathcal{L}_{e,\omega}^{\text{LI}} = eR - 2e\Lambda + \cdots. \tag{55}$$

The first term in this expression is the Einstein-Hilbert Lagrangian $\mathcal{L}_{\rm EH}$ in Riemann-Cartan spacetime, while the second contains the cosmological constant Λ. When coupled to matter and gauge fields with energy-momentum and spin-density tensors defined as in Eq. (4), these two terms generate field equations of the form

$$G^{\mu\nu}+\Lambda g^{\mu\nu}=\kappa T_e{}^{\mu\nu},$$
$$\hat{T}^{\lambda\mu\nu}=\kappa S_\omega{}^{\lambda\nu\mu} \tag{56}$$

for the Riemann-Cartan spacetime, where the trace-corrected torsion $\hat{T}^{\lambda\mu\nu}$ is defined in Eq. (A10) of Appendix A. In the Lorentz-invariant Lagrangian (55), the ellipsis represents possible higher-order terms in curvature, torsion, and covariant derivatives. These terms generate corrections to the field equations (56), and they can produce independently propagating vierbein and spin-connection modes corresponding to dynamical torsion and curvature. Note that terms with mass dimension greater than two typically lead to higher-derivative conditions. The complexity of the Lagrangian series is already considerable at second order in the curvature and torsion [44]. However, the explicit form of the higher-order Lorentz-invariant terms is unnecessary for present purposes.

Following the discussion in Sec. II A, each term in the Lorentz-violating Lagrangian $\mathcal{L}^{\rm LV}_{e,\omega}$ is constructed by combining coefficients for Lorentz violation with gravitational field operators to produce a quantity that is both local observer Lorentz invariant and general observer coordinate invariant. The relevant field operators are formed from the vierbein, the spin connection, and their derivatives. It is convenient to express these operators in terms of the curvature, torsion, and covariant derivatives wherever possible. The Lagrangian $\mathcal{L}^{\rm LV}_{e,\omega}$ can then also be written as a series:

$$\begin{aligned}\mathcal{L}^{\rm LV}_{e,\omega}=&e(k_T)^{\lambda\mu\nu}T_{\lambda\mu\nu}+e(k_R)^{\kappa\lambda\mu\nu}R_{\kappa\lambda\mu\nu}\\&+e(k_{TT})^{\alpha\beta\gamma\lambda\mu\nu}T_{\alpha\beta\gamma}T_{\lambda\mu\nu}\\&+e(k_{DT})^{\kappa\lambda\mu\nu}D_\kappa T_{\lambda\mu\nu}+\cdots.\end{aligned} \tag{57}$$

In this equation, all the coefficients for Lorentz violation are real, and they inherit the symmetries of the associated Lorentz-violating operators. The coefficient $(k_T)^{\lambda\mu\nu}$ has dimensions of mass, while the others listed are dimensionless. The ellipsis represents higher-order terms in the curvature, torsion, and covariant derivatives, along with other possible higher-order terms such as the gravitational analogue of the Chern-Simons terms (51) in the SME gauge sector [45]. At low energies, the leading-order terms displayed explicitly in Eq. (57) describe dominant effects of Lorentz violation. As the relevant energies increase towards the Planck scale, higher-order terms represented by the ellipsis in Eq. (57) are expected to play an increasingly significant role.

Note that any coefficients for Lorentz violation in $\mathcal{L}^{\rm LV}_{e,\omega}$ with an even number of indices can also yield Lorentz-invariant contributions to the Lagrangian (54), since they can contain pieces proportional to products of $g^{\mu\nu}$ and $\epsilon^{\kappa\lambda\mu\nu}$. Similarly, by direct contraction with $g^{\mu\nu}$ and $\epsilon^{\kappa\lambda\mu\nu}$, any coefficients for Lorentz violation with an even number of indices can contribute to a position-dependent term of the same general form as the cosmological-constant term. The net effective cosmological constant may therefore be partially or entirely due to Lorentz violation and may vary with spacetime position. It is conceivable that a simple model could be found featuring a realistically small cosmological constant tied to small Lorentz violation.

The Lagrangian series (55) and (57) can be organized according to the mass dimension of the operators or directly in powers of the fields. In any case, several potential simplifications can be considered. First, appropriate use of the Bianchi identities for the curvature and torsion may eliminate some combinations of operators. Second, partial integrations on operators with covariant derivatives can be used to interrelate terms if total derivatives are disregarded. In this way, for instance, the coefficient $(k_{DT})^{\kappa\lambda\mu\nu}$ in Eq. (57) can be converted into a special case of the coefficient $(k_{TT})^{\alpha\beta\gamma\lambda\mu\nu}$. Also, general topological results such as the Gauss-Bonnet theorem imply that under suitable circumstances some combinations of terms form topological invariants and so could be removed in the classical action.

The Lorentz-violating terms in the Lagrangian (57) introduce spacetime anisotropies in the gravitational field equations, which in turn could trigger various physical consequences of theoretical and experimental relevance. Standard gravitational solutions such as those for black holes, cosmology, gravitational waves, and post-Newtonian physics are all expected to be corrected by terms depending on the coefficients for Lorentz violation in Eq. (57). These effects would be independent of ones induced by Lorentz violation in the matter and gauge sectors of the SME. Both for gravitational quanta and for other fundamental particles in the SME, the ensuing Lorentz-violating behavior can depend on momentum magnitude and orientation, spin magnitude and orientation, and the particle species and *CPT* properties.

