

Topological Representation of Canonicity for Varieties of Modal Algebras

Saleh Aliyari

Submitted to the faculty of the University Graduate School

in partial fulfillment of the requirements

for the degree

Doctor of Philosophy

in the Department of Mathematics

Indiana University

April, 2010

Accepted by the Graduate Faculty, Indiana University, in partial fulfillment of the
requirements for the degree of Doctor of Philosophy

Doctoral Committee

Lawrence S. Moss, Ph.D. (Chair)

Darrell Haile, Ph.D.

J. Michael Dunn, Ph.D.

Amr Sabry, Ph.D.

April 19, 2010

© 2010

Saleh Aliyari

ALL RIGHTS RESERVED

iii

To whom it may concern,

Acknowledgements

First I would like to thank people without whose help this dissertation would never have been completed. I thank my parents Zahra Jabbari and M.Sadegh Aliyari, and my siblings for their loving support throughout my life. My advisor Larry Moss's support and encouragement started before I started as a student at Indiana university. Larry has remained ever so helpful to this day. I thank him for all he has done for me, for his generosity with his time, and for giving me freedom in my research. Jennifer Bowen of the office of international services has been most helpful and caring. She is yet another person without whom this dissertation would never see the light of the day. Kevin Zumbun in his capacity as the graduate advisor and later the chairperson in the math department repeatedly went beyond the call of duty to help me. I am truly grateful to him for his support. I cannot thank my most loyal friend Shakila Shayan enough for her invaluable support. Without her help I would not even have started working on this dissertation.

I feel abundant gratitude to the faculty members on my research committee. J. M. Dunn, D. Haile, A. Sabry were generous with their time and most encouraging. My special thanks to J.M. Dunn for many enlightening discussions and great suggestions. I have enjoyed the help of other faculty and students as well. My thanks to I. Viglizzo, J. Kim, D. Mc Cartey, M. Larson, and E.Haghverdi for so many interesting discussions. I am specially thankful to Guram Bezhanishvili with whom I had numerous interesting

discussions about my research. He provided invaluable help, had great suggestions and his vast knowledge of the subject was more than inspiring. I would like to thank Y. Venema and R. Goldblatt for their help through discussions and correspondence.

Many people in the math department at Indiana university helped provide a friendly environment for us graduate students. I am thankful to faculty and staff of the math department for their help. In particular my thanks go to J.Dadok, C. Livingston, P.Sternberg, S.Andersson, L.McKinely, G. Peters, W. Wheeler, C. Wright, J. Taylor and E. Henline. My special thanks to M. Cummings without whose help I would be totally lost in the rules and regulations.

I enjoyed the support and company of many good friends during the hard as well as good times. My thanks to H.Hamidi, Sh.Mokhtari, M. Hashemi, M. Emami, A. Strang, F. Erdinc, C. Larson, M. Thomas, and S. Grimes. I am also thankful to Ekbia family who were in so many ways like my own.

Staff of the international services were most helpful during my years at Indiana University. So were the faculty and staff at the college of arts and science. In particular I would like to thank M. Wailes D. Ruddick, L. Di Pietro, and M. Byler. B. Noel our great librarian has been more than helpful with anything one could hope to find in a library and more. I greatly appreciate his help.

Finally I should thank some great individuals semi anonymously. Many thanks to Judy K., Susan S., George C., and Mark C. for their care and help.

Preface

Much of what is done in this dissertation is based on the connections between modal logic and algebraic logic. Not every logic can be studied from an algebraic point of view. This leads to the notion of an algebraizable logic. Normal modal logics which are the only types of modal logics we consider can be studied using the tools available in universal algebra. In other words normal modal logics are algebraizable, to use Blok and Pigozzi's terminology. In their work Blok and Pigozzi consider many logics and in many cases they show that the logical concepts can be thoroughly captured by appropriate types of algebras. Such logics are called algebraizable. C.f. [BPA].

A particular question we consider in this dissertation is what we call the Countable Canonicity Conjecture, or the CCC, whose roots can be found in the work of K. Fine. Fine's result on the connection between first order logic and modal logic is quite well known. In 1975 Fine published a paper in which he showed that any class of Kripke frames that is first order definable has a canonical modal logic. C.f. [FFM]. He then laid out a question, known as Fine's Canonicity Conjecture, that remained open until it was negatively answered by Goldblatt, Hodkinson, and Venema in [GHV]. It stipulated that any class of Kripke frames defined by a canonical logic is first order definable. We should point out that in this context one only considers modal logics with countably many atomic propositions. R. Goldblatt has made many significant contributions to the study of modal logic and in particular the connection between modal and algebraic logic clearing the way for the resolution of Fine's conjecture. In

1989 Goldblatt published a paper titled: "Varieties of Complex Algebras", [GVC], in which he defined the notion of a canonical variety of boolean algebras with operators. One of the results in [GVC] is that any variety of modal algebras which is closed under canonical extensions is defined by a canonical modal logic. This follows from his observation that the canonical frame of a logic Σ , is isomorphic to the ultrafilter frame of the Lindenbaum-Tarski algebra of Σ . The term Canonical Variety was introduced by Goldblatt to refer to such a variety, c.f. [GVC]. The concept of the canonical extension was first introduced by L. Henkin, J.D. Monk, and A. Tarski in [HMA] under the name of a "Canonical Embedding Algebra". Another result by Goldblatt shows that a class of Kripke frames is closed under ultrafilter extensions if it is closed under ultraproducts. This result was later strengthened by J. van Benthem in [VBUE] where he proves the closure under ultrapowers is indeed enough. Goldblatt also showed that if a class of Kripke frames is closed under ultrafilter extensions the variety of modal algebras generated by that class is a canonical variety. C.f. [GVC]. As a result Goldblatt's work in [GVC] Fine's canonicity conjecture effectively creates two new problems that can be stated as follows.

1. Is any class of Kripke frames whose modal algebras of subsets generate a canonical variety first order definable?
2. Is any variety of modal algebras defined by a canonical logic a canonical variety?

The converses of both questions are proved true by Goldblatt's work. C.f. [GVC].

Therefore Fine’s conjecture would have been true if both questions were answered positively. However, this was not what the research in the area revealed. The first question above was negatively answered in 2003. C.f. [GHV]. The second question remains open to date. So far as we know Goldblatt has not actually stipulated that the answer to the latter question should be positive. This is the only reason we refrain from using the term Goldblatt’s Canonicity conjecture. A combination of some of our results, that we will refer to, lead us to surmise that conjecture must be true. First are corollary 3.3, and proposition 3.3. These two results give an impression, admittedly vaguely, of the preservation of the structure of sv-frames (which are the main structures we study) in the canonical frame of any logic they validate. Our result on coproduct preservation, lemma 4.11 and proposition 5.1 also play a role in persuading us to hope for a positive answer to the CCC. We are certainly not the first to believe that the conjecture is likely to have a positive answer. Yet, we have not been able to find the first instance of a clear assertion of the CCC as a conjecture in the existing literature. Therefore, regretfully, we are not able to give the due credit to whomever who is deserving of it.

Almost all the definitions and basic results that are included in this dissertation are from [BRV] which does include extensive citations of original references. Let us briefly mention some major contributors to the topics discussed in this dissertation. This is not an exhaustive list of course for many bright thinkers have contributed to practically everything that appears in our work.

The work of A. Tarski, B. Jónsson, L. Henkin has been instrumental to the development of algebraic logic. One can even say they had significant contributions to the creation of algebraic logic. P. Halmos and E.J. Lemmon are also major contributors to this field. The work of M. Stone and L. Vietoris is fundamental to our work here as it is apparent in our terminology. Needless to say without S. Kripke's semantics little would be left for us to say. R. Goldblatt and J.M. Dunn have also made many contributions that are mainly responsible for our choice of a dissertation topic. The work of many others who are not named here has been inspirational to us and we feel sincerely grateful for all who did so much excellent work. We have cited some of the references at the end of the dissertation.

Topological Representation of Canonicity for Varieties of Modal Algebras

The main subject of this dissertation is to approach the question of countable canonicity of varieties of modal algebras from a topological and categorical point of view. The category of coalgebras of the Vietoris functor on the category of Stone spaces provides a class of frames we call sv-frames. We show that the semantic of this frames is equivalent to that of modal algebras so long as we are limited to certain valuations called sv-valuations. We show that the canonical frame of any normal modal logic which is directly constructed based on the logic is an sv-frame. We then define the notion of canonicity of a logic in terms of varieties and their dual classes. We will then prove that any morphism on the category of coalgebras of the Vietoris functor whose codomain is the canonical frame of the minimal normal modal logic are exactly the ones that are invoked by sv-valuations. We will then proceed to reformulate canonicity of a variety of modal algebras determined by a logic in terms of properties of the class of sv-frames that correspond to that logic. We define ultrafilter extension as an operator on the category of sv-frames, prove a coproduct preservation result followed by some equivalent forms of canonicity. Using Stone duality the notion of co-variety of sv-frames is defined. The notion of validity of a logic on a frame is presented in terms of ranges of theory maps whose domain is the given frame. Partial equivalent results on co-varieties of sv-frames are proved. We classify theory maps which are

maps invoked by a valuation on a Kripke frame using the classification of sv-theory maps and properties of ultrafilter extension. A negative categorical result concerning the existence of an adjoint functor for ultrafilter extension is also proved.

Contents

1	Introduction	1
2	Basic Concepts	15
2.1	Modal Algebras	15
2.2	Kripke Frames	18
2.3	Connecting Kripke Frames and Modal Algebras	19
2.4	Syntax and Semantics of Normal Modal Logics	26
2.4.1	Modal Semantics for Modal Algebras	29
2.4.2	Modal Semantics for Kripke Frames	33
2.4.3	Models Over Kripke frames	34
2.5	Modal Semantics in Categories of Algebras and Frames	37
2.6	Stone Spaces and sv -Frames	39
2.7	Soundness	55
2.8	Semantics Equivalence Lemma	57
2.9	Varieties and Co-Varieties	67
2.9.1	Varieties of Modal Algebras	67
2.9.2	Co-Varieties of sv -Frames	73
2.10	Descriptive General Frames	78
2.11	Canonicity	81
2.12	Canonicity in Dual Form	85
2.13	The Countable Canonicity Conjecture	88

3	Theory Maps and Their Properties	93
3.1	Classification of sv -Theory Maps	93
3.2	Hennessy-Milner Property and sv -Frames	102
4	Properties of SV-Frames and Ultrafilter Extension	108
4.1	Natural Injections and Surjections	108
4.2	sv -Frames as Modal Frames	113
4.2.1	Two Observations	121
4.3	Coproducts in the KF and SV	125
4.4	Characterization of Ultrafilter Extensions	133
4.5	Negative Results	142
5	Canonicity, Definability, and Completeness	146
5.1	Canonicity Results	146
5.2	Modal definability and Completeness	151

1 Introduction

In this introduction we assume some knowledge of modal logic; in particular the notions of truth and validity on Kripke frames. These notions are introduced in detail in chapter 1. We only talk about normal modal logics, i.e. those with one modal operator. We freely use the term logic, or modal logic to refer to a normal modal logic. Throughout the dissertation we will assume familiarity with point set topology. However we will use a very narrow set of results from topology. We will also use some facts from category theory. In neither case have we included the basic material. Except for basic facts from topology and category theory, we have included every definition/fact that is needed for our purposes; albeit in a brief manner. Definitions and theorems that fall in the scope of universal algebra are adapted to the particular case of modal algebras.

The Countable Canonicity Conjecture, or the CCC, is a question about varieties of modal algebras, or more generally, of boolean algebras with operators. The goal of this dissertation is to discuss this conjecture and its connection to the notion of modal definability of classes of Kripke frames. To this end, we will prove the equivalence of the conjecture with other statements mainly formulated for what we will call sv-frames. An sv-frame is a Kripke frame whose underlying set is a special topological space. That is, a Stone space; a compact, Hausdorff, totally disconnected topological space.

The CCC is usually stated as follows:

If Σ is a canonical modal logic in a countable language then V_Σ , the variety of modal algebras defined by Σ , is a canonical variety.

Let us go over the definitions we need to understand the statement of the CCC.

A canonical modal logic Σ is a logic which is valid, i.e true under all valuations, on a special frame \mathfrak{F}^Σ . \mathfrak{F}^Σ is called the canonical frame of Σ . The canonical frame of Σ is closely related to Σ and is constructed by using maximal consistent sets of formulae in Σ . In contrast to canonical modal logics, any modal logic is globally true on its canonical model which is the canonical frame equipped with a particular, canonical, valuation. Intuitively, the canonical model of a logic Σ is the model built following the *rules* of Σ . Therefore the least one would expect from a standard Henkin style completeness proof is for Σ to be globally true on its canonical model. Canonicity of Σ eliminates the dependence of global truth on the canonical valuation. So to emphasize the notion:

Σ is canonical if and only if Σ is valid on its canonical frame.

An algebra is a set equipped with some operators and possibly distinguished elements.

Algebras that have the same number of operators and distinguished elements are said to be of the same type. By a class of algebras we mean a class of algebras of the same type unless otherwise specified. An equationally definable class of algebras is the class of algebras that satisfy a fixed set of equations. We do not get into details of equational logic. Let us just say that the notion of satisfaction of an equation in an algebra is the general form of an equation in a group, or a ring. All variables in such an equation are assumed to be universally quantified. A class of algebras closed under homomorphic images, subalgebras, and arbitrary products is called a variety. Given a class C , HC , SC , and PC , denote the closure of C under homomorphic images, subalgebras, and arbitrary products, respectively.

