# Supersymmetry and a time-dependent Landau system

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A general technique is outlined for investigating supersymmetry properties of a charged spin-$\frac{1}{2}$ quantum particle in time-varying electromagnetic fields. The case of a time-varying uniform magnetic induction is examined and shown to provide a physical realization of a supersymmetric quantum-mechanical system. Group-theoretic methods are used to factorize the relevant Schrödinger equations and obtain eigensolutions. The supercoherent states for this system are constructed.

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## I. INTRODUCTION

The quantum behavior of a nonrelativistic charged spin-$\frac{1}{2}$ particle in the presence of a constant and uniform magnetic induction is of importance in many physical contexts. The wave functions for this system are solutions of the Pauli equation, which has two components corresponding to the two possible orientations of the spin. Each component has an energy spectrum consisting of a tower of equally spaced levels called Landau levels [1]. The two sets of Landau levels are degenerate, except for the ground state. This system is known to provide a physical realization of supersymmetric quantum mechanics [2,3]. The supersymmetry generators act to reverse the particle spin, thereby mapping one tower of Landau levels into the other.

The classical motion of a point charge in a constant and uniform magnetic induction is rotation about a circle in the plane perpendicular to the magnetic field. This motion is most closely reproduced in the quantum system by coherent states [4–6], for which the expectation values of the charge's coordinates follow the classical cyclotron motion. Coherent states can also be introduced for the spin-$\frac{1}{2}$ Landau system. The presence of the supersymmetry makes possible an extension of these states, resulting in supercoherent states [7].

It is natural to ask whether the notions of supersymmetry and of supercoherent states can be introduced in the context of the motion of a charged spin-$\frac{1}{2}$ particle in more general electromagnetic fields. As the construction presented in Ref. [7] relies on the factorization of the Hamiltonian, it is not apparent *a priori* how to handle more complicated situations. The present paper addresses this issue. We demonstrate that a group-theoretic analysis can provide the key to a supersymmetric factorization. Here, we focus primarily on the case of a uniform but time-dependent magnetic induction as an explicit example. However, the formulation of the problem and the methods used are applicable in a broader context.

We also use the results to obtain supercoherent states. Our construction extends the previously developed coherent states for a spinless charge in time-dependent magnetic (and electric) fields [8–10]. This earlier approach used time-dependent integrals of the motion satisfying an oscillator algebra and the standard displacement-operator method [11–13].

In Sec. II, we establish our notation and perform a first separation of variables, using the invariance of the Schrödinger operator under translations along the direction of the magnetic induction. A rotated variable set is introduced that simplifies much of the subsequent analysis. The group-theoretic analysis of the resulting equations is presented in Sec. III. We seek symmetries of the problem using the methods detailed in Ref. [14] and applied in Refs. [15–17]. The complexified symmetry algebra is constructed in Sec. III B. This generalizes the dynamical symmetry group for the constant-induction case, presented in Ref. [18].

We use these results to develop several factorization schemes, which are given in Sec. IV. The solution to the relevant Schrödinger equation is obtained in Sec. V, using group-theoretic techniques and some representation theory taken from Ref. [19]. In Sec. VI, we extend these expressions to solutions of the Pauli equation, and in Sec. VII the supersymmetry is explicitly identified. Finally, we construct the relevant supercoherent states in Sec. VIII, thereby completing the generalization of the problem treated in Ref. [7]. The coherent states for a single tower of levels, allowing for the time variation, are con-

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tained as a limit of these supercoherent states and are closely related to those constructed from integrals of motion in Ref. [20]. The Appendix demonstrates the reduction of all these results to the time-independent case.

## II. TIME-DEPENDENT LANDAU PROBLEM

Consider a nonrelativistic spin-$\frac{1}{2}$ particle of mass $M$ and charge $e$ moving with momentum $\mathbf{p}$ in a time-dependent electromagnetic field with four-vector potential ($\phi$, $\mathbf{A}$). The Pauli equation for this system is

$$\left[\frac{1}{2M}\{\boldsymbol{\sigma}\cdot[\mathbf{p}-e\,\mathbf{A}(r,t)]\}^2+e\phi(\mathbf{r},t)\right]\Psi_3(\mathbf{r},t) = i\partial_t\Psi_3(\mathbf{r},t) \ , \quad (1)$$

where $\Psi_3$ is a two-component wave function in three space dimensions and the quantity $\boldsymbol{\sigma}=(\sigma_x,\sigma_y,\sigma_z)$ is a vector consisting of the three Pauli matrices.

In this paper, we shall assume that the scalar potential $\phi(\mathbf{r},t)$ is zero and that the vector potential $\mathbf{A}(\mathbf{r},t)$ describes a uniform, time-dependent magnetic induction $\mathbf{B}$. For convenience, we work *in vacuo* where the magnetic induction $\mathbf{B}$ is related to the magnetic field $\mathbf{H}$ by $\mathbf{B}=\mu_0\mathbf{H}$, and we choose the cylindrical gauge

$$A_x=-\tfrac{1}{2}By \ , \quad A_y=\tfrac{1}{2}Bx \ . \quad (2)$$

Writing the upper and lower components of $\Psi_3$ as $\Psi_{3+}$ and $\Psi_{3-}$, the Pauli equation reduces to the two equations

$$(T_x^2+T_y^2+T_z^2+2Me\phi\mp eB-2iM\partial_t)\Psi_{3\pm}=0 \ , \quad (3)$$

where

$$T_x=p_x+eBy/2 \ , \quad T_y=p_y-eBx/2 \ , \quad T_z=p_z \ . \quad (4)$$

Substituting for $p_x$, $p_y$, and $p_z$ the usual operator forms, these equations can be rewritten as

$$S_{3\pm}\Psi_{3\pm}=0 \ , \quad (5)$$

where

$$S_{3\pm}=\partial_{xx}+\partial_{yy}+\partial_{zz}+iwL_z-\tfrac{1}{4}w^2(x^2+y^2)\mp w+2i\partial_\tau \quad (6)$$

are called the Schrödinger operators in three space dimensions and where we have set

$$\hbar=1 \ , \quad \tau=t/M \ , \quad eB(\tau)=w(\tau) \ , \quad L_z=y\partial_x-x\partial_y \ . \quad (7)$$

The operators $S_{3\pm}$ commute with the $z$-translation operator $\partial_z$. This implies [14] that we can separate Eq. (5) with respect to $z$. Set

$$\Psi_{3\pm}(\mathbf{r},\tau)=\Psi_\pm(x,y,\tau)Z_{\kappa\pm}(z) \ . \quad (8)$$

The functions $Z_{\kappa\pm}(z)$ satisfy the eigenvalue problems

$$-i\partial_z Z_{\kappa\pm}=\kappa_\pm Z_{\kappa\pm} \ , \quad (9)$$

with the usual plane-wave solutions. This procedure reduces Eq. (5) to

$$S_{2\pm}\Psi_\pm=0 \ , \quad (10)$$

where the Schrödinger operators $S_{2\pm}$ in two space dimensions are

$$S_{2\pm}=\partial_{xx}+\partial_{yy}+iwL_z+2i\partial_\tau-2h_2(x^2+y^2)-2h_{0\pm} \ , \quad (11)$$

with $2h_{0\pm}=\mp w+\kappa_\pm^2$ and $2h_2=w^2/4$. These new equations depend only on the variables $x$, $y$, and $\tau$. The quantities $\kappa_\pm$ are the constants of separation.

It is convenient for our analysis to eliminate the $iwL_z$ term from (11). Introduce the operator

$$R=\exp[\eta(\tau)L_z] \ , \quad (12)$$

where $\eta(\tau)$ is to be chosen below to eliminate $iw(\tau)L_z$ from the expressions for $S_{2\pm}$. Define rotated solutions $\Theta_\pm$ by

$$\Psi_\pm=R^{-1}\Theta_\pm \ . \quad (13)$$

The rotated Schrödinger operators $\mathcal{S}_{2\pm}$ are given by

$$\begin{aligned}\mathcal{S}_{2\pm}&=RS_{2\pm}R^{-1}=\exp[\eta L_z]S_{2\pm}\exp[-\eta L_z]\\&=\partial_{xx}+\partial_{yy}+iw(\tau)L_z+2i\partial_\tau-2i\dot{\eta}L_z\\&\quad-2h_2(x^2+y^2)-2h_{0\pm} \ . \end{aligned} \quad (14)$$

Setting

$$\eta(\tau)=\tfrac{1}{2}\int_\tau w(\mu)d\mu \quad (15)$$

simplifies the expressions to

$$\mathcal{S}_{2\pm}=\partial_{xx}+\partial_{yy}+2i\partial_\tau-2h_2(x^2+y^2)-2h_{0\pm} \ . \quad (16)$$

These operators are the Schrödinger operators for time-dependent isotropic harmonic oscillators. In the rotated frame, Eq. (10) becomes

$$\mathcal{S}_{2\pm}\Theta_\pm=0 \ . \quad (17)$$

Throughout the remainder of this paper, we use the usual italic letters to represent operators in the original space of the problem (10), and script letters to represent operators associated with the rotated equations (17).

## III. LIE SYMMETRIES

If $w=eB$ is time independent, the solution of Eq. (17) is possible by a direct treatment. In Sec. IV of [7], the separation of variables was performed by setting $p_z=0$ and $\Psi_\pm(x,y,\tau)=\psi_\pm(x,y)e^{-iE_\pm\tau}$, and the resulting differential equation was factored into raising and lowering operators. However, when $w$ varies with time it is no longer immediately apparent how to separate variables or how to identify appropriate raising and lowering operators into which the differential equation (10) can be factored. Instead, we proceed with a systematic approach that makes use of symmetries of the Schrödinger operators $\mathcal{S}_{2\pm}$ in Eq. (16).

To simplify the expressions in this section and in Secs. IV and V, we write equations only for the case associated with the Schrödinger operator $\mathcal{S}_{2+}$, rather than both $\mathcal{S}_{2\pm}$

at once. We also denote $\mathcal{S}_{2+}$ by $\mathcal{S}_2$, and $h_{0+}$ by $h_0$. Analogous expressions for the problem with $\mathcal{S}_{2-}$ can be found by replacing all occurrences of $h_0=h_{0+}$ by $h_{0-}$. This means that all the results obtained here and in the subsequent two sections have duplicate forms. We discuss the role and significance of this duality beginning in Sec. VI.

