View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by IUScholarWorks

# Systems of Natural Logic with Adjectives and Their Completeness

*Elizabeth Kammer*

**Abstract**

In natural logic, the goal is to create a system of logic that is as similar to natural language as possible. In order to build a natural logic, simple sentence forms are considered, slowly incorporating more language throughout time. As background, we first consider the fragments of the form *All X are Y* and then add *No X are Y*, where $X$ and $Y$ are nouns. We will look at the rules of logic and completeness of their proof systems. Next, we will introduce the idea of intersecting adjectives which are adjectives with meaning separate from the noun being modifying. For example, the plural noun red cars means the intersection of everything red and everything that is a car. This paper will then present versions of the simple systems that contain intersecting adjectives and will discuss their completeness.

## 1 Introduction

Natural logic is concerned with fragments of natural language and the logic derived from systems using only these sentences. In particular, we are interested in the completeness of these systems of logic. We will begin with a language consisting of one sentence and slowly adding more phrases and looking at their completeness theorems as background. Then we will move on to systems including intersecting adjectives and discuss the completeness of their system.

Historically, logic was largely a study of syllogisms. The classic example of a syllogism is as follows:

$$\frac{\begin{array}{l}\textit{All men are mortal.}\\ \textit{Socrates is a man.}\end{array}}{\textit{Socrates is mortal.}}$$

In this syllogism, whatever makes the two statements above the line true will also make the statement below the line true. It is also important to notice that if we have any three sentences of these particular forms, models which make the first two true will also make the third sentence true. We can show this as:

$$\frac{\begin{array}{l}\textit{All } m \textit{ are } n.\\ S \textit{ is a } m.\end{array}}{S \textit{ is } n.}$$

We explore this type of idea within our systems of natural logic. We look at simple sentence forms with variables as placeholders for nouns. We create a system of rules based on what can be derived using only these sentences, and study the completeness of these rules.

First we look at the simple systems as done by Larry Moss[1]. These systems initially contain only the fragments of the form *All* $X$ *are* $Y$ and then add those like *No* $X$ *are* $Y$, where $X$ and $Y$ are nouns. We consider at the rules of logic and completeness of their proof systems. The following systems of logic involve intersecting adjectives. This type of adjective includes those which have meanings separate from the nouns they modify. For example, it Jane is a female student, and Jane is an athlete, we can deduce that Jane is a female athlete. Intersecting adjectives include things such as colors, but does not include adjectives such as tall or short. For intersecting adjectives, we assume that the adjectives and nouns are sets of things described by that particular word, and an intersecting adjectives is the set of things in the intersection of the adjectives and nouns. Intersecting adjectives include things such as colors, but does not include adjectives such as tall or short. If an object is a red car, we take the intersection of things that are red and things that are cars; however, if something is a short child, this is not the intersection of all short things and children because a short five year old is very different from a short twelve year old. Also note that we can add adjectives productively, meaning that a noun could have multiple adjectives modifying it. An example of this would be that many countries have red, white, and blue flags. We will allow finitely many adjectives to modify a noun in our system. After looking at some systems for background, we will look at those including adjectives and their completeness.

# 2 Background

Before looking at a particular system, there are two concepts, syntax and semantics, which we have in all systems. The work presented in the background is not original, but will help develop an understanding of the systems which we are considering. We will discuss ideas within each of these that we will use throughout the rest of the paper. Then we will begin to look at systems done by Larry Moss[1], building up from the simplest to the more complicated systems.

For the first idea, syntax, we have narrow forms that sentences can take. When working with basic nouns we use $X, Y, ...$ to represent nouns and when dealing with adjectives we denote the intersecting adjectives either by $red$ or when used productively as $c_1, c_2, ..., c_k$. When representing nouns, possibly with or without adjectives, attached we use the variables $m, n$. In any of our systems $\Gamma$ denotes a sets of statements in that particular logic that can take the form of the sentences in our syntax. In the syntax for our system, we also deal with sentences of the forms *All* $X(n)$ *are* $Y(m)$, *No* $X(n)$ *are* $Y(m)$, and *Some* $X(n)$ *are* $Y(m)$, using only some of these sentences depending on our particular system. In the syntax, $\Gamma \vdash S$ means that there is a proof tree with leaves from $\Gamma$ and has the root $S$ that only follows the rules of our system.

