On the Combinatorics of Schubert Calculus

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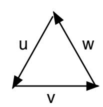
1 Introduction

We construct a lattice by choosing three unit vectors u, v, w in the plane such that $u + v + w = 0$.



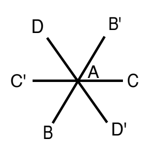
The points $iu + jv$ with i, j integers will be called *lattice points*, and a segment joining two nearest lattice points will be called a *small edge*. We consider positive measures m which are supported by a union of small edges, that satisfy the following properties:

(1) The restriction of m to each small edge is a multiple of a linear measure. This multiple is called the *density* of m on the small edge.

(2) m satisfies the balance condition

$$m(AB) - m(AB') = m(AC) - m(AC') = m(AD) - m(AD')$$

whenever A is a lattice point and the neighboring lattice points B, C', D, B', C, D' are in cyclic order around A.



The density of a measure m will be considered to be zero on segments outside its support. A lattice point incident to at least three small edges in the support of m is called a *branch point* of the measure m. We only consider measures with at least one branch point.

Fix an integer $r \geq 1$, and denote by Δ_r the (closed) triangle with vertices $0, ru, ru + rv = -rw$. We use the notation

$$A_j = ju, \quad B_j = ru + jv, \quad C_j = (r - j)w,$$

for $j = 0, 1, \dots, r$, for the lattice points on the boundary of Δ_r. We also set

$$X_j = A_j + w, \quad Y_j = B_j + u, \quad Z_j = C_j + v$$

for $j = 0, 1, 2, \dots, r$.

We denote by $\mathcal{M}_r$ the collection of all measures m satisfying conditions (1) and (2), whose branch points are contained in Δ_r, and such that

$$m(A_j X_{j+1}) = m(B_j Y_{j+1}) = m(C_j Z_{j+1}) = 0, \quad j = 0, 1, \dots, r.$$

The numbers

$$\alpha_j = m(A_j X_j), \quad \beta_j = m(B_j Y_j), \quad \gamma_j = m(C_j Z_j),$$

where $j = 0, 1, \dots, r$, will be called the *exit densities* of m. A measure $m \in \mathcal{M}_r$ is said to be rigid if there is no other measure $m' \in \mathcal{M}_r$ with the same exit points and exit densities as m. In other words, a rigid measure is entirely determined by its exit densities.

Given a measure $m \in \mathcal{M}_r$, we define its weight $w(m) \in \mathbb{R}_+$ to be

$$w(m) = \sum_{j=0}^{r} m(A_j X_j) = \sum_{j=0}^{r} m(B_j Y_j) = \sum_{j=0}^{r} m(C_j Z_j).$$

The equality of the three sums giving $w(m)$ is an easy consequence of the balance condition.

The remainder of the paper is organized as follows. In Section 2 we formulate the Littlewood-Richardson Rule in terms of measures. In Section 3 we focus our discussion of measures on a special kind - tree measures. This leads us to our main results in Section 4, where we develop a set of rules for constructing rigid tree measures. We conclude the paper with possible directions for future research in Section 5.

2 The Littlewood-Richardson Rule

We can describe the Littlewood-Richardson rule in terms of measures, and this turns out to be a very useful way to study intersections of Schubert varieties. Given integers $n, 1 \leq k \leq n-1$, the Grassmanian manifold $Gr(n,k)$ is defined to be

$$Gr(n,k) = \{k\text{-dimensional linear vector subspaces of } \mathbb{C}^n\}$$

For every flag

$$\mathcal{E} = \{\{0\} = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = \mathbb{C}^n\},$$

where E_j is a subspace of dimension j, $Gr(n,k)$ can be written as a union of Schubert varieties described as follows. For each set

$$I = \{i_1 < i_2 < \cdots < i_r\} \subset \{1, 2, \dots, n\},$$

one defines the Schubert variety

$$S(\mathcal{E}, I) = \{M \in Gr(n,k) : \dim(M \cap E_{i_x}) \geq x, x = 1, 2, \ldots, k\}.$$

Given sets $I, J, K \subset \{1, 2, \ldots, n\}$ of cardinality k such that

$$\sum_{l=1}^{k} (i_l + j_l + k_l - 3l) = 2k(n-k)$$

the Littlewood-Richardson rule provides a non-negative integer c_{IJK} with the property that the set

$$S(\mathcal{E}, I) \cap S(\mathcal{F}, J) \cap S(\mathcal{G}, K)$$

has a finite intersection, equal to c_{IJK}. The integer c_{IJK} is called the Littlewood-Richardson coefficient, and c_{IJK} can be defined in terms of measures.