Formulae of modal logics can be translated into equations over a particular kind of algebra, a modal algebra. One can use this translation in order to consider a modal logic as an equational logic. This enables us to apply the classic Birkhoff Variety Theorem below to modal logics. A modal algebra is a boolean algebra with an additional unary operator. The unary operator of a modal algebra satisfies some properties that we will discuss in detail in chapter 1.

The Birkhoff Variety Theorem shows that a class of algebras is a variety if and only if it is definable by a set of equations. Furthermore, to check whether a class C , of algebras is a variety one only needs to check C for closure under HSP in this particular

order. This implies that for any class C , HSPC is a variety which is called the variety generated by C . The variety generated by the singleton $\{\mathfrak{A}\}$ is called the variety generated by \mathfrak{A} . Given the translation of modal logics into equations we can apply the theorem to modal algebras. Any variety V_Σ of modal algebras is generated by any $\mathfrak{A} \in V_\Sigma$ whose theory (the set of formulae valid on \mathfrak{A}) is Σ . That is, any algebra in V_Σ is a homomorphic image of a subalgebra of a power of \mathfrak{A} (a product where all the components are isomorphic to \mathfrak{A}), where \mathfrak{A} is an algebra that validates Σ and nothing more than Σ . In particular there is an algebra called the Lindenbaum-Tarski algebra of Σ and denoted by L_Σ , that generates V_Σ . That is, $V_\Sigma = \text{HSP}\{L_\Sigma\}$. For a modal logic L_Σ is a quotient algebra which is defined by an equivalence relation on the set of all formulae in the language of Σ . The equivalence relation is the equivalence according to Σ . In other words two formulae are considered equivalent (are in the same equivalence class in the quotient algebra) if and only if Σ proves them to be equivalent (as formulae of modal logic).

For any modal algebra \mathfrak{A} the set of ultrafilters of \mathfrak{A} can be equipped with a relation, based on the modal operator of \mathfrak{A} , that turns it into a Kripke frame. This frame is denoted by \mathfrak{A}_+ and is called the ultrafilter frame of \mathfrak{A} . Let us mention that \mathfrak{A}_+ is not just a Kripke frame; it can be equipped with a Stone topology determined by \mathfrak{A} . Under this topology \mathfrak{A}_+ is what we call an sv-frame. This fact is used later in this introduction.

For each Kripke frame $F = \langle W, R \rangle$, the modal algebra of subsets of F is denoted by F^+ . F^+ is the boolean algebra of all subsets of W equipped with an operator defined by

$$f_R(A) = \{x \in W \mid \exists y \in A \ x R y\}.$$

The canonical extension of a modal algebra \mathfrak{A} is the modal algebra of subsets of the ultrafilter frame of \mathfrak{A} . In other words the canonical extension of \mathfrak{A} is defined by

$$\mathfrak{Cm} \mathfrak{A} = (\mathfrak{A}_+)^+.$$

A canonical variety is one which is closed under the operation of forming canonical extensions. This means that CCC can be stated as follows:

Given a canonical logic Σ and an arbitrary $\mathfrak{A} \in V_\Sigma$, $\mathfrak{Cm} \mathfrak{A}$ belongs to V_Σ .

What follows should make it apparent that CCC relates to both algebraic and frame semantics for modal logics. We use the notations $\mathfrak{A} \models \Sigma$, $F \Vdash \Sigma$, to denote that a set of formulae Σ is valid on an algebra \mathfrak{A} , a Kripke frame F , respectively.

It is a well known fact that for any set of formulae Σ ,

$$F \Vdash \Sigma \quad \text{iff} \quad F^+ \models \Sigma.$$

In general the validity of a formula ϕ (or a logic Σ) on \mathfrak{A} does not imply the validity of ϕ (Σ) on $\mathfrak{Cm} \mathfrak{A}$. In chapter 2 we will see that if a modal formula ϕ is valid on a

modal algebra \mathfrak{A} , then ϕ is valid on \mathfrak{A}_+ , under a different definition of validity which is weaker than Kripke validity. We refer to this weaker notion of validity as sv-validity. The definitions are given in chapter 1. We will however return to this notion here. The notion of sv-validity is an equivalent form of validity on descriptive general frames, which are usually considered for providing a complete semantics for normal modal logics in the standard literature.

DGF, the category of descriptive general frames, is known to be dually equivalent to the category of modal algebras. A DGF, $D = \langle W, R, A \rangle$, is a structure where $\langle W, R \rangle$ is a Kripke frame and A is a special set of subsets of W called admissible sets. The conditions on admissible sets in the definition of a DGF can be described in the context of categories of certain topological spaces. Therefore instead of working with DGF we consider a category SV which is isomorphic to DGF. So we can skip the specifics of the definition of admissible sets. Instead we consider a functor on Stone, the category of Stone spaces, that assigns a Stone space VX to any given Stone space X . The points of VX are closed subsets of X , and VX is given a topology that makes VX a Stone space. VX is the analogous to the power set construction for Stone spaces.

Here is what we need to know about sv-frames at this point:

An sv-frame is a triple $S = \langle W, R, \tau \rangle$, where $\langle W, R \rangle$ is a Kripke frame, and τ is a Stone topology on W such that

1. The R-image of each node is a closed set of τ . The R-image of x is defined by

$$R[x] = \{y \mid Rxy\}.$$

2. If $K[W]$, the set of closed subsets of τ is equipped with the Vietoris topology based on τ

then

$$\gamma_R(x) = R[x],$$

is a continuous function.

Kripke frames can be seen as coalgebras of the power set functor on the category Sets. sv-frames can be seen as coalgebras of the Vietoris functor over the category Stone in much the same way. ClgV is the category of coalgebras of the Vietoris functor on the category of Stone spaces. As shown in [KKV] ClgV is dually equivalent to MA, the category of modal algebras, and isomorphic to DGF. Since sv-frames can be seen as coalgebras of the Vietoris functor on Stone, we can consider SV, the category of sv-frames, also dually equivalent to MA, and isomorphic to DGF. In chapter 1 we will see that morphisms of ClgV (and hence SV) are identifiable with continuous maps between their corresponding frames that are also bounded morphisms of Kripke frames. Our choice to work with sv-frames, as opposed to descriptive general frames, is strictly notational. We could avoid introducing sv-frames and work with DGF's. Yet the choice made here significantly simplifies the terminology used for more detailed discussions.

Since the Stone topology on S is fixed, for all practical purposes we do not need to work with specifics of the topology on S . All we need to know is that the topology on S is compact, Hausdorff, and totally disconnected and satisfies the properties mentioned in the definition of an sv-frame. So when there is no ambiguity (practically throughout the entire discussion) we do not mention the topology on $S = \langle W, R, \tau \rangle$. We refer to S and its underlying Kripke frame $\langle W, R \rangle$ by the same name. The need to make a distinction arises when we consider validity of formulae on sv-frames. Hence we will introduce the notion of sv-validity. To distinguish between validity of a formula on the underlying Kripke frame of S and sv-validity on S we will introduce the notation

$$S \Vdash_{sv} \phi,$$

which means ϕ is globally true on S under all valuations whose values are restricted to clopens of the Stone topology of S . We use the notation $S \Vdash_{sv} \Sigma$ to denote that Σ is sv-valid (true under all sv-valuations) on S . Obviously, Kripke validity implies sv-validity but not vice versa.

The duality of ClgV as discussed in [KKV] is based on The Stone Representation Theorem. It is proved that the ultrafilter frame of each modal algebra is an sv-frame (coalgebra of V) and each sv-frame is the ultrafilter frame of an algebra, the modal algebra of the clopen subsets of its Stone topology.

As mentioned before, the ultrafilter frame of a modal algebra is an sv-frame. In chap-

ter 2, using the duality mentioned above we will show

$$\mathfrak{A} \models \Sigma \quad \text{iff} \quad \mathfrak{A}_+ \Vdash_{\text{sv}} \Sigma .$$

Therefore given a modal algebra \mathfrak{A} in V_Σ , i.e given an algebra \mathfrak{A} such that $\mathfrak{A} \models \Sigma$ we have

$$\mathfrak{A}_+ \Vdash_{\text{sv}} \Sigma .$$

A restatement of CCC is the following.:

If Σ is a canonical modal logic, not only

$$\mathfrak{A}_+ \Vdash_{\text{sv}} \Sigma \quad \text{for all } \mathfrak{A} \in V_\Sigma ,$$

but also

$$\mathfrak{A}_+ \Vdash \Sigma \quad \text{for all } \mathfrak{A} \in V_\Sigma .$$

To see this, recall that $\mathfrak{Cm} \mathfrak{A} = ((\mathfrak{A})_+)^+$, and that for any Kripke frame F

$$F \Vdash \Sigma \quad \text{iff} \quad F^+ \models \Sigma .$$

In particular

$$(\mathfrak{A})_+ \Vdash \Sigma \quad \text{iff} \quad ((\mathfrak{A})_+)^+ \models \Sigma .$$

Therefore

$$(\mathfrak{A})_+ \Vdash \Sigma \quad \text{iff} \quad \mathfrak{Cm} \mathfrak{A} \models \Sigma \quad \text{iff} \quad \mathfrak{Cm} \mathfrak{A} \in V_\Sigma .$$

Another known fact is that $L_{\Sigma+} \simeq \mathfrak{F}^\Sigma$. Therefore

$$\mathfrak{Cm} L_\Sigma \simeq ((L_\Sigma)_+)^+ \simeq (\mathfrak{F}^\Sigma)^+$$

On the other hand, Σ is valid on \mathfrak{F}^Σ iff it is valid on $(\mathfrak{F}^\Sigma)^+$. This implies that Σ is a canonical logic iff

$$\mathfrak{Cm} L_\Sigma \models \Sigma .$$

Put differently Σ is a canonical logic iff

$$\mathfrak{Cm} L_\Sigma \in V_\Sigma ,$$

Recalling that L_Σ is always in V_Σ , we have just shown that Σ is a canonical logic if V_Σ is closed under forming the canonical extension of a particular algebra, i.e. L_Σ . On the other hand V_Σ is a canonical variety if it is closed under forming canonical extensions of arbitrary algebras. It should now be clear that Σ is a canonical logic if V_Σ is a canonical variety.

CCC is the reverse of this statement.

Since we approach the problem of canonicity at the level of frames, we are interested in a dual statement of CCC. Note that in essence, we deal with the frames which are ultrafilter frames of some modal algebra. These are sv-frames or DGF's. For convenience we can define the ultrafilter extension of a frame F by

$$ue F = (F^+)_+ ,$$

which is an equivalent form of the standard definition. We extend this definition to sv-frames. Basically the ultrafilter extension of an sv- frame is the ultrafilter extension

of its underlying Kripke frame.

We also define

$$SV^\Sigma = V_{\Sigma+} = \{\mathfrak{A}_+ \mid \mathfrak{A} \in V_\Sigma\} = \{\mathfrak{A}_+ \mid \mathfrak{A} \models \Sigma\}.$$

or

$$SV^\Sigma = V_{\Sigma+} = \{S \in SV \mid \Vdash_{SV} \Sigma\}.$$

We can then show that canonicity of V_Σ is equivalent to closure of $(V_\Sigma)_+$ under ultrafilter extensions. But first we have a look at the dual operations on the category SV . By duality, an injective (surjective) map of modal algebras is associated with a surjective (injective) map between the corresponding sv -frames. So subalgebras translate to quotient sv -frames and homomorphic images translate to generated sv -subframes. Let S^* denote the modal algebra of clopen subsets of S , which is the dual object to S based on The Stone Representation Theorem. We can define the coproduct of a family of sv -frames by letting it be the ultrafilter frame of the product of the associated algebras of the sv -frames of the family. That is,

$$\bigoplus_I^{sv} S_i = [\prod_I ((S_i)^*)]_+.$$

It is easily seen that

$$\prod_I ((S_i)^*) \models \phi \quad \text{iff} \quad \bigoplus_I^{sv} S_i \Vdash_{SV} \phi$$

or equivalently

$$\prod_I (\mathfrak{A}_i) \models \phi \quad \text{iff} \quad \bigoplus_I^{sv} (\mathfrak{A}_i)_+ \Vdash_{SV} \phi.$$

The Birkhoff Variety Theorem describes varieties as classes of algebras closed under HSP. By duality, a class of sv-frames is defined by a logic (where the notion of validity is sv-validity) iff it is closed under the dual operations denoted by GQC.

Mapping a Kripke frame to its modal algebra of subsets provides a contravariant functor, as is the case if we map a modal algebra to its ultrafilter frame. Hence both canonical and ultrafilter extensions can be seen as functors, as they are compositions of two functors. Since by their definitions both canonical and ultrafilter extensions are the result of applying two contravariant functors in a row, both preserve injective and surjective morphisms.

Since any $\mathfrak{A} \in V_\Sigma$ is a homomorphic image of a subset of a power of any generator of V_Σ (L_Σ for example), for $\mathfrak{Cm} \mathfrak{A}$ (arbitrary \mathfrak{A}) to belong to V_Σ it is enough to show

$$\mathfrak{Cm} \prod_i L_\Sigma \in V_\Sigma .$$

L_Σ can be replaced by any generator, of course.

Dually, closure of SV^Σ under ultrafilter extensions. (ue-closed-ness) is reduced to the case of coproducts.