When looking at the semantics of a system, we first create a model $\mathcal{M}(M, [\![\ ]\!])$, which contains a set $M$, and a subset $[\![X]\!] \subseteq M$ for each variable $X$ and $[\![red]\!] \subseteq M$ for each adjective. In semantics, we get the following:

$$\begin{array}{lll} \mathcal{M} \models \textit{All } X \textit{ are } Y & \textit{iff} & [\![X]\!] \subseteq [\![Y]\!] \\ \mathcal{M} \models \textit{Some } X \textit{ are } Y & \textit{iff} & [\![X]\!] \cap [\![Y]\!] \neq \emptyset \\ \mathcal{M} \models \textit{No } X \textit{ are } Y & \textit{iff} & [\![X]\!] \cap [\![Y]\!] = \emptyset \end{array}$$

We allow $[\![X]\!]$ to be empty, and in this case, $\mathcal{M} \models \textit{All } X \textit{ are } Y$ vacuously. For a $\Gamma$, we use $\mathcal{M} \models \Gamma$ to mean that $\mathcal{M} \models S$ for all $S \in \Gamma$. Additionally, we have the following semantic meaning in our systems with adjectives:

$$[\![c_1 c_2 ... c_k X]\!] = [\![c_1]\!] \cap [\![c_2]\!] \cap ... \cap [\![c_k]\!] \cap [\![X]\!]$$

For our final semantic definition, we write $\Gamma \models S$ to mean that every model with makes all sentences in $\Gamma$ true also make $S$ true.

Next we present the idea of the Soundness of our systems.

**Lemma 2.1** (Soundness)**.** *If* $\Gamma \vdash S$*, then* $\Gamma \models S$*.*

For a proof, see work by Larry Moss[1]. That all of our systems are sound tells us that anything we can prove in our systems, will also be true semantically. This guarantees that our systems do not produce nonsensical results. We would also like the converse of this statement to be true.

**Theorem 2.2** (Completeness)**.** *If* $\Gamma \models S$*, then* $\Gamma \vdash S$*.*

Completeness guarantees that any result we can prove semantically, will also be provable in our system. We leave the proofs of completeness to the individual systems.

## 2.1 All

We begin with the simplest system, $\mathcal{L}(all)$, which utilizes only sentences of the form *All* $X$ *are* $Y$. The rules of logic for this system are shown in Figure 1. And present a proof of completeness. First, we present an example to display the difference between semantics and syntactical proofs.

*Example* 2.3. Let $\Gamma =$ All $A$ are $B$, All $B$ are $C$, All $C$ are $D$.

Claim 1: $\Gamma \models$ All $A$ are $D$.

From $\Gamma$, we have the following

$$[\![A]\!] \subseteq [\![B]\!], [\![B]\!] \subseteq [\![C]\!], [\![C]\!] \subseteq [\![D]\!].$$

So, we have

$$[\![A]\!] \subseteq [\![D]\!].$$

Claim 2: $\Gamma \vdash$ All $A$ are $D$.

$$\dfrac{\dfrac{\text{All } A \text{ are } B \quad \text{All } B \text{ are } C}{\text{All } A \text{ are } C} \quad \text{All } C \text{ are } D}{\text{All } A \text{ are } D}$$

$$\frac{}{\textit{All } X \textit{ are } Y} \qquad \frac{\textit{All } Y \textit{ are } Z \quad \textit{All } X \textit{ are } Z}{\textit{All } X \textit{ are } Y}$$

Figure 1: The Rules of Logic for $\mathcal{L}(all)$.

And now we turn to the completeness of $\mathcal{L}(all)$.