Assume that $m \in \mathcal{M}_r$ assigns integer densities to all small edges. Let $\alpha_n, \beta_n, \gamma_n$ be the exit densities of m. We can then define an integer

$$n = r + w(m),$$

and sets $I, J, K \subset \{1, 2, \ldots, n\}$ of cardinality r by setting $I = \{i_1, i_2, \ldots, i_r\}$, where

$$i_l = l + \sum_{n=0}^{l-1} \alpha_n, \quad l = 1, 2, \ldots, r, \tag{1}$$

$$j_l = l + \sum_{n=0}^{l-1} \beta_n, \quad l = 1, 2, \ldots, r, \tag{2}$$

$$k_l = l + \sum_{n=0}^{l-1} \gamma_n, \quad l = 1, 2, \ldots, r. \tag{3}$$

These are precisely the triples of sets (I, J, K) which satisfy the Littlewood-Richardson rule. The *Littlewood-Richardson coefficient* c_{IJK} equals the number of measures $m \in \mathcal{M}_r$ with integer densities which satisfy (1), (2), and (3), i.e. which have the same exit densities as m.

Given a measure $m \in \mathcal{M}_r$, we formulate the associated Schubert intersection problem. A measure m determines sets I, J, K as above. The problem is to compute explicit elements in the intersection of three Schubert varieties,

$$S(\mathcal{E}, I) \cap S(\mathcal{F}, J) \cap S(\mathcal{G}, K),$$

where $\mathcal{E}, \mathcal{F}, \mathcal{G}$ are generic flag varieties. An explicit solution of the Schubert intersection problem associated with a measure can be produced in the rigid case, and the method is described in [2].

Example. In Figure 1, the measure m assigns density 1 to the thickened edges and density 0 to the other edges in the triangle.

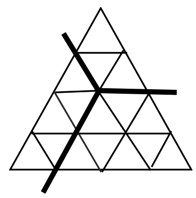


Figure 1: This measure has one branch point.

The sets I, J, K are determined by m as follows:

$$I = \{1, 3, 4, 5\}$$
$$J = \{1, 3, 4, 5\}$$
$$K = \{1, 2, 4, 5\}$$

The subspace M of $\mathbb{C}^5$ in the intersection of three Schubert varieties

$$S(\mathcal{E}, I) \cap S(\mathcal{F}, J) \cap S(\mathcal{G}, K)$$

is exactly $M = E_1 + F_1 + G_2$, where E_1, F_1, G_2 are elements of the flags $\mathcal{E}, \mathcal{F}, \mathcal{G}$, respectively.

3 Trees and Measures

Some measures $m \in \mathcal{M}_r$ have an underyling tree structure which we describe next. We consider trees embedded in the plane $\mathbb{R}^2$ such that

(T.1) each edge of the tree is a straight line segment of unit length,

(T.2) each vertex has order 2 or 3, and

(T.3) there are only finitely many vertices of order 3.

These conditions imply that a tree is infinite, but has a finite number of *ends*. Ends are sequences of vertices of the form $V_0V_1\ldots$, where V_0 is a branch point, V_j has order 2 for $j \geq 1$, and V_jV_{j+1} is an edge for each $j \geq 0$. We will require one more condition on our trees.

(T.4) The shortest path joining two different ends contains an odd number of vertices of order 3.

All trees discussed in the sequel satisfy the above four properties.

An *immersion* of a tree $T \subset \mathbb{R}^2$ is a continuous map $f : T \to \mathbb{R}^2$ which satisfies the following properties:

(1) f is isometric on each edge.

(2) If VA and VB are two edges meeting at a vertex of order 2, then

$$2f(V) = f(A) + f(B).$$

(3) If VA, VB, VC are three edges meeting at a vertex of order 3, then

$$3f(V) = f(A) + f(B) + f(C),$$

and the restriction of f to $VA \cup VB \cup VC$ preserves the orientation of the tree.

