Adiabatic theorem for quantum systems with spectral degeneracy

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By stating the adiabatic theorem of quantum mechanics in a clear and rigorous way, we establish a necessary condition and a sufficient condition for its validity, where the latter is obtained employing our recently developed adiabatic perturbation theory. Also, we simplify further the sufficient condition into a useful and simple practical test at the expense of its mathematical rigor. We present results for the most general case of quantum systems, i.e., those with degenerate energy spectra. These conditions are of upmost importance for assessing the validity of practical implementations of non-Abelian braiding and adiabatic quantum computation. To illustrate the degenerate adiabatic approximation, and the necessary and sufficient conditions for its validity, we analyze in depth an exactly solvable time-dependent degenerate problem.

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Introduction. The adiabatic theorem [1] has played, and still plays, a fundamental role in practical quantum physics applications. Indeed, the ability to determine how the *slow* dynamics of external probes coupled to a system affect its time evolution has applications ranging from the notion of thermal equilibrium and nonequilibrium phenomena [2] to the conditions under which an adiabatic quantum computer can reliably operate [3]. Useful and practical quantitative conditions for the validity of the adiabatic theorem are also relevant to the important current problem of assessing the feasibility of any information processing scheme that uses the concept of fractional exchange statistics and non-Abelian braiding [4].

General physical principles dictate that, in three space dimensions, elementary particles can only obey fermionic or bosonic statistics. Kinematic constraints do not allow for fractional exchange statistics: electrons are spin-1/2 fermions and photons are spin-1 bosons. Nonetheless, fractional statistics *particles* or *modes* may emerge from the collective behavior of elementary particles, i.e., collective excitations of a quantum field, as a result of a dynamical process. The latter requires special circumstances and constraints that should be analyzed on a case by case basis. For instance, for two localized degenerate Majorana modes to realize a non-Abelian braiding process, we need to design the physical Hamiltonians realizing the braiding that do not lift the degeneracy and can be implemented adiabatically. If those constraints are not met experimentally, then the braiding operation is faulty. Physical systems where such fractional statistics emerges have a highly degenerate energy spectrum, thus justifying a careful statement of the adiabatic theorem and the precise conditions for its validity.

Despite its practical importance, no consensual and rigorous necessary and sufficient conditions for the validity of the adiabatic theorem have been given. Only recently a proof that the commonly used textbook condition [1] is necessary for nondegenerate Hamiltonians [5], but not sufficient [6], was given. For degenerate systems, even a clear presentation of the theorem is lacking, let alone necessary and sufficient conditions. It is this article's intention to fill that gap.

With that in mind, our goal is three fold. First, using techniques developed in [8] and [9], we aim at providing a clear and rigorous version of the adiabatic theorem for Hamiltonians with nondegenerate and degenerate spectra using a single formalism. We want to be as precise as possible in stating the adiabatic theorem to avoid common misunderstandings [7], mainly due to a lack of quantitative rigor in the way the theorem is usually presented. Second, we prove necessary and sufficient conditions for the validity of the rigorous version of the adiabatic theorem presented here. The necessary condition for degenerate spectra reduces to the one in [5] when no degeneracy is present. To obtain a sufficient condition, we rely on the adiabatic perturbation theory developed in [8] and [9]. Finally, we apply these ideas to an exactly solvable time-dependent degenerate problem [9], where we show that the necessary and sufficient conditions developed here provide the correct conditions under which the adiabatic theorem holds.

To properly formulate the degenerate adiabatic theorem (DAT), we first need to introduce the degenerate adiabatic approximation (DAA). As we will see, the DAT is essentially a statement about the mathematical conditions for the validity of the DAA. This understanding of the essence of the adiabatic theorem is akin to those of Berry [10] and Tong [5], for nondegenerate systems, and to those of Wilczek and Zee (WZ) [11] and Wilczek [12], for degenerate systems.