**Theorem 2.4.** *The logic of $\mathcal{L}(all)$ is complete.*

*Proof.* Suppose $\Gamma \models S$ and let $S$ be All $X$ are $Y$. We begin by making a model $\mathcal{M}(\Gamma)$. Let

$$M = \text{all variables in } \Gamma$$

and we set the semantics of any variable be

$$[\![V]\!] = \{W : \Gamma \vdash \text{All } W \text{ are } V\}$$

Claim: $\mathcal{M} \models \Gamma$ Let All $A$ are $B \in \Gamma$. We must show $[\![A]\!] \subseteq [\![C]\!]$. Let $P \in [\![A]\!]$. Then we have that $\Gamma \vdash$All $P$ are $A$. And we can get the following proof tree

$$\frac{\begin{matrix}\vdots \\ \text{All } P \text{ are } A\end{matrix} \quad \text{All } A \text{ are } B}{\text{All } P \text{ are } B}$$

Since $\Gamma \vdash$ All $P$ are $B$, $B \subseteq [\![*B*]\!]$. From this we can conclude that $[\![A]\!] \subseteq [\![C]\!]$.

Since $\mathcal{M} \models \Gamma$ and $\Gamma \models$ All $X$ are $Y$, $\mathcal{M} \models$ All $X$ are $Y$. Therefore, we have $[\![X]\!] \subseteq [\![Y]\!]$. Since we have

$$\frac{}{\text{All } X \text{ are } Y}$$

we know that $X \in [\![X]\!]$ and $X \in [\![Y]\!]$. Therefore, $\Gamma \vdash$ All $X$ are $Y$. □

Thus we have the completeness of $\mathcal{L}(all)$ of our simplest proof system. From here, we move on to systems that add complexity to this system.

## 2.2 All and No

We expand our language to also contain sentences of the form *No X are Y*. Note that *No X are X* means that there are no $X$. And, in addition to the rules of $\mathcal{L}(all)$, the system $\mathcal{L}(all, no)$ also contains the rules listed in Figure 2. Again, we are interested in the completeness of the system.

**Theorem 2.5.** *The logic of $\mathcal{L}(all, no)$ is complete.*

*Proof.* Let $\Gamma$ be a set of sentences in $\mathcal{L}(all, no)$ Suppose $\Gamma \models S$. We consider the model M($\Gamma$) where $M$ = set of sets $a$ such that the following are true

if $V \in a$ and $\Gamma \vdash$ All $V$ are $W$, then $W \in a$
if $V, W \in a$, then $\Gamma \nvdash$ No $V$ are $W$

we set the semantics of any variable to be

$$[\![Z]\!] = \{a \in \mathcal{M} : Z \in a\}$$

We claim that $\mathcal{M} \models \Gamma$. By our first condition, we have that if *All X are Y* belongs to $\Gamma$, then $[\![X]\!] \subseteq [\![Y]\!]$. If *No X are Y* belongs to $\Gamma$, then let $a \in [\![X]\!]$, so $X \in a$. By the second condition, $Y \notin a$ and therefore, $a \notin [\![Y]\!]$. Which shows that $[\![X]\!] \cap [\![Y]\!] = \emptyset$. So we have that $\mathcal{M} \models \Gamma$.

Since we have that $\mathcal{M} \models \Gamma$ and $\Gamma \models S$, we have $\mathcal{M} \models S$. First we consider the case where $S$ is *All X are Y*. Let

$$a = \{V : \Gamma \vdash \text{All } X \text{ are } V\}$$

Case I: $a \notin \mathcal{M}$ Then there must be some $A, B \in a$ such that $\Gamma \vdash$ *No A are B*. Then we get the following proof tree

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \text{All } X \text{ are } B\end{array} \quad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \text{All } X \text{ are } A\end{array} \quad \begin{array}{c}\vdots\\ \text{No } A \text{ are } B\end{array}}{\text{No } X \text{ are } B}}{\text{No } B \text{ are } X}}{\text{No } X \text{ are } X}}{\text{All } X \text{ are } Y}$$

Case II: $a \in \mathcal{M}$. Then since $a \in [\![X]\!]$, we have $a \in [\![Y]\!]$. Therefore, $Y \in a$, and $\Gamma \vdash \text{All } X \text{ are } Y$.