The effects of Lorentz violation are likely to be large only in regions of large curvature and torsion, such as near black holes or in the early Universe, or in certain cosmological contexts such as those involving the cosmological constant, dark matter, or dark energy. Nonetheless, Lorentz-violating effects could be detectable in various situations. For example, the homogeneous Friedman-Robertson-Walker cosmological solutions may acquire anisotropic corrections, potentially leading to a realistic anisotropic cosmology with observable signals. Candidate Lorentz-violating cosmological effects include the alignment anomalies on large angular scales reported in the Wilkinson microwave anisotropy probe (WMAP) data [46], which are theoretically problematic in conventional scenarios [47]. Another example is provided by the gravitational-wave equations, which acquire corrections from the coefficients for Lorentz violation in Eq. (57). The resulting effects are compounded in certain scenarios for Lorentz violation. For instance, the Goldstone modes arising from spontaneous Lorentz violation are known to affect the propagating degrees of freedom [4,48]. Spacetime-anisotropic features of gravitational modes may eventually be detectable in Earth- or space-based gravitational-wave experiments [49]. For suitable astrophysical sources, compari-

sons of the speed of gravitational waves with the speed of light and neutrinos may also eventually be feasible, which would represent direct sensitivity to a combination of coefficients for Lorentz violation in the gravitational, photon, and matter sectors of the SME. Similarly, Lorentz violation may be detectable in laboratory and space-based experiments studying post-Newtonian gravitational physics, such as tests of the inverse square law [50] or of gravitomagnetic effects, including geodetic precession and the dragging of inertial frames [51]. The detailed exploration of all these effects would be of definite interest but lies beyond the scope of the present work.

Experiments sensitive to Lorentz violation in the matter and gauge sectors of the SME [13,17–30] suggest that the coefficients for Lorentz violation are minuscule, which is consistent with the notion that they arise as Planck-suppressed effects. If this feature extends to the gravitational sector as expected, it is likely that the many existing standard experimental tests of gravity [42] would lack sufficient sensitivity to detect Lorentz violation, although a few may exhibit the necessary exceptional sensitivity. For the analysis of these experiments in the context of metric theories of gravity, a widely applicable test framework exists, called the parametrized post-Newtonian (PPN) formalism [52,53]. A standard version of this formalism [42] that is relevant for solar system experiments assumes a Riemann spacetime asymptotic to Minkowski spacetime, a perfect fluid obeying conventional equations for the covariant conservation of energy momentum and for electrodynamic fields, and conventional geodesic equations for test particles. This PPN formalism contains ten parameters, and bounds on them have been obtained in a variety of experiments. Under suitable assumptions on the SME matter sector and in the zero-torsion limit, an explicit connection between the SME coefficients for Lorentz violation and the PPN parameters should exist. Although beyond the scope of the present work, determining this connection would also be of definite interest.

B. Riemannian limit

The Lorentz-violating extension of Einstein's theory of general relativity is contained in the results of the previous subsection as the limit in which the torsion vanishes. This Riemann-spacetime limit is of interest both its own right and also as a case in which the field equations remain comparatively simple. Even in a Riemann-Cartan spacetime with nonzero torsion, the relevant dominant Lorentz-violating effects can under suitable circumstances be extracted from the zero-torsion limit because in realistic situations torsion effects are typically heavily suppressed compared to curvature effects.

The remainder of this subsection assumes that quantities such as the curvature tensor, its contractions, covariant derivatives, and the Einstein tensor are all evaluated in the zero-torsion limit. For simplicity, the tilde notation for these quantities adopted elsewhere in the present work is suppressed throughout this subsection.

The leading-order Lagrangian terms for this zero-torsion theory consist of the Einstein-Hilbert and cosmological-constant terms, together with the curvature-linear Lorentz-violating piece of Eq. (57). In fact, the resulting action could also be obtained directly by starting from general relativity and imposing plausible constraints on the form of allowed Lorentz-violating terms. It is convenient to expand the coefficient $(k_R)^{\kappa\lambda\mu\nu}$ for Lorentz violation in Eq. (57) and to write the action in the form

$$S_{e,\omega,\Lambda}=\frac{1}{2\kappa}\int d^4x[e(1-u)R-2e\Lambda+es^{\mu\nu}R_{\mu\nu} +et^{\kappa\lambda\mu\nu}R_{\kappa\lambda\mu\nu}]. \tag{58}$$

The introduction of the coefficients $s^{\mu\nu}$, $t^{\kappa\lambda\mu\nu}$, u explicitly distinguishes unconventional effects involving the Riemann, Ricci, and scalar curvatures and so can simplify the consideration of certain special models. As an example, consider the action (B3) of the curvature-coupled bumblebee model described in Appendix B. With the field $B^\mu=b^\mu+\delta B^\mu$ expanded about its Lorentz-violating vacuum value, this theory incorporates only a coefficient for Lorentz violation of the $s^{\mu\nu}$ type:

$$s_B^{\mu\nu}=\xi b^\mu b^\nu-+\frac{1}{4}\xi b^2 g^{\mu\nu}. \tag{59}$$

In this equation, the trace has been absorbed into a u rescaling of R, although this could be avoided by adding an extra term $-\frac{1}{4}\xi eB^2R$ to the Lagrangian (B3). In general, if indeed there is Lorentz violation in nature, coefficients for Lorentz violation of only the $s^{\mu\nu}$ or only the $t^{\kappa\lambda\mu\nu}$ type might well emerge as the result of a comparatively simple mechanism at the Planck scale.

The coefficients for Lorentz violation $s^{\mu\nu}$ and $t^{\kappa\lambda\mu\nu}$ appearing in the action (58) are real and dimensionless. By definition, $s^{\mu\nu}$ inherits the symmetries of the Ricci tensor and $t^{\kappa\lambda\mu\nu}$ inherits those of the Riemann tensor. In considering the full theory (58), the saturated traces of these coefficients could be assumed to vanish, $s^\mu{}_\mu=t^{\mu\nu}{}_{\mu\nu}=0$, since any nonzero values could be absorbed into the Lorentz-invariant coefficient u. Moreover, single traces of $t^{\kappa\lambda\mu\nu}$ such as $t^{\lambda\mu}{}_\lambda{}^\nu$ could also be assumed zero, since nonzero contributions could be absorbed into $s^{\mu\nu}$. It follows that the theory (58) involves 19 independent Lorentz-violating degrees of freedom, nine controlled by the trace-free coefficient $s^{\mu\nu}$ and ten controlled by the trace-free $t^{\kappa\lambda\mu\nu}$. Only one combination of these 19 coefficients, given in a local frame by $s^0{}_0\equiv-s^j{}_j$, is locally rotation invariant. Note that the vanishing-trace assumptions are equivalent to replacing $s^{\mu\nu}$ and $t^{\kappa\lambda\mu\nu}$ with their irreducible Ricci and Weyl pieces, whereupon the Lorentz-violating part of the Lagrangian for the action (58) could be written in the form

$$\mathcal{L}_{e,\omega,\Lambda}\supset es^{\mu\nu}R^T_{\mu\nu}+et^{\kappa\lambda\mu\nu}C_{\kappa\lambda\mu\nu}, \tag{60}$$

where $R^T_{\mu\nu}$ is the trace-free Ricci tensor and $C_{\kappa\lambda\mu\nu}$ is the Weyl tensor.