### A. Symmetry operators

The symmetries we seek for the Schrödinger operator $\mathcal{S}_2$ have the form [14,15]

$$\tilde{\mathcal{L}}=\mathcal{A}(x,y,\tau)\partial_\tau+\mathcal{B}^1(x,y,\tau)\partial_x+\mathcal{B}^2(x,y,\tau)\partial_y+C(x,y,\tau) . \tag{18}$$

These operators generate space-time transformations. For these space-time transformations to be symmetries of (16), they must satisfy the commutator relation [14,15]

$$[\mathcal{S}_2,\tilde{\mathcal{L}}]=\Lambda(x,y,\tau)\mathcal{S}_2 , \tag{19}$$

where $\Lambda(x,y,\tau)$ is some function of the space-time variables. In the unrotated space of Eq. (10), this equation takes the form

$$[S_2,\tilde{L}]=\lambda(x,y,\tau)S_2 , \tag{20}$$

where we have defined

$$\tilde{L}=R^{-1}\tilde{\mathcal{L}}R=\tilde{A}(x,y,\tau)\partial_\tau+\tilde{B}^1(x,y,\tau)\partial_x+\tilde{B}^2(x,y,\tau)\partial_y+\tilde{C}(x,y,\tau) \tag{21}$$

and

$$\lambda(x,y,\tau)=R^{-1}\Lambda(x,y,\tau)R . \tag{22}$$

We next proceed to establish and solve a set of differential equations that determine the explicit form of Eq. (18).

Substituting (16) for $\mathcal{S}_2$ and (18) for $\tilde{\mathcal{L}}$ in Eq. (19), we obtain a system of partial differential equations for the coefficients $\mathcal{A}$, $\mathcal{B}^1$, $\mathcal{B}^2$, and $\mathcal{C}$:

$$\begin{aligned}
&\mathcal{A}_x=\mathcal{A}_y=0 , \quad 2\mathcal{B}^1_x=2\mathcal{B}^2_y=\Lambda , \quad \mathcal{B}^1_y+\mathcal{B}^2_x=0 ,\\
&\mathcal{A}_{xx}+\mathcal{A}_{yy}+2i\mathcal{A}_\tau=2i\Lambda ,\\
&\mathcal{B}^1_{xx}+\mathcal{B}^1_{yy}+2i\mathcal{B}^1_\tau+2\mathcal{C}_x=0 ,\\
&\mathcal{B}^2_{xx}+\mathcal{B}^2_{yy}+2i\mathcal{B}^2_\tau+2\mathcal{C}_y=0 .
\end{aligned} \tag{23}$$

We solve (23) in the usual manner [15] to obtain

$$\begin{aligned}
&\mathcal{A}=\mathcal{A}(\tau) , \quad \Lambda=\dot{\mathcal{A}} ,\\
&\mathcal{B}^1=\tfrac{1}{2}\dot{\mathcal{A}}x+\beta^1_2y+\epsilon^1(\tau) ,\\
&\mathcal{B}^2=-\beta^1_2x+\tfrac{1}{2}\dot{\mathcal{A}}_y+\epsilon^2(\tau) ,\\
&\mathcal{C}=-\frac{i}{4}\ddot{\mathcal{A}}(x^2+y^2)-i\dot{\epsilon}^1x-i\dot{\epsilon}^2y+\epsilon(\tau) ,
\end{aligned} \tag{24}$$

where the $\tau$-dependent coefficients $\mathcal{A}$, $\epsilon^1$, $\epsilon^2$, and $\epsilon$ satisfy

$$\dddot{\mathcal{A}}+8h_2\dot{\mathcal{A}}+4\dot{h}_2\mathcal{A}=0 , \tag{25}$$

$$\ddot{\epsilon}^1+2h_2\epsilon^1=0 , \tag{26}$$

$$\ddot{\epsilon}^2+2h_2\epsilon^2=0 , \tag{27}$$

$$i\dot{\epsilon}-\frac{i}{2}\ddot{\mathcal{A}}+\dot{k}_0\mathcal{A}+h_0\dot{\mathcal{A}}=0 . \tag{28}$$

Since Eqs. (26) and (27) have the same form, they are satisfied by particular solutions $\chi_1(\tau)$ and $\chi_2(\tau)$ that also have the same form. The coefficient of $\dot{\epsilon}_1$ is zero in (26), so the Wronskian $W(\chi_1,\chi_2)$ must be constant [15]. We choose to scale the two solutions so that

$$W(\chi_1,\chi_2)=\chi_1\dot{\chi}_2-\dot{\chi}_1\chi_2=1 . \tag{29}$$

The general solutions then have the form

$$\begin{aligned}
&\epsilon^1(\tau)=\beta^{11}\chi_1(\tau)+\beta^{12}\chi_2(\tau) ,\\
&\epsilon^2(\tau)=\beta^{21}\chi_1(\tau)+\beta^{22}\chi_2(\tau) .
\end{aligned} \tag{30}$$

It is known [15] that if $\chi_1$ and $\chi_2$ are solutions of (26), then

$$\varphi_1(\tau)=(\chi_1)^2 , \quad \varphi_2(\tau)=(\chi_2)^2 , \quad \varphi_3(\tau)=2\chi_1\chi_2 \tag{31}$$

are particular solutions of (25). The general solution is the linear combination

$$\mathcal{A}(\tau)=\beta^1\{\varphi_1(\tau)\}+\beta^2\{\varphi_2(\tau)\}+\beta^3\{\varphi_3(\tau)\} , \tag{32}$$

where $\beta^1$, $\beta^2$, and $\beta^3$ are real constants.

At this state, Eq. (28) can be integrated to yield

$$\epsilon(\tau)=\beta^1\{\tfrac{1}{2}\dot{\varphi}_1+ik_0\varphi_1\}+\beta^2\{\tfrac{1}{2}\dot{\varphi}_2+ik_0\varphi_2\}+\beta^3\{\tfrac{1}{2}\dot{\varphi}_3+ik_0\varphi_3\}+\beta^4\{i\} . \tag{33}$$

The remaining coefficients in the Lie derivative are then found to be

$$\begin{aligned}
\mathcal{B}^1=&\beta^1\{\tfrac{1}{2}\dot{\varphi}_1x\}+\beta^2\{\tfrac{1}{2}\dot{\varphi}_2x\}+\beta^3\{\tfrac{1}{2}\dot{\varphi}_3x\}+\beta^1_2\{y\}+\beta^{11}\{\chi_1\}+\beta^{21}\{\chi_2\} ,\\
\mathcal{B}^2=&-\beta^1_2\{x\}+\beta^1\{\tfrac{1}{2}\dot{\varphi}_1y\}+\beta^2\{\tfrac{1}{2}\dot{\varphi}_2y\}+\beta^3\{\tfrac{1}{2}\dot{\varphi}_3y\}+\beta^{21}\{\chi_1\}+\beta^{22}\{\chi_2\} ,\\
\mathcal{C}=&\beta^1\left\{-\frac{i}{4}\ddot{\varphi}_1(x^2+y^2)+\tfrac{1}{2}\dot{\varphi}_1+ih_0\varphi_1\right\}+\beta^2\left\{-\frac{i}{4}\ddot{\varphi}_2(x^2+y^2)+\tfrac{1}{2}\dot{\varphi}_2+ih_0\varphi_2\right\}\\
&+\beta^3\left\{-\frac{i}{4}\ddot{\varphi}_3(x^2+y^2)+\tfrac{1}{2}\dot{\varphi}_3+ih_0\varphi_3\right\}+\beta^{11}\{-ix\dot{\chi}_1\}+\beta^{12}\{-ix\dot{\chi}_2\}\\
&+\beta^{21}\{-iy\dot{\chi}_1\}+\beta^{22}\{-iy\dot{\chi}_2\}+\beta^4\{i\} .
\end{aligned} \tag{34}$$

Finally, the generators of the symmetry group of the Schrödinger operators $\mathcal{S}_2$ can be obtained by direct substitution. Three of them have the form

$$\tilde{\mathcal{L}}_j=\varphi_j\partial_\tau+\tfrac{1}{2}\dot{\varphi}_j(x\partial_x+y\partial_y)-\frac{i}{4}\ddot{\varphi}_j(x^2+y^2)+\tfrac{1}{2}\dot{\varphi}_j+ih_0\varphi_j , \quad (35)$$

for $j=1,2,3$. These operators satisfy the commutation relations of an sl(2,$\mathbb{R}$) Lie algebra [15]:

$$[\tilde{\mathcal{L}}_3,\tilde{\mathcal{L}}_1]=-2\tilde{\mathcal{L}}_1 , \quad [\tilde{\mathcal{L}}_3,\tilde{\mathcal{L}}_2]=2\tilde{\mathcal{L}}_2 , \quad [\tilde{\mathcal{L}}_1,\tilde{\mathcal{L}}_2]=\tilde{\mathcal{L}}_3 . \quad (36)$$

Another generator is $L_z$, which spans an o(2) algebra and commutes with the $\tilde{\mathcal{L}}_j$:

$$[L_z,\tilde{\mathcal{L}}_j]=0 , \quad j=1,2,3 . \quad (37)$$

The remaining five generators span a Heisenberg-Weyl algebra $w_2$ in two-dimensional space [15]. These operators have the forms

$$\begin{aligned}&E=i ,\\ &\tilde{\mathcal{P}}_1=\chi_1\partial_x-ix\dot{\chi}_1 , \quad \tilde{\mathcal{P}}_2=\chi_2\partial_x-ix\dot{\chi}_2 ,\\ &\tilde{\mathcal{K}}_1=\chi_1\partial_y-iy\dot{\chi}_1 , \quad \tilde{\mathcal{K}}_2=\chi_2\partial_y-iy\dot{\chi}_2 .\end{aligned} \quad (38)$$

Their nonzero commutation relations are

$$[\tilde{\mathcal{P}}_1,\tilde{\mathcal{P}}_2]=-\mathcal{E} , \quad [\tilde{\mathcal{K}}_1,\tilde{\mathcal{K}}_2]=-\mathcal{E} . \quad (39)$$