Degenerate adiabatic approximation. Consider an explicitly time-dependent Hamiltonian $\mathbf{H}(t)$ with orthonormal eigenvectors $|n^{g_n}(t)\rangle$, where $g_n = 0,1,\ldots,d_n-1$ labels states of the degenerate eigenspace $\mathcal{H}_n$ of dimension d_n and eigenenergy $E_n(t)$, $\mathbf{H}(t)|n^{g_n}(t)\rangle = E_n(t)|n^{g_n}(t)\rangle$, and assume that d_n does not change during the total time evolution, $t \in [0,T]$. An arbitrary state at $t=0$ can be written as $|\Psi^{(0)}(0)\rangle = \sum_n \sum_{g_n=0}^{d_{n-1}} b_n(0) U^n_{h_n g_n}(0)|n^{g_n}(0)\rangle$, where $|b_n(0)|^2$ gives the probability of the system being in eigenspace $\mathcal{H}_n$ and $|b_n(0)U^n_{h_n g_n}(0)|^2$ the probability of measuring a specific eigenstate. A given initial condition within an eigenspace is characterized by one value of $h_n = 0,1,\ldots,d_n-1$. A compact way of representing all possible initial conditions spanning the orthonormal eigenspace $\mathcal{H}_n$ is [9] $|\mathbf{\Psi}^{(0)}(0)\rangle = \sum_{n=0} b_n(0)\mathbf{U}^n(0)|\mathbf{n}(0)\rangle$, where $|\mathbf{n}(t)\rangle = (|n^0(t)\rangle, |n^1(t)\rangle, \ldots, |n^{d_n-1}(t)\rangle)$ is a column vector, and $\mathbf{U}^n(0)$ is a $d_n \times d_n$ unitary matrix, $\mathbf{U}^n(0)[\mathbf{U}^n(0)]^\dagger = \mathbb{1}$. A particular initial state corresponds to choosing the corresponding element of the column vector $|\mathbf{\Psi}^{(0)}(0)\rangle$.

Then the most general way of writing the DAA is

$$|\mathbf{\Psi}^{(0)}(t)\rangle = \sum_{n=0} e^{-i\omega_n(t)} b_n(0)\mathbf{U}^n(t)|\mathbf{n}(t)\rangle, \quad (1)$$

where $\omega_n(t) = \int_0^t E_n(t')dt'/\hbar$ is the dynamical phase, and the unitary matrix $\mathbf{U}^n(t) = \mathbf{U}^n(0)\mathcal{T}\exp\left(\int_0^t \mathbf{A}^{nn}(t')dt'\right)$ is the non-Abelian WZ phase. Here $\mathcal{T}$ denotes a time-ordered operator, and $A^{nm}_{h_n g_m}(t) = (M^{nm}_{h_n g_m}(t))^*$ a $d_n \times d_n$ matrix defined as

$$[\mathbf{M}^{mn}(t)]_{g_m h_n} = M^{nm}_{h_n g_m}(t) = \langle n^{h_n}(t)|\dot{m}^{g_m}(t)\rangle, \quad (2)$$

with the dot meaning the time derivative. For example, for a system starting at the ground eigenspace ($b_n(0) = \delta_{n0}$), $|\mathbf{\Psi}^{(0)}(t)\rangle = e^{-i\omega_0(t)}\mathbf{U}^0(t)|\mathbf{0}(t)\rangle$.

The time evolution of an informationally isolated quantum system is dictated by the Schrödinger equation (SE) $i\hbar|\dot{\mathbf{\Psi}}(t)\rangle = \mathbf{H}(t)|\mathbf{\Psi}(t)\rangle$. What are the constraints on the rate of change of $\mathbf{H}(t)$ under which the system's evolved state $|\mathbf{\Psi}(t)\rangle$ gets *close* to the DAA? The adiabatic theorem we formulate next sets the conditions under which the DAA holds. In other words, it precisely states when the system's dynamics can be approximated by the DAA.

Adiabatic theorem. If a system's Hamiltonian $\mathbf{H}(t)$ changes *slowly* during the course of time, say from $t = 0$ to $t = T$, and the system is prepared in an arbitrary superposition of eigenstates of $\mathbf{H}(t)$ at $t = 0$, say $|\mathbf{\Psi}^{(0)}(0)\rangle$, then the transitions between eigenspaces $\mathcal{H}_n$ of $\mathbf{H}(t)$ during the interval $t \in [0,T]$ are *negligible* and the system *evolves* according to DAA.