A tree is endowed with an arclength measure. Given an immersion f of T, we consider the push-forward m_f of this measure. That is, if $f(T)$ is contained in the small edges of a lattice determined by the vectors u, v, w, then m_f assigns to each edge a density equal to the number of its preimages in T. The resulting measure satisfies the balance condition at all vertices. Since T has a finite number of ends, we can arrange f so that m_f belongs to $\mathcal{M}_r$ for sufficiently large r.

A measure m will be called a *tree measure* if $m = m_f$ for some immersion f of a tree. In the next section, we will construct a set of rules for which an immersion produces a rigid tree measure.

4 Results

There is a certain class of loops which indicate non-rigidity if they exist in the support of a measure. Let $A_1A_2 \ldots A_kA_1$ be a loop consisting of small edges A_jA_{j+1} contained in the support of a measure $m \in \mathcal{M}_r$. We will say that this loop is *evil* if each three consecutive points $A_{j-1}A_jA_{j+1} = ABC$ forms an *evil turn*, i.e. one of the following situations occurs:

(E.1) $C = A$, and the small edges BX, BY, BZ which are $120°, 180°$, and $240°$ clockwise from AB are in the support of m.

(E.2) BC is $120°$ clockwise from AB.

(E.3) $C \neq A$ and A, B, C are collinear.

(E.4) BC is $120°$ counterclockwise from AB and the edge BX which is $120°$ clockwise from AB is in the support of m.

(E.5) BC is $60°$ counterclockwise from AB and the edges BX, BY which are $120°$ and $180°$ clockwise from AB are in the support of m.

The existence of an evil loop in the support of a measure implies non-rigidity. This was proven in [1]:

Theorem 4.1. *A measure $m \in \mathcal{M}_r$ is rigid if and only if its support contains none of the following configurations:*

(1) Six edges meeting at one lattice point.

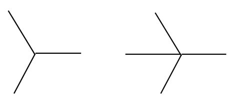
(2) An evil loop.

Our main result is to prove the following proposition:

Proposition 4.2. *Let T be a tree that satisfies properties (T.1) through (T.4), and f an immersion $f : T \to \mathbb{R}^2$. Suppose f satisfies the following conditions:*

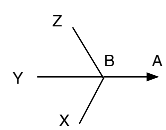
(1) There is a vertex $A \in T$ (called the root of T) such that $f^{-1}(f(A)) = \{A\}$.

(2) The only branch points of $f(T)$ are of the following forms (up to rotation):



(3) f has consistent orientation. That is, suppose X_1X_2 and Y_1Y_2 are edges of T such that X_1 and Y_1 are closer to the root of T than X_2 and Y_2, respectively, and $f(X_1X_2) = f(Y_1Y_2)$. Then $f(X_1) = f(Y_1)$ (and consequently, $f(X_2) = f(Y_2)$).

(4) If four edges meet at a lattice point B, then the orientation of one of the edges is determined as follows. Let AB be the small edge such that the other small edges BX, BY, BZ are located $120°, 180°, 240°$ clockwise from AB. Then the orientation of AB must point from B to A.



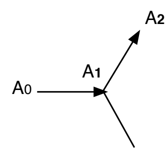
Then m_f is a rigid tree measure.

Proof. Suppose $f : T \to \mathbb{R}^2$ satisfies the conditions of the hypothesis. By Rule (2), a branch point in $f(T)$ cannot have six surrounding edges all belonging to $f(T)$. By Theorem 4.1, it is enough to show that $f(T)$ contains no evil loops. We show that if an evil loop exists in $f(T)$, then the loop lifts to a single branch of T. This will lead to a contradiction of property T.3.

By the conditions imposed on the immersion f, the only evil turns that can arise in $f(T)$ are E.1, E.2, E.3 and E.4. The evil turn E.5 cannot arise because of Rule (2). The turns E.2 and E.4 are reversible under our conditions, in the sense that they are evil either way we traverse them.