The full symmetry algebra is the Schrödinger algebra in two space dimensions, $[\mathrm{sl}(2,\mathbb{R})\oplus\mathrm{o}(2)]\oplus\!\!\!\!\supset w_2$. The remaining commutation relations are

$$\begin{aligned}&[\tilde{\mathcal{L}}_1,\tilde{\mathcal{P}}_1]=0 , \quad [\tilde{\mathcal{L}}_2,\tilde{\mathcal{P}}_1]=-\tilde{\mathcal{P}}_2 , \quad [\tilde{\mathcal{L}}_3,\tilde{\mathcal{P}}_1]=-\tilde{\mathcal{P}}_1 ,\\ &[\tilde{\mathcal{L}}_1,\tilde{\mathcal{P}}_2]=\tilde{\mathcal{P}}_1 , \quad [\tilde{\mathcal{L}}_2,\tilde{\mathcal{P}}_2]=0 , \quad [\tilde{\mathcal{L}}_3,\tilde{\mathcal{P}}_2]=\tilde{\mathcal{P}}_2 ,\\ &[\tilde{\mathcal{L}}_1,\tilde{\mathcal{K}}_1]=0 , \quad [\tilde{\mathcal{L}}_2,\tilde{K}_1]=-\tilde{\mathcal{K}}_2 ,\\ &[\tilde{\mathcal{L}}_3,\tilde{\mathcal{K}}_1]=-\tilde{\mathcal{K}}_1 , \quad [\tilde{\mathcal{L}}_1,\tilde{\mathcal{K}}_2]=\tilde{\mathcal{K}}_1 ,\\ &[\tilde{\mathcal{L}}_2,\tilde{\mathcal{K}}_2]=0 , \quad [\tilde{\mathcal{L}}_3,\tilde{\mathcal{K}}_2]=\tilde{\mathcal{K}}_2 ,\\ &[L_z,\tilde{\mathcal{P}}_\alpha]=\tilde{\mathcal{K}}_\alpha , \quad [L_z,\tilde{\mathcal{K}}_\alpha]=\tilde{\mathcal{P}}_\alpha , \quad \alpha=1,2 .\end{aligned} \quad (40)$$

### B. Complexification of the symmetry algebra

To work with Hermitian or Hermitian-conjugate operators, we need to complexify the symmetry algebra [16]. We begin by looking at the solutions of the differential equations (26) and (27). The real solutions to these equations were denoted by $\chi_1(\tau)$ and $\chi_2(\tau)$. We can obtain complex solutions from these by defining

$$\begin{aligned}&\xi(\tau)=\frac{1}{\sqrt{2}}[\chi_1(\tau)+i\chi_2(\tau)] ,\\ &\bar{\xi}(\tau)=\frac{1}{\sqrt{2}}[\chi_1(\tau)-i\chi_2(\tau)] .\end{aligned} \quad (41)$$

In this case, the general solutions to Eqs. (26) and (27) can be written

$$\epsilon^1(\tau)=\beta^{11}\{\xi\}+\beta^{12}\{\bar{\xi}\} , \quad \epsilon^2(\tau)=\beta^{21}\{\xi\}+\beta^{22}\{\bar{\xi}\} . \quad (42)$$

The Wronskian of these solutions is

$$W(\xi,\bar{\xi})=\xi\dot{\bar{\xi}}-\dot{\xi}\bar{\xi}=-iW(\chi_1,\chi_2)=-i . \quad (43)$$

Complex solutions to the differential equation (25) can be written in analogy to the real solutions (31). We have

$$\varphi_1=\xi^2 , \quad \varphi_2=\bar{\xi}^2 , \quad \varphi_3=2\xi\bar{\xi} . \quad (44)$$

Then, in terms of these complex solutions, three of the generators are

$$\hat{\mathcal{L}}_j=\varphi_j\partial_\tau+\tfrac{1}{2}\dot{\varphi}_j(x\partial_x+y\partial_y)-\frac{i}{4}\ddot{\varphi}_j(x^2+y^2)+\tfrac{1}{2}\dot{\varphi}_j+ih_0\varphi_j , \quad (45)$$

where $j=1,2,3$. We can express these operators as linear combinations of the original operators (35), which form the basis of the sl(2,$\mathbb{R}$) subalgebra in the following way:

$$\begin{aligned}&\hat{\mathcal{L}}_1=\tfrac{1}{2}(\tilde{\mathcal{L}}_1-\tilde{\mathcal{L}}_2+i\tilde{\mathcal{L}}_3) , \quad \hat{\mathcal{L}}_2=\tfrac{1}{2}(\tilde{\mathcal{L}}_1-\tilde{\mathcal{L}}_2-i\tilde{\mathcal{L}}_3) ,\\ &\hat{\mathcal{L}}_3=\tilde{\mathcal{L}}_1+\tilde{\mathcal{L}}_2 .\end{aligned} \quad (46)$$

It is more convenient to define the operators

$$\mathcal{M}_3=i\hat{\mathcal{L}}_3 , \quad \mathcal{M}_+=\hat{\mathcal{L}}_2 , \quad \mathcal{M}_-=-\hat{\mathcal{L}}_1 . \quad (47)$$

These operators satisfy the commutation relations

$$\begin{aligned}&[\mathcal{M}_+,\mathcal{M}_-]=-\mathcal{M}_3 , \quad [\mathcal{M}_3,\mathcal{M}_+]=2M_+ ,\\ &[\mathcal{M}_3,\mathcal{M}_-]=-2\mathcal{M}_- .\end{aligned} \quad (48)$$

The operators (47) form a basis for an su(1,1) algebra.

The generator $L_z$ spans an o(2) algebra, as before.

Finally, there are the five generators of a Heisenberg-Weyl algebra $w_2$:

$$\begin{aligned}&\mathcal{P}_-=\xi\partial_x-ix\dot{\xi} , \quad \mathcal{P}_+=-\bar{\xi}\partial_x+ix\dot{\bar{\xi}} ,\\ &\mathcal{K}_-=\xi\partial_y-iy\dot{\xi} , \quad \mathcal{K}_+=-\bar{\xi}\partial_x+iy\dot{\bar{\xi}} ,\\ &I=1 .\end{aligned} \quad (49)$$

These operators satisfy the nonzero commutation relations

$$[\mathcal{P}_-,\mathcal{P}_+]=I , \quad [\mathcal{K}_-,\mathcal{K}_+]=I . \quad (50)$$

The operators $\mathcal{P}_\pm$ and $\mathcal{K}_\pm$ can also be expressed in terms of the operators $\mathcal{P}_\alpha$ and $\mathcal{K}_\alpha$, $\alpha=1,2$, as follows:

$$\begin{aligned}&\mathcal{P}_-=\frac{1}{\sqrt{2}}(\tilde{\mathcal{P}}_1+i\tilde{\mathcal{P}}_2) , \quad \mathcal{P}_+=\frac{1}{\sqrt{2}}(-\tilde{\mathcal{P}}_1+i\tilde{\mathcal{P}}_2) ,\\ &\mathcal{K}_-=\frac{1}{\sqrt{2}}(\tilde{\mathcal{K}}_1+i\tilde{\mathcal{K}}_2) , \quad \mathcal{K}_+=\frac{1}{\sqrt{2}}(-\tilde{\mathcal{K}}_1+i\tilde{\mathcal{K}}_2) .\end{aligned} \quad (51)$$

The remaining commutation relations of the full Schrödinger algebra are

$$
\begin{aligned}
&[L_z,\mathcal{M}_3]=[L_z,M_+]=[L_z,M_-]=0\ ,\\
&[L_z,\mathcal{J}_-]=\mathcal{K}_-\ ,\quad [L_z,\mathcal{J}_+]=\mathcal{K}_+\ ,\\
&[L_z,\mathcal{K}_-]=-\mathcal{J}_-\ ,\quad [L_z,\mathcal{K}_+]=-\mathcal{J}_+\ ,\quad [\mathcal{M}_-,\mathcal{J}_-]=0\ ,\\
&[\mathcal{M}_+,\mathcal{J}_-]=-i\mathcal{J}_+\ ,\quad [\mathcal{M}_3,\mathcal{J}_-]=-\mathcal{J}_-\ ,\\
&[\mathcal{M}_-,\mathcal{J}_+]=-i\mathcal{J}_-\ ,\quad [\mathcal{M}_+,\mathcal{J}_+]=0\ ,\quad [\mathcal{M}_3,\mathcal{J}_+]=\mathcal{J}_+\ ,\\
&[\mathcal{M}_-,\mathcal{K}_-]=0\ ,\quad [\mathcal{M}_+,\mathcal{K}_-]=-i\mathcal{K}_+\ ,\\
&[\mathcal{M}_3,\mathcal{K}_-]=-\mathcal{K}_-\ ,\quad [\mathcal{M}_-,\mathcal{K}_+]=-i\mathcal{K}_-\ ,\\
&[\mathcal{M}_+,\mathcal{K}_+]=0\ ,\quad [\mathcal{M}_3,\mathcal{K}_+]=\mathcal{K}_+\ .
\end{aligned}
\tag{52}
$$

It is also useful to have the expressions for the operators and equations in the original, unrotated frame. The operators of the Schrödinger algebra for (10) can be obtained from (21) and are given by

$$
\begin{aligned}
j_-&=R^{-1}\mathcal{J}_-R=\mathcal{J}_-\cos(\eta)-\mathcal{K}_-\sin(\eta)\ ,\\
j_+&=R^{-1}\mathcal{J}_+R=\mathcal{J}_+\cos(\eta)-\mathcal{K}_+\sin(\eta)\ ,\\
k_-&=R^{-1}\mathcal{K}_-R=\mathcal{K}_-\cos(\eta)+\mathcal{J}_-\sin(\eta)\ ,\\
k_+&=R^{-1}\mathcal{K}_+R=\mathcal{K}_+\cos(\eta)+\mathcal{J}_+\sin(\eta)\ ,
\end{aligned}
\tag{53}
$$

and

$$
\begin{aligned}
m_3&=R^{-1}\mathcal{M}_3R=\mathcal{M}_3+\frac{i}{2}\varphi_3 wL_z\\
&=i\left\{\varphi_3(\partial_\tau+\tfrac12 wL_z)+\tfrac12\dot\varphi_3(x\partial_x+y\partial_y)\right.\\
&\qquad\left.-\frac{i}{4}\ddot\varphi_3(x^2+y^2)+\tfrac12\dot\varphi_3+ih_0\varphi_3\right\}\ ,\\
m_+&=R^{-1}\mathcal{M}_+R=\mathcal{M}_++\tfrac12\varphi_2 wL_z\\
&=\varphi_2(\partial_\tau+\tfrac12 wL_z)+\tfrac12\dot\varphi_2(x\partial_x+y\partial_y)-\frac{i}{4}\ddot\varphi_2(x^2+y^2)\\
&\qquad+\tfrac12\dot\varphi_2+ih_0\varphi_2\ ,\\
m_-&=R^{-1}\mathcal{M}_-R=\mathcal{M}_--\tfrac12\varphi_1 wL_z\\
&=-\varphi_1(\partial_\tau+\tfrac12 wL_z)-\tfrac12\dot\varphi_1(x\partial_x+y\partial_y)\\
&\qquad+\frac{i}{4}\ddot\varphi_1(x^2+y^2)-\tfrac12\dot\varphi_1-ih_0\varphi_1\ .
\end{aligned}
\tag{54}
$$

The commutation relations (47), (50), and (52) are preserved by the transformations (53) and (54).