The three important concepts, *slow, negligible*, and *evolved states*, need further explanation. First, DAA is based on the assumption that the rate of change of $\mathbf{H}(t)$ is slow. A crucial matter is then to establish the meaning of slow precisely. Intuitively, the latter notion can be understood as a relation between a characteristic *internal* time of the evolved system T_i, encoded in $\mathbf{H}(t)$, and the total evolution time T, such that $T_i/T \ll 1$. For a fixed and finite T_i, one can always choose an evolution time T that satisfies this condition. This state of affairs, however, is not satisfactory from a mathematical standpoint. Indeed, a main source of controversy in the literature arises from the lack of a precise quantification of the term *slow*. By using the degenerate adiabatic perturbation theory (DAPT) [9], a generalization of the APT [8], we can give a precise meaning to this notion of slowness, which is the key ingredient to the derivation of the sufficient condition of the DAT. Second, to establish the necessary condition, we follow Tong [5] and others [10–12] and assume that if the system's state is well described by the DAA, then all measurements performed on the system at *any* time must indeed be consistent with this assumption. This has a profound implication on the approximate dynamics the system obeys [5]. The following necessary and sufficient conditions provide the mathematical rigor required to make those concepts precise.

Necessary condition. There is no unique way of establishing how *close* two quantum states are, implying that there is no unique distance measure between states. A popular choice in the context of quantum information is the fidelity measure. We stress, though, that the DAT is not a statement about the fidelity between the true time-dependent state $|\Psi(t)\rangle$ and the DAA $|\Psi^{(0)}(t)\rangle$ being close to 1, i.e., $|\langle\Psi(t)|\Psi^{(0)}(t)\rangle| \sim 1$. It is more than that; it is a statement about the DAA expectation value of *any* observable being *close* to the exact ones. This notion is crucial for defining geometric phases, and thus for particle exchange statistics, and is crucial for the philosophy behind DAPT and the proof of necessity that now follows.

If the DAA is an accurate description of the time evolution of a degenerate system starting, with no loss of generality, in its ground eigenspace $[b_n(0) = \delta_{n0}]$, then $|\mathbf{\Psi}(t)\rangle = |\mathbf{\Psi}^{(0)}(t)\rangle + O(1/T) \approx |\mathbf{\Psi}^{(0)}(t)\rangle$, with $\|O(1/T)\|_{\max} \ll 1$, where $\|\cdot\|_{\max}$ is the max norm (the absolute value of the greatest element of a given vector or matrix). It immediately follows that (a) the system approximately satisfies the SE $i\hbar|\dot{\mathbf{\Psi}}^0(t)\rangle \approx \mathbf{H}(t)|\mathbf{\Psi}^0(t)\rangle$, which *implies* [5] $|\dot{\mathbf{\Psi}}(t)\rangle \approx |\dot{\mathbf{\Psi}}^{(0)}(t)\rangle$, and that (b) transitions to excited *eigenspaces* are negligible [13], $\|\langle\mathbf{n}(t)|^T|\mathbf{\Psi}(t)\rangle^T\|_{\max} \ll 1$, $n \neq 0$.

Now, using (a) and (b) and defining $\Delta_{nm}(t) = E_n(t) - E_m(t)$, we note that for $n \neq 0$ [14],

$$\begin{aligned}\langle\mathbf{n}(t)|^T|\mathbf{\Psi}(t)\rangle^T &= \frac{\langle\mathbf{n}(t)|^T(\mathbf{H}(t) - E_0(t))|\mathbf{\Psi}(t)\rangle^T}{\Delta_{n0}(t)} \\ &= \frac{\langle\mathbf{n}(t)|^T(i\hbar|\dot{\mathbf{\Psi}}(t)\rangle^T - E_0(t)|\mathbf{\Psi}(t)\rangle^T)}{\Delta_{n0}(t)} \\ &\approx \frac{i\hbar\langle\mathbf{n}(t)|^T|\dot{\mathbf{\Psi}}^{(0)}(t)\rangle^T}{\Delta_{n0}(t)} \\ &= i\hbar e^{-i\omega_0(t)}\frac{\langle\mathbf{n}(t)|^T[\mathbf{U}^0(t)|\dot{\mathbf{0}}(t)\rangle]^T}{\Delta_{n0}(t)},\end{aligned}$$