Recalling $\mathfrak{F}^\Sigma \simeq (L_\Sigma)_+$ we can see that \mathfrak{F}^Σ generates the dual class SV^Σ . So the canonicity of SV^Σ is equivalent to

$$\text{ue} \bigoplus_i^{\text{sv}} \mathfrak{F}^\Sigma \in SV^\Sigma .$$

We will prove that this statement, and hence CCC can be reformulated in terms of semantical equivalence of certain structures, in a sense we will define in detail. We will also prove certain results that suggest the connection between canonicity and modal definability might have been understated in standard literature. We will prove some properties of (mainly definable) classes of Kripke frames that are related to $\mathfrak{u}\varepsilon$ -closure in an indirect way.

Everything we have introduced so far is either entirely included in the current literature or equivalent to some already existing definitions/results. However some of the concepts above were independently developed though they turned out to be equivalent to previously known definitions/results. Expanding the definition of ultrafilter extension and its consequent equivalent forms of the CCC is done here for the first time to the best of our knowledge. So are the definitions of theory maps and their subsequent classification. Theory maps provide a new approach for studying canonicity, or other questions concerning validity for that matter. We will define certain maps that we call theory maps, and we show that validity of a formula ϕ on an sv-frame S is equivalent to the inclusion of ranges of all theory maps with domain S , in \mathfrak{F}^Σ , where $\Sigma \vdash \phi$.

A couple of other results that are new to this dissertation are the following. First we prove that continuous theory maps into \mathfrak{F}^k , and continuous bounded frame morphisms into \mathfrak{F}^k , are one and the same. Here, \mathfrak{F}^k is the canonical frame of the minimal

normal modal logic \mathbf{K} .

As a consequence of the above classification theorem we obtain a the following re-statement of the CCC :

For any canonical logic Σ , continuous bounded frame morphisms with the domain

$$u\epsilon \bigoplus_{\mathbf{I}}^{\text{sv}} \mathfrak{F}^{\Sigma}$$

land in \mathfrak{F}^{Σ} (i.e. their range is included in \mathfrak{F}^{Σ}). We will not however prove any of the results related to canonicity using the theory map formulation.

There are two other new results we have proved in the last chapter. That is, the non-existence of a right adjoint for the ultrafilter extension functor, and the canonicity of a particular logic which is our proposition 5.8.

2 Basic Concepts

2.1 Modal Algebras

Definition 2.1.1. (Modal Algebras) An algebra $\mathfrak{A} = \langle A, \vee, -, 0, 1, f \rangle$, is a modal algebra provided that $\langle A, \vee, -, 0, 1 \rangle$ is a boolean algebra, and $f : A \rightarrow A$ is a map with the following properties:

$$f(0) = 0 \quad \text{and} \quad f(a \vee b) = f(a) \vee f(b) \quad (1)$$

The map f is called the modal operator of \mathfrak{A} .

Definition 2.1.2. (Subalgebras of Modal Algebras) A modal algebra \mathfrak{B} is a subalgebra of a modal algebra \mathfrak{A} , notation: $\mathfrak{B} \preceq \mathfrak{A}$, provided that the underlying boolean algebra of \mathfrak{B} is a subalgebra of the underlying boolean algebra of \mathfrak{A} , and the modal operator of \mathfrak{B} is the restriction of that of \mathfrak{A} to the underlying set of \mathfrak{B} .

Definition 2.1.3. (Homomorphisms of Modal Algebras) Suppose $\mathfrak{A} = \langle A, \vee, -, 0, 1, f \rangle$, and $\mathfrak{B} = \langle B, \vee, -, 0, 1, g \rangle$, are modal algebras. A map $h : A \rightarrow B$, is called a homomorphism of modal algebras provided that the following conditions hold:

1. $h(0) = 0$ and $h(1) = 1$
2. $h(a \vee b) = h(a) \vee h(b)$
3. $h(-a) = -h(a)$

$$4. h(f(a)) = g(h(a))$$

Differently put, a homomorphism of modal algebras is a boolean algebra homomorphism that preserves the modal operator as well. We write $h : \mathfrak{A} \rightarrow \mathfrak{B}$ to denote that h is a modal algebra morphism.

an injective homomorphism of modal algebras is called an embedding.

Definition 2.1.4. (Category of Modal Algebras) \mathbf{MA} , the category of modal algebras, is the category with modal algebras as objects, and homomorphisms of modal algebras as morphisms.

Terminology We use the following commonly used notations

$$a \wedge b = -(-a \vee -b),$$

and

$$a \leq b \quad \text{iff} \quad a \vee b = b.$$

Definition 2.1.5. (Filters and Ultrafilters of Boolean Algebras) Given a boolean algebra

$$\mathfrak{A} = \langle A, \vee, -, 0, 1 \rangle$$

a subset U is called a (non-trivial) filter of \mathfrak{A} provided that for any elements a and b of \mathfrak{A} ,

1. $a \wedge b$ is in U if both a and b belong to U .
2. b is in U if a is in U and $a \leq b$.
3. 0 does not belong to U .

A filter is called an ultrafilter provided that it is maximal, i.e. not included any other filter. In the context of boolean algebras ultrafilters and prime filters are one and the same. A filter is a prime filter provided the following condition holds.

4. Either a or b is in U if $a \vee b$ is in U .

We use the term $UF(\mathfrak{A})$ to denote the set of all ultrafilters of \mathfrak{A} .

Definition 2.1.6. (Principal Filters) For any boolean/modal algebra \mathfrak{A} and any nonzero $a \in |\mathfrak{A}|$ there is a filter called the principal filter generated by a , denoted by Π_a or $\uparrow a$, and defined by

$$\{b \in \mathfrak{A} \mid a \leq b\},$$

The fact that $\uparrow a$ is a filter is obvious.

Remark If \mathfrak{A} is the boolean algebra of subsets of a set X and $a \in X$ the principal filter generated by the singleton $\{a\}$, which by an abuse of notation is also denoted by Π_a , or $\uparrow a$, is an ultrafilter. In this case $\uparrow a$ is called the principal ultrafilter generated by a . This is an instance of a more general fact. $\uparrow a$ is an ultrafilter if and only if a is an

atom, where an atom is an element a such that $a \not\leq b$ for any b in \mathfrak{A} .

2.2 Kripke Frames

Definition 2.2.1. (Kripke Frames) A Kripke frame $F = \langle W, R \rangle$ is a relational structure on a set W , called the universe or the underlying set of F . R is a binary relation on W , which is called the accessibility relation of F . We can also use the notation $a \rightarrow b$ which means b is accessible from a , (that is, Rab) whenever there is no ambiguity.

Definition 2.2.2. (Bounded Homomorphisms) Suppose $F = \langle W_0, R_0 \rangle$, and $G = \langle W_1, R_1 \rangle$, are Kripke frames. A function

$$f : W_0 \longrightarrow W_1$$

is called a bounded homomorphism provided that,

$$\text{for all } a \text{ and } b \text{ in } W_0, \quad R_1 f(a)f(b) \quad \text{if} \quad R_0 ab \quad (2)$$

and

$$\text{if } R_1 f(a)x, \text{ then there is } b \text{ in } W_0 \text{ such that } R_0 ab \text{ and } f(b) = x. \quad (3)$$

Definition 2.2.3. (Category of Kripke Frames) The category KF , of Kripke frames, is the category with Kripke frames as objects and bounded homomorphisms as morphisms. Morphisms of KF are called frame morphisms.

We use the term *bounded morphism*, or *frame morphism* to refer to a bounded homomorphism of Kripke frames.

Notation We use $|\mathfrak{A}|$ to denote the underlying boolean algebra of the modal algebra \mathfrak{A} , and $|F|$ to denote the underlying (carrier) set of the frame F .

Definition 2.2.4. (Subframes) Suppose $F = \langle W, R \rangle$ is a Kripke frame. A Kripke frame $G = \langle W_0, R_0 \rangle$ is called a (generated) subframe of F , notation: $G \preceq F$, provided that

$$W_0 \subseteq W \text{ and } R_0 = R \upharpoonright W_0$$

and

$$b \in W_0 \text{ if } R a b \text{ and } a \in W_0$$

Differently put, a (generated) subframe of a frame F is a relational substructure of F which is closed under the accessibility relation of F .

2.3 Connecting Kripke Frames and Modal Algebras

Every set determines a boolean algebra. That is, the boolean algebra of its subsets. When it comes to Kripke frames, there is a relation on the carrier set of a Frame. As a result, the boolean algebra of subsets of the carrier set of a Kripke frame is more than a mere boolean algebra.

Definition 2.3.1. (Modal Algebra of Subsets) Let $F = \langle W, R \rangle$ be a Kripke frame. The power set boolean algebra of F can be turned into a modal algebra

$$F^+ = \langle PW, \cup, \emptyset, W, \neg, f_R \rangle,$$

where f_R is defined by

$$f_R[A] = \{x \mid R[x] \cap A \neq \emptyset\} = \{x \mid \exists y \in A \ x \rightarrow y\}.$$

Here, and every where else in this dissertation,

$$R[x] = \{y \mid Rxy\} = \{y \mid x \rightarrow y\}.$$

F^+ , specially in the context of modal logics with multiple modal operators, is often referred to as the complex modal algebra of F .

Definition 2.3.2. We can define a contravariant functor

$$(_)^+ : KF \longrightarrow MA$$

as follows.

1. On objects

$$(_)^+(F) = F^+.$$

2. On morphisms

$$(_)^+(f) = f^{-1},$$

where for a Kripke frame morphism

$$f : F \longrightarrow G ,$$

the modal algebra homomorphism

$$f^{-1} : G^+ \longrightarrow F^+ .$$

is defined by taking inverse images of subsets. That is, for $B \subseteq |G|$ we define

$$f^+(B) = f^{-1}[B] = \{x \in |F| \mid f(x) \in B\}$$

Verifying that the definition 2.3.2 above, does in fact define a contravariant functor is straightforward. Note that

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1} .$$

Remark If f is an injective(surjective) frame morphism f^+ is a surjective(injective) homomorphism of modal algebras. C.f.[BRV].

Definition 2.3.3. (Ultrafilter Frames) For a modal algebra $\mathfrak{A} = \langle A, f \rangle$, the ultrafilter frame of \mathfrak{A} , notation \mathfrak{A}_+ , or $UF(\mathfrak{A})$, is a Kripke frame $F = \langle UF(\mathfrak{A}), R_f \rangle$, where $UF(\mathfrak{A})$ is the set of ultrafilters of $|\mathfrak{A}|$, and

$$R_f UW \text{ iff } W \subseteq f^{-1}[U].$$

Definition 2.3.4. We can define a contravariant functor

$$(_)_{+} : MA \longrightarrow KF$$

as follows.

1. On objects

$$(_)_+(\mathfrak{A}) = \mathfrak{A}_+.$$

2. On morphisms, given a modal homomorphism of

$$g : \mathfrak{A} \longrightarrow \mathfrak{B}$$

and an arbitrary ultrafilter Q of \mathfrak{B} , the Kripke frame morphism $(_)_+(g) = g_+$ is defined by:

$$g_+(Q) = \{a \in \mathfrak{A} \mid g(a) \in Q\}.$$

One has to verify that this defines an ultrafilter of \mathfrak{A} .

Remark If g is an injective(surjective) homomorphism of modal algebras g_+ is a surjective(injective) frame morphism. C.f.[BRV].

Next, we introduce an operation, in an admittedly non-standard way. An operation that assigns a Kripke frame to each Kripke frame defining a covariant functor on KF . Later, we will elaborate on the reason we define this operation and the functor it defines in the particular manner we have chosen here.

Definition 2.3.5. (Ultrafilter Extensions) *The ultrafilter extension of a frame*

$$F = \langle W, R \rangle$$

is the frame

$$\mathfrak{u}\mathfrak{e} F = \langle \text{UF}(|P(W)|), R^{\mathfrak{u}\mathfrak{e}} \rangle ,$$

where

$$R^{\mathfrak{u}\mathfrak{e}} Q_0 Q_1$$

iff

$$\forall A \in Q_1 \ (\{x \in W \mid \exists y \in A \ R_{xy}\} \in Q_0) .$$

Here we have defined the ultrafilter extension of a Kripke frame F as another Kripke frame. This does not suggest a new possibility of ambiguous definition. The reason is that $\mathfrak{u}\mathfrak{e} F$ is an sv-frame since it is an ultrafilter frame. That is, it is the ultrafilter frame of the modal algebra of subsets of $|F|$. This can be routinely verified following the definitions.

Our definitions of ultrafilter frames, and ultrafilter extensions, although non-standard, are more suitable for the approach employed in our work.

Fact 2.1. (Ultrafilter Extension as a Functor) *The following facts are easy to verify.*

1.

$$\mathfrak{u}\mathfrak{e} F \simeq (F^+)_+ .$$

2. *There is a covariant functor*

$$\mathfrak{u}\mathfrak{e} : \mathbf{KF} \longrightarrow \mathbf{KF} ,$$

defined by composing the functors defined in definitions 2.3.2 and 2.3.4 above.

3. $\mathfrak{u}\mathfrak{e}$ preserves injective/surjective frame morphisms.

Proof: The first statement follows from the definition. The composition of two contravariant functors is obviously covariant. Since the functors whose composition defines $\mathfrak{u}\mathfrak{e}$ turn injective(surjective) morphisms into surjective(injective) morphisms the last claim above becomes obvious. \dashv

Remark Our use of the notation $\mathfrak{u}\mathfrak{e}$ for the functor

$$((_)^+)_+ : \mathbf{KF} \longrightarrow \mathbf{KF} ,$$

is justified by part 1 of 2.1 above.