A more convenient choice for a basis for $w_2$ can be made. Define the operators

$$
\begin{aligned}
a_-&=\frac{1}{\sqrt2}(j_-+ik_-)=\frac{e^{i\eta}}{\sqrt2}(\mathcal{J}_-+i\mathcal{K}_-)\\
&=\frac{e^{i\eta}}{\sqrt2}[\xi(\partial_x+i\partial_y)-i(x+iy)\dot\xi]\ ,\\
c_-&=\frac{1}{\sqrt2}(j_--ik_-)=\frac{e^{-i\eta}}{\sqrt2}(\mathcal{J}_--i\mathcal{K}_-)\\
&=\frac{e^{-i\eta}}{\sqrt2}[\xi(\partial_x-i\partial_y)-i(x-iy)\dot\xi]\ ,
\end{aligned}
\tag{55}
$$

and their conjugates $a_+$ and $c_+$. These operators are related to the raising and lowering operators $a$, $a^\dagger$, $c$, and $c^\dagger$ introduced in Ref. [7]. The Appendix contains an explicit demonstration of the connection between the two sets of operators for the specific case of a charged particle moving in a constant magnetic induction.

The nonzero commutation relations for the complexified Schrödinger algebra in this new basis are

$$
\begin{aligned}
&[m_+,m_-]=-m_3\ ,\quad [m_3,m_+]=2m_+\ ,\\
&[m_3,m_-]=-2m_-\ ,
\end{aligned}
\tag{56}
$$

$$[a_-,a_+]=I\ ,\quad [c_-,c_+]=I\ ,\tag{57}$$

$$[m_3,a_\pm]=\pm a_\pm\ ,\quad [m_3,c_\pm]=\pm c_\pm\ ,\tag{58}$$

$$
\begin{aligned}
&[m_-,a_+]=-ic_-\ ,\quad [m_-,c_+]=-ia_-\ ,\\
&[m_+,a_-]=-ic_+\ ,\quad [m_+,c_-]=-ia_+\ ,
\end{aligned}
\tag{59}
$$

$$[\mathcal{L}_z,a_\pm]=\mp a_\pm\ ,\quad [\mathcal{L}_z,c_\pm]=\pm c_\pm\ .\tag{60}$$

## IV. FACTORIZATION OF THE SCHRÖDINGER EQUATION

In this section, we continue to work with the Schrödinger operator $\mathcal{S}_2=\mathcal{S}_{2+}$ as noted at the beginning of the previous section. Our next goal is to establish candidate factorizations for this operator.

It suffices to work with the subalgebra $\mathcal{G}=\{m_3,a_\pm,c_\pm,I\}\oplus\{\mathcal{L}_z\}$ of the Schrödinger algebra. If we denote the oscillator subalgebra generated by $\{m_3,a_\pm,c_\pm,I\}$ as os(2), then $\mathcal{G}=\mathrm{os}(2)\oplus\mathrm{o}(2)$, where $\mathcal{L}_z=iL_z$ is the generator of o(2). The span of the oscillator subalgebra satisfies the nonzero commutation relations (57), (58), and (60).

The first step is to calculate the operators $a_+a_-$ and $c_+c_-$. We find

$$
\begin{aligned}
a_+a_-&=\tfrac12\left[-\tfrac12\varphi_3(\partial_{xx}+\partial_{yy})+\frac{i}{2}\dot\varphi_3(x\partial_x+y\partial_y)\right.\\
&\qquad\left.-i(y\partial_x-x\partial_y)+\dot\xi\dot{\bar\xi}(x^2+y^2)+\frac{i}{2}\dot\varphi_3-1\right]\\
&=\tfrac12[-\tfrac12\varphi_3\mathcal{S}_2+\mathcal{M}_3-\mathcal{L}_z-1]\\
&=\tfrac12[-\tfrac12\varphi_3S_2+m_3-\mathcal{L}_z-1]
\end{aligned}
\tag{61}
$$

and

$$
\begin{aligned}
c_+c_-&=\tfrac12\left[-\tfrac12\varphi_3(\partial_{xx}+\partial_{yy})+\frac{i}{2}\dot\varphi_3(x\partial_x+y\partial_y)\right.\\
&\qquad\left.+i(y\partial_x-x\partial_y)+\dot\xi\dot{\bar\xi}(x^2+y^2)+\frac{i}{2}\dot\varphi_3-1\right]\\
&=\tfrac12[-\tfrac12\varphi_3\mathcal{S}_2+\mathcal{M}_3+\mathcal{L}_z-1]\\
&=\tfrac12[-\tfrac12\varphi_3S_2+m_3+\mathcal{L}_z-1]\ .
\end{aligned}
\tag{62}
$$

Rearranging Eqs. (61) and (62), we obtain

$$-\tfrac12\varphi_3\mathcal{S}_2=2a_+a_--\mathcal{M}_3+\mathcal{L}_z+1=2c_+c_--\mathcal{M}_3-\mathcal{L}_z+1\ ,\tag{63}$$

$$-\tfrac{1}{2}\varphi_3 S_2 = 2a_+a_- - m_3 + \mathcal{L}_z + 1 = 2c_+c_- - m_3 - \mathcal{L}_z + 1 , \tag{64}$$

where

$$\mathcal{M}_3 = i\left\{\varphi_3\partial_\tau + \tfrac{1}{2}\dot{\varphi}_3(x\partial_x + y\partial_y) - \frac{i}{4}\ddot{\varphi}_3(x^2+y^2) + \tfrac{1}{2}\dot{\varphi}_3 + ih_0\varphi_3\right\} , \tag{65}$$

and $m_3$ is given by (54). Equation (63) provides two different factorization schemes for the Schrödinger operator for the two-dimensional time-dependent harmonic oscillator, while Eq. (64) represents two factorization schemes for the Schrödinger operator for an electron in a time-dependent uniform magnetic induction. The equations are related to one another by the rotation $R$ of (12).

We remind the reader again that similar factorizations can be obtained for the Schrödinger operators $S_{2-}$ and $\mathcal{S}_{2-}$. Note also that the operators $a_\pm$, $c_\pm$, and $L_z$ are unaffected by the replacement of $h_{0+}$ by $h_{0-}$.

## V. SOLUTION OF THE TIME-DEPENDENT SCHRÖDINGER EQUATION

In this section, we present solutions of the two-dimensional Schrödinger equations obtained in Sec. II. The results are found by group-theoretic techniques, and as such can be extended to more general situations than the one considered here. Once again, we remind the reader that to minimize notational confusion, results are presented only for the operator $\mathcal{S}_2 = \mathcal{S}_{2+}$, the wave function $\Psi = \Psi_+$, and rotated forms. The analogous expression for $\mathcal{S}_{2-}$ and $\Psi_-$ can be found by replacing $h_0 = h_{0+}$ with $h_{0-}$.

### A. Further separation

Denote the solution space of the Schrödinger equation $\mathcal{S}_2\Psi = 0$ by $Q_{\mathcal{S}_2}$. The generators of space-time transformations for the Schrödinger equation, say $\tilde{L}$ of (21), must satisfy Eq. (20). Then, for $\Psi \in Q_{\mathcal{S}_2}$, we see that

$$[\mathcal{S}_2, \tilde{L}]\Psi = \lambda\mathcal{S}_2\Psi = 0 . \tag{66}$$

This means the generators are constants of the motion [21]. In particular, the generators $\mathcal{L}_z$ and $m_3$ are two commuting constants of the motion. Therefore, we can find a set of common eigenfunctions $\Psi_{u,l}$ labeled by the eigenvalues $u$ and $l$ of $m_3$ and $\mathcal{L}_z$, respectively:

$$m_3\Psi_{u,l} = u\Psi_{u,l} , \quad \mathcal{L}_z\Psi_{u,l} = l\Psi_{u,l} . \tag{67}$$

Recall from (13) that $\Psi_{u,l} = e^{-\eta L_z}\Theta_{u,l}$. Hence, we have

$$\mathcal{M}_3\Theta_{u,l} = e^{\eta L_z} m_3 e^{-\eta L_z}\Theta_{u,l} = u\Theta_{u,l} . \tag{68}$$

Similarly, we obtain

$$\mathcal{L}_z\Theta_{u,l} = l\Theta_{u,l} \tag{69}$$

for the second eigenvalue equation.

In Eq. (67), substitute the operator (65), where $\varphi_3$ is given by (44). The resulting equation is a first-order partial differential equation for $\Theta_{u,l}$, which can be solved by the method of characteristics. The general solution has the form

$$\Theta_{u,l} = \exp^{\{i\mathcal{R}(\zeta_1,\zeta_2,\mu)\}}\psi_{u,l}(\zeta_1,\zeta_2)T_u(\mu) , \tag{70}$$

where the separable coordinates are

$$\zeta_1 = \frac{x}{\varphi_3^{1/2}} , \quad \zeta_2 = \frac{y}{\varphi_3^{1/2}} , \quad \mu = \tau , \tag{71}$$

and where

$$\mathcal{R} = \tfrac{1}{4}\dot{\varphi}_3(\zeta_1^2+\zeta_2^2) , \quad T_u(\mu) = \varphi_3^{-1/2}\left[\frac{\bar{\xi}}{\xi}\right]^{u/2}\exp(-iK_0) ,$$
$$\int_\mu ds\, h_0 = K_0 . \tag{72}$$

For details of the integration, see the Appendix in Ref. [16]. Note that the function $\mathcal{R}$ cannot be written as a sum $\mathcal{R}_1(\zeta_1)+\mathcal{R}_2(\zeta_2)+\mathcal{R}_3(\mu)$ of arbitrary functions. The solutions $\Theta_{u,l}$ are called $R$-separable solutions [14] of the equation $\mathcal{S}_2\Theta_{u,l}=0$.

Similarly, the functions $\Psi_{u,l}$ that solve the first eigenvalue problem in (67) also solve the Schrödinger equation $S_2\Psi=0$. Since $e^{-\eta L_z}$ commutes with $e^{i\mathcal{R}}$, we obtain the solution to (10):

$$\Psi_{u,l} = e^{i\mathcal{R}}e^{-\eta L_z}\psi_{u,l}(\zeta_1,\zeta_2)T_u(\mu) , \tag{73}$$

where $\mathcal{R}$ and $T_u$ are given by (72). Substituting for $\Psi_{u,l}$ in (10) and suppressing the $u$ and $l$ labels, we obtain

$$\psi_{\zeta_1\zeta_1} + \psi_{\zeta_2\zeta_2} - (\zeta_1^2+\zeta_2^2)\psi + 2q\psi = 0 , \tag{74}$$

which is the eigenvalue problem for a two-dimensional harmonic oscillator.