where $\langle\mathbf{n}(t)|^T|\mathbf{0}(t)\rangle^T = \mathbf{0}$. Taking the max norm on both sides and using (b), we get the necessary condition $\hbar\|\langle\mathbf{n}(t)|^T[\mathbf{U}^0(t)|\dot{\mathbf{0}}(t)\rangle]^T/\Delta_{n0}(t)\|_{\max} \ll 1$, $\neq 0$, $t \in [0,T]$. Finally, using that $\|\mathbf{U}^n(t)\|_{\max} \leqslant 1$ leads to a stronger WZ phase-free necessary condition,

$$\hbar\left\|\frac{\mathbf{M}^{n0}(t)}{\Delta_{n0}(t)}\right\|_1 \ll 1, \quad n \neq 0, \quad t \in [0,T], \quad (3)$$

where $\|A\|_1 = \max_{1\leqslant j\leqslant p}\sum_{i=1}^q |a_{ij}|$ for a $p \times q$–dimensional matrix A. When the spectrum is nondegenerate ($d_n = 1$), Eq. (3) reduces to the necessary condition of Ref. [5].

Sufficient condition. The first step in establishing the sufficient condition is to prove the convergence of the DAPT in its full generality. Intrinsic to the formulation of the DAPT is a Taylor series expansion in terms of the parameter $v = 1/T$ and a necessary rescaling of time according to $s = vt$ with $s \in [0,1]$ [9]. For small enough v, one can always make the DAPT converge [cf. Eq. (6)].

Inserting the ansatz

$$|\Psi(s)\rangle = \sum_{n=0}\sum_{p=0}^{\infty}\mathbf{C}_n^{(p)}(s)|\mathbf{n}(s)\rangle \quad (4)$$

into the SE with $\mathbf{C}_n^{(p)}(s) = e^{-\frac{i}{v}\omega_n(s)}v^p\mathbf{B}_n^{(p)}(s)$ and $\mathbf{B}_n^{(p)}(s) = \sum_{m=0} e^{\frac{i}{v}\omega_{nm}(s)}\mathbf{B}_{mn}^{(p)}(s)$, the DAPT gives recursive equations for $\mathbf{B}_{mn}^{(p)}(s)$ in terms of a lower order in p coefficients [9]. The zeroth order is exactly the DAA, with the WZ phase naturally appearing as a requirement for the consistency of the series expansion. Note that for each n we have a series involving the matrix $\mathbf{C}_n^{(p)}(s)$, $p = 0,1,\ldots,\infty$. The matrix element $[\mathbf{C}_n^{(p)}(s)]_{h_n g_n}$ is the coefficient giving the contribution to order p of the state $|n^{g_n}(s)\rangle$ to the solution to the SE. Here h_n handles different initial conditions, and for definiteness we pick the case $h_n = 0$, $\forall n$. Applying the ratio test for series expansions, if the condition

$$\lim_{p\to\infty} \left|\left[\mathbf{C}_n^{(p+1)}(s)\right]_{0g_n} / \left[\mathbf{C}_n^{(p)}(s)\right]_{0g_n}\right| < 1, \quad \forall n, g_n, \tag{5}$$

is satisfied for all coefficients, then we guarantee convergence of the DAPT. We can simplify (5) further by invoking the comparison test [14],

$$\lim_{p\to\infty} \frac{v\sum_{m=0}\left|\left[\mathbf{B}_{mn}^{(p+1)}(s)\right]_{0g_n}\right|}{\sum_{m=0}\left|\left[\mathbf{B}_{mn}^{(p)}(s)\right]_{0g_n}\right|} < 1, \quad \forall n, g_n. \tag{6}$$