Definition 2.3.6. (Canonical Extension of a Modal Algebra) Given a modal algebra \mathfrak{A} , the canonical extension algebra of \mathfrak{A} , denoted by $\mathfrak{Cm}\mathfrak{A}$, is the modal algebra $(\mathfrak{A}_+)^+$. That is, $\mathfrak{Cm}\mathfrak{A}$, is the modal algebra of subsets of the ultrafilter frame of \mathfrak{A} .

Proposition 2.1. (Canonical Extension as a Functor) Extending the definition of \mathfrak{Cm} to modal homomorphisms by defining

$$\mathfrak{Cm}(f) = (f_+)^+ ,$$

yields a covariant functor

$$\mathfrak{Cm} : \mathbf{MA} \longrightarrow \mathbf{MA} ,$$

which preserves injective/surjective modal homomorphisms.

Proof: The first part of the proposition is clearly true by examining definitions 2.3.3 and 2.3.4, above. To see that injective/surjective morphisms are preserved note that

1. If h is an injective/surjective homomorphism of modal algebras then h_+ is a surjective/injective bounded frame morphism.
2. If h is a surjective/injective bounded frame morphism then h^+ is an injective/surjective homomorphism of modal algebras.

⊢

Next is a citation of a result by Jónsson and Tarski. We will use this map and its properties later.

Proposition 2.2. (Jónsson-Tarski Embedding Theorem) *For any modal algebra \mathfrak{A} , there is an embedding J from \mathfrak{A} into $\mathcal{Cm} \mathfrak{A}$.*

Proof The map

$$J : \mathfrak{A} \rightarrow \mathcal{Cm} \mathfrak{A},$$

defined by

$$J(a) = \langle a \rangle, \quad \text{where } \langle a \rangle = \{U \in \mathfrak{A}_+ \mid a \in U\},$$

is the desired embedding. See [BRV] for a proof.

⊢

Notation: We use the notation, $|F|$, to refer to the carrier set W of the frame $F = \langle W, R \rangle$. Also for a modal algebra, $\mathfrak{A} = \langle A, +, \cdot, 0, 1, f \rangle$, $|\mathfrak{A}|$ refers to the set A .

Moreover by abuse of notation we might write $F = \langle F, R \rangle$ thereby using the same name for the frame F and its carrier set $|F|$. Finally, we use R_F to refer to the accessibility relation of F .

2.4 Syntax and Semantics of Normal Modal Logics

In what follows we briefly go over the definition of basic modal languages and modal logics. [BRV] contains all we need for our discussions here, and much more.

To develop the syntax of modal logic, we fix a countable set of atomic propositions, denoted by AtProp . The modal language L over AtProp is defined inductively, as follows.

1. Any atomic proposition p is a modal formula.
2. If ϕ and ψ are two modal formulae, $\phi \vee \psi$, $\neg\phi$, and $\Diamond\phi$ are modal formulae.

Boolean operations \wedge and \Rightarrow are defined in terms of \vee , and \neg as in basic propositional logic. Modal operator \Box is defined by:

$$\Box\phi \equiv \neg(\Diamond\neg\phi)$$

The notion of derivation, or logical deduction, is pretty much the same as in proposi-

tional logic, except we have an extra rule **N** introduced below.

Definition 2.4.1. (Normal Modal Logics) A normal modal logic in the language \mathcal{L} , is a set Σ of modal formulae of \mathcal{L} , including all propositional tautologies, all instances of the axiom

$$\Box(\phi \Rightarrow \psi) \Rightarrow (\Box\phi \Rightarrow \Box\psi) \quad (\mathbf{K}),$$

and closed under uniform substitution of formulae . The rules of deduction are *modus ponens* and the following necessitation rule:

$$\frac{\phi}{\Box\phi} \quad (\mathbf{N}).$$

Uniform substitution is defined by simultaneous replacement of all instances of any propositional variable by an arbitrary formulae.

The minimal modal normal logic which contains no modal axioms but **K**, is also called **K**.

Terminology Throughout the dissertation the term modal logic refers to a normal modal logic unless otherwise specified.

Definition 2.4.2. Notion of Proof

1. Given a modal logic Σ and a modal formula ϕ we say ϕ is provable in Σ , or ϕ is a consequence of Σ , or there is a Σ -proof for ϕ , provided that there is a finite sequence

$$\langle \phi_i \rangle_{i \leq n},$$

of modal formulae, such that $\phi_n = \phi$ and for any $i < n$, ϕ_i is either in Σ , is the result of uniform substitution of a formula for a propositional variable in a formula ϕ_j of the sequence ($j < i$), or is obtained from an application of a deduction rule to one or more elements of the sequence, with indices smaller than i .

We denote this by

$$\frac{}{\Sigma} \vdash \phi \quad \text{or} \quad \Sigma \vdash \phi.$$

2. Given a set of formulae Γ , $\mathbf{K}\Gamma$ is the smallest normal modal logic that contains Γ . such a modal logic is easily seen to always exist. Moreover if Σ is a modal logic one can see that $\mathbf{K}\Sigma$ is exactly the set of all formulae provable in Σ . That is,

$$\frac{}{\Sigma} \vdash \phi \quad \text{iff} \quad \phi \in \mathbf{K}\Sigma.$$

3. If Γ is a set of modal formulae, Σ a modal logic, and ϕ a modal formula we say ϕ is a consequence of Γ in Σ , or ϕ is provable from Γ in Σ , or ϕ is provable in Σ

assuming Γ , provided that ϕ is in $\mathbf{K}(\Sigma \cup \Gamma)$. Notation

$$\Gamma \vdash_{\Sigma} \phi.$$

4. A set Γ of modal formulae Σ -consistent provided that

$$\Gamma \not\vdash_{\Sigma} \phi \wedge \neg\phi.$$

Consistent (no logic specified) means \mathbf{K} -consistent.

Terminology

We use $\vdash_{\Gamma} \phi$ and $\Gamma \vdash \phi$ interchangeably. The former is usually used when Γ is a modal logic, while the latter notation is more often used when Γ is any set of modal formulae. Unless the logic of deduction is indicated as a subscript we assume that the proof is in \mathbf{K} , the minimal (normal) modal logic. If the logic of deduction Σ is different from \mathbf{K} we always specify it.

2.4.1 Modal Semantics for Modal Algebras

To define the modal semantics, we define valuations on modal algebras and Kripke frames. A valuation on a frame specifies what propositional variables are true at any given node of a frame. In contrast, a valuation on a modal algebra assigns an element of the algebra as the value of each atomic proposition. Values under a valuation are then assigned to all modal formulae, inductively. Formal definitions follow.

Definition 2.4.3. (Modal Algebra Valuations) A valuation on a modal algebra is a map $\mu : \text{AtProp} \rightarrow |\mathfrak{A}|$. The set of all valuations on \mathfrak{A} is denoted by $\text{Val}(\mathfrak{A})$.

Given a valuation μ on a modal algebra $\mathfrak{A} = \langle A, \vee, -, 0, 1, f \rangle$, an extension $\bar{\mu}(\phi) : L \rightarrow |\mathfrak{A}|$ of μ is defined inductively, as follows.

1. $\bar{\mu}(p) = \mu(p)$
2. $\bar{\mu}(\phi \vee \psi) = \bar{\mu}(\phi) \vee \bar{\mu}(\psi)$
3. $\bar{\mu}(\Diamond \phi) = f(\bar{\mu}(\phi))$

The extension of μ from AtProp to L can be shown to be unique. This justifies an abuse of notation. That is, we use μ instead of $\bar{\mu}$. The uniqueness of the extension above is best explained in an algebraic context. To see this let us define a special algebra.

Definition 2.4.4. (Lindenbaum (Modal) Algebra) Given a modal logic Σ , in the language L define an equivalence relation on formulae of L by

$$\phi \sim_{\Sigma} \psi \quad \text{iff} \quad \Sigma \vdash \phi \Leftrightarrow \psi.$$

We drop the subscript Σ as there is no confusion. If $[\phi]_{\sim}$ is the equivalence class containing ϕ , and L/Σ is the set of equivalence classes of \sim . The Lindenbaum algebra of Σ , notation: L_{Σ} , is defined as

$$L_{\Sigma} = \langle L/\Sigma, \neg, \vee, \perp, \top, f_{\Diamond} \rangle$$

Here \top is the equivalence class containing all formulae of Σ , \perp is the equivalence class of the negations of all theorems of Σ , $\neg[\phi]_{\sim} = [\neg\phi]_{\sim}$, $[\phi]_{\sim} \vee [\psi]_{\sim} = [\phi \vee \psi]_{\sim}$, and finally $f_{\diamond}([\phi]_{\sim}) = [\diamond\phi]_{\sim}$.

One can verify that L_{Σ} is a modal algebra for L . A very special algebra at that as we will see below.

Definition 2.4.5. (Notion of Truth for Modal Algebras) If \mathfrak{A} is a modal algebra and $\mu \in \text{Val}(\mathfrak{A})$ is a valuation on \mathfrak{A} , we say \mathfrak{A} satisfies a modal formula ϕ under μ , or ϕ is true in \mathfrak{A} under μ , notation: $\mathfrak{A}, \mu \models \phi$, provided that $\mu(\phi) = 1$.

A modal formula is valid on \mathfrak{A} , $\mathfrak{A} \models \phi$, provided that $\mathfrak{A}, \mu \models \phi$, for all $\mu \in \text{Val}(\mathfrak{A})$.

We say a set of formulae Γ is true under μ on \mathfrak{A} provided that any formulae in Γ is true under μ on \mathfrak{A} . Notation

$$\mathfrak{A}, \mu \models \Gamma.$$

Similarly Γ is valid on \mathfrak{A} provided that any formula in Γ is valid on \mathfrak{A} . Notation

$$\mathfrak{A} \models \Gamma.$$

Here also We mainly follow the terminology and definitions in [BRV].

Remark Lindenbaum algebras for different logics (modal or not) are ubiquitous in universal algebra. The fact that for a modal logic Σ , L_Σ of definition 2.4.4 is a modal algebra over which Σ is valid is standard material of Algebraic Logic. In fact the Lindenbaum algebra of any logic has this property. One has to check that the operations defined on L_Σ are well-defined and prove, inductively, $L_\Sigma \models \Sigma$.

We state the following proposition without proof.

Proposition 2.3. *The following facts hold for L_Σ .*

1. $L_\Sigma \models \phi$ iff $\Sigma \vdash \phi$.

2. L_Σ is a free algebra for the class of all algebras that validate Σ . This means that for any algebra $\mathfrak{A} \models \Sigma$, any $\mu : \text{AtProp} \rightarrow |\mathfrak{A}|$ uniquely extends to an algebra morphism $\bar{\mu} : L_\Sigma \rightarrow \mathfrak{A}$. This of course makes sense under the identification of p and $[p]_\sim$ for any $p \in \text{AtProp}$. The converse is also true; if any $\mu \in \text{Val}(\mathfrak{A})$, extends to an algebra morphism $\bar{\mu}$ from L_Σ into \mathfrak{A} then $\mathfrak{A}, \mu \models \Sigma$.

Remark The approach we have chosen here describes the notion of truth for modal formulae over an algebra. An alternative, yet equivalent, approach is to define the truth in terms of equations over algebras.

Proofs for this and other facts above are usually covered in a standard course on universal algebra; in a more general context. That is analogous results hold for general classes of algebras, not just modal algebras. [BRV] includes a sufficient yet artfully concise discussion of Algebraic Logic; adequately covering the background material on universal algebra that we refer to here.

2.4.2 Modal Semantics for Kripke Frames

Definition 2.4.6. (Frame Valuations) Assume L is a modal language over AtProp . A valuation on $F = \langle W, R \rangle$ is a function

$$\mu : \text{AtProp} \rightarrow \prod_{w \in W} \{0, 1\} = 2^W.$$

So the value of a propositional variable under μ on F can be viewed as a subset of W , the set of all nodes at which p is true, under μ . The set of all valuations on a frame F is denoted by $\text{Val}(F)$. A valuation μ on F can be uniquely extended to a map $\bar{\mu} : L \rightarrow |F|$, also called a valuation on F . Again we use μ in place of $\bar{\mu}$. Here is the inductive definition :

1. $\mu(\phi \wedge \psi) = \mu(\phi) \cap \mu(\psi)$.
2. $\mu(\neg\phi) = W \setminus \mu(\phi)$.
3. $\mu(\Diamond\phi) = \{a \in W \mid \exists b \in W \text{ such that } Rab\}$.

Remark As mentioned above, the extension of a valuation on a frame from atomic propositions to all modal formulae is unique. This can be seen using the fact that the power set of $|F|$ is a modal algebra and a valuation on F is a modal algebra valuation for the modal algebra of subsets of F .

2.4.3 Models Over Kripke frames

Definition 2.4.7. (Models over Frames) let $F = \langle W, R \rangle$ be a Kripke frame. A model (over F) is triple $\langle W, R, \mu \rangle$, where μ is a valuation on F . In other words a model over F is a structure $\mathfrak{M} = \langle F, \mu \rangle$, obtained by fixing a valuation $\mu \in \text{Val}(F)$.

Definition 2.4.8. (Model Morphisms) Assume $\mathfrak{M} = \langle F, \mu \rangle$, and $\mathfrak{N} = \langle G, \rho \rangle$, are models. A map $f : \mathfrak{M} \rightarrow \mathfrak{N}$, is a morphism of Kripke models provided that $f : F \rightarrow G$ is a

frame morphism such that

$$f(a) \in \rho(p) \quad \text{if} \quad a \in \mu(p).$$

Differently put, a frame morphism f between the underlying frames of two models \mathfrak{M} and \mathfrak{N} , is also a morphism of models provided that the value of μ on each node a of \mathfrak{M} agrees with the value of ρ on $f(a)$ for all propositional variables (and hence all modal formulae).