The above properties of $s^{\mu\nu}$ and $t^{\kappa\lambda\mu\nu}$ are reminiscent of those for the coefficient $(k_F)_{\kappa\lambda\mu\nu}$ in the QED extension or the *CPT*-even coefficients in the gauge sector of the SME.

This is because $s^{\mu\nu}$ and $t^{\kappa\lambda\mu\nu}$ are extracted from the coefficient $(k_R)^{\kappa\lambda\mu\nu}$ in Eq. (57), which like $(k_F)_{\kappa\lambda\mu\nu}$ has the symmetries of the Riemann tensor. Among the consequences is that the coefficient $s^{\mu\nu}$ can under suitable circumstances be moved to other sectors of the SME by redefining the coordinates and fields, following the discussion at the end of Sec. III A.

Since the theory (58) is torsion free, the gravitational field equations can be obtained directly by varying with respect to the metric while treating the spin connection as a dependent variable. Restricting attention for simplicity on the case with $u=\Lambda=0$ but making no assumptions about the traces of $s^{\mu\nu}$ and $t^{\kappa\lambda\mu\nu}$, the variation of the action can be written as

$$\delta S_{e,\omega}=\frac{1}{2\kappa}\int d^4x e[-G^{\mu\nu}+(T^{Rst})^{\mu\nu}]\delta g_{\mu\nu}+eR^{\mu\nu}\delta s_{\mu\nu} + eR^{\kappa\lambda\mu\nu}\delta t_{\kappa\lambda\mu\nu}. \tag{61}$$

The variations $\delta s_{\mu\nu}$ and $\delta t_{\kappa\lambda\mu\nu}$ are included in this expression for completeness. They contribute to the variational equations fixing the coefficients $s^{\mu\nu}$, $t^{\kappa\lambda\mu\nu}$ for Lorentz violation. In Eq. (61), the quantity $(T^{Rst})^{\mu\nu}$ is defined by

$$\begin{aligned}(T^{Rst})^{\mu\nu}\equiv&\frac{1}{2}s^{\alpha\beta}R_{\alpha\beta}g^{\mu\nu}-s^{\mu\alpha}R_{\alpha}{}^{\nu}-s^{\nu\alpha}R_{\alpha}{}^{\mu}+\frac{1}{2}D_\alpha D^\mu s^{\alpha\nu}\\&+\frac{1}{2}D_\alpha D^\nu s^{\alpha\mu}-\frac{1}{2}D^2 s^{\mu\nu}-\frac{1}{2}g^{\mu\nu}D_\alpha D_\beta s^{\alpha\beta}\\&-\frac{3}{2}t^{\alpha\beta\gamma\mu}R_{\alpha\beta\gamma}{}^{\nu}-\frac{3}{2}t^{\alpha\beta\gamma\nu}R_{\alpha\beta\gamma}{}^{\mu}\\&+\frac{1}{2}t^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}g^{\mu\nu}-D_\alpha D_\beta t^{\mu\alpha\nu\beta}-D_\alpha D_\beta t^{\nu\alpha\mu\beta}.\end{aligned} \tag{62}$$

Then, denoting by $T_g{}^{\mu\nu}$ the symmetric energy-momentum tensor arising from varying the matter sector with respect to the metric $g_{\mu\nu}$, the field equations following from the variation (61) are found to be

$$G^{\mu\nu}-(T^{Rst})^{\mu\nu}=\kappa T_g{}^{\mu\nu}. \tag{63}$$

These 10 extended Einstein equations incorporate the leading-order effects of Lorentz violation in general relativity, and they reduce as expected to the usual Einstein equations when $s^{\mu\nu}$ and $t^{\kappa\lambda\mu\nu}$ vanish. Although beyond the scope of the present work, it would be of interest and appears feasible to study the Cauchy initial-value problem for these extended equations. The presence of coefficients for Lorentz violation can be expected to modify the conventional analysis [54].

The extended Einstein equations (63) imply several other results. Tracing with the metric gives

$$R-D_a D_\beta s^{\alpha\beta}-R_{\alpha\beta\gamma\delta}t^{\alpha\beta\gamma\delta}=-\kappa T_g, \tag{64}$$

where $T_g\equiv g_{\mu\nu}T_g{}^{\mu\nu}$. This expression is comparatively simple because several terms vanish as a consequence of the symmetries of $s^{\mu\nu}$ and $t^{\kappa\lambda\mu\nu}$. The result (64) in turn can be used to obtain the trace-reversed version of Eq. (63):

$$R^{\mu\nu}=\kappa\left(T_g{}^{\mu\nu}-\frac{1}{2}g^{\mu\nu}T_g\right)+(T^{Rst})^{\mu\nu} + \frac{1}{2}g^{\mu\nu}(D_\alpha D_\beta s^{\alpha\beta}+R_{\alpha\beta\gamma\delta}t^{\alpha\beta\gamma\delta}). \tag{65}$$

The presence of nonzero $s^{\mu\nu}$ and $t^{\kappa\lambda\mu\nu}$ also allows some qualitatively different types of trace condition. For example, contracting $s^{\mu\nu}$ with Eq. (63) yields

$$s_{\mu\nu}G^{\mu\nu}\approx\kappa s_{\mu\nu}T_g{}^{\mu\nu} \tag{66}$$

to first order in the small coefficients for Lorentz violation.