Next we consider the case when $S$ is *No X are Y*. Here, we let

$$a = \{V : \Gamma \vdash \text{All } X \text{ are } V or \Gamma \vdash \text{All } Y \text{ are } V\}$$

Notice that $X, Y \in a$. We claim that $a \notin \mathcal{M}$. To see this notice that if $a \in \mathcal{M}$, we have that $a \in [\![X]\!] \cap [\![Y]\!]$ . Which implies that $[\![X]\!] \cap [\![Y]\!] \neq \emptyset$ which contradicts $\mathcal{M} \models$ *No X are Y*. So we do in fact have that $a \notin \mathcal{M}$, implying that there are some $V, W \in a$ such that $\Gamma \vdash$ *No V are W*. There are four possible cases, depending on if $\Gamma \vdash$ *All X are V* or $\Gamma \vdash$ *All Y are V* and $\Gamma \vdash$ *All X are W* or $\Gamma \vdash$ *All Y are W*. We first explore the case where $\Gamma \vdash$ *All Y are V* and $\Gamma \vdash$ *All X are W*.

$$\dfrac{\begin{array}{c}\vdots\\ \text{All } X \text{ are } W\end{array} \quad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \text{All } Y \text{ are } V\end{array} \quad \begin{array}{c}\vdots\\ \text{No } V \text{ are } W\end{array}}{\text{No } Y \text{ are } W}}{\text{No } W \text{ are } Y}}{\text{No } X \text{ are } Y}$$

The proof tree for $\Gamma \vdash$ *All X are V* and $\Gamma \vdash$ *All Y are W* is similar. Next we consider the case where $\Gamma \vdash$ *All X are V* and $\Gamma \vdash$ *All X are W*.

$$\frac{\textit{All } X \textit{ are } Y \quad \textit{No } Y \textit{ are } Z}{\textit{No } X \textit{ are } Z} \qquad \frac{\textit{No } X \textit{ are } Y}{\textit{No } Y \textit{ are } X} \qquad \frac{\textit{No } X \textit{ are } X}{\textit{All } X \textit{ are } Y}$$

Figure 2: The rules logic of $\mathcal{L}(all, no)$ when combined with the rules of logic for $\mathcal{L}(all)$.

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ \text{All } X \text{ are } W\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\vdots\\ \text{All } X \text{ are } V\end{matrix} \quad \begin{matrix}\vdots\\ \text{No } V \text{ are } W\end{matrix}}{\text{No } X \text{ are } W}}{\text{No } W \text{ are } X}}{\text{No } X \text{ are } X}}{\text{All } X \text{ are } Y} \qquad \begin{matrix}\vdots\\ \text{No } X \text{ are } X\end{matrix}}{\text{No } X \text{ are } Y}$$

The case where $\Gamma \vdash \textit{All } Y \textit{ are } V$ and $\Gamma \vdash \textit{All } Y \textit{ are } W$ follows similarly. Therefore, we have that $\Gamma \vdash S$. $\square$

We also have know that $\mathcal{L}(all, no)$ is complete.

## 2.3 All and Intersecting Adjectives

In this section, we look at the simplest logic including adjectives. There are many similarities between this case and $\mathcal{L}(all)$; however, we add a few rules concerning adjectives that are not derivable from the our original logic. Additionally, the proof of completeness of $\mathcal{L}(all, adjectives)$ is similar to that of our first completeness proof.

**Theorem 2.6.** *The logic of $\mathcal{L}(all, adjectives)$ is complete.*

*Proof.* Suppose $\Gamma \models S$ and let $S$ be All $n$ are $m$. We begin by making a model $\mathcal{M}(\Gamma)$. Let

$$M = \text{all nouns in } \Gamma$$

and we set the semantics of any variable be

$$[\![V]\!] = \{n : \Gamma \vdash \text{All } n \text{ are } V\}$$

$$[\![red]\!] = \{m : \Gamma \vdash \text{All } n \text{ are } red\ p \text{ for some } p \in M\}$$

Claim: $\mathcal{M} \models \Gamma$ Let All $n$ are $m \in \Gamma$. There are a number of cases, dependent on $n, m$, but we will only explore a few as others are similar.