In our discussions, we rarely use models. Our main interest is the study of frames, as will become clear later. However, let us define the notion of *bisimulation* between two models. This can be quite useful in semantical arguments.

Definition 2.4.9. (Bisimulations) Suppose $\langle F_0, \mu \rangle$ and $\langle F_1, \rho \rangle$ are models. A binary relation $B \subseteq |F_0| \times |F_1|$ is called a bisimulation between these two models provided that

1. If x is in $|F|$, y is in $|G|$ and, Bxy then for any atomic proposition p , $x \in \mu(p)$ iff $y \in \rho(p)$.
2. If $R_0x_0x_1$, and Bx_0y_0 , then there is y_1 in $|F_1|$ such that $R_1y_0y_1$, and Bx_1y_1 .
3. If $R_1y_0y_1$, and Bx_0y_0 , then there is x_1 in $|F_0|$ such that $R_0x_0x_1$, and Bx_1y_1 .

The notation $\langle F_0, x \rangle \longleftrightarrow \langle F_1, y \rangle$ is used to indicate that x and y are related by a bisimulation. Two nodes are called bisimilar provided they are related by a bisimulation.

Remark The relation \leftrightarrow defined above is itself a bisimulation. To be more specific the relation defined by

$$a \leftrightarrow b \text{ iff there is a bisimulation } B \text{ such that } aBb,$$

is a bisimulation. Equivalently \leftrightarrow can be defined as the union of all bisimulations between any two models. This is the largest bisimulation between those models.

It is a routine task to show that if there is a bisimulation between F_0 and F_1 relating x and y as in the above definition then for any modal formula ϕ

$$F_0, \mu, x \Vdash \phi \text{ iff } F_1, \rho, y \Vdash \phi.$$

The converse is not in general true. However, for certain classes of models, one can prove the converse.

Remark Any bounded morphism

$$f : \mathfrak{M} \longrightarrow \mathfrak{N},$$

of Kripke models, considered as a relation, is a bisimulation which relates a node and its image under f , by the very definition of bounded morphisms of models.

Definition 2.4.10. (Global Truth and Validity for Frames)

1. A modal formula ϕ is said to be globally true on F , under μ , for $\mu \in \text{Val}(F)$, notation $F, \mu \Vdash \phi$, provided that $F, \mu, a \Vdash \phi$ for all $a \in |F|$.
2. A modal formula ϕ is valid on F , notation $F \Vdash \phi$, provided that $F, \mu \Vdash \phi$ for all $\mu \in \text{Val}(F)$.
3. A set of formulae Γ , is valid on a frame F provided that any $\phi \in \Gamma$ is valid on F .
4. A modal formula ϕ is valid on a model $\mathfrak{M} = \langle F, \mu \rangle$, notation $\mathfrak{M} \Vdash \phi$, provided that $F, \mu \Vdash \phi$.
5. A set of formulae Γ , is valid on model \mathfrak{M} provided that any $\phi \in \Gamma$ is valid on \mathfrak{M} .

Note that our choice of notation simply reflects the equivalence of the notion of validity on a model $\mathfrak{M} = \langle F, \mu \rangle$, and global truth on F under μ .

2.5 Modal Semantics in Categories of Algebras and Frames

We will now address the specific choice of morphisms in the categories \mathbf{MA} and \mathbf{KF} .

The main point is that these particular choices preserve major properties of truth and validity, when expected. [BRV] is the major reference for missing details.

Proposition 2.4. *The following properties are obvious by definitions of the semantics for Kripke frames and modal algebras.*

1. For any modal algebra morphism

$$f : \mathfrak{A} \longrightarrow \mathfrak{B},$$

(a) if f is injective, and for $\mu \in \text{Val}(\mathfrak{A})$,

$\mu' \in \text{Val}(\mathfrak{B})$, and any $p \in \text{AtProp}$, $\mu'(p) = f(\mu(p))$ then

$$\mathfrak{B}, \mu \models \phi \quad \text{if} \quad \mathfrak{A}, \mu' \models \phi.$$

This implies that for a subalgebra $\mathfrak{B} \preceq \mathfrak{A}$,

$$\text{if } \mathfrak{A} \models \phi \quad \text{then } \mathfrak{B} \models \phi.$$

(b) If f is surjective, and $\mu \in \text{Val}(\mathfrak{B})$, and μ' in $\text{Val}(\mathfrak{A})$ is such that $f(\mu'(p)) = \mu(p)$

for all $p \in \text{AtProp}$ then

$$\mathfrak{B}, \mu \models \phi \quad \text{if} \quad \mathfrak{A}, \mu' \models \phi.$$

Therefore

$$\mathfrak{B} \models \phi \quad \text{if} \quad \mathfrak{A} \models \phi.$$

2. Likewise for any frame morphism

$$f : G \longrightarrow F.$$

(a) If f is injective and for $\mu \in \text{Val}(F)$, $\mu' \in \text{Val}(G)$, any $x \in |G|$, and any $p \in$

AtProp , $x \in \mu'(p)$ if $f(x) \in \mu(p)$ then for any modal formula ϕ

$$G, \mu', x \Vdash \phi \quad \text{if} \quad F, \mu, f(x) \Vdash \phi.$$

This implies that if $G \preceq F$ and $F \Vdash \phi$ then $G \Vdash \phi$

(b) If $f : F_0 \twoheadrightarrow F_1$, is a surjective frame morphism, $\mu \in \text{Val}(G)$, and $\mu' \in \text{Val}(F)$, such that for any $x \in |F|$ and any $p \in \text{AtProp}$,

$$x \in \mu'(p) \quad \text{if} \quad f(x) \in \mu(p)$$

then for any modal formula ϕ

$$G, \mu, f(x) \Vdash \phi \quad \text{if} \quad F, \mu, x \Vdash \phi.$$

This implies $F \Vdash \phi$, then $G \Vdash \phi$.

Based on the facts above it is also easy to conclude

(c) If $f : F_0 \rightarrow F_1$, any formula valid in F_0 or F_1 is valid in $f[F_0]$, the image (quotient) of F_0 under f . This is true because $f[F_0]$, is a (generated) subframe of F_1 , as well as an image of F_0 .

2.6 Stone Spaces and sv-Frames

Definition 2.6.1. (Stone Spaces) Stone is the category of Stone spaces, topological spaces that are compact, Hausdorff, and totally disconnected, with continuous functions as morphisms. Instead of the more traditional definition of a totally disconnected space we use the equivalent property the space has a basis of clopen sets, sets that are both open and closed.

$\text{UF}(\mathfrak{A})$, the set of ultrafilters of any boolean algebra \mathfrak{A} , has a natural topology which makes it into a Stone space. More specifically $\text{UF}(\mathfrak{A})$ is equipped with the, Stone,

topology generated by $\{\langle a \rangle\}_{a \in |\mathfrak{A}|}$. Here $\langle a \rangle$ is defined as the set of those ultrafilters of \mathfrak{A} that contain a . Whenever we refer to *the topology* of $\text{Clp}(\mathfrak{A})$ we mean the topology just described, unless otherwise specified.

The Stone Representation Theorem states that any boolean algebra \mathfrak{A} , is isomorphic to the boolean algebra $\text{Clp}(\text{UF}(\mathfrak{A}))$, of clopen subsets of the ultrafilter frame of \mathfrak{A} . Conversely any Stone space X is homeomorphic to $\text{UF}(\mathfrak{A})$ the space of ultrafilters of $\mathfrak{A} = \text{Clp}(X)$. Let us start with a description of the content of the theorem before stating it more formally.

The homeomorphism between a Stone space X and the space of ultrafilters of clopens of X is established by mapping a point x in X to

$$U_x = \cap \{U \in \text{UF}(\text{Clp}(X)) \mid 0 \in U \text{ and } x \in 0\}.$$

Note that in a Stone space every point is the unique element in the intersection of all clopens to which it belongs.

The boolean isomorphism between a boolean algebra \mathfrak{A} and the boolean algebra of clopens of its ultrafilter space is the function that maps an element a in $|\mathfrak{A}|$ to a clopen

$$O_a = \{U \in \text{UF}(\mathfrak{A}) \mid a \in U\}.$$

Note that because of the homeomorphism between X and $\text{UF}(\text{Clp}(X))$ we can assume that the clopens of a Stone space are exactly the basis elements generating the topol-

ogy of the space. This is true by construction of the basis elements of the space $\text{Clp}(\text{UF}(\mathfrak{A}))$. If O is a clopen subset of a Stone space $X \simeq \text{UF}(\text{Clp}(X))$, O can be written as

$$O = \bigcup_{i \in I} \langle a_i \rangle, \text{ where } \{a_i\}_{i \in I} \text{ is a subset of } |\mathfrak{A}|$$

By compactness of O , after rearranging indices, for some natural number n we have

$$O = \bigcup_{i=1}^n \langle a_i \rangle = \langle \bigvee_{i=1}^n a_i \rangle.$$

So O is a basis element in the homeomorphic copy of X .

Our main reference here is [KKV], where detailed proofs of basic properties of Stone are laid out.

Proposition 2.5. (Stone Representation Theorem) *The category Stone of stone spaces and continuous maps is dually equivalent to BA the category of boolean algebras and boolean homomorphisms.*

Here we just sketch the proof. For a stone space X the set $\text{Clp}(X)$ of clopens of X is obviously a boolean algebra of sets. Conversely it is easy to see that the space $\text{UF}(\mathfrak{A})$, of all ultrafilters of the boolean algebra \mathfrak{A} , equipped with the topology generated by the basis, $\{\langle a \rangle\}_{a \in |\mathfrak{A}|}$ is a stone space.

Next, if $f : X \rightarrow Y$ is a continuous map between two topological spaces X , and Y , then $f^{-1} : \text{Clp}(Y) \rightarrow \text{Clp}(X)$ is a boolean homomorphism between $\text{Clp}(Y)$ and $\text{Clp}(X)$.

Similarly, suppose $f : \mathfrak{A} \rightarrow \mathfrak{B}$ is a boolean homomorphism. Since the inverse image of an ultrafilter of \mathfrak{B} is an ultrafilter of \mathfrak{A} , it is then easy to verify that $f^{-1} : \text{UF}(\mathfrak{B}) \rightarrow \text{UF}(\mathfrak{A})$ is a continuous map.

To complete the proof one should verify

$$X \simeq \text{UF}(\mathfrak{A}), \quad (4)$$

and

$$\mathfrak{A} \simeq \text{Clp}(\text{UF}(\mathfrak{A})). \quad (5)$$

The map $f : X \rightarrow \text{UF}(\text{Clp}(X))$ defined by

$$f(x) = \langle \{x\} \rangle$$

is the homeomorphism we need in (4) and the map

$$g(a) = \langle a \rangle$$

is the boolean homomorphism in (5). ⊢

We will not verify all the facts used in the following definition. Statements for which a proof is not provided are routine facts of general topology.

Definition 2.6.2. (Vietoris Functor) *The Vietoris functor is defined on, Stone, the category of Stone spaces and continuous functions.*

For any object X in Stone, $V(X)$ is $K(x) = \{K \mid K \subseteq X \text{ is closed}\}$ where the topology of VX is defined by applying an operator B , defined below, to the class of open sets of X .

For any set X and any class Q of subsets of X , ($Q \subseteq P(X)$), the operator B on $P(X)$ is defined by

$$B(Q) = \{\{K \subseteq X \mid K \cap U \neq \emptyset\}\}_{U \in Q} \cup \{\{K \subseteq X \mid K \subseteq U\}\}_{U \in Q}$$

When X is a topological space, $B(Q)$ forms a sub-basis for a topology on $K(X)$, the set of closed subsets of X . $(K(X), \tau_V)$ is called the Vietoris space generated by (X, τ) where the topology on $K(X)$ is the topology generated by $B(\tau)$. One has to verify that $V(X)$ is in fact a Stone space with the topology just defined.

On morphisms the Vietoris functor is defined by taking forward images. That is, given

$$f : X \longrightarrow Y ,$$

the image of f under V is the (continuous) map

$$Vf : VX \longrightarrow VY .$$

defined by

$$Vf(F) = f(F) .$$

It is easy to see that Vf is in fact a continuous map between the corresponding Stone spaces.

Given an arbitrary category \mathbb{C} and a functor the category $\text{Clg } F$ is defined as follows.

Definition 2.6.3. (Categories of Coalgebras) *Let \mathbb{C} be a category and*

$$F : \mathbb{C} \longrightarrow \mathbb{C},$$

be a functor on \mathbb{C} .

1. *A coalgebra of F is a pair $\langle C, \gamma \rangle$, also denoted by γ_C , where*

$$\gamma : C \longrightarrow F(C),$$

is a morphism in the category \mathbb{C} .

2. *The category of coalgebras of F , denoted by $\text{Clg } F$ is the category whose objects are coalgebras of F and a morphism of whose consists of a pair, $\langle \gamma_C, \gamma_D \rangle$, of coalgebras of F and a morphism, f ,*

$$f : C \longrightarrow D$$

of \mathbb{C} such that the following diagram commutes

$$\begin{array}{ccc} C & \xrightarrow{\gamma_C} & F(C) \\ f \downarrow & & \downarrow F(f) \\ D & \xrightarrow{\gamma_D} & F(D) \end{array}$$

When f is the morphism above we refer to the coalgebra morphism above by f as well.