Acting with D_μ on the extended Einstein equations (63) and imposing the trace Bianchi identity $D_\mu G^{\mu\nu}=0$ yields the condition

$$\begin{aligned}\kappa D_\mu T_g{}^{\mu}{}_{\nu}&=-D_\mu(T^{Rst})^{\mu}{}_{\nu}\\&=-\frac{1}{2}R^{\alpha\beta}D_\nu s_{\alpha\beta}+R^{\alpha\beta}D_\beta s_{\alpha\nu}+\frac{1}{2}s_{\alpha\nu}D^\alpha R\\&\quad-\frac{1}{2}R^{\alpha\beta\gamma\delta}D_\nu t_{\alpha\beta\gamma\delta}+2R^{\alpha\beta\gamma\delta}D_\delta t_{\alpha\beta\gamma\nu}\\&\quad-4t_{\alpha\beta\gamma\nu}D^\alpha R^{\beta\gamma}.\end{aligned} \tag{67}$$

This condition can be interpreted as the statement of covariant conservation of total energy-momentum, including both the matter energy-momentum tensor $T_g{}^{\mu\nu}$ and the energy-momentum contribution from the curvature couplings associated with $s^{\mu\nu}$, $t^{\kappa\lambda\mu\nu}$. The same result would also follow by direct calculation of $D_\mu T_g{}^{\mu\nu}$ using the matter-sector action, followed by substitution of the complete variational equations for $s^{\mu\nu}$ and $t^{\kappa\lambda\mu\nu}$. Since by definition $T_g{}^{\mu\nu}$ is independent of the Lorentz-violating curvature couplings involving $s^{\mu\nu}$ and $t^{\kappa\lambda\mu\nu}$, all the terms on the right-hand side of Eq. (67) would then arise from the latter step. Note that Eq. (67) implies the matter energy-momentum tensor can be covariantly conserved by itself, $D_\mu T_g{}^{\mu\nu}=0$, under suitable circumstances. For example, this is the case for any solution to the equations of motion obeying the conditions $R_{\mu\nu}=0$ and $D_\alpha s_{\beta\gamma}=D_\alpha t_{\beta\gamma\delta\epsilon}=0$.

An illustrative example of the above considerations is provided by the zero-torsion limit of the curvature-coupled bumblebee model described in Appendix B. This model involves a traceless coefficient $s_B^{\mu\nu}$ given in Eq. (59), but the relevant calculations in this case can be performed for the full theory. The matter energy-momentum tensor $T^B_{\mu\nu}$ obtained from the action (B3) is

$$T^B_{\mu\nu}=-B_{\mu\alpha}B^{\alpha}{}_{\nu}-\frac{1}{4}B_{\alpha\beta}B^{\alpha\beta}g_{\mu\nu}-Vg_{\mu\nu}+2V'B_\mu B_\nu, \tag{68}$$

where the prime denotes differentiation with respect to the argument, as usual. The equations of motion are the extended Einstein equations,

$$G_{\mu\nu}=\kappa T^{B}_{\mu\nu}+\xi\Big[\tfrac{1}{2}B^{\alpha}B^{\beta}R_{\alpha\beta}g_{\mu\nu}-B_{\mu}B^{\alpha}R_{\alpha\nu}-B_{\nu}B^{\alpha}R_{\alpha\mu}+\frac{1}{2}D_{\alpha}D_{\mu}(B^{\alpha}B_{\nu})+\frac{1}{2}D_{\alpha}D_{\nu}(B^{\alpha}B_{\mu})-\frac{1}{2}D^{2}(B_{\mu}B_{\nu})-\frac{1}{2}g_{\mu\nu}D_{\alpha}D_{\beta}(B^{\alpha}B^{\beta})\Big], \tag{69}$$

and the equations for the bumblebee field,

$$D_{\mu}B^{\mu\nu}=2V'B^{\nu}-\frac{\xi}{\kappa}B_{\mu}R^{\mu\nu}. \tag{70}$$

The latter imply the covariant current-conservation law

$$D_{\nu}(2\kappa V'B^{\nu})=D_{\nu}(\xi B_{\mu}R^{\mu\nu}). \tag{71}$$

The covariant conservation law for the energy-momentum tensor is

$$\kappa D^{\mu}T^{B}_{\mu\nu}=\xi D^{\beta}(R_{\alpha\beta}B^{\alpha}B_{\nu})-\tfrac{1}{2}\xi R^{\alpha\beta}D_{\nu}(B_{\alpha}B_{\beta}), \tag{72}$$

and it can be obtained at least two ways. One follows the derivation of Eq. (67), taking the covariant derivative of the extended Einstein equations (69) and applying the trace Bianchi identity. The other applies the procedure outlined below Eq. (67), involving the direct calculation of $D^{\mu}T^{B}_{\mu\nu}$ from the defining equation (68), followed by substitution of the equations of motion (70).

C. Geometry

This subsection contains some remarks about the compatibility of explicit Lorentz violation with the geometry of a Riemann-Cartan spacetime. For simplicity, the arguments are presented allowing for torsion but restricting Lorentz violation to the matter sector. They can be extended to other situations, including the presence of Lorentz-violating curvature and torsion couplings, and they contain as a special limit the case of general relativity coupled to a Lorentz-violating matter sector.

The basic chain of reasoning is as follows. The geometry of a Riemann-Cartan theory with local Lorentz and general coordinate invariance can be regarded as a bundle of frames over a base spacetime manifold endowed with a metric and with structure group being the Lorentz group. This framework offers the freedom to define certain geometrical quantities, notably the curvature and torsion, prior to specification of the equations of motion that fix the spacetime. The curvature and torsion are required by the geometrical structure to satisfy two sets of Bianchi identities. The curvature and torsion and hence the Riemann-Cartan spacetime are fixed by demanding that they also solve certain other differential equations, the field equations. The Bianchi identities impose certain conditions on the sources of the field equations, and the compatibility of these conditions with properties of the sources is a necessary requirement for the theory to be self-consistent. However, for sources exhibiting explicit Lorentz violation, it turns out that these conditions are typically incompatible with the covariant conservation laws for the energy-momentum and spin-density tensors.