Case I: $n = X, m = red\ Y$ We must show $[\![X]\!] \subseteq [\![red]\!] \cap [\![Y]\!]$. Let $p \in [\![X]\!]$. Then we have that $\Gamma \vdash$All $p$ are $X$. And we can get the following proof trees

$$\dfrac{\begin{matrix}\vdots\\ \text{All } p \text{ are } X\end{matrix} \quad \text{All } A \text{ are } red\ Y}{\text{All } p \text{ are } red\ Y}$$

$$\frac{}{\textit{All } n \textit{ are } m} \qquad \frac{\textit{All } m \textit{ are } l \quad \textit{All } n \textit{ are } l}{\textit{All } n \textit{ are } m}$$

$$\frac{}{\textit{All red } n \textit{ are } n} \qquad \frac{\textit{All } n \textit{ are } m \quad \textit{All } n \textit{ are red } m}{\textit{All } n \textit{ are red } l}$$

Figure 3: The rules of logic for $\mathcal{L}(all, adjectives)$.

$$\frac{\begin{matrix}\vdots\\ \text{All } p \text{ are } red\ Y\end{matrix} \quad \frac{}{\text{All } red\ Y \text{ are } Y}}{\text{All } p \text{ are } Y}$$

It follows that $p \in [\![red]\!]$ and $p \in [\![Y]\!]$. So we have that $[\![X]\!] \subseteq [\![red]\!] \cap [\![Y]\!]$.

Case II: $n = red\ blue\ X, m = green\ Y$ We must show $[\![red]\!] \cap [\![blue]\!] \cap [\![X]\!] \subseteq [\![green]\!] \cap [\![Y]\!]$. Let $p \in [\![red]\!] \cap [\![blue]\!] \cap [\![X]\!]$, then $p \in [\![red]\!], p \in [\![blue]\!]$, and $p \in [\![X]\!]$, so $\Gamma \vdash$ All $p$ are $red\ a$, $\Gamma \vdash$ All $p$ are $blue\ b$, and $\Gamma \vdash$ All $p$ are $X$. And get the following trees

$$\frac{\frac{\begin{matrix}\vdots\\ \text{All } p \text{ are } red\ a\end{matrix} \quad \frac{\begin{matrix}\vdots\\ \text{All } p \text{ are } blue\ b\end{matrix} \quad \begin{matrix}\vdots\\ \text{All } p \text{ are } X\end{matrix}}{\text{All } p \text{ are } blue\ X}}{\text{All } p \text{ are } red\ blue\ X} \qquad \text{All } red\ blue\ X \text{ are } green\ Y}{\text{All } p \text{ are } green\ Y}$$

$$\frac{\begin{matrix}\vdots\\ \text{All } p \text{ are } green\ Y\end{matrix} \quad \frac{}{\text{All } green\ Y \text{ are } Y}}{\text{All } p \text{ are } Y}$$

Other cases are done similarly. It follows that $\mathcal{M} \models \Gamma$.

Since $\mathcal{M} \models \Gamma$ and $\Gamma \models$ All $n$ are $m$, $\mathcal{M} \models$ All $n$ are $m$. Here again, we have many cases, but will look at one to understand the general method.

$n = red\ X, m = blue\ green\ Y$ Therefore we have $[\![red]\!] \cap [\![X]\!] \subseteq [\![blue]\!] \cap [\![green]\!] \cap [\![Y]\!]$. We know that $red\ X \subset [\![red]\!] \cap [\![X]\!]$ because we have the following trees

$$\frac{}{\text{All } red\ X \text{ are } red\ X}$$

$$\frac{}{\text{All } red\ X \text{ are } X}$$

we know that $X \in [\![red]\!] \cap [\![X]\!]$ and $X \in [\![blue]\!] \cap [\![green]\!] \cap [\![Y]\!]$. Therefore, $\Gamma \vdash$ All $red\ X$ are $blue\ green\ Y$. □

The proof of the completeness of $\mathcal{L}(all, adjectives)$ concludes our introduction to systems of logic their completeness.

$$\frac{All\ n\ are\ m \quad No\ m\ are\ l}{No\ n\ are\ l} \qquad \frac{No\ n\ are\ m}{No\ m\ are\ n}$$

$$\frac{No\ n\ are\ n}{All\ n\ are\ m} \qquad \frac{No\ n\ are\ red\ m}{No\ m\ are\ red\ n}$$

Figure 4: The rules logic of $\mathcal{L}(all, adjectives, no)$ when combined with the rules of logic for $\mathcal{L}(all, adjectives)$.