An example of a category of coalgebras, defined in 2.6.3 above, is $\text{Clg } P$, the category of the power set functor on Sets . The power set endofunctor on morphisms is defined by taking images. That is, for a function

$$f : A \longrightarrow B ,$$

and a subset C of A ,

$$P(f) : P(A) \longrightarrow P(B)$$

is defined as

$$P(f)(C) = f[C] .$$

It was noted by Jan Rutten, C.f. [RUC] that Kripke frames can be seen as coalgebras of the power set endofunctor, P , on Sets .

In what follows we take a closer look at the correspondence just mentioned and use it to define a class of frames, called *sv*-frames. There is no essential novelty in defining these frames. They just happen to provide us with a convenient terminology to approach the question of canonicity.

Kripke Frames as Coalgebras, and *sv*-Frames.

A coalgebra $\gamma : W \rightarrow P(W)$ of $P : \text{Sets} \rightarrow \text{Sets}$ determines a binary relation $R \subseteq W \times W$, as follows:

$$\forall u, w \in W \quad R u w \quad \text{iff} \quad w \in \gamma(u).$$

Conversely, a Binary relation, R , on a set W , defines a coalgebra of the power set functor. Given $R \subseteq W \times W$, $\gamma : W \rightarrow P(W)$ is defined by:

$$\gamma(u) = R[u].$$

The choice of bounded morphisms is necessary for the equivalence of Kripke frames and coalgebras of the power set functor. Let us verify that bounded frame morphisms correspond to coalgebra morphisms of $\text{Clg } P$ and vice versa.

Assume

$$\gamma : W_0 \longrightarrow P(W_0)$$

and

$$\alpha : W_1 \longrightarrow P(W_1)$$

are coalgebras, and

$$f : \langle W_0, \gamma \rangle \longrightarrow \langle W_1, \alpha \rangle$$

is a coalgebra morphism. That is, the following diagram commutes.

$$\begin{array}{ccc} W_0 & \xrightarrow{\gamma} & P(W_0) \\ f \downarrow & & \downarrow Pf \\ W_1 & \xrightarrow{\alpha} & P(W_1) \end{array}$$

Suppose F_0 and F_1 be frames determined by $\langle W_0, \gamma \rangle$, and $\langle W_1, \alpha \rangle$, respectively. We prove that f is a frame morphism. Let a be a node in W_0 , and $R_1 [f(a)y]$. Therefore

$$y \in R_1 [f(a)] = \alpha(f(a)) = Pf(\gamma(a)) = f[\gamma(a)].$$

Therefore $y = f(x)$ for some $x \in f[\gamma(a)] = f[R_0 [a]]$. That is, $y = f(x)$ for some x such that $R_0 ax$; so f is a frame morphism.

Conversely, let

$$f : F_0 \longrightarrow F_1$$

be a frame morphism, a a node in W_0 , and $R_1 f(a)y$. By the definition of R_1 we have

$$y \in R_1 [f(a)] = \alpha[f(a)]$$

By the definition of frame morphism we conclude

$$\exists x \in R_0[a] = \gamma(a) \quad \text{such that} \quad y = f(x).$$

So $y \in f(\gamma(a)) = \text{Pf}(\gamma(a))$. This proves

$$\alpha[f(a)] \subseteq \text{Pf}(\gamma(a)).$$

Similarly, we can prove

$$\text{Pf}(\gamma(a)) \subseteq \alpha(f(a)).$$

So $\text{Pf}(\gamma(a)) = f[\alpha[f(a)]]$, and the diagram above commutes.

⊢

Making a Case for \mathbf{sv} -Frames

Given a stone space X , VX , as a set, is a subset of the power set of X . A closed subset is a subset after all. As a functor, the value of V on morphisms is defined by taking images exactly as it is done for the power set functor. Therefore a Vietoris coalgebra morphism can be seen as coalgebra morphism of the power set functor, forgetting the topologies on the corresponding Stone spaces.

Hence a similar correspondence holds for coalgebras of the Vietoris functor. Although for any Stone space X , and any coalgebra

$$\gamma : X \longrightarrow VX,$$

of the Vietoris functor on Stone induces a binary relation on X defined by

$$a R_\gamma b \quad \text{iff} \quad b \in \gamma(a),$$

One should not forget that R_γ is not just any binary relation. The corresponding map, γ , to this binary relation is a continuous map between two Stone spaces W_0 and $K(W_0)$.

The following sums up our discussion above. Let

$$\gamma : X_0 \longrightarrow V(X_0)$$

and

$$\alpha : X_1 \longrightarrow V(X_1)$$

be coalgebras of V , and

$$f : \langle X_0, \gamma \rangle \longrightarrow \langle X_1, \alpha \rangle$$

a morphism of $\text{Clg } V$. Commutativity of the diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{\gamma} & V(X_0) \\ f \downarrow & & \downarrow Vf \\ X_1 & \xrightarrow{\alpha} & V(X_1) \end{array}$$

implies that the corresponding map

$$f : \langle W_0, R_0 \rangle \longrightarrow \langle W_1, R_1 \rangle ,$$

is a bounded morphism, which is also continuous under Stone topologies on W_0 and W_1 .

Our discussion above shows that the following two definitions can be used interchangeably.

Definition 2.6.4. (sv-Frames) *SV*, is the category of coalgebras of the Vietoris functor on Stone, the category of stone spaces and continuous maps. Following the terminology used in the theory of coalgebras this can be written as $SV = \text{Clg } V$.

Alternatively, we can define an sv-frame as follows

Definition 2.6.5. (sv-Frames) *(Alternative Definition)* An sv-frame is a Kripke frame

$$S = \langle W, R \rangle .$$

where W can be equipped with a Stone topology, τ , such that the binary relation R has the following properties:

1. $R[x]$ is closed in τ .
2. Once the set of closed subsets of W is equipped with the Vietoris topology, the map corresponding to R

$$\gamma_R : W \longrightarrow \mathcal{V}W ,$$

defined by

$$\gamma_R(x) = R[x] ,$$

is continuous.

SV is the category with sv-frames as objects and continuous bounded homomorphisms as morphisms.

Definition 2.6.6. (sv-Subframes) A generated subframe S' of a an sv-frame S , is called an sv-subframe provided that

1. S' is a generated subframe of S as a Kripke frame.
2. S' is an sv-frame; the topology of S' is the subspace topology inherited from S .

Definition 2.6.7. (sv-Semantics) *If $S = \langle W, R \rangle$ is an sv-frame, the semantics of modal formulae is defined exactly as it is for Kripke frames, except we only consider valuations whose values on propositional variables are basis elements (clopens) of W . The set of all such valuation on S , called sv-valuations on S , is denoted by $SVal(S)$. So for an sv-frame S we can consider two distinct ways of interpreting modal formulae. Either we consider S as a Kripke frame, and use the same notation as we do for a Kripke frame, or we consider sv-valuations only. We define, and introduce different notations for, the latter case, below.*

Remark When considering an sv-frame, we can consider it with the sv-semantics just defined, or simply as a Kripke frame, with Kripke semantics. Because we use distinct notations, there is no ambiguity as to which semantics is used in a particular discussion.

For an sv-frame S , any $a \in |S|$, any $\mu \in SVal(S)$, and any modal formula ϕ , the definition of truth at the node a of S under μ remains the same, as defined for Kripke frames. S is a Kripke frame and the notion of truth is defined for Kripke frames already. The notation is also inherited.

The notion of Global truth remains the same as well. A formula is globally true under an sv-valuation μ , if it is true at all nodes of S . However if we wish to emphasize that μ is an sv-valuation under which a formula ϕ is true at $a \in S$, we use the notation

$$S, \mu \Vdash_{sv} \phi.$$

Likewise given μ , we might say ϕ is globally sv-true in S under μ , just to emphasize that μ is a sv-valuation. Here are the definitions:

Definition 2.6.8. (sv-Global Truth and sv-Validity) For any $\mu \in \text{SVal}(S)$, and any modal formula ϕ

$$S, \mu \Vdash_{\text{sv}} \phi \quad \text{iff} \quad S, \mu, a \Vdash \phi \text{ for all } a \in |S|.$$

We say ϕ is sv-valid on S , or S sv-validates ϕ , notation: $S \Vdash_{\text{sv}} \phi$, provided that

$$S, \mu \Vdash_{\text{sv}} \phi \text{ for all } \mu \in \text{SVal}(S).$$

Remark Unlike the previous two cases, there is a significant difference between sv-validity and validity in the Kripke sense. Obviously if a formula ϕ is valid on an sv-frame S , it is also sv-valid on S .

We will use the terms full-validity, or Kripke-validity, to contrast validity under all valuations on a sv-frame S , to sv-validity on S .

Semantics in the Category of sv-Frames

The following proposition which is the analogous to proposition is obvious by definition of sv-semantics.

Proposition 2.6. *For any sv-frame morphism*

$$f : S_0 \longrightarrow S_1 .$$

1. *If f is injective and for $\mu_0 \in \text{SVal}(S_0)$, $\mu_1 \in \text{SVal}(S_1)$, any $x \in |S_1|$, and any $p \in \text{AtProp}$, $x \in \mu_1(p)$ if $f(x) \in \mu_0(p)$ then for any modal formula ϕ*

$$S_1, \mu_1, x \Vdash \phi \quad \text{if} \quad S_0, \mu_0, f(x) \Vdash \phi .$$

This implies that if $S_1 \preceq S_0$ and $S_0 \Vdash_{\text{sv}} \phi$ then $S_1 \Vdash_{\text{sv}} \phi$.

2. *If $f : S_0 \twoheadrightarrow S_1$, is a surjective sv-frame morphism, $\mu_1 \in \text{SVal}(S_1)$, and $\mu_0 \in \text{SVal}(S_0)$, such that for any $x \in |S_0|$ and any $p \in \text{AtProp}$,*

$$x \in \mu_1(p) \quad \text{if} \quad f(x) \in \mu_0(p) ,$$

then for any modal formula ϕ

$$S_1, \mu, f(x) \Vdash \phi \quad \text{if} \quad S_0, \mu, x \Vdash \phi .$$

This implies $S_1 \Vdash_{\text{sv}} \phi$, then $S_0 \Vdash_{\text{sv}} \phi$.

Based on the facts above it is also easy to conclude

3. *If $f : S_0 \rightarrow S_1$ is an sv-frame morphism, any formula sv-valid in S_0 or S_1 is valid in $f[S_0]$, the image (quotient) of S_0 under f . This is true because $f[S_0]$, is a (generated) sv-subframe of S_1 , as well as a continuous image (an sv-quotient) of S_0 .*

The duality between Stone spaces and boolean algebras extends to a duality between SV and MA. The underlying Stone space of an sv-frame S corresponds to the boolean

algebra of clopen subsets of S , denoted by S^* . The accessibility relation of S defines the modal operator on S^* , in exactly the same way as defined for the modal algebra of subsets. A direct proof is not difficult however we will defer this discussion for a while and we will see this equivalence through descriptive general frame later. For the moment let us refer to [KKV] and assume a duality between MA and SV. Using the notation just introduced we have the following isomorphisms for any sv-frame S and any modal algebra \mathfrak{A}

$$S \simeq S_+^* \quad \text{and} \quad \mathfrak{A} \simeq (\mathfrak{A})_+^*.$$

This sums up the main part of the basic semantical facts we will use in our discussions of sv-frames.

2.7 Soundness

A natural property that one expects from any class of structures that provides a semantics for a logic is soundness. Whatever the notion of truth is, we would expect the consequences of a set of formulae to be true on a structure if the set of formulae itself is true on the structure.

Definition 2.7.1. *A set Σ of modal formulae is valid on a class \mathbb{C} of structures, i.e. Kripke frames, sv-frames, models, or algebras provided that it is valid on every structure*

in \mathbb{C} . The notation we use is

$$\mathbb{C} \Vdash \Sigma,$$

when \mathbb{C} is a class of Kripke frames, sv-frames, or models. We use the notation

$$\mathbb{C} \models \Sigma,$$

when \mathbb{C} is a class of algebras.

We use $\mathbb{C} \Vdash \phi$ (or $\mathbb{C} \models \phi$ for a class of algebras) when $\Sigma = \{\phi\}$.

Definition 2.7.2. (Soundness) A modal logic Σ is sound with respect to a class \mathbb{C} of frames, models, provided that for any modal formula ϕ ,

$$\mathbb{C} \Vdash \phi \quad \text{if} \quad \frac{}{\Sigma} \phi \quad \text{and} \quad \mathbb{C} \Vdash \Sigma.$$

Similarly for a class \mathbb{C} of modal algebras, Σ is sound with respect to \mathbb{C} provided that

$$\mathbb{C} \models \phi \quad \text{if} \quad \frac{}{\Sigma} \phi \quad \text{and} \quad \mathbb{C} \models \Sigma.$$

We state the next proposition without proof.

Proposition 2.7. Every normal modal logic Σ is sound with respect to any class \mathbb{C} of Kripke frames, Kripke models, sv-frames, sv-models, or modal algebras.

Proof Standard inductive argument provides a routine proof of this proposition starting with validity on a single structure. Once this is proved the statement for a class of structures follows readily by definition. \dashv

2.8 Semantics Equivalence Lemma

We have defined the modal algebra F^+ of subsets of a given Kripke frame F . A simple fact, sometimes left as an exercise in text-books points out a form of semantics equivalence between a frame and its modal algebra of subsets. Here we state it as a proposition without proof.