Imposing that $\sum_{p=0}^{\infty}|[\mathbf{C}_n^{(p+1)}(s)]_{0g_n}| \ll |[\mathbf{C}_n^{(0)}(s)]_{0g_n}|, \forall n, g_n$, meaning that the zeroth order dominates, is equivalent to

$$\sum_{p=0}^{\infty}\sum_{m=0} v^{p+1}\left|\left[\mathbf{B}_{mn}^{(p+1)}(s)\right]_{0g_n}\right| \ll \sum_{m=0}\left|\left[\mathbf{B}_{mn}^{(0)}(s)\right]_{0g_n}\right|, \tag{7}$$

which, together with Eq. (6), is the rigorous sufficient conditions for the validity of the DAA. In practice it is extremely difficult to compute the previous limit when $p \to \infty$ and all orders p. We can come up, nevertheless, with some practical condition of convergence by looking at the ratio for a couple of finite orders p. Working with increasing p we get more and more conditions that, in the nondegenerate case, can become stronger than those in [15]. In its simplest form, we may consider only $p = 0$. In this case both expressions merge into one and we demand it to be *much smaller* than the smallest *non-null* term appearing on the right-hand side of (7). Thus, the practical sufficient test reads

$$v\sum_{m=0}\left|\left[\mathbf{B}_{mn}^{(1)}(s)\right]_{0g_n}\right| \ll \min_{n,g_n}\sum_{m=0}\left|\left[\mathbf{B}_{mn}^{(0)}(s)\right]_{0g_n}\right|. \tag{8}$$

Using [9] $\mathbf{B}_{mn}^{(0)}(s) = b_n(0)\mathbf{U}^n(s)\delta_{mn}$ and the fact that at $t = 0$ the initial state is $|0^0(0)\rangle$ $[b_n(0) = \delta_{n0}]$, we get

$$v\sum_{m=0}\left|\left[\mathbf{B}_{mn}^{(1)}(s)\right]_{0g_n}\right| \ll \min_{g_0}\left(\left|[\mathbf{U}^0(s)]_{0g_0}\right|\right), \quad \forall n, g_n, \tag{9}$$

which is our intuitive and practical sufficient condition. Indeed, noting that $v\sum_{m=0} e^{-\frac{i}{v}\omega_m(s)}[\mathbf{B}_{mn}^{(1)}(s)]_{0g_n}$, with $n \neq 0$, gives the first-order contribution of the excited state $|n^{g_n}(s)\rangle$ to the wave equation, and that for $n = 0$ it is related to the first-order correction to the WZ phase [9], it is clear that they must be much smaller than the smallest coefficient appearing in the zeroth order if we want the DAA to hold.

Equation (9) also depends on $\mathbf{U}^n(s)$ because $\mathbf{B}_{mn}^{(1)}(s)$ depends on $\mathbf{U}^n(s)$. However, a calculation similar to the one done for the necessary condition gets rid of these unitary matrices, leading to [14]

$$D_{g_n}^n(t) \ll \min_{g_0}\left(\left|[\mathbf{U}^0(t)]_{0g_0}\right|\right), \quad t \in [0,T], \tag{10}$$

where, for $n = 0$ and $\forall g_0$, we have $D_{g_0}^0(t)$ equals

$$\hbar d_0 \int_0^t dt' \sum_{n=1}\left\{\frac{\sum_{k_0,i_0=0}^{d_0-1}|[\mathbf{M}^{0n}(t')(\mathbf{M}^{0n}(t'))^\dagger]_{k_0 i_0}|}{|\Delta_{0n}(t')|}\right\}, \tag{11}$$

and for $n \neq 0$ and $\forall g_n$, $D_{g_n}^n(t)$ is given by

$$\frac{\hbar}{|\Delta_{n0}(0)|}\left\{\sum_{k_0=0}^{d_0-1}\left|[\mathbf{M}^{0n}(t)]_{k_0 g_n}\right| + d_n\sum_{k_0,l_n=0}^{d_0-1,d_n-1}\left|[\mathbf{M}^{0n}(0)]_{k_0 l_n}\right|\right\}. \tag{12}$$