# 3 All, No, and Adjectives

Next we look at the combination of sentences of the forms *All n are m* and *No n are m*, where $n$ and $m$ are nouns with or without adjectives. The rules for this system can be found in Figure 4, in addition to those rules for $\mathcal{L}(all, adjectives)$. We claim that $\mathcal{L}(all, no, adjectives)$ is also complete.

**Theorem 3.1.** *The logic of $\mathcal{L}(all, no, adjectives)$ is complete.*

*Proof.* Let $\Gamma$ be a set of sentences in $\mathcal{L}(all, no)$ Suppose $\Gamma \models S$. We consider the model M($\Gamma$) where $M$ = set of sets $a$ such that the following are true

if $n \in a$ and $\Gamma \vdash$ All $n$ are $m$, then $m \in a$
if $n, m \in a$, then $\Gamma \nvdash$ No $V$ are $W$
if $n, red\ m \in a$, then $red\ n \in a$

and we set the semantics of any variable and adjective to be

$$[\![Z]\!] = \{a \in \mathcal{M} : Z \in a\}$$

$$[\![red]\!] = \{b \in \mathcal{M} : red\ p \in a, p \text{ some noun} \in \Gamma\}$$

We claim that $\mathcal{M} \models \Gamma$.

Case I: All $n$ are $m \in \Gamma$ It should be noted that in the case where *All X are Y* belongs to $\Gamma$, then by the same argument used from $\mathcal{L}(all, no)$ that $[\![X]\!] \subseteq [\![Y]\!]$. Now we look at the general case with finitely many colors. If *All* $c_1...c_jX$ *are* $d_1...d_kY \in \Gamma$, we must show that

$$[\![c_1]\!] \cap [\![c_2]\!] \cap ... \cap [\![c_j]\!] \cap [\![X]\!] \subseteq [\![d_1]\!] \cap [\![d_2]\!] \cap ... \cap [\![d_k]\!] \cap [\![Y]\!]$$

Let $a \in [\![c_1]\!] \cap [\![c_2]\!] \cap ... \cap [\![c_j]\!] \cap [\![X]\!]$. Then $c_1p_1, ..., c_jp_j, X \in a$. By our third condition since $c_jp_j \in a$ and $X \in a$, it follows that $c_jX \in a$. Repeating this arguement $j - 1$ more times, we hae that $c_1...c_jX \in a$. Since $\Gamma \vdash$ *All* $c_1...c_jX$ *are* $d_1...d_kY$, we know $d_1...d_kY \in a$. And therefore $a \in [\![d_1]\!]$. And we can get the following proof tree

$$\frac{}{\text{All } d_1...d_kY \text{ are } d_2...d_kY}$$

Which tells us that $d_2...d_kY \in a$. Repeating this argument, we find that $a \in [\![d_1]\!] \cap [\![d_2]\!] \cap ... \cap [\![d_k]\!] \cap [\![Y]\!]$. And therefore we have

$$[\![c_1]\!] \cap [\![c_2]\!] \cap ... \cap [\![c_j]\!] \cap [\![X]\!] \subseteq [\![d_1]\!] \cap [\![d_2]\!] \cap ... \cap [\![d_k]\!] \cap [\![Y]\!]$$

Case II: *No* $n$ *are* $m$ belongs to $\Gamma$. Notice that in this case, if *No* $X$ *are* $Y$ $\in \Gamma$, the proof from $\mathcal{L}(all, no)$ shows that that $[\![X]\!] \cap [\![Y]\!] = \emptyset$. We look at *No* $c_1...c_jX$ *are* $d_1...d_kY \in \Gamma$, we must show that

$$[\![c_1]\!] \cap [\![c_2]\!] \cap ... \cap [\![c_j]\!] \cap [\![X]\!] \cap [\![d_1]\!] \cap [\![d_2]\!] \cap ... \cap [\![d_k]\!] \cap [\![Y]\!] = \emptyset$$