Proposition 2.8. *For any Kripke frame F and any valuation μ on F*

$$F, \mu \Vdash \phi \quad \text{iff} \quad F^+, \mu^+ \models \phi.$$

Here $\mu^+(p) = \mu(p)$. The value on the left is an element of the modal algebra F^+ where as the value on the right is a subset of $|F|$. The two valuations have the same value of course. The notation marks a difference just to point out that the valuations are on different structures. \dashv

When it comes to the ultrafilter frame however we do not have the exact equivalence. That is the following statement **does not hold**.

$$\mathfrak{A} \models \phi \quad \text{iff} \quad \mathfrak{A}_+ \Vdash \phi.$$

In what follows we show that the modification of the notion of truth for sv-frames proves an analogous statement. In the existing literature this is done in different ways. We believe the choice of sv-frames is a very natural way for a comparative study of semantics, contrasting Kripke frames and modal algebras.

Our main tool is the following lemma.

Lemma 2.9. (Semantics Equivalence Lemma) *Given a modal algebra \mathfrak{A} , and μ in $\text{Val}(\mathfrak{A})$, there exists a unique valuation μ_+ in $\text{SVal}(\mathfrak{A}_+)$ such that for any modal formula ϕ ,*

$$\mathfrak{A}, \mu \models \phi \quad \text{iff} \quad \mathfrak{A}_+, \mu_+ \Vdash \phi. \quad (6)$$

Moreover, for any sv-frame S , the modal algebra $S^* = \text{Clp}(\mathfrak{A})$ has the following property

$$\forall \mu \in \text{SVal}(S) \exists! \mu^* \in \text{Val}(\mathfrak{A}) \text{ such that, } \mu = (\mu^*)_+. \quad (7)$$

To prove this equivalence, we rely on some basic facts about ultrafilter frame of a modal algebra. We refer to [BRV] for details of these properties.

Proof. Given a valuation $\mu \in \text{Val}(\mathfrak{A})$, we define $\mu_+ \in \text{SVal}(\mathfrak{A}_+)$ by

$$\mu_+(p) = \langle \mu(p) \rangle. \quad (8)$$

That is, the value of p under μ_+ is the set of all ultrafilters which contain $\mu(p)$. By The Stone Representation Theorem, $\mu_+(p)$ is a clopen subset of $|S| = |\mathfrak{A}_+|$, and so an sv-valuation on S .

We show that in fact for any modal formula ϕ , $\mu_+(\phi)$ is a clopen subset of $|\mathfrak{A}_+|$.

More specifically, for any modal formula ϕ

$$\mu_+(\phi) = \langle \mu(\phi) \rangle. \quad (9)$$

Claim If the equality (9) above holds then

$$\mathfrak{A}, \mu \models \phi \quad \text{iff} \quad \mathfrak{A}_+, \mu_+ \Vdash \phi.$$

.

Proof of Claim:

$$\mathfrak{A}, \mu \models \phi \quad (10)$$

iff

$$\mu(\phi) = 1$$

iff

$$\forall U \in |\mathfrak{A}_+| \mu(\phi) \in U$$

iff

$$\langle \mu(\phi) \rangle = |\mathfrak{A}_+|$$

iff

$$\mu_+(\phi) = |\mathfrak{A}_+| \quad \text{by 9.}$$

iff

$$\mathfrak{A}_+, \mu_+ \Vdash \phi. \quad (11)$$

As we just proved that (9) implies the main statement of the lemma.

It remains to show that (9) is true.

Proof of (9) We prove (9) by induction on the complexity of modal formulae.

In case of propositional variables, (9) is simply the definition of μ_+ . Induction steps for boolean operators are obvious. We only consider the case of the modal operator.

Assume $\mathfrak{A} = \langle A, \vee, \neg, 0, 1, f \rangle$, is a modal algebra, ϕ is a modal formula, μ is a valuation on \mathfrak{A} , and $\mu_+(\phi) = \langle \mu(\phi) \rangle$.

By the definition of truth, for all $U \in |\mathfrak{A}|_+$:

$$U \in \mu_+(\Diamond \phi),$$

iff

$$\exists W \in |\mathfrak{A}|_+, R_f U W \text{ and } W \in \mu_+(\phi)$$

iff

$$\exists W \in |\mathfrak{A}|_+, W \subseteq f^{-1}[U] \text{ and } W \in \mu_+(\phi),$$

iff

$\exists W \in |\mathfrak{A}|_+, W \subseteq f^{-1}[U]$ and $W \in \langle \mu(\phi) \rangle$. By I.H.

iff

$\exists W \in |\mathfrak{A}|_+, W \subseteq f^{-1}[U]$ and $\mu(\phi) \in W$ Hence $f(\mu(\phi)) \in U$,

iff

$\exists W \in |\mathfrak{A}|_+, f(\mu(\phi)) \in U$ and $\bigcap_{\mu(\phi)} W \subseteq f^{-1}[U]$, ($\bigcap_a = \bigcap \{U \mid a \in U\}$)

iff

$f(\mu(\phi)) \in U$ and $\forall a \in \mathfrak{A}, f(a) \in U$ or $f(-a) \in U$

(By properties of ultrafilters)

iff

$f(\mu(\phi)) \in U$ and $\forall a \in \mathfrak{A}, f(a \vee -a) = f(1) \in U$

iff

$f(\mu(\phi)) \in U$. ($\mu(\phi) \leq 1$, and $f(\mu(\phi)) \in U$.)

iff

$\mu(\Diamond \phi) = f(\mu(\phi)) \in U$

iff

$U \in \langle \mu(\phi) \rangle$.

The first statement of the lemma is therefore proved.

The second statement of the lemma, i.e. (7) above, is proved simply by invoking The Stone Representation Theorem. Given an sv-frame S , $S^* = \text{Clp}(|S|)$ is a modal algebra, and $(S^*)_+$ is homeomorphic to $|S|$, and any choice of clopens for values of $\mu \in \text{SVal}(S)$ is of the form $\langle a \rangle$ for some $a \in \mathfrak{A}$. \dashv

Remark Obvious corollaries of the semantics equivalence lemma can be quite useful in stating and proving properties of frames and algebras. For example, it is obvious that the lemma implies that for any modal algebra \mathfrak{A} , and any modal formula ϕ

$$\mathfrak{A} \models \phi \quad \text{iff} \quad \mathfrak{A}_+ \Vdash_{\text{sv}} \phi. \quad \dashv$$

Here you can find a series of definitions mainly intended to facilitate stating results and proofs, shortening some otherwise lengthy arguments.

Definitions (Theories and Theory Maps)

1. For a modal algebra \mathfrak{A} , and a valuation $\mu \in \text{Val}(\mathfrak{A})$, the theory of \mathfrak{A} under μ , notation: $\text{Th}(\mathfrak{A}, \mu)$ is defined by

$$\text{Th}(\mathfrak{A}, \mu) = \{\phi \mid \mathfrak{A}, \mu \models \phi\}.$$

2. For any frame F , any node $a \in F$, and any valuation $\mu \in \text{Val}(F)$, the theory of F under μ at a , which is denoted by $T(F, \mu, a)$, or $T_\mu(F, a)$, or even $T_\mu(a)$ when there is no ambiguity, is defined by

$$T(F, \mu, a) = T_\mu(F, a) = T_\mu(a) = \{\phi \mid F, \mu, a \Vdash \phi\}.$$

Note that there is no need to specify whether or not μ is an sv-valuation. As discussed in definition 2.6.8, the definition of truth is the same for all valuations, sv or otherwise. Hence there is no distinction between theory of a Kripke frames versus that of an sv-frame for a given valuation μ , in what follows.

3. The theory of a frame F under a valuation $\mu \in \text{Val}(F)$, notations $T(F, \mu)$, or $T_\mu(F)$ is defined by

$$T_\mu(F) = T(F, \mu) = \{\phi \mid \text{for all } a \in F, F, \mu, a \Vdash \phi\}.$$

We call T_μ above a theory map. If S is an sv-frame and $\mu \in \text{SVal}(S)$ T_μ is called an sv-theory map or a continuous theory map.

4. The Kripke theory of a frame F , denoted by $\text{Tk}(F)$ is the set of formulae valid on F .
5. The sv-theory of an sv-frame S , denoted by $\text{Tsv}(S)$, is the set of all formulae sv-valid on S .

6. For a class \mathbb{C} of modal algebras $\text{Th}(\mathbb{C})$ is the set of all formulae valid on every element of \mathbb{C} .
7. For a class \mathbb{C} of Kripke frames/modles $\text{Tk}(\mathbb{C})$ is the set of all formulae valid on every element of \mathbb{C} .
8. For a class \mathbb{C} of sv-frames/models $\text{Tsv}(\mathbb{C})$ is the set of all formulae sv-valid on every element of \mathbb{C} .

Remark For a class \mathbb{C} of structures, i.e. modal algebras, Kripke frames/modles, sv-frames/models we have

$$\begin{aligned}\text{Th}(\mathbb{C}) &= \bigcap_{\mathfrak{A} \in \mathbb{C}} \text{Th}(\mathfrak{A}), \\ \text{Tk}(\mathbb{C}) &= \bigcap_{F \in \mathbb{C}} \text{Tk}(F),\end{aligned}$$

or

$$\text{Tsv}(\mathbb{C}) = \bigcap_{S \in \mathbb{C}} \text{Tsv}(S),$$

depending on the type of structures in the class.

Definition 2.8.1. (Notions of Equivalence)

1. Two modal algebras \mathfrak{A} and \mathfrak{B} are said to be (modally) equivalent, notation: $\mathfrak{A} \equiv \mathfrak{B}$, provided that for any modal formula ϕ ,

$$\mathfrak{A} \models \phi \quad \text{iff} \quad \mathfrak{B} \models \phi.$$

2. Two Kripke frames F_0 and F_1 are said to be (modally) equivalent, notation: $F_0 \equiv F_1$, provided that for any modal formula ϕ ,

$$F_0 \Vdash \phi \quad \text{iff} \quad F_1 \Vdash \phi.$$

3. Two sv-frames S_0 and S_1 are (modally) sv-equivalent, notation: $S_0 \equiv_{sv} S_1$, provided that for all modal formula ϕ ,

$$S_0 \Vdash_{sv} \phi \quad \text{iff} \quad S_1 \Vdash_{sv} \phi.$$

Proposition 2.10. *The semantics equivalence lemma implies*

$$\mathfrak{A} \models \phi \quad \text{iff} \quad \mathfrak{A}_+ \Vdash_{sv} \phi.$$

Therefore for arbitrary modal algebras \mathfrak{A} and \mathfrak{B}

$$\mathfrak{A} \equiv \mathfrak{B} \quad \text{iff} \quad \mathfrak{A}_+ \equiv_{sv} \mathfrak{B}_+.$$

⊢

Proposition 2.11. *For any frame F , and any modal formula ϕ ,*

$$F \Vdash \phi \quad \text{iff} \quad u \vDash F \Vdash_{sv} \phi.$$

Proof By proposition 2.8 above

$$F \Vdash \phi \quad \text{iff} \quad F^+ \models \phi.$$

By the Semantics Equivalence Lemma

$$F^+ \models \phi \quad \text{iff} \quad (F^+)_+ \Vdash_{\text{sv}} \phi.$$

On the other hand

$$(F^+)_+ = \text{ue } F.$$

Hence

$$F \Vdash \phi \quad \text{iff} \quad \text{ue } F \Vdash_{\text{sv}} \phi,$$

as required. \dashv

Remark The proposition just proved can be restated as follows.

$$\text{Tk}(F) = \text{Tsv}(\text{ue } F).$$

Direct product of a family of algebras is defined as the cartesian product of the carrier sets with component-wise operations. We state the following without proof.

Fact 2.2. *For a family $\{\mathfrak{A}_i\}_{i \in I}$ of algebras, and any modal formula ϕ ,*

$$\prod_I \mathfrak{A}_i \models \phi \quad \text{iff} \quad \forall i \in I \quad \mathfrak{A}_i \models \phi.$$

Using the notations introduced above

$$\text{Th}(\prod_I \mathfrak{A}_i) = \bigcap_{i \in I} \text{Th}(\mathfrak{A}_i).$$

2.9 Varieties and Co-Varieties

2.9.1 Varieties of Modal Algebras

In what we have discussed so far, we have seen different notions of validity of a modal logic over different structures. We have also seen that if a modal logic Σ is valid over a class \mathbb{C} of structures, all consequences of Σ are also valid over \mathbb{C} . This property, i.e. soundness, is almost always expected for any semantics of a logic. In contrast one might ask whether a given consistent logic is valid somewhere, on some structure. This is part of the question of completeness.

In case of modal logic, when it comes to modal algebras the answer is strongly affirmative. By proposition 2.3 any modal logic Σ is valid on its Lindenbaum algebra. Indeed there is a stronger result for algebras as we will discuss below. We will see that the answer in case of *sv*-frames is affirmative as well. This we establish by way of the duality between *sv*-frames and modal algebras. When it comes to Kripke frames the general answer is negative. We will not consider this question for Kripke frames in detail. Let us just mention that there are consistent poly-modal logics that are not valid over any class of frames. However for modal logics which are the subject of our focus any logic is valid on a class of frames. C.f. [GMM] and [BRV] for details. We shall discuss some cases of completeness briefly in the last chapter. There are many subtleties to the question of completeness that fall outside the scope of this dissertation.