A further separation of variables can now be made. We solve the eigenvalue equation (69), where $\mathcal{L}_z$ is given by

$$\mathcal{L}_z = i(y\partial_x - x\partial_y) = i(\zeta_2\partial_{\zeta_1} - \zeta_1\partial_{\zeta_2}) , \tag{75}$$

and $\Theta_{u,l}$ by (70). Since $\mathcal{L}_z$ commutes with $\mathcal{R}$, we obtain the eigenvalue problem

$$\mathcal{L}_z\psi_{u,l} = l\psi_{u,l} , \tag{76}$$

which yields directly the first-order partial differential equation

$$\zeta_2\frac{\partial\psi}{\partial\zeta_1} - \zeta_1\frac{\partial\psi}{\partial\zeta_2} = -il\psi . \tag{77}$$

Solving this equation by the method of characteristics, we get the solution

$$\psi_{u,l} = \Upsilon_{u,l}(\rho)e^{il\theta} , \tag{78}$$

where the variables of separation $\rho$ and $\theta$ form a polar coordinate system with

$$\rho^2 = \zeta_1^2+\zeta_2^2 , \quad \theta = \tan^{-1}\left[\frac{\zeta_2}{\zeta_1}\right] \tag{79}$$

and

$$\zeta_1=\rho\cos\theta\ ,\quad \zeta_2=\rho\sin\theta\ . \tag{80}$$

The eigenfunction $\Theta_{u,l}$ has the form

$$\Theta_{u,l}(\rho,\theta,\mu)=e^{i\mathcal{R}}\Upsilon(\rho)e^{il\theta}T_u(\mu)\ , \tag{81}$$

where $T_u(\mu)$ is given by (72) and

$$\mathcal{R}=\tfrac{1}{4}\dot{\varphi}_3\rho^2\ . \tag{82}$$

The wave function $\Psi_{u,l}\in Q_{\mathcal{S}_2}$ has the form

$$\Psi_{u,l}=e^{i\mathcal{R}}e^{i\eta\mathcal{L}_z}\Upsilon_{u,l}(\rho)e^{il\theta}T_u(\mu)$$
$$=e^{i\mathcal{R}}\Upsilon_{u,l}(\rho)e^{il(\theta+\eta)}T_u(\mu)\ , \tag{83}$$

and $T_u$ is given by (72).

### B. Rotation of the operators

To obtain the wave function $\Upsilon_{u,l}(\rho)$ using the ladder operators $a_\pm$ and $c_\pm$, we transform the generators of the os(2) subalgebra into a form that can be written as a product of a time-dependent function and an operator depending only on $\rho$ and $\theta$. Let $o$ be some operator which acts on the manifold of solutions $Q_{\mathcal{S}_2}$. Then, we have

$$o\Psi=oe^{i\mathcal{R}}e^{-\eta L_z}\psi(\zeta_1,\zeta_2)T(\mu)$$
$$=e^{i\mathcal{R}}e^{-\eta L_z}e^{\eta L_z}e^{-i\mathcal{R}}oe^{i\mathcal{R}}e^{-\eta L_z}\psi(\zeta_1,\zeta_2)T(\mu)$$
$$=e^{i\mathcal{R}}e^{-\eta L_z}O\psi(\zeta_1,\zeta_2)T(\mu)\ , \tag{84}$$

where the new operator $O$ is defined to be

$$O=e^{\eta L_z}e^{-i\mathcal{R}}oe^{i\mathcal{R}}e^{-\eta L_z}\ , \tag{85}$$

with $\mathcal{R}$ given by (72) or (82). The operator $O$ acts on the product space of functions denoted by $\{\Upsilon_{u,l}(\rho)e^{il\theta}T_u(\mu)\}$. Note that the generator $\mathcal{L}_z$ of the o(2) algebra is invariant under this transformation.

Let the operator $o$ be a generator of the Heisenberg-Weyl algebra $w_2$. For $a_\pm$, given by (55), we get

$$e^{\eta L_z}e^{-i\mathcal{R}}a_-e^{i\mathcal{R}}e^{-\eta L_z}=\frac{1}{2}\left[\frac{\bar{\xi}}{\xi}\right]^{-1/2}e^{i\theta}\left[\partial_\rho+\rho+\frac{i}{\rho}\partial_\theta\right]$$
$$=A_- \tag{86}$$

and

$$e^{\eta L_z}e^{-i\mathcal{R}}a_+e^{i\mathcal{R}}e^{-\eta L_z}=\frac{1}{2}\left[\frac{\bar{\xi}}{\xi}\right]^{1/2}e^{-i\theta}\left[-\partial_\rho+\rho+\frac{i}{\rho}\partial_\theta\right]$$
$$=A_+\ . \tag{87}$$

For the remaining operators $c_\pm$ of the $w_2$ subalgebra, we obtain

$$e^{\eta L_z}e^{-i\mathcal{R}}c_-e^{i\mathcal{R}}e^{-\eta L_z}=\frac{1}{2}\left[\frac{\bar{\xi}}{\xi}\right]^{-1/2}e^{-i\theta}\left[\partial_\rho+\rho-\frac{i}{\rho}\partial_\theta\right]$$
$$=C_- \tag{88}$$

and

$$e^{\eta L_z}e^{-i\mathcal{R}}c_+e^{i\mathcal{R}}e^{-\eta L_z}=\frac{1}{2}\left[\frac{\bar{\xi}}{\xi}\right]^{1/2}e^{i\theta}\left[\partial_\rho+\rho-\frac{i}{\rho}\partial_\theta\right]$$
$$=C_+\ . \tag{89}$$

In later applications, it is also useful to introduce the operators

$$a=\frac{1}{2}e^{i\theta}\left[\partial_\rho+\rho+\frac{i}{\rho}\partial_\theta\right]\ ,\quad a^\dagger=\frac{1}{2}e^{-i\theta}\left[-\partial_\rho+\rho+\frac{i}{\rho}\partial_\theta\right]\ ,$$
$$c=\frac{1}{2}e^{-i\theta}\left[\partial_\rho+\rho-\frac{i}{\rho}\partial_\theta\right]\ ,\quad c^\dagger=\frac{1}{2}e^{i\theta}\left[\partial_\rho+\rho-\frac{i}{\rho}\partial_\theta\right]\ , \tag{90}$$

where

$$A_-=\left[\frac{\bar{\xi}}{\xi}\right]^{-1/2}a\ ,\quad A_+=\left[\frac{\bar{\xi}}{\xi}\right]^{1/2}a^\dagger\ ,$$
$$C_-=\left[\frac{\bar{\xi}}{\xi}\right]^{-1/2}c\ ,\quad C_+=\left[\frac{\bar{\xi}}{\xi}\right]^{1/2}c^\dagger\ . \tag{91}$$

The operators $a$, $a^\dagger$, $c$, and $c^\dagger$ are the analogs of the corresponding operators in [7]. See the Appendix.

Finally, the generator $m_3$ transforms as

$$e^{\eta L_z}e^{-i\mathcal{R}}m_3e^{i\mathcal{R}}e^{-\eta L_z}=i\{\varphi_3\partial_\mu+\tfrac{1}{2}\dot{\varphi}_3+ih_0\varphi_3\}$$
$$=M_3\ . \tag{92}$$

Note that all the spatial dependence has been removed in $M_3$. It is now a purely $\mu$-dependent operator.

### C. Reduction to a radial equation

To complete the solution of Eqs. (10), we shall exploit the representation theory of the Lie algebra $\mathcal{G}=\mathrm{os}(2)\oplus\mathrm{o}(2)$. It is more convenient to work in the new basis

$$\{d_3,f_3,a_\pm,c_\pm,I\}\ , \tag{93}$$

where the operators $d_3$ and $f_3$ are defined to be

$$d_3=\tfrac{1}{2}(m_3+\mathcal{L}_z)\ ,\quad f_3=\tfrac{1}{2}(m_3-\mathcal{L}_z)\ . \tag{94}$$

The operators in this algebra satisfy the nonzero commutation relations:

$$[m_3,a_\pm]=\pm a_\pm\ ,\quad [a_-,a_+]=I\ ,$$
$$[m_3,c_\pm]=\pm c_\pm\ ,\quad [c_-,c_+]=I\ , \tag{95}$$
$$[\mathcal{L}_z,a_\pm]=\mp a_\pm\ ,\quad [\mathcal{L}_z,c_\pm]=\pm c_\pm\ .$$

The detailed representation theory for this Lie algebra is presented elsewhere [19]. The specific representation that is of interest to us is denoted $\uparrow_{-1/2}\otimes\uparrow_{-1/2}$. The representation space is spanned by a set of vectors $\{|n,m\rangle, n,m\in\mathbb{Z}^+\}$, where $\mathbb{Z}^+$ is the set of non-negative integers. The base for the Lie algebra $\mathcal{G}$ acts on this space in the following way:

$$
\begin{aligned}
&f_3|n,m\rangle=(\tfrac{1}{2}+n)|n,m\rangle\ ,\quad d_3|n,m\rangle=(\tfrac{1}{2}+m)|n,m\rangle\ ,\\
&a_+|n,m\rangle=\sqrt{n+1}|n+1,m\rangle\ ,\\
&a_-|n,m\rangle=\sqrt{n}\,|n-1,m\rangle\ ,\\
&c_+|n,m\rangle=\sqrt{m+1}|n,m+1\rangle\ ,\\
&c_-|n,m\rangle=\sqrt{m}\,|n,m-1\rangle\ ,\\
&I|n,m\rangle=|n,m\rangle\ .
\end{aligned}
\tag{96}
$$

The representation space is a normed space and we have

$$\langle n',m'|n,m\rangle=\delta_{n'n}\delta_{m'm}\ . \tag{97}$$

We identify the solution spaces $Q_{S_2}$ and $Q_{\mathcal{S}_2}$ with the unrotated and rotated representation spaces of the irreducible representation $\uparrow_{-1/2}\otimes\uparrow_{-1/2}$.