Suppose not, and let $a \in [\![c_1]\!] \cap [\![c_2]\!] \cap ... \cap [\![c_j]\!] \cap [\![X]\!] \cap [\![d_1]\!] \cap [\![d_2]\!] \cap ... \cap [\![d_k]\!] \cap [\![Y]\!]$, then by our previous arguments $c_1...c_jX \in a$, $d_1...d_kY \in a$ and $a \in \mathcal{M}$. But By our third condition, since *blue* $q$ and $X \in a$, *blue* $X \in a$. Similarly since *red* $p$ and *blue* $X \in a$, we have *red blue* $X \in a$. By the same argument, we can conlude that *green yellow* $Y \in a$. But we have that $\Gamma \vdash$ *No* $c_1...c_jX$ *are* $d_1...d_kY$, which contradicts the second condition for $a \in \mathcal{M}$. So we have that $[\![c_1]\!] \cap [\![c_2]\!] \cap ... \cap [\![c_j]\!] \cap [\![X]\!] \cap [\![d_1]\!] \cap [\![d_2]\!] \cap ... \cap [\![d_k]\!] \cap [\![Y]\!] = \emptyset$ and therefore also that $\mathcal{M} \models \Gamma$.

Since we have that $\mathcal{M} \models \Gamma$ and $\Gamma \models S$, we have $\mathcal{M} \models S$. First we consider the case where $S$ is of the form *All* $n$ *are* $m$. Here we consider the general case where $S$ is *All* $c_1...c_jX$ *are* $d_1...d_kY$. Let

$$a = \{n : \Gamma \vdash \text{All } c_1...c_jX \text{ are } n\}$$

Case I: $a \notin \mathcal{M}$ Then there must be some $m, l \in a$ such that $\Gamma \vdash$ *No* $m$ *are* $l$. Then we get the following proof tree

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \text{All } c_1...c_jX \text{ are } l\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \text{All } c_1...c_jX \text{ are } m\end{array} \quad \begin{array}{c}\vdots\\ \text{No } m \text{ are } l\end{array}}{\text{No } c_1...c_jX \text{ are } l}}{\text{No } l \text{ are } c_1...c_jX}}{\text{No } c_1...c_jX \text{ are } c_1...c_jX}}{\text{All } c_1...c_jX \text{ are } d_1...d_kY}$$

So we have that $\Gamma \vdash$ *All* $c_1...c_jX$ *are* $d_1...d_kY$.

Case II: $a \in \mathcal{M}$. Then since $a \in [\![c_1]\!] \cap ... \cap [\![c_j]\!] \cap [\![X]\!]$, we have $a \in [\![d_1]\!] \cap ... \cap [\![d_k]\!] \cap [\![Y]\!]$. Therefore, $d_1p_1, ..., d_kd_k, Y \in a$. Using previous arguments we have that $d_1...d_kY \in a$, so $\Gamma \vdash$ *All* $c_1...c_jX$ *are* $d_1...d_kY$.

Next we consider the case when $S$ is of the form *No* $n$ *are* $m$. Specifically we will consider *No* $c_1...c_jX$ *are* $d_1...d_kY$. Here, we let

$$a = \{n : \Gamma \vdash \text{All } c_1...c_jX \text{ are } nor \Gamma \vdash \text{All } d_1...d_kY \text{ are } n\}$$