To demonstrate this, it is convenient to start with the Bianchi identities in the form given in Eq. (A14) of Appendix A. Some manipulation, which includes taking traces, converts the first of these into the form

$$D_{\mu}G^{\mu\nu}=\frac{1}{2}T_{\mu\alpha\beta}R^{\alpha\beta\mu\nu}-T^{\lambda\mu\nu}R_{\mu\lambda}. \tag{73}$$

From this expression, it is straightforward to prove the identity

$$(D_{\mu}-T^{\lambda}{}_{\lambda\mu})G^{\mu\nu}+T_{\lambda\mu}{}^{\nu}G^{\mu\lambda}+\frac{1}{2}R^{\alpha\beta\mu\nu}\hat{T}_{\mu\beta\alpha}=0, \tag{74}$$

where the trace-corrected torsion $\hat{T}^{\lambda\mu\nu}$ is defined in Eq. (A10) of Appendix A. Similarly, tracing the second Bianchi identity and extracting the antisymmetric part of the Einstein tensor yields

$$G_{\mu\nu}-G_{\nu\mu}=D_{\mu}T^{\alpha}{}_{\alpha\nu}-D_{\nu}T^{\alpha}{}_{\alpha\mu}-D^{\alpha}T_{\alpha\mu\nu}+T^{\beta}{}_{\beta\alpha}T^{\alpha}{}_{\mu\nu}, \tag{75}$$

from which follows the identity

$$G^{\mu\nu}-G^{\nu\mu}=-(D_{\alpha}-T^{\beta}{}_{\beta\alpha})\hat{T}^{\alpha\mu\nu}. \tag{76}$$

Note that the results (74) and (76) are a strict consequence of the original two Bianchi identities (A14), following from basic tensorial manipulation alone.

The identities (74) and (76) have been written so that direct substitution of the field equations yields conditions on the sources in the form of covariant conservation laws. Taking Λ to be zero for simplicity, the field equations (56) become $G^{\mu\nu}=\kappa T_{e}{}^{\mu\nu}$ and $\hat{T}^{\lambda\mu\nu}=\kappa S_{\omega}{}^{\lambda\nu\mu}$. Substitution immediately gives

$$(D_{\mu}-T^{\lambda}{}_{\lambda\mu})T_{e}{}^{\mu}{}_{\nu}+T^{\lambda}{}_{\mu\nu}T_{e}{}^{\mu}{}_{\lambda}+\frac{1}{2}R^{ab}{}_{\mu\nu}S_{\omega}{}^{\mu}{}_{ab}=0,$$
$$T_{e}{}^{\mu\nu}-T_{e}{}^{\nu\mu}-(D_{\alpha}-T^{\beta}{}_{\beta\alpha})S_{\omega}{}^{\alpha\mu\nu}=0. \tag{77}$$

These two equations have the same form as the covariant conservation laws (6), (9), except that the terms in the latter two that depend on the coefficients k_x for explicit Lorentz violation are missing in Eq. (77). The two sets of equations are therefore incompatible unless these terms vanish identically.

The incompatibility arises from the special geometrical structure of the gravitational bundle of frames, which ties the Bianchi identities to the equations of motion in a nontrivial way. This can already be seen in the context of conventional general relativity without torsion, where the Bianchi identities are $D_{\mu}G^{\mu\nu}=0$, the Einstein equations are $G^{\mu\nu}=\kappa T^{\mu\nu}$, and substitution of the Einstein equations into the Bianchi identities yields the constraint $D_{\mu}T^{\mu\nu}=0$ on the energy-momentum source. In contrast, the geometrical description of a local gauge theory lacks this feature. For example, the

geometry of a theory such as QED with U(1) gauge invariance is based on a principal fiber bundle with U(1) structure group over a base spacetime manifold. The curvature of the bundle is the antisymmetric field strength $F_{\mu\nu}$, obeying the Bianchi identities $\partial_\lambda F_{\mu\nu}+\partial_\mu F_{\nu\lambda}+\partial_\nu F_{\lambda\mu}=0$. The field strength and hence the bundle geometry are fixed by imposing equations of motion, say $\partial_\mu F^{\mu\nu}=j^\nu$. In this instance, direct attempts to substitute the equations of motion into the Bianchi identities fail to yield the current-conservation law $\partial_\nu j^\nu=0$, which instead follows immediately from the equations of motion by virtue of the antisymmetry of the curvature $F^{\mu\nu}$. The current source j^ν can therefore incorporate explicit Lorentz violation without incompatibility.

The above clash between geometry and symmetry violation occurs for explicit Lorentz breaking but not for spontaneous Lorentz breaking. As discussed in Sec. II B, Eq. (77) is indeed valid when Lorentz symmetry is spontaneously broken. For example, no difficulties are encountered in the treatment of the bumblebee model in the previous subsection. Since in a suitable limit the effects of spontaneous symmetry breaking can be approximated by terms in the action with explicit symmetry breaking, it is interesting to consider how in this limit the results (6) and (9) are recovered from the law (77). Suppose the spontaneous Lorentz violation occurs when a set of fields f_x acquire nonzero vacuum values k_x. The limit in question requires discarding all modes of f_x representing fluctuations about k_x, including massive modes and Goldstone modes or their Higgs equivalents. Discarding the massive modes has no untoward consequences in the low-energy limit. However, in the case of spontaneous Lorentz violation, it is known that the Goldstone modes are absorbed into the gravitational fields without generating a mass for the graviton $h_{\mu\nu}$ [4,48]. Discarding the Goldstone modes therefore changes certain degrees of freedom in the curvature and torsion, and so it is unsurprising that the condition (77) becomes modified in this limit. It would be of some interest to demonstrate this limiting procedure in a simple model, including the explicit recovery of Eqs. (6) and (9), but this lies outside the scope of the present work.