It is advantageous to transform $d_3$ and $f_3$ into operators acting on the solution space $Q_{\mathcal{S}_2}$ of the oscillator. We have

$$\mathcal{D}_3=e^{\eta L_z}d_3e^{-\eta L_z}=\tfrac{1}{2}(\mathcal{M}_3+\mathcal{L}_z)\ , \tag{98}$$

$$\mathcal{F}_3=e^{\eta L_z}f_3e^{-\eta L_z}=\tfrac{1}{2}(\mathcal{M}_3-\mathcal{L}_z)\ . \tag{99}$$

According to (96), the spectra of these operators are

$$S(d_3)=S(\mathcal{D}_3)=\{m+\tfrac{1}{2}:m\in\mathbb{Z}^+\}\ , \tag{100}$$

$$S(f_3)=S(\mathcal{F}_3)=\{n+\tfrac{1}{2}:n\in\mathbb{Z}^+\}\ . \tag{101}$$

Let $\Theta_{n,m}\in Q_{\mathcal{S}_2}$ be the coordinate representation of the vector $|n,m\rangle$ in the representation space $\uparrow_{-1/2}\otimes\uparrow_{-1/2}$. Then, we have

$$\mathcal{F}_3\Theta_{n,m}=(n+\tfrac{1}{2})\Theta_{n,m}\ ,\quad \mathcal{D}_3\Theta_{n,m}=(m+\tfrac{1}{2})\Theta_{n,m}\ . \tag{102}$$

We have already seen that the polar coordinate system $(\rho,\theta,\mu)$ is a natural coordinate system for this problem. Therefore, we express $\mathcal{D}_3$ and $\mathcal{F}_3$ in these coordinates:

$$
\begin{aligned}
\mathcal{F}_3&=\frac{i}{2}[\varphi_3\partial_\mu-\frac{i}{4}\ddot{\varphi}_3\varphi_3\rho^2+\tfrac{1}{2}\dot{\varphi}_3+ih_0+\partial_\theta]\ ,\\
\mathcal{D}_3&=\frac{i}{2}[\varphi_3\partial_\mu-\frac{i}{4}\ddot{\varphi}_3\varphi_3\rho^2+\tfrac{1}{2}\dot{\varphi}_3+ih_0-\partial_\theta]\ ,
\end{aligned}
\tag{103}
$$

where $\mathcal{L}_z=-i\partial_\theta$. Substituting (103) into (102) and solving the resulting first-order partial differential equations, we obtain the result

$$\Theta_{n,m}=e^{i\mathcal{R}}\Upsilon_{n,m}(\rho)e^{i\theta(m-n)}T_{n,m}(\mu)\ , \tag{104}$$

where $\mathcal{R}$ is given by (82) and

$$T_{n,m}=\varphi_3^{-1/2}\left[\frac{\bar{\xi}}{\xi}\right]^{(n+m+1)/2}e^{-iK_0}\ . \tag{105}$$

The radial functions $\Upsilon_{n,m}(\rho)$ are unknown at this point. If we compare the functions (81) with (104) and (72) with (105), then we see that they are equivalent solutions if we make the identification

$$u=n+m\ ,\quad l=m-n\ . \tag{106}$$

Note that these solutions are normed. The norm is

$$
\begin{aligned}
\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}dx\,dy\,\overline{\Psi}_{n,m}(x,y)\Psi_{n,m}(x,y)
&=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}d\zeta_1 d\zeta_2\bar{\psi}_{0,0}(\zeta_1,\zeta_2)\psi_{0,0}(\zeta_1,\zeta_2)\\
&=\int_0^\infty\int_0^{2\pi}\rho\,d\rho\,d\theta\,\overline{\Upsilon}_{n,m}(\rho)\Upsilon_{n,m}(\rho)\ .
\end{aligned}
\tag{107}
$$

### D. Solution of the radial equation

We can use the algebraic structure of $\mathcal{G}$ in the basis (93) and the structure of the irreducible representation $\uparrow_{-1/2}\otimes\uparrow_{-1/2}$ to obtain an explicit form for the radial functions. Since the representation $\uparrow_{-1/2}\otimes\uparrow_{-1/2}$ is bounded below with respect to the spectra of both $f_3$ and $d_3$, there exists an extremal state

$$
\begin{aligned}
\Psi_{0,0}(\rho,\theta,\mu)&=e^{-\eta L_z}\Theta_{0,0}(\rho,\theta,\mu)\\
&=e^{-\eta L_z}e^{i\mathcal{R}}\psi_{0,0}(\rho,\theta)T_{0,0}(\mu)\\
&=e^{-\eta L_z}e^{i\mathcal{R}}\Upsilon_{0,0}(\rho)T_{0,0}(\mu)\ .
\end{aligned}
\tag{108}
$$

Since

$$a_-\Psi_{0,0}=c_-\Psi_{0,0}=0\Longrightarrow a\Upsilon_{0,0}=c\Upsilon_{0,0}=0\ , \tag{109}$$

we obtain

$$\Upsilon_{0,0}=\frac{1}{\sqrt{\pi}}e^{-\rho^2/2}\ , \tag{110}$$

after normalization. Therefore, the extremal eigenstates are

$$\Theta_{0,0}=\frac{1}{\sqrt{\pi}}e^{i\mathcal{R}}e^{-\rho^2/2}\ ,\quad \Psi_{0,0}=\frac{1}{\sqrt{\pi}}e^{i\mathcal{R}}e^{-\eta L_z}e^{-\rho^2/2}\ . \tag{111}$$

This means $\Theta_{0,0}=\Psi_{0,0}$.

According to Eqs. (96), we have

$$\Psi_{n,m}=\frac{1}{\sqrt{n!}\sqrt{m!}}a_+^n c_+^m\Psi_{0,0}\ . \tag{112}$$

After some calculation [19], the complete wave function can be written as

$$\Psi_{n,m}=e^{i\mathcal{R}}e^{-\eta L_z}\Upsilon_{n,m}(\rho)e^{i\theta(m-n)}T_{n,m}(\mu)\ , \tag{113}$$

in which the radial wave function $\Upsilon_{n,m}(\rho)$ is

$$\Upsilon_{n,m}(\rho)=\frac{(-)^k}{\sqrt{\pi}}\frac{k!}{\sqrt{n!}\sqrt{m!}}L_k^{(|m-n|)}(\rho^2)\rho^{|m-n|}e^{-\rho^2/2}\ , \tag{114}$$

where

$$k=\tfrac{1}{2}[n+m-|m-n|]\ , \tag{115}$$

and the $\mu$-dependent function has the form

$$T_{n,m}(\mu)=\varphi_3^{-1/2}\left[\frac{\bar{\xi}}{\xi}\right]^{(n+m+1)/2}e^{-iK_0}\ . \tag{116}$$

The last equation may be compared to $T_u(\mu)$ in (72). Note that for each $n$ there is an infinite degeneracy in $m$.

Recall that $m_3$ and $\mathcal{L}_z$ are diagonal in the basis $\{\Psi_{n,m}\}$, since they are linear combinations of $d_3$ and $f_3$. From (94), we see that

$$m_3=d_3+f_3 \ , \quad \mathcal{L}_z=d_3-f_3 \ . \tag{117}$$

Therefore, the eigenvalues of $m_3$ and $\mathcal{L}_z$ are

$$m_3\Psi_{n,m}=(m+n+1)\Psi_{n,m} \tag{118}$$

and

$$\mathcal{L}_z\Psi_{n,m}=(m-n)\Psi_{n,m} \ , \tag{119}$$

where $n,m \in \mathbb{Z}^+$. This implies that the spectrum of $m_3$ is

$$S(m_3)=\{m+n+1{:}n,m\in\mathbb{Z}^+\} \ , \tag{120}$$

where each eigenvalue is $(n+m+1)$-fold degenerate. The spectrum of $\mathcal{L}_z$ is

$$S(\mathcal{L}_z)=\{m-n{:}n,m\in\mathbb{Z}^+\}=\{0,\pm1,\pm2,\dots\} \ , \tag{121}$$

which is as expected since the operator $\mathcal{L}_z$ generates the group O(2). Each state can also be characterized by the eigenvalues of $m_3$ and $\mathcal{L}_z$. Since each state with eigenvalue $n+m+1$ is $(n+m+1)$-fold degenerate, the degenerate states are classified or labeled by the eigenvalues of $\mathcal{L}_z$. They are

$$\pm(m+n),\pm(m+n-2),\pm(m+n-4),\dots,\pm1 \ \text{ or } 0 \ , \tag{122}$$

according to whether $n+m+1$ is even or odd, respectively.

## VI. SOLUTIONS TO THE PAULI EQUATION

At this stage, we can return to the original problem discussed in Sec. I. The solutions of the Schrödinger equation found in Sec. V and the symmetry structure developed in parallel make possible the solution of the Pauli equation and the identification of the supersymmetry. In this section, we present the solutions of the Pauli equation in a useful form.

The two-dimensional Pauli equation is reduced in Sec. I to Eq. (10),

$$S_{2\pm}\Psi_\pm=0 \ , \tag{123}$$

where $S_{2\pm}$ are given in Eq. (11). The symmetry analysis above for each of $S_{2+}$ and $S_{2-}$ yields the same time-dependent coefficients $\xi$, $\varphi_1$, $\varphi_2$, and $\varphi_3$. Multiplying $S_{2\pm}$ by $-\frac{1}{2}\varphi_3$ and using $[a_-,a_+]=I$ in Eq. (64) and its $h_{0+}\to h_{0-}$ partner gives

$$\begin{aligned} -\tfrac{1}{2}\varphi_3 S_{2\pm}&=2a_\pm a_\mp+\mathcal{L}_z-m_{3\pm}\pm1 \ , \\ &=2a_\pm a_\mp-2f_{3\pm}\pm1 \ , \end{aligned} \tag{124}$$

where $m_{3\pm}$ and $f_{3\pm}$ are given by (54) and (94) and their partners. Equation (10) can therefore be rewritten as

$$(2a_\pm a_\mp-2f_{3\pm}\pm1)\Psi_\pm=0 \ . \tag{125}$$

From Sec. V, the solutions to this equation can be taken as

$$\begin{aligned}\Psi_\pm&=\Psi_{n_\pm,m_\pm}(x,y,\tau) \\ &=e^{i\mathcal{R}}e^{-\eta L_z}\psi_{n_\pm,m_\pm}(\zeta_1,\zeta_2)T_{\kappa_\pm;n_\pm,m_\pm}(\mu) \ , \end{aligned} \tag{126}$$

where $\Psi_{n_+,m_+}(x,y,\tau)$ is given by (113) and $\Psi_{n_-,m_-}(x,y,\tau)$ is its partner. The function $\psi_{n,m}$ is a function of either of the variables of separation $(\zeta_1,\zeta_2)$ or $(\rho,\theta)$. The result is a double eigenvalue problem:

$$a_\pm a_\mp\Psi_{n_\pm,m_\pm}=\begin{bmatrix} n_+ \\ n_-+1\end{bmatrix}\Psi_{n_\pm,m_\pm} \ . \tag{127}$$

Using (126) for the solution $\Psi_{n_+,m_+}$ and a similar expression for its partner, and noting (86) and (87), we obtain the result

$$A_\pm A_\mp\psi_{n_\pm,m_\pm}T_{n_\pm,m_\pm}=\begin{bmatrix} n_+ \\ n_-+1\end{bmatrix}\psi_{n_\pm,m_\pm}T_{\kappa_\pm;n_\pm,m_\pm} \ . \tag{128}$$

Finally, since the operators $A_\pm$ contain no time derivatives and because of Eqs. (91), the two-component Pauli equation can be written as

$$\begin{bmatrix} a^\dagger a & 0 \\ 0 & aa^\dagger\end{bmatrix}\begin{bmatrix}\psi_{n_+,m_+} \\ \psi_{n_-,m_-}\end{bmatrix}=\begin{bmatrix} n_+ & 0 \\ 0 & n_-+1\end{bmatrix}\begin{bmatrix}\psi_{n_+,m_+} \\ \psi_{n_-,m_-}\end{bmatrix} , \tag{129}$$

where the operators $a^\dagger a$ and $aa^\dagger$ are given in the coordinate representation by

$$\begin{aligned} a^\dagger a&=\tfrac{1}{4}[-(\partial_{\zeta_1\zeta_1}+\partial_{\zeta_2\zeta_2})-2i(\zeta_2\partial_{\zeta_1}-\zeta_1\partial_{\zeta_2}) \\ &\quad+(\zeta_1^2+\zeta_2^2)-2] \ , \\ aa^\dagger&=\tfrac{1}{4}[-(\partial_{\zeta_1\zeta_1}+\partial_{\zeta_2\zeta_2})-2i(\zeta_2\partial_{\zeta_1}-\zeta_1\partial_{\zeta_2}) \\ &\quad+(\zeta_1^2+\zeta_2^2)+2] \ . \end{aligned} \tag{130}$$

Equation (129) is the generalization of the factorization obtained in Ref. [7].