Notice that $c_1...c_jX, d_1...d_kY \in a$. We claim that $a \notin \mathcal{M}$. To see this notice that if $a \in \mathcal{M}$, we have that $a \in [\![c_1]\!] \cap ... \cap [\![c_j]\!] \cap [\![X]\!] \cap [\![d_1]\!] \cap ... \cap [\![d_k]\!] \cap [\![Y]\!]$ . Which implies that $[\![c_1]\!] \cap ... \cap [\![c_j]\!] \cap [\![X]\!] \cap [\![d_1]\!] \cap ... \cap [\![d_k]\!] \cap [\![Y]\!] \neq \emptyset$. So we do in fact have that $a \notin \mathcal{M}$, implying that there are some $n, m \in a$ such that $\Gamma \vdash$ *No* $n$ *are* $m$. There are four possible cases, depending on if $\Gamma \vdash$ *All* $c_1...c_jX$ *are* $n$ or $\Gamma \vdash$ *All* $d_1...d_kY$ *are* $n$ and $\Gamma \vdash$ *All* $c_1...c_jX$ *are* $m$ or $\Gamma \vdash$ *All* $d_1...d_kY$ *are*

$$\frac{\textit{All } n \textit{ are } m \quad \textit{Some } n \textit{ are } l}{\textit{Some } m \textit{ are } l} \qquad \frac{\textit{Some } n \textit{ are } m}{\textit{Some } m \textit{ are } n}$$

$$\frac{\textit{Some } n \textit{ are } m}{\textit{Some } n \textit{ are } n} \qquad \frac{\textit{Some } n \textit{ are red } m}{\textit{Some } m \textit{ are red } n}$$

Figure 5: The rules logic of $\mathcal{L}(all, some, adjectives)$ when combined with the rules of logic for $\mathcal{L}(all, adjectives)$.

$m$. We first explore the case where $\Gamma \vdash$ *All* $d_1...d_kY$ *are* $n$ and $\Gamma \vdash$ *All* $c_1...c_jX$ *are* $m$.

$$\frac{\begin{matrix}\vdots\\ \text{All } c_1...c_jX \text{ are } m\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\vdots\\ \text{All } d_1...d_kY \text{ are } n\end{matrix} \quad \begin{matrix}\vdots\\ \text{No } n \text{ are } m\end{matrix}}{\text{No } d_1...d_kY \text{ are } m}}{\text{No } m \text{ are } d_1...d_kY}}{\text{No } c_1...c_jX \text{ are } d_1...d_kY}$$

We also have the case where $\Gamma \vdash$ *All blue green Y are m* and $\Gamma \vdash$ *All red X are n* which has a similar proof tree. Next we explore the case where $\Gamma \vdash$ *All* $c_1...c_jX$ *are* $n$ and$\Gamma \vdash$ *All* $c_1...c_jX$ *are* $m$.

$$\frac{\dfrac{\dfrac{\begin{matrix}\vdots\\ \text{All } c_1...c_jX \text{ are } m\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\vdots\\ \text{All } c_1...c_jX \text{ are } n\end{matrix} \quad \begin{matrix}\vdots\\ \text{No } n \text{ are } m\end{matrix}}{\text{No } c_1...c_jX \text{ are } m}}{\text{No } m \text{ are } c_1...c_jX}}{\text{No } c_1...c_jX \text{ are } c_1...c_jX}}{\text{All } c_1...c_jX \text{ are } d_1...d_kY} \qquad \begin{matrix}\vdots\\ \text{No } c_1...c_jX \text{ are } c_1...c_jX\end{matrix}}{\text{No } c_1...c_jX \text{ are } d_1...d_kY}$$

Similarly, we have the case where *All* $d_1...d_kY$ *are* $n$ and *All* $d_1...d_kY$ *are* $n$. Therefore, we know that $\Gamma \vdash S$. □

We concluded with the completeness of the combinations of systems we have thus far discussed.

# 4 All, Some, Adjectives

As discussed by Moss[1], there is also a language which combines sentences of the form *All X are Y* and *Some X are Y*. A direction that we began to explore but can be continued is to add adjectives to $\mathcal{L}(all, some)$. We present here what we believe to be the complete rules of logic for $\mathcal{L}(all, some, adjectives)$.

# 5 Acknowledgments

I'd like to thank Professor Larry Moss for his support and guidance throughout this summer. Addition, many thanks to Professor Kevin Pilgrim for organizing and making this experience possible.

# References

[1] L. S. Moss: Completeness Theorems for Syllogistic Fragments.