Another interesting question is whether there exists an alternative to the geometry of the Riemann-Cartan bundle of frames that would yield consistent Bianchi identities in the presence of explicit Lorentz violation. Intuitively, the clash described above arises because the Riemann-Cartan geometry is predicated upon the existence throughout the bundle of certain geometrical quantities like the curvature and torsion. Incorporating a coefficient for Lorentz violation corresponds geometrically to introducing another quantity that couples to the existing ones but that originates outside the Riemann-Cartan framework and hence disrupts it. However, it is reasonable to conjecture that a more general geometrical framework can be constructed in which the basic geometrical entities implement directional dependences at each spacetime point corresponding to nonzero coefficients for explicit Lorentz violation. One option might be to generalize the notion of metric to include a dependence on direction, as occurs in Finsler geometries [55].

VI. SUMMARY

In this work, the gravitational couplings in the Lorentz- and *CPT*-violating standard-model extension (SME) have been studied. A general framework is discussed for treating Lorentz violation in the context of a Riemann-Cartan spacetime with curvature and torsion. This allows the description of gravitational couplings involving matter fields for bosons and fermions, with the general-relativistic and Minkowski-spacetime cases recovered as special limits.

The Lorentz- and *CPT*-violating QED extension incorporating gravitational couplings is constructed, and the dominant terms in the low-energy effective action are explicitly given. The partial action in the fermion sector can be found in Eq. (12). Many of the properties and physical implications are similar to those of the Minkowski-spacetime limit, but some new features emerge in the presence of nonzero curvature and torsion. The leading terms in the photon partial action for the QED extension are given in Eq. (25), and some consequences of the gravitational coupling are deduced.

The action for the matter and gauge sector of the SME with gravitational couplings is considered in Sec. IV. First, the conventional standard model of particle physics is embedded in a Riemann-Cartan spacetime. Then, the Lagrangian terms expected to dominate Lorentz- and *CPT*-violating physics at low energies are explicitly given for the case of $SU(3)\times SU(2)\times U(1)$ invariance. Up to possible coordinate and field redefinitions, each term in the SME offers a distinct way for Lorentz symmetry to be violated. The presence of gravitational couplings enhances the options for experimental tests.

The pure-gravity sector of the SME is considered in Sec. V. The leading-order terms in the Lagrangian are given in Eqs. (55) and (57). These terms suggest several interesting directions for theoretical and experimental study. The special limit of zero torsion, which is the Lorentz-violating extension of general relativity, is comparatively simple. The Lorentz-violating physics is dominated by the action (58), which contains 19 independent coefficients for Lorentz violation. The presence of Lorentz-violating curvature couplings has several physical implications, such as curvature-dependent modifications to the covariant conservation law. In Sec. V C, some geometrical issues associated with explicit Lorentz breaking in the effective field theory are addressed. Explicit Lorentz breaking is shown to clash with the geometry of Riemann-Cartan spacetime, but spontaneous Lorentz violation encounters no difficulty.

In conclusion, relativity violations provide candidate low-energy signals for a unified quantum theory of gravity and other forces. The SME is the appropriate general framework for describing the associated Lorentz- and *CPT*-violating effects. The gravitational couplings presented in this work offer promising directions for exploration, with the potential ultimately to offer insight into physics at the Planck scale.

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APPENDIX A: CONVENTIONS

The Minkowski metric η_{ab} in a local Lorentz frame is

$$\eta_{ab}=\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}. \tag{A1}$$

Note that this metric convention involves a sign relative to that adopted for the original discussion of the SME in Ref. [3]. The antisymmetric tensor in this frame is fixed by $\epsilon_{0123}=+1$. The Dirac matrices in this frame are taken to satisfy

$$\{\gamma^a,\gamma^b\}=-2\,\eta^{ab}, \tag{A2}$$

with the additional definition

$$\sigma^{ab}=\frac{1}{2}i[\,\gamma^a,\gamma^b]. \tag{A3}$$

Latin indices are used to label local Lorentz coordinates, while Greek indices are used for spacetime coordinates. However, *x*,*y* denote generic (composite) indices spanning an irreducible representation $(X_{[ab]})^x{}_y$ of the local Lorentz group. The commutation relations for the Lorentz algebra are

$$[X_{[ab]},X_{[cd]}]=\eta_{ac}X_{[bd]}-\eta_{ad}X_{[bc]}-\eta_{bc}X_{[ad]}+\eta_{bd}X_{[ac]}. \tag{A4}$$

For example, for the spinor representation $X_{[ab]}=-i\sigma_{ab}/2$, while for the vector representation $(X_{[ab]})^c{}_d=-\eta_a{}^c\eta_{bd}+\eta_{ad}\eta_b{}^c$.

The Minkowski metric is related to the curved-spacetime metric $g_{\mu\nu}$ by the vierbein $e_\mu{}^a$:

$$g_{\mu\nu}=e_\mu{}^a e_\nu{}^b\eta_{ab}\,. \tag{A5}$$

The determinant of the vierbein is denoted e. To avoid confusion, the charge on the electron is denoted by $-q$. The symbol D is used for all covariant derivatives, including spacetime, internal, and mixed covariant derivatives, with the meaning understood from the context or otherwise specified. For the spacetime covariant derivative, the connection is assumed to be metric:

$$D_\lambda g_{\mu\nu}=0,\quad D_\lambda e_\mu{}^a=0. \tag{A6}$$

The spacetime covariant derivative corrects local Lorentz indices with the spin connection $\omega_\mu{}^{ab}$. Thus, acting on a field f^y, it takes the matrix form

$$(D_\mu)^x{}_y f^y=\left[\delta^x{}_y\partial_\mu-\frac{1}{2}\omega_\mu{}^{ab}(X_{[ab]})^x{}_y\right]f^y. \tag{A7}$$

The covariant derivative of the conjugate representation f_x is given by the same equation with f^y replaced by f_x and the minus sign replaced by a plus sign.