Example. We now apply the previous ideas to a doubly degenerate four-level system subjected to a rotating magnetic field of constant magnitude $\mathbf{B}(t) = B\mathbf{r}(t)$ and in spherical coordinates $\mathbf{r}(t) = (\sin\theta\cos wt, \sin\theta\sin wt, \cos\theta)$, with $w > 0$ and $0 \leqslant \theta \leqslant \pi$ being the polar angle. The Hamiltonian describing this system is [9,16] $\mathbf{H}(t) = \hbar b\,\mathbf{r}(t)\cdot\mathbf{\Gamma}/2$, where $b > 0$ is proportional to the coupling between the field and the system and $\mathbf{\Gamma} = (\Gamma_x, \Gamma_y, \Gamma_z)$ are the Dirac matrices $\Gamma_j = \sigma_x \otimes \sigma_j$, $j = x,y,z$. Here σ_j are the standard Pauli matrices implying the following algebra for Γ_j, $\{\Gamma_i,\Gamma_j\} = 2\delta_{ij}\boldsymbol{I}_4, [\Gamma_i,\Gamma_j] = 2i\epsilon_{ijk}\Pi_k$, where $\boldsymbol{I}_4$ is the identity matrix of dimension 4, δ_{ij} the Kronecker δ, ϵ_{ijk} the Levi-Civita symbol, and $\Pi_k = \boldsymbol{I}_2 \otimes \sigma_k$. Starting at the ground state $|0^0(0)\rangle$, the time-dependent solution in terms of the snapshot eigenstates is [9] $|\Psi(t)\rangle = e^{iwt/2}[(1+\cos\theta)A_-(t) + (1-\cos\theta)A_+(t)]/2|0^0(t)\rangle + e^{-iwt/2}\sin\theta[A_+(t) - A_-(t)]/2|0^1(t)\rangle + e^{iwt/2}\sin^2\theta[B_+(t) + B_-(t)]/2|1^0(t)\rangle + e^{-iwt/2}\sin\theta[(1+\cos\theta)B_-(t) - (1-\cos\theta)B_+(t)]/2|1^1(t)\rangle$, where $A_\pm(t) = \cos(\Omega_\pm t/2) + i(b \pm w\cos\theta)\sin(\Omega_\pm t/2)/\Omega_\pm$, $B_\pm(t) = iw\sin(\Omega_\pm t/2)/\Omega_\pm$, and $\Omega_\pm^2 = w^2 + b^2 \pm 2wb\cos\theta$.

Necessary condition. Since in this example Eq. (2) is $[\mathbf{M}^{10}(t)]_{11} = -[\mathbf{M}^{10}(t)]_{00} = iw\sin^2(\theta)/2$ and $[\mathbf{M}^{10}(t)]_{10} = -[\mathbf{M}^{10}(t)]_{01}^* = -iw\sin(2\theta)e^{iwt}/4$, the necessary condition, (3), becomes $w\sin\theta|\sin\theta + \cos\theta|/(2b) \ll 1$. Our task now is to look at the exact solution, impose that the DAA holds, and see if it implies the necessary condition above. If the DAA holds, then the absolute values of the coefficients multiplying $|1^0(t)\rangle$ and $|1^1(t)\rangle$ must be negligible. This leads to [14] $w\sin\theta f(\theta)/(2b) \ll 1$, with $f(\theta) = |b/\Omega_+ + b/\Omega_- + \cos\theta(b/\Omega_- - b/\Omega_+)|$. Noting that $f(\theta)$ has a global minimum at $\theta = \pi/2$ equal to $2b/\sqrt{b^2+w^2}$, it is not difficult to see that if $w < b$, then $f(\theta) \geqslant \sqrt{2}$. Hence, $1 \gg w\sin\theta f(\theta)/(2b) \geqslant w\sin\theta\sqrt{2}/(2b) \geqslant w\sin\theta|\sin\theta + \cos\theta|/(2b)$, which is exactly the necessary condition. When $w \geqslant b$ we have $f(\theta) \geqslant b\sqrt{2}/w$, which leads to $w\sin\theta f(\theta)/(2b) \geqslant \sqrt{2}\sin(\theta)/2 \approx \sin\theta$. Since $\sin\theta \approx 1$, the DAA is not a faithful approximation to the exact state for general θ when $w \geqslant b$. This is expected since the rotating frequency w of the magnetic field must be much smaller than the coupling constant b (natural frequency of the system) for the DAA to hold. The pathological situation where $\sin\theta \to 0$ and the fidelity of the state approaches unity even though $w \geqslant b$ does not lead to a state evolving according to the DAA [7].