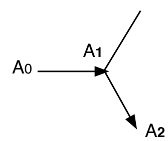
Let $A_0A_1A_2 \cdots A_n$ be an evil loop in $f(T)$ (where $A_0 = A_n$). Since an evil loop must have a turn of the form E.2, E.4, or E.5 (because collinear "turns" alone cannot form a loop), we may choose A_0 so that the evil turn $A_0A_1A_2$ is of the form E.2 or E.4. Since the turns E.2 and E.4 are reversible, we may assume that A_0A_1 assumes the orientation of the tree (if not, we traverse the evil loop in the reverse order). That is, we can assume that there is a lift B_0B_1 to the tree of the edge A_0A_1. We show that A_1A_2 can also be lifted by showing that the orientation of A_1A_2 matches the tree orientation. The possible configurations of $A_0A_1A_2$ are:

(E.2) A_1A_2 is 120° clockwise from A_0A_1.
The edge A_1X which is 240° clockwise from A_0A_1 must also be in the support of $f(T)$.



The edge A_1A_2 must assume the orientation of the tree by consistency of orientation of the immersion f.

(E.4) A_1A_2 is 120° counterclockwise from A_0A_1 and the edge A_1X which is 120° clockwise from A_0A_1 is in the support of $f(T)$.



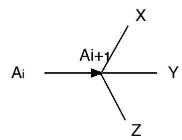
The edge A_1A_2 must assume the orientation of the tree by consistency of orientation of the immersion f.

Thus, given a lift B_0B_1 of A_0A_1, there is a vertex B_2 such that B_1B_2 lifts A_1A_2. So the evil turn $A_0A_1A_2$ lifts to a simple path $B_0B_1B_2$ on T, consistent with the orientation of T.

We inductively show that each edge in the evil loop $A_0 \dots A_n$ lifts to an edge of the tree. For simplicity, we let $A_{k+mn} = A_k$ for all $m \in \mathbb{N}$, $0 \leq k < n$.

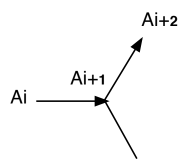
Suppose there is a lift B_iB_{i+1} of the edge A_iA_{i+1} so that A_iA_{i+1} assumes the orientation of the tree. We show that there is also a lift $B_{i+1}B_{i+2}$ of the edge $A_{i+1}A_{i+2}$. The possible configurations of the evil turn $A_iA_{i+1}A_{i+2}$ are:

(E.1) $A_{i+2} = A_i$, and the small edges $A_{i+1}X, A_{i+1}Y, A_{i+1}Z$ which are $120°, 180°$, and $240°$ clockwise from A_iA_{i+1} are in the support of $f(T)$.



The orientation of the edge A_iA_{i+1} in this evil turn contradicts Rule (4) of the immersion f, so this evil turn does not arise in $f(T)$.

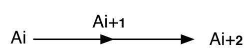
(E.2) $A_{i+1}A_{i_2}$ is $120°$ clockwise from A_iA_{i+1}.
The edge $A_{i+1}X$ which is $240°$ clockwise from A_iA_{i+1} must also be in the support of $f(T)$.



The edge $A_{i+1}A_{i+2}$ must assume the orientation of the tree by consistency of orientation of the immersion f.

(E.3) $A_{i+2} \neq A_i$ and A_i, A_{i+1}, A_{i+2} are collinear. There are two cases:

(a) The lattice point A_{i+1} has only two surrounding edges, A_iA_{i+1} and $A_{i+1}A_{i+2}$, in the support of $f(T)$.



Then the edge $A_{i+1}A_{i+2}$ must assume the orientation of the tree by consistency of orientation of the immersion f.

(b) The lattice point A_{i+1} has four surrounding edges in the support of $f(T)$.
By Rule (4), the edge $A_{i+1}A_{i+2}$ must assume the orientation of the tree.

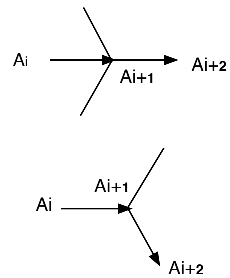
(E.4) $A_{i+1}A_{i+2}$ is $120°$ counterclockwise from A_iA_{i+1} and the edge $A_{i+1}X$ which is $120°$ clockwise from A_iA_{i+1} is in the support of $f(T)$.

The edge $A_{i+1}A_{i+2}$ must assume the orientation of the tree by consistency of orientation of the immersion f.

Thus, given a lift B_iB_{i+1} of A_iA_{i+1}, there is a vertex B_{i+2} such that $B_{i+1}B_{i+2}$ lifts $A_{i+1}A_{i+2}$, completing the induction.