Curved-spacetime indices are corrected with the Cartan connection $\Gamma^\lambda{}_{\mu\nu}$, while mixed objects acquire both types of correction. For example,

$$D_\mu e_\nu{}^a=\partial_\mu e_\nu{}^a-\Gamma^a{}_{\mu\nu}e_\alpha{}^a+\omega_\mu{}^a{}_b e_\nu{}^b. \tag{A8}$$

The Cartan connection is a combination of the Levi-Civita connection and the torsion tensor

$$\Gamma^\lambda{}_{\mu\nu}=\Gamma^\lambda{}_{(\mu\nu)}+\tfrac{1}{2}T^\lambda{}_{\mu\nu}=\left\{\begin{matrix}\lambda\\ \mu\nu\end{matrix}\right\}-T_{(\mu\nu)}{}^\lambda+\tfrac{1}{2}T^\lambda{}_{\mu\nu}, \tag{A9}$$

where the first term after the second equality is the Christoffel symbol and $T^\lambda{}_{\mu\nu}=-T^\lambda{}_{\nu\mu}$ is the torsion tensor. Parentheses enclosing pairs of indices denote symmetrization with a factor of $\frac{1}{2}$.

In practical applications, the trace-corrected torsion tensor defined by

$$\hat{T}^{\lambda\mu\nu}\equiv T^{\lambda\mu\nu}+T^\alpha{}_\alpha{}^\mu g^{\lambda\nu}-T^\alpha{}_\alpha{}^\nu g^{\lambda\mu} \tag{A10}$$

is often useful. Also, equations involving torsion are sometimes more profitably expressed in terms of the contortion tensor $K^\lambda{}_{\mu\nu}$, defined as

$$K^\lambda{}_{\mu\nu}=\tfrac{1}{2}(T^\lambda{}_{\mu\nu}-T_\mu{}^\lambda{}_\nu-T_\nu{}^\lambda{}_\mu). \tag{A11}$$

The inverse relation is $T^\lambda{}_{\mu\nu}=K^\lambda{}_{\mu\nu}-K^\lambda{}_{\nu\mu}$. The contortion tensor obeys $K_{\lambda\mu\nu}=-K_{\nu\mu\lambda}$. Note that $K^\lambda{}_{\lambda\nu}=T^\lambda{}_{\lambda\nu}$.

The curvature tensor is defined as

$$\begin{aligned} R^\kappa{}_{\lambda\mu\nu}&\equiv(\partial_\mu\Gamma^\kappa{}_{\nu\lambda}+\Gamma^\kappa{}_{\mu\alpha}\Gamma^\alpha{}_{\nu\lambda})-(\mu\leftrightarrow\nu)\\ &=\tilde{R}^\kappa{}_{\lambda\mu\nu}+[(D_\mu K^\kappa{}_{\nu\lambda}+K^\alpha{}_{\mu\nu}K^\kappa{}_{\alpha\lambda}+K^\alpha{}_{\mu\lambda}K^\kappa{}_{\nu\alpha})\\ &\quad-(\mu\leftrightarrow\nu)], \end{aligned} \tag{A12}$$

where $\tilde{R}^\kappa{}_{\lambda\mu\nu}$ is the usual Riemann curvature tensor in the absence of torsion, given by replacing the Cartan connections in the first expression above with the corresponding Christoffel symbols. The Ricci tensor $R_{\mu\nu}$, the curvature scalar R, and the Einstein tensor $G_{\mu\nu}$ are defined as

$$\begin{aligned} R_{\mu\nu}&\equiv R^\kappa{}_{\mu\kappa\nu},\\ R&\equiv g^{\mu\nu}R_{\mu\nu},\\ G_{\mu\nu}&\equiv R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R. \end{aligned} \tag{A13}$$

The reader is cautioned that the presence of nonzero torsion in a generic Riemann-Cartan spacetime means that these three quantities also differ from their Riemann-spacetime counterparts $\tilde{R}$, $\tilde{R}_{\mu\nu}$, and $\tilde{G}_{\mu\nu}$.

The curvature and torsion tensors satisfy symmetry properties that follow directly from their definition. They also obey the two sets of Bianchi identities

$$\sum_{(\lambda\mu\nu)} [D_\nu R^\xi{}_{\kappa\lambda\mu} + T^\alpha{}_{\lambda\mu} R^\xi{}_{\kappa\alpha\nu}] = 0,$$

$$\sum_{(\lambda\mu\nu)} [D_\nu T^\kappa{}_{\lambda\mu} + T^\alpha{}_{\lambda\mu} T^\kappa{}_{\alpha\nu} - R^\kappa{}_{\nu\lambda\mu}] = 0. \tag{A14}$$

In these equations, the summation symbol is understood to represent the sum over cyclic permutations of the indices in parentheses.

The definition (A5) and the condition (A6) fix the relationship between the spin connection and the torsion or contortion. The basic variables can be taken as the vierbein and the spin connection, and all other variables such as curvature and torsion can then be expressed in terms of these. For example, the Cartan connection is

$$\Gamma^\lambda{}_{\mu\nu} = e^{\lambda a}(\partial_\mu e_{\nu a} - \omega_\mu{}^b{}_a e_{\nu b}), \tag{A15}$$

while the torsion is

$$T_{\lambda\mu\nu} = e_\lambda{}^a[(\partial_\mu e_{\nu a} + \omega_{\mu ab} e_\nu{}^b) - (\mu \leftrightarrow \nu)], \tag{A16}$$

and the curvature is

$$R^\kappa{}_{\lambda\mu\nu} = e^\kappa{}_a e_\lambda{}^b[(\partial_\mu \omega_\nu{}^a{}_b + \omega_\mu{}^a{}_c \omega_\nu{}^c{}_b) - (\mu \leftrightarrow \nu)]. \tag{A17}$$

Another useful expression is the relationship between the spin connection and the vierbein:

$$\omega_\mu{}^{ab} = \frac{1}{2} e^{\nu a}(\partial_\mu e_\nu{}^b - \partial_\nu e_\mu{}^b) - \frac{1}{2} e^{\nu b}(\partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a) - \frac{1}{2} e^{\alpha a} e^{\beta b} e_\mu{}^c (\partial_\alpha e_{\beta c} - \partial_\beta e_{\alpha c}) + K_{\nu\mu\lambda} e^{\nu a} e^{\lambda b}. \tag{A18}$$

In the limiting case of Riemann geometry relevant for Einstein gravity, the torsion and contortion are zero. This equation then fixes the spin connection in terms of the metric. Using these expressions, the standard Riemann-spacetime covariant derivative $\tilde{D}_\mu$ involving a symmetric connection and the Christoffel symbols emerges as the zero-torsion limit of the covariant derivative in Eq. (A7).