## VII. SUPERSYMMETRY

In this section, we identify a supersymmetry associated with a nonrelativistic charged spin-$\frac{1}{2}$ particle moving in a time-varying uniform magnetic induction $\mathbf{B}(\tau)=B(\tau)\hat{\mathbf{z}}$. This supersymmetry generalizes that discussed in Refs. [2,3,7].

The relevant time-dependent Pauli equation for this system is Eq. (1), with $\phi(\mathbf{r},t)=0$ and the gauge choice in Eq. (2). For more generality, we do not take $p_z$ as zero but instead separate variables with respect to $z$. The momentum in the $z$ direction is then represented in the expressions below by its eigenvalue $\kappa_\pm$.

As described in the previous sections, the solution space $Q_{S_{2+}}$ spanned by the functions $\{\Psi_{n_+,m_+}:n_+,m_+\in\mathbb{Z}^+\}$ forms a basis for a representation space for the irreducible representation $\uparrow_{-1/2}\otimes\uparrow_{-1/2}$ of the Lie algebra of operators $\{d_{3+},f_{3+},a_\pm,c_\pm,I\}$. A partner space $Q_{S_{2-}}$ spanned by the partner functions $\{\Psi_{n_-,m_-}:n_-,m_-\in\mathbb{Z}^+\}$ also exists and forms a basis for the same irreducible representation of the isomorphic Lie algebra of operators $\{d_{3-},f_{3-},a_\pm,c_\pm,I\}$. The spectra are

$$S(f_{3\pm})=\{n_\pm+\tfrac{1}{2}:n_\pm\in\mathbb{Z}^+\}\ ,$$
$$S(d_{3\pm})=\{m_\pm+\tfrac{1}{2}:m_\pm\in\mathbb{Z}^+\}\ . \tag{131}$$

From Eq. (129) we see that the spectrum of the operator $a^\dagger a$ is

$$S(a^\dagger a)=\{n_+:n_+\in\mathbb{Z}^+\}=\{0,1,2,\ldots\}\ , \tag{132}$$

while the spectrum of $aa^\dagger$ is

$$S(aa^\dagger)=\{n_-+1:n_-\in\mathbb{Z}^+\}=\{1,2,3\ldots\}\ . \tag{133}$$

In effect, the two operators $a^\dagger a$ and $aa^\dagger$ have identical spectra, except that the latter is missing the ground state. This suggests the existence of a supersymmetry.

Following Ref. [7], we introduce a unified notation that permits the simultaneous handling of the two spaces. We define a parameter $\nu$ that takes the value 0 for the "bosonic" space $Q_{S_{2+}}$ and 1 for the "fermionic" space $Q_{S_{2-}}$. It distinguishes the upper and lower components of the two-component Pauli equation. States in the two spaces can then be denoted by $|n,m;\nu\rangle$, where $n=0,1,2,\ldots$. For the bosonic space $n=n_+$ and $m=m_+$, while for the fermionic space $n=n_-$ and $m=m_-$. In two-component notation, we have

$$|n,m;\nu\rangle=\psi_{n,m}(\zeta_1,\zeta_2)\begin{bmatrix}\delta_{0\nu}\\ \delta_{1\nu}\end{bmatrix}\ . \tag{134}$$

The action of the operators $a$ and $c$ and their conjugates becomes

$$\begin{aligned}a|n,m;\nu\rangle&=\sqrt{n}\,|n-1,m;\nu\rangle\ ,\\ a^\dagger|n,m;\nu\rangle&=\sqrt{n+1}|n+1,m;\nu\rangle\ ,\\ c|n,m;\nu\rangle&=\sqrt{m}\,|n,m-1;\nu\rangle\ ,\\ c^\dagger|n,m;\nu\rangle&=\sqrt{m+1}|n,m+1;\nu\rangle\ .\end{aligned} \tag{135}$$

These expressions are the natural time-varying extensions of the results obtained in Ref. [7].

We can also define raising and lowering operators $b$ and $b^\dagger$ for the index $\nu$. By definition, these operators satisfy the anticommutation relations

$$\{b,b^\dagger\}=I\ ,\quad \{b,b\}=\{b^\dagger,b^\dagger\}=0\ . \tag{136}$$

Their two-component form is

$$b=\begin{bmatrix}0&1\\0&0\end{bmatrix}\ ,\quad b^\dagger=\begin{bmatrix}0&0\\1&0\end{bmatrix}\ . \tag{137}$$

Their action on the ket $|n,m;\nu\rangle$ is given by

$$b|n,m;\nu\rangle=\delta_{1\nu}|n,m;0\rangle\ ,\quad b^\dagger|n,m;\nu\rangle=\delta_{0\nu}|n,m;1\rangle\ . \tag{138}$$

We can now introduce the time-dependent operator $\hat{H}$, defined by

$$\hat{H}=a^\dagger a+b^\dagger b\ . \tag{139}$$

Note that although $\hat{H}$ is an integral of the motion, there is implicit time dependence in $a$ and $a^\dagger$. When normalized as $H=eB\hat{H}/M$, this operator is a time-dependent extension of the usual Hamiltonian for the time-dependent Landau problem. It directly generalizes the operator denoted by $\hat{H}$ in Ref. [7]. Using Eq. (139) and the new ket notation, the factorized Schrödinger equation (129) becomes

$$\hat{H}|n,m;\nu\rangle=(n+\nu)|n,m;\nu\rangle\ . \tag{140}$$

This expression shows that the states $|n_+,m_+;0\rangle$ and $|n_-=n_+-1,m_-;1\rangle$ are degenerate, except for the unique ground state $|0,0;0\rangle$, which has zero eigenvalue for $\hat{H}$ (as required for unbroken supersymmetry).

The framework for the supersymmetry is now almost complete. It remains merely to introduce appropriate supersymmetry generators mapping degenerate states into one another. These operators are defined by

$$Q=ab^\dagger\ ,\quad Q^\dagger=a^\dagger b\ . \tag{141}$$

They satisfy the graded commutation relations of the supersymmetry quantum-mechanical algebra sqm(2):

$$\{Q,Q\}=\{Q^\dagger,Q^\dagger\}=0\ ,\quad \{Q,Q^\dagger\}=\hat{H}\ ,$$
$$[\hat{H},Q]=[\hat{H},Q^\dagger]=0\ . \tag{142}$$

Explicitly, the action of the supersymmetry generators is

$$\begin{aligned}Q|n,m;\nu\rangle&=\sqrt{n}\,\delta_{0\nu}|n-1,m;1\rangle\ ,\\ Q^\dagger|n,m;\nu\rangle&=\sqrt{n+1}\delta_{1\nu}|n+1,m;0\rangle\ .\end{aligned} \tag{143}$$

The operator $Q$ maps bosonic states into fermionic ones, while its conjugate does the reverse.

## VIII. SUPERCOHERENT STATES

At this stage, we are in a position to construct the supercoherent states of the time-varying Landau system. To do so requires a supersymmetric generalization of the standard displacement-operator method. A natural approach to this was introduced in Ref. [7], using the supermanifold formalism developed in Ref. [22] and the techniques for the Baker-Campbell-Hausdorff relations of Ref. [23]. For a description of the general procedure and examples of its application, see the above references and Refs. [24,25].

In the present case, the relevant superalgebra $\mathcal{G}_s$ is the one obtained by extending the Lie algebra $\mathcal{G}$ as follows:

$$\mathcal{G}_s=\{a^\dagger a,c^\dagger c,a,a^\dagger,c,c^\dagger,1;b^\dagger b,b,b^\dagger\}\ . \tag{144}$$

A fixed state is required by the construction. We choose it as the ground state $|0,0;0\rangle$. The subalgebra consisting of the operators $\{a^\dagger a,c^\dagger c,1;b^\dagger b\}$ leaves this state fixed, i.e., the ground state is an eigenvector of these operators. According to the procedure of Ref. [7], the supercoherent states are to be defined via the action of the operators in the quotient algebra $\{a,a^\dagger,c,c^\dagger;b,b^\dagger\}$ on the fixed state.

It is convenient to define a super-Hermitian basis for the quotient algebra:

$$X_1=a+a^\dagger\ ,\quad X_2=i(a-a^\dagger)\ ,\quad X_3=c+c^\dagger\ ,$$

$$X_4=i(c-c^\dagger)\ ,\quad X_5=i(b+b^\dagger)\ ,\quad X_6=b^\dagger-b\ . \tag{145}$$

Constructing a unitary representation of the supergroup for an element in the quotient algebra [7,25], we obtain

$$\begin{aligned} T(g)=&(iA_1X_1+iA_2X_2+iC_3X_3+iC_4X_4\\ &+i\theta_1X_5+i\theta_2X_6)\ ,\\ =&\exp(-\bar{A}a+Aa^\dagger-\bar{C}c+Cc^\dagger+\theta b^\dagger+\bar{\theta}b)\ , \end{aligned} \tag{146}$$

where $A=A_2+iA_1$, $C=C_2+iC_1$, and $\theta=-\theta_1+i\theta_2$. This means that $A,C\in{}^0B_L$ and $\theta\in{}^1B_L$ are complex Grassmann-valued variables.