We have constructed inductively an infinite path

$$B_0B_1B_2 \cdots B_nB_{n+1} \cdots$$



of the tree T with the following properties:

(1) $f(B_j) = A_j$

(2) $B_j B_{j+1}$ is oriented away from the root.

We claim that infinitely many of the vertices B_j have order 3.

By construction, the vertex A_1 is a branch point of m_f. Thus lifted vertex $B_1 \in T$ with $f(B_1) = A_1$ is a vertex of order 3, because the immersion f cannot map a vertex of order 2 to a turn of the form E.2 or E.4.

Moreover, each vertex B_{1+mn}, $m \in \mathbb{N}$, has order 3 because

$$f(B_{1+mn}) = A_{1+mn} = A_1.$$

Thus, infinitely many vertices B_j have order 3. But this contradicts property T.3 of the tree. Thus, we've shown that en evil loop cannot exist in $f(T)$, and therefore m_f is a rigid tree measure. □

The above proposition guarantees that if we construct a tree measure following the four stated rules, then the resulting measure will be rigid.

5 Future Research

The finiteness of the number of vertices of order 3 (i.e. property T.3) was crucial in the proof of Proposition 4.2. Our next question is whether Proposition 4.2 still applies to trees with an infinite number of vertices of order 3. That is, if a tree satisfies properties T.1, T.2, and T.4, and if an immersion $f : T \to \mathbb{R}^2$ satisfies the four rules of Proposition 4.2, then will the resulting measure $f(T)$ be rigid?

Another question we investigated is the following: Given a triangle of size r, what is the maximum weight that a rigid tree measure $m \in \mathcal{M}_r$ could have? We have a lower bound on this number: for large r, the maximum weight of a measure in $\mathcal{M}_r$ is at least $2^{\lfloor r/3 \rfloor}$. The lower bound can be easily explained with the help of a couple of figures.

Given a measure $m \in \mathcal{M}_r$ of weight w, Figures 2 and 3 demonstrate how to produce a measure $m' \in \mathcal{M}_{r+3}$ of weight $2w$. By branching the exit densities on one side of Δ_r in the illustrated way, one exit branch of density w is added

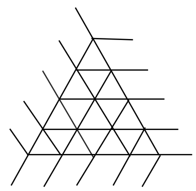


Figure 2: A measure $m \in \mathcal{M}_4$ of weight w

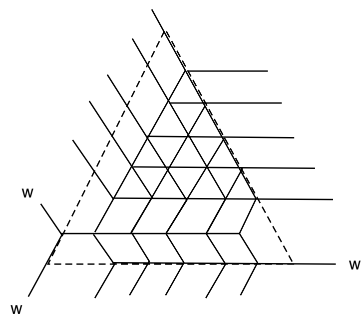


Figure 3: A measure $m' \in \mathcal{M}_7$ of weight $2w$

to each side of the triangle, thereby increasing the total weight by a factor of 2. In this process, we also increase the size of the triangle by 3. Thus, for large r, the maximum weight of $m \in \mathcal{M}_r$ is $\geq 2^{\lfloor r/3 \rfloor}$. We are interested in finding a least upper bound on the maximum weight of a measure.

A tree measure of weight w has $3w - 2$ branch points. This relation allows us to investigate the maximal weight problem using a different approach. The problem becomes that of finding the maximum number of branch points of a rigid tree measure in $\mathcal{M}_r$.

Thus, we attempted to characterize trees with rigid immersions. That is, we are interested in the types of trees that can be immersed onto the plane to produce rigid measures. Does every tree have a rigid immersion? Are there some tree configurations that absolutely prohibit a rigid immersion? By characterizing trees with rigid immersions, we might be able to find an upper bound on the number of branch points that a rigid tree measure $m \in \mathcal{M}_r$ can have, and ultimately find the maximum weight that such a measure could have.

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References

[1] H. Bercovici, B. Collins, K. Dykema, W. S. Li and D. Timotin, *Intersections of Schubert varieties and eigenvalue inequalities in an arbitrary finite factor*, Journal of Functional Analysis 258 (2010) 1579-1627.

[2] H. Bercovici, W. S. Li, and D. Timotin, *A family of reductions for Schubert intersection problems.* ArXiv 0909.0908 (2009): http://adsabs.harvard.edu/abs/2009arXiv0909.0908B.