Various special cases of the general Riemann-Cartan spacetimes (which have $R^\kappa{}_{\lambda\mu\nu}$, $T^\lambda{}_{\mu\nu}$ both nonzero) are of interest. They include the Riemann spacetimes of general relativity mentioned above, with $T^\lambda{}_{\mu\nu} = 0$. The Weitzenböck spacetimes [56] are defined by $R^\kappa{}_{\lambda\mu\nu} = 0$. The term "flat" is reserved for spacetimes with $\tilde{R}^\kappa{}_{\lambda\mu\nu} = 0$, which may have nonzero torsion. Finally, the Minkowski spacetimes have $R^\kappa{}_{\lambda\mu\nu} = T^\lambda{}_{\mu\nu} = 0$.

It is sometimes useful to work in a Minkowski-spacetime background containing weak gravitational fields. Then, the metric can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{A19}$$

where the metric fluctuation $h_{\mu\nu}$ is symmetric. At leading order, spacetime and local Lorentz indices can be treated as equivalent, and the vierbein and spin connection can be expressed in terms of small quantities:

$$e_{\mu a} = \eta_{\mu a} + \epsilon_{\mu a} \approx \eta_{\mu a} + \tfrac{1}{2} h_{\mu a} + \chi_{\mu a},$$
$$e \approx 1 + \tfrac{1}{2} h,$$
$$\omega_{\mu ab} \approx -\tfrac{1}{2}\partial_a h_{\mu b} + \tfrac{1}{2}\partial_b h_{\mu a} + \partial_\mu \chi_{ab} + K_{a\mu b}. \tag{A20}$$

Here, the antisymmetric part of the vierbein fluctuation is denoted $\chi_{\mu a}$. This variable can be viewed as containing the six extra degrees of freedom in the vierbein relative to the metric that transform under local Lorentz rotations, so fixing $\chi_{\mu a}$ can be regarded as a gauge choice.

Throughout most of this work, natural units with $\hbar = c = \epsilon_0 = 1$ are adopted.

APPENDIX B: BUMBLEBEE MODEL

Models in which the Lorentz violation arises from the dynamics of a single vector or axial-vector field B_μ, called the bumblebee field, are of particular interest because they have a comparatively simple form but encompass interesting features, including rotation, boost, and *CPT* violations. In a Riemann-Cartan spacetime, the field strength corresponding to B_μ can be defined either as

$$B_{\mu\nu} \equiv D_\mu B_\nu - D_\nu B_\mu + T^\lambda{}_{\mu\nu} B_\lambda = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{B1}$$

or as

$$B_{\mu\nu} \equiv D_\mu B_\nu - D_\nu B_\mu. \tag{B2}$$

The former is U(1) gauge invariant even in the presence of torsion while the latter is not, so the two definitions involve qualitatively different physics. However, they coincide in Riemann or Minkowski spacetimes.

As an example, consider the simple model with action

$$S_B = \int d^4x \left[\frac{1}{2\kappa}(eR + \xi e B^\mu B^\nu R_{\mu\nu}) - \tfrac{1}{4} e B^{\mu\nu} B_{\mu\nu} - eV(B^\mu B_\mu \pm b^2) \right], \tag{B3}$$

where ξ is a real coupling constant controlling a nonminimal curvature-coupling term, and b^2 is a real positive constant. The potential V driving Lorentz and *CPT* violation can be chosen to have a minimum at $B^\mu B_\mu \pm b^2 = 0$. A simple choice for $V(x)$ is $V(x) = \frac{1}{2}\lambda x^2$, where λ is a real coupling constant. Another simple choice with similarities to a sigma model is $V(x) = \lambda x$, where now λ is a Lagrange-multiplier field. Note that the form of the potential ensures breaking of the U(1) symmetry, irrespective of the definition (B1) or (B2) adopted for $B_{\mu\nu}$.

In a region where the curvature and torsion vanish, the potential drives a nonzero vacuum value $B^\mu = b^\mu$, where $b^\mu b_\mu = \mp b^2$. The quantity b_μ is a coefficient for Lorentz and *CPT* violation. In a local Lorentz frame the condition be-

comes $B_a B^a = b^2$, and the local Lorentz coefficient b_a can be taken to have a preferred form as discussed in Sec. II A. This holds in an asymptotically flat spacetime and also in the Minkowski-spacetime limit, although the effects of the potential may be masked for certain matter couplings and in regions of strong curvature and torsion.

The physical insights offered by this theory are remarkably rich. The special limit of Minkowski spacetime and the Lagrange-multiplier potential is equivalent to a theory studied many years ago by Nambu [57], who obtained an elegant proof that it is equivalent to electrodynamics in a nonlinear gauge. The case without Lorentz violation and zero potential V but with nonzero ξ has been used as an alternative theory of gravity in a Riemann spacetime by Will and Nordtvedt [42,53,58]. The theories with $\xi = 0$ were introduced in Ref. [4] to illustrate some ideas about spontaneous Lorentz violation, and these and related models have been explored further in recent works [16,59,60]. In particular, if one or more fermion fields also appear in the action, the covariant axial coupling to the bumblebee field induces terms with coefficients for Lorentz and *CPT* violation of the type b_μ in the fermion sector of the SME [16]. The action (B3) with a potential V and nonzero curvature coupling ξ is used as an illustrative example in parts of the present work.

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