Sufficient condition. Equations (10)–(12) become, for $t \in [0,T]$, $w^2 t\sin^2(\theta)/b \ll \min_{g_0}(|[\mathbf{U}^0(t)]_{0g_0}|)$ and $5w\sin\theta(|\cos\theta| + \sin\theta)/(2b) \ll \min_{g_0}(|[\mathbf{U}^0(t)]_{0g_0}|)$, where $|[\mathbf{U}^0(t)]_{00}| = \{1 - \sin^2\theta\sin^2[wt\cos(\theta)/2]\}^{1/2}$ and $|[\mathbf{U}^0(t)]_{01}| = \sin\theta|\sin[wt\cos(\theta)/2]|$, with $g_0 = 0,1$. Note that the sufficient condition here is stronger than the necessary one because $5w\sin\theta(|\cos\theta| + \sin\theta)/(2b) \geqslant (w\sin\theta|\cos\theta + \sin\theta|/(2b)$. Moreover, looking at Eqs. (3) and (9), and, in particular, (12), we can show that in general the practical sufficient condition implies the necessary one whenever the gap is constant. Since for this example the natural choice for the perturbative parameter v is the rotating frequency of the field ($v = w$) [9], we have $wt \leqslant 1$ for $t \in [0,T]$. This implies that $|[\mathbf{U}^0(t)]_{00}| \geqslant |[\mathbf{U}^0(t)]_{01}|$ and $5w\sin\theta(|\cos\theta| + \sin\theta)/(2b) \geqslant w^2 t \sin^2(\theta)/b$ during the whole evolution of the state. Hence, the sufficient condition boils down to only one equation, $5w\sin\theta(|\cos\theta| + \sin\theta)/(2b) \ll \sin\theta|\sin(wt\cos(\theta)/2)|$, leading to $5w/(2b) \ll |\sin(wt\cos(\theta)/2)|/(|\cos\theta| + \sin\theta)$. Note that when $t \approx 0$ and/or $\theta \approx \pi/2$, $|[\mathbf{U}^0(t)]_{01}| \approx 0$ and we must work with the non-null coefficient $|[\mathbf{U}^0(t)]_{00}|$. In this case the sufficient condition is $5w/(2b) \ll 1$.

It is important to remark now that if $w \geqslant b$, we cannot satisfy the sufficient condition, no matter what the value of $\sin\theta$ is. Indeed, since both terms appearing on the right-hand side of the sufficient conditions are <1, assuming $w \geqslant b$ leads to a left-hand side >1. The sufficient conditions are then consistent with the cases where the necessary condition fails. We cannot have $\sin\theta \approx 0$ and $w \geqslant b$ as an instance in which the DAA holds.

Our last task is to show that for $w < b$ these conditions imply the DAA. In other words, we must use them to show that the absolute values of the coefficients multiplying $|1^0(t)\rangle$ and $|1^1(t)\rangle$ of the exact solution are negligible. Working with the largest of those, this is equivalent to showing that [14] $wg(\theta)/b \ll 1$, with $g(\theta) = \sin\theta\,(b/\Omega_+ + b/\Omega_-)$. Using that $g(\theta)$ has a maximum, for $\theta \in [0,\pi]$, at $\theta = \pi/2$ given by $2b/(b^2 + w^2)^{1/2}$, we get $wg(\theta)/b \leqslant 2w/(b^2 + w^2)^{1/2} \leqslant 2w/b$. Hence, if the sufficient conditions imply that $2w/b \ll 1$, we are done. But noting that $|\sin(wt\cos(\theta)/2)|/(|\cos\theta| + \sin\theta) < 1/2$, the sufficient conditions reduce to $5w/b \ll 1$, which obviously implies $2w/b \ll 1$.

Summary. We have established one rigorous necessary condition and two sufficient conditions, one rigorous and one practical, for the validity of the quantum adiabatic theorem for systems with degenerate spectra. Concepts such as "slowly or adiabatically changing Hamiltonians" and the "adiabatic approximation" for degenerate systems, of greatest importance for the implementation of adiabatic and topological quantum computation as well as non-Abelian fractional statistics, have been quantitatively stated. It is this quantitative specification that allows for a precise and rigorous formulation of the adiabatic theorem. Finally, we have applied the adiabatic theorem to an exactly solvable degenerate problem and provided a complete characterization of the mathematical conditions under which the DAA holds.

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