Using a suitable Baker-Campbell-Hausdorff relation for the supergroup element and Lemma 1 of Ref. [23], we can construct the analogs of the supercoherent states in Ref. [7]. The states are parameterized by three Grassman-valued parameters $A$, $C$, and $\theta$, and are given as

$$\begin{aligned} |Z\rangle=&\exp(\tfrac{1}{2}\theta\bar{\theta})\exp(-\tfrac{1}{2}|A|^2)\exp(-\tfrac{1}{2}|C|^2)\\ &\times\sum_{n,m}\frac{A^nC^m}{\sqrt{n!}\sqrt{m!}}(|n,m;0\rangle+\theta|n,m;1\rangle)\ . \end{aligned} \tag{147}$$

Recall that the operators $a$, $a^\dagger$, $c$, and $c^\dagger$ have an implicit time dependence in them through their dependence upon the variables $\zeta_1=x/\varphi_3^{1/2}(\tau)$ and $\zeta_2=y/\varphi_3^{1/2}(\tau)$. The supercoherent states defined above are the natural time-dependent generalizations of the supercoherent states derived in [7].

In Ref. [26], several theorems about time-dependent integrals of the motion are given. The first states that any function of integrals of the motion is itself an integral of the motion. The second states that eigenvalues of time-dependent integrals of the motion do not depend on time. The third states that applying an integral of the motion on a solution to a wave equation (either a Schrödinger or a Pauli time-dependent equation) yields a function that itself is a solution to the same wave equation. Since $a$, $b$, $c$, and their conjugates are integrals of the motion, so too is the operator $T(g)$. Furthermore, since $|0,0,0\rangle$ is a solution to the Pauli equation, $|Z\rangle$ is also a solution to the same equation. All the formulas in Ref. [7] may be rewritten identically for the properties of the present supercoherent states, with obvious replacements. Note that some aspects of integrals of motion in simple problems with supersymmetric quantum mechanics have been discussed in Ref. [27].

If one introduces a time-dependent electric field in addition to the present magnetic field, the basic algebraic structure derived here can again be applied. Provided this leads to a supersymmetric formulation, this provides a means of obtaining the explicit form of supercoherent states for this more general case. It is plausible that supercoherent states for a relativistic electron moving in an electromagnetic field of arbitrary configuration could be found using these methods combined with coherent states in the proper-time formalism, as discussed in Ref. [28].

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## APPENDIX: CONSTANT MAGNETIC INDUCTION

In this appendix, we demonstrate explicitly the connection between our results and the special case of constant induction treated in Ref. [7]. When the magnetic induction $B$ is constant and uniform, $w=eB$, the differential equations (26) and (27) take the form

$$\ddot{\epsilon}^\alpha+\tfrac{1}{4}w^2\epsilon^\alpha=0\ ,\quad \alpha=1,2\ , \tag{A1}$$

and have real solutions

$$\chi_1=\left[\frac{2}{w}\right]^{1/2}\cos\left[\frac{w\tau}{2}\right]\ ,\quad \chi_2=\left[\frac{2}{w}\right]^{1/2}\sin\left[\frac{w\tau}{2}\right]\ , \tag{A2}$$

satisfying the Wronskian condition $W(\chi_1,\chi_2)=1$. The complex solutions (41) become

$$\xi=\frac{1}{\sqrt{2}}(\chi_1+i\chi_2)=\frac{1}{\sqrt{w}}e^{iw\tau/2}\ ,\quad \bar{\xi}=\frac{1}{\sqrt{w}}e^{-iw\tau/2}\ , \tag{A3}$$

with Wronskian $W(\xi,\bar{\xi})=-i$.

The complex solutions to Eq. (25) are given by (44):

$$\varphi_1=\frac{1}{w}e^{iw\tau}\ ,\quad \varphi_2=\frac{2}{w}e^{-iw\tau}\ ,\quad \varphi_3=\frac{2}{w}\ ,\quad \eta=\tfrac{1}{2}w\tau\ . \tag{A4}$$

Using the equations derived in Sec. V B, we can write down the generators of the Schrödinger algebra

$$m_-=\frac{1}{w}e^{iw\tau}\left[-\partial_\tau-\frac{w}{2}L_z-\frac{iw}{2}(x\partial_x+y\partial_y)-\frac{i}{4}w^2(x^2+y^2)-\frac{iw}{2}-ih_0\right],$$

$$m_+=\frac{1}{w}e^{-iw\tau}\left[\partial_\tau+\frac{w}{2}L_z-\frac{iw}{2}(x\partial_x+y\partial_y)+\frac{i}{4}w^2(x^2+y^2)-\frac{iw}{2}+ih_0\right],$$

$$m_3=\frac{2i}{w}\left[\partial_\tau+\frac{1}{w}L_z+ih_0\right],\quad \mathcal{L}_z=iL_z=i(y\partial_x-x\partial_y),$$

$$a_-=\frac{e^{iw\tau}}{\sqrt{2w}}\left[\partial_x+i\partial_y+\frac{w}{2}(x+iy)\right], \tag{A5}$$

$$a_+=\frac{e^{-iw\tau}}{\sqrt{2w}}\left[-(\partial_x-i\partial_y)+\frac{w}{2}(x-iy)\right],$$

$$c_-=\frac{1}{\sqrt{2w}}\left[\partial_x-i\partial_y+\frac{w}{2}(x-iy)\right],$$

$$c_+=\frac{1}{\sqrt{2w}}\left[-(\partial_x+i\partial_y)+\frac{w}{2}(x+iy)\right],$$

$$I=1\ .$$

The generators of the subalgebra $\mathcal{G}$ are $\mathcal{G}=\{d_3,f_3,a_\pm,c_\pm,I\}$. We have

$$f_3=\frac{1}{w}(i\partial_\tau-h_0),\quad d_3=\frac{1}{w}(i\partial_\tau+w\mathcal{L}_z-h_0), \tag{A6}$$

along with the operators (A5)

From the results of Sec. V, the solutions (113) are

$$\Psi_{n,m}=e^{-\eta L_z}\Upsilon_{n,m}(\rho)e^{i\theta(m-n)}T_{n,m}(\mu), \tag{A7}$$

where $\Upsilon_{n,m}(\rho)$ is given by (114) and

$$T_{n,m}=\left[\frac{w}{2}\right]^{1/2}e^{-i(n+m+1)w\mu/2}e^{-ih_0\mu}, \tag{A8}$$

where $n,m\in\mathbb{Z}^+$. The polar coordinate system is related to the Cartesian system through the transformation

$$\rho^2=\zeta_1^2+\zeta_2^2,\quad \theta=\tan^{-1}(\zeta_2/\zeta_1), \tag{A9}$$

where

$$\zeta_1=x\left[\frac{w}{2}\right]^{1/2},\quad \zeta_1=x\left[\frac{w}{2}\right]^{1/2},\quad \mu=\tau, \tag{A10}$$

following Eqs. (71).

By direct calculation, we see that the action of $f_3$ and $d_3$ on $\Psi_{n,m}$ is

$$f_3\Psi_{n,m}=(n+\tfrac{1}{2})\Psi_{n,m},\quad d_3\Psi_{n,m}=(m+\tfrac{1}{2})\Psi_{n,m}. \tag{A11}$$

Also, the ladder operators $a_\pm$ and $c_\pm$ raise and lower the quantum numbers $n$ and $m$, respectively. Thus

$$a_-\Psi_{n,m}=\sqrt{n}\,\Psi_{n-1,m},\quad a_+\Psi_{n,m}=\sqrt{n+1}\Psi_{n+1,m},$$
$$c_-\Psi_{n,m}=\sqrt{m}\,\Psi_{n,m-1},\quad c_+\Psi_{n,m}=\sqrt{m+1}\Psi_{n,m+1}. \tag{A12}$$

In the special case of constant field, the $\tau$ or $\mu$ variable may be separated and we can work with the Cartesian-like coordinates $(\zeta_1,\zeta_2)$ or with the polar coordinates $(\rho,\theta)$, as defined in (A10) and (A9), respectively. In the former case, we use the wave function $\psi_{n,m}(\zeta_1,\zeta_2)$, and in the latter we use $\psi_{n,m}(\rho,\theta)=\Upsilon_{n,m}(\rho)e^{i\theta(m-n)}$. On these solution spaces, the operators $f_3$ and $d_3$ are replaced by $N_a=a^\dagger a$ and $N_c=c^\dagger c$, respectively. The ladder operators $a$ and $c$ are defined by

$$a=\tfrac{1}{2}[(\partial_{\zeta_1}+i\partial_{\zeta_2})+(\zeta_1+i\zeta_2)]=\frac{1}{\sqrt{2w}}\left[(\partial_x+i\partial_y)+\frac{w}{2}(x+iy)\right],$$
$$c=\tfrac{1}{2}[(\partial_{\zeta_1}-i\partial_{\zeta_2})+(\zeta_1-i\zeta_2)]=\frac{1}{\sqrt{2w}}\left[(\partial_x-i\partial_y)+\frac{w}{2}(x-iy)\right], \tag{A13}$$

where we have made use of (A10). In polar coordinates (A9), the operators and their conjugates are given in Eqs. (90).

The operators in (A13) are exactly the $a$ and $c$ operators obtained in Ref. [7]. Their time-dependent extensions are $a_-$ and $c_-$, respectively.

The number operators obey the nonzero commutation relations

$$[N_a,a]=-a,\quad [N_a,a^\dagger]=+a^\dagger,\quad [a,a^\dagger]=I,$$
$$[N_c,c]=-c,\quad [N_c,c^\dagger]=+c^\dagger,\quad [c,c^\dagger]=I. \tag{A14}$$

The action of these operators on the manifold of states $\{\psi_{n,m}\}$ is

$$N_a\psi_{n,m}=n\psi_{n,m},\quad a\psi_{n,m}=\sqrt{n}\,\psi_{n-1,m},$$
$$a^\dagger\psi_{n,m}=\sqrt{n+1}\psi_{n+1,m},\ N_c\psi_{n,m}=m\psi_{n,m},$$
$$c\psi_{n,m}=\sqrt{m}\,\psi_{n,m-1},\quad c^\dagger\psi_{n,m}=\sqrt{m+1}\psi_{n,m+1}. \tag{A15}$$

Substituting (A13) for $a$ in $N_a$, we obtain

$$N_a=\tfrac{1}{4}[-\partial_{\zeta_1\zeta_1}+\partial_{\zeta_2\zeta_2}+2\mathcal{L}_z+\zeta_1^2+\zeta_2^2-2]=\frac{1}{2w}\left[-(\partial_{xx}+\partial_{yy}+w\mathcal{L}_z)+\frac{w^2}{4}(x^2+y^2)-w\right], \tag{A16}$$

which is proportional to the Hamiltonian [7].

This completes the explicit demonstration that our formalism contains as a special case the description of the motion of a nonrelativistic spin-$\frac{1}{2}$ charged particle in a constant and uniform magnetic induction.

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