Definition 2.9.1. (Classes of Structures Defined by a Logic) *Given a modal logic Σ , the class \mathbb{C} of structures defined by \mathbb{C} is the class of structures on which Σ is valid. We could also use the term \mathbb{C} is definable by Σ .*

Proposition 2.3 shows that the class of algebras defined by a modal logic is always non-empty. There is a stronger result for modal algebras which is a special case of a theorem for algebras in general. We will consider this result next. When a modal logic Σ defines a class \mathbb{C} of structures, it is very natural to ask about the relation between Σ and $\text{Th}(\mathbb{C})$, (or $\text{Tk}(\mathbb{C})$ or $\text{Tsv}(\mathbb{C})$ when \mathbb{C} is a class of frames or models) Obviously Σ is a subset of the theory of a class of structures it defines. Can the theory of a class of structures include Σ properly? We should be asking this question about $K\Sigma$ indeed, since the theory of any class of structures is closed under logical deduction by soundness. So to reformulate, can the theory of a class of structures defined by Σ include anything besides Σ and whatever Σ can prove? The Birkhoff Variety Theorem answers this question. Here we state the theorem for modal algebras with some changes for the sake of brevity, after a definition we need.

Definition 2.9.2. (Varieties of Algebras) *A class \mathbb{C} of algebras is called a variety iff it is closed under formation of direct products, homomorphic images, and subalgebras. That is, \mathbb{C} is a variety iff*

1. *For all \mathfrak{A} , if \mathfrak{A} is in \mathbb{C} and $\mathfrak{B} \preceq \mathfrak{A}$ then \mathfrak{B} is in \mathbb{C} .*

2. If \mathfrak{A} is in \mathbb{C} and $f : \mathfrak{A} \rightarrow \mathfrak{B}$, then \mathfrak{B} is in \mathbb{C} .

3. If $\{\mathfrak{A}_i\}_{i \in I}$ is a family of algebras in \mathbb{C} , $\prod_I \mathfrak{A}_i$ is in \mathbb{C} .

We use the following notations

1. $H(\mathbb{C}) = \{\mathfrak{B} \mid \mathfrak{A} \rightarrow \mathfrak{B} \text{ \& } \mathfrak{A} \in \mathbb{C}\}$

2. $S(\mathbb{C}) = \{\mathfrak{B} \mid \mathfrak{B} \hookrightarrow \mathfrak{A} \text{ \& } \mathfrak{A} \in \mathbb{C}\}$

3. $P(\mathbb{C}) = \{\mathfrak{B} \mid \mathfrak{A} \simeq \prod_I \mathfrak{A}_i\}, \text{ where } \{\mathfrak{A}_i\}_{i \in I} \text{ is a family of algebras in } \mathbb{C}$

We drop extra parentheses when using these notations.

Theorem 2.12. (The Birkhoff Variety Theorem) *A class of modal algebras is definable by a modal logic if and only if it is a variety.* ⊢

Remark Embedded in the proof of the variety theorem is an underlying fact. For any class \mathbb{C} of modal algebras the closure of a \mathbb{C} under the operations H, S , and P , is evidently a variety by definition. However one needs only apply these operators in the specific order HSP . This is not a trivial observation. What this amounts to, is to say any algebra obtained by repeated application of H, S , and P to members of \mathbb{C} in arbitrary order, can be obtained by applying HSP to elements of \mathbb{C} at most once in the specific order, H, S, P . This fact, which justifies our next definition, will be assumed throughout the rest of this dissertation. Unfortunately including a proof of the Birkhoff Variety Theorem in any shortened form does not do the theorem justice.

Hence we just refer to [UA1],[UA2], and [UA3] for proofs.

In view of this theorem we could use V_Σ to denote the an arbitrary variety, as it must be defined by some modal logic Σ . We mentioned some changes in the statement of the theorem. The theorem in its general form is stated for definability by a set of equations, which are in this context equivalent to modal logics. [BRV] contains details of the correspondence between equations and modal formulae, and the Birkhoff Variety Theorem, to the degree we need to use here.

Definition 2.9.3. (HSP) *Since the closure of any class of algebras under direct products, subalgebras, and homomorphic images is a variety, by definition, every class \mathbb{C} , of algebras generates a variety. denoted by $\text{HSP}(\mathbb{C})$, or $\langle \mathbb{C} \rangle$.*

Now we can explain how the variety theorem implies that the theory of V_Σ is $\mathbf{K}\Sigma$.

Proposition 2.13. *For any modal logic Σ*

$$\text{Th}(V_\Sigma) = \Sigma.$$

Proof Since Σ is sound with respect to V_Σ it is obvious that $\mathbf{K}\Sigma \subseteq \text{Th}(V_\Sigma)$. To show the equality we consider the Lindenbaum algebra of Σ . Essentially, our task is to show that the variety generated by L_Σ is V_Σ . However let us focus on the theories of structures, as this approach is helpful later.

Consider $\text{HSP}(\mathbf{L}_\Sigma)$. It is obvious that applying the operations HSP does not eliminate any formulae from $\text{Th}(\mathbf{L}_\Sigma)$. Hence $\text{Th}(\text{HSP}(\mathbf{L}_\Sigma)) = \text{Th}(\mathbf{L}_\Sigma) = \mathbf{K}\Sigma$. It is also obvious that \mathbf{L}_Σ belongs to \mathbf{V}_Σ . Therefore by proposition 2.3

$$\mathbf{K}\Sigma \subseteq \text{Th}(\mathbf{V}_\Sigma) \subseteq \text{Th}(\mathbf{L}_\Sigma) = \mathbf{K}\Sigma.$$

Therefore

$$\text{Tk}(\text{HSP}(\mathbf{L}_\Sigma)) = \mathbf{K}\Sigma,$$

as desired. ⊢

We can summarize the discussion completed by the proposition 2.13 above as follows

$$\langle \mathbb{C} \rangle = \mathbf{V}_{\text{Th}(\mathbb{C})} = \{ \mathfrak{B} \mid \mathfrak{B} \models \text{Th}(\mathbb{C}) \} = \text{HSP}(\mathbb{C}),$$

where \mathbb{C} is any non empty class of algebras. In terms of Σ

$$\phi \in \text{Th}(\mathbf{V}_\Sigma) \quad \text{iff} \quad \vdash_\Sigma \phi \quad \text{iff} \quad \mathbf{L}_\Sigma \models \phi,$$

or equivalently

$$\mathbf{V}_{\text{Th}(\mathbf{L}_\Sigma)} = \mathbf{V}_\Sigma = \langle \mathbf{L}_\Sigma \rangle.$$

In what we discussed above the so called algebraic completeness, i.e that fact that the set of formulae valid on \mathbf{L}_Σ is exactly $\mathbf{K}\Sigma$ is essential. A similar result does not hold for Kripke frames in general. The reason is that a modal algebra \mathfrak{A} is not semantically equivalent to its ultrafilter frame when the notion of validity is Kripke validity. The modal algebra of a Kripke frame F does validate the same formulae that F does.

However not every modal algebra is a modal algebra of subsets of some frame. Therefore the result does not automatically apply to Kripke frames and we do not have a theorem analogous to the variety theorem for Kripke frames. This is why a general completeness result for Kripke frames cannot be proved. As mentioned above in the general context of modal logics with more than one modal operator, this is not even true.

The dual equivalence of MA and SV however, makes it possible to obtain a dual theorem for sv-frames as follows.

By the semantics equivalence lemma, lemma 2.9, there is an equivalence between \mathfrak{A} and \mathfrak{A}_+ , once we change the notion of validity with the more restricted sv-validity. We are going to use this equivalence to demonstrate a dual of the Birkhoff variety Theorem. This result follows from existing facts in the modal algebra literature. However we have included it as the particular approach we have chosen makes the statements more natural and proofs almost trivial.

Let us define some terminology before we consider further questions. We always assume, without loss of generality, that classes of algebras and frames we consider are closed under isomorphism.

2.9.2 Co-Varieties of sv -Frames

Definition 2.9.4. (Dual classes) Suppose \mathbb{D} is a class of sv -frames. The dual class of \mathbb{D} , \mathbb{D}^* is defined by

$$\mathbb{D}^* = \{S^* \mid S \in \mathbb{D}\}.$$

Similarly, suppose \mathbb{C} is a class of modal algebras. The dual class of \mathbb{C} , $(\mathbb{C})_+$ is defined by

$$\mathbb{C}_+ = \{\mathfrak{A}_+ \mid \mathfrak{A} \in \mathbb{C}\}.$$

Recall that the duality between SV and MA and that $(\mathfrak{A}_+)^* \simeq \mathfrak{A}$ and $(S^*)_+ \simeq S$. It becomes obvious that

$$(\mathbb{C}_+)^* = \mathbb{C},$$

and

$$(\mathbb{D}^*)_+ = \mathbb{D}.$$

Note that we assume \mathbb{C} and \mathbb{D} are closed under isomorphism.

To find the operation analogous to the direct product, we need the following definition.

Definition 2.9.5. (sv-Coproducts) The sv -coproduct of a family $\{S_i\}_{i \in I}$ of sv -frames, is defined as

$$\bigoplus_I^{sv} S_i = \left(\prod_I S_i^* \right)_+.$$

Here, S_i^* is the modal algebra of clopens of S_i .

Remark By the Semantics Equivalence Lemma it is easy to see that

$$\bigoplus_{\mathbf{I}}^{\text{sv}} S_i \Vdash_{\text{sv}} \phi \quad \text{iff} \quad \forall i \in \mathbf{I} \quad S_i \Vdash_{\text{sv}} \phi.$$

This is true because

$$\bigoplus_{\mathbf{I}}^{\text{sv}} S_i \Vdash_{\text{sv}} \phi,$$

iff

$$(\prod_{\mathbf{I}} S_i^*) \models \phi,$$

iff

$$\text{for all } i \in \mathbf{I} \quad S_i^* \models \phi,$$

iff

$$\text{for all } i \in \mathbf{I} \quad S_i \Vdash_{\text{sv}} \phi.$$

The first and last equivalence hold by definition, and the second equivalence holds by characteristic property of products of algebras. \dashv

Note that the sv-coproduct of a family of sv-frames is in fact a coproduct of the family, in the sense of category theory. This is true by duality, and the fact that the product of a family of algebras is the categorical product of the algebras in the family.

Definition 2.9.6. (GQC) For a class \mathbb{D} of sv-frames we adapt the following notations

$$1. \quad G(\mathbb{D}) = \{S' \in \text{SV} \mid S \twoheadrightarrow S' \ \& \ S \in \mathbb{D}\}$$

$$2. Q(\mathbb{D}) = \{S' \in SV \mid S' \mapsto S \ \& \ S \in \mathbb{D}\}$$

$$3. C(\mathbb{D}) = \{S \in SV \mid S \simeq \bigoplus_I^{sv} S_i\}, \text{ where } \{S_i\}_{i \in I} \text{ is any family of sv-frames in } \mathbb{D}$$

Because validity on modal algebras and sv-validity on sv-frames are equivalent, it is obvious that

$$(V_\Sigma)_+ = \{\mathfrak{A}_+ \mid \mathfrak{A} \in V_\Sigma\} = \{S \mid S^* \in V_\Sigma\}$$

is determined by Σ . That is

$$SV^\Sigma = \{S \mid S \Vdash_{sv} \Sigma\}.$$

Terminology (Covarieties Of sv-Frames)

We use the term *the covariety defined/generated by Σ* to refer to the class SV^Σ . Also, if \mathbb{C} is a class of sv-frames with Σ as its sv-theory, we use the term *the covariety generated by \mathbb{C}* .

Definition 2.9.7. (Generated Varieties) For a class K of Kripke frames, the variety generated by K , notation: V_K , is defined by

$$V_K = \langle K^+ \rangle = HSP(K^+) = HSP(\{F^+ \mid F \in K\}).$$

Similarly, the covariety generated by K , notation: SV^K , is defined by

$$SV^K = \langle \mathfrak{u}K \rangle = GQC(\mathfrak{u}K) = GQC(\{\mathfrak{u}F \mid F \in K\}).$$

In what we just laid out \mathbb{K} is any nonempty class of Kripke frames. A priori there is no modal logic affiliated with this class. Varieties and covarieties on the other hand are determined by modal logics. By duality between classes of modal algebras and sv-frames we can easily conclude the following:

Since the variety V_{Σ} of algebras is closed under HSP, SV^{Σ} is closed under the *dual operations*, GQC.

It is obvious, by duality, that (generated) subframes of sv-frames correspond to homomorphic images of the corresponding algebras, and quotients of sv-frames correspond to subalgebras of the corresponding algebras.

The discussion above shows that a class \mathbb{D} of sv-frames is closed under the operations GQC, iff \mathbb{D} is a covariety. It should also be clear that these operations can be applied in the specific order mentioned here, as this is the case for HSP.

Proposition 2.14. (Classification of Co-Varieties) *A class \mathbb{D} of sv-frames is definable by a modal logic (via sv-validity) iff it is closed under generated sv-subframes, quotients, and sv-coproducts.*

Proof If \mathbb{D} is a class of sv-frames then the class corresponding class \mathbb{D}^* of modal algebras, is closed under HSP if and only if \mathbb{D} is closed under GQC, by duality. This is

true because

1. If $f : \mathfrak{A} \rightarrow \mathfrak{B}$ is surjective then $f^{-1} : \mathfrak{B}_+ \rightarrow \mathfrak{A}_+$ is injective.
2. If $f : \mathfrak{A} \rightarrow \mathfrak{B}$ is injective then $f^{-1} : \mathfrak{B}_+ \rightarrow \mathfrak{A}_+$ is surjective.
3. If $\mathfrak{A} = \prod_I \mathfrak{A}_i$ then $\mathfrak{A}_+ = \bigoplus_I^{\text{sv}} (\mathfrak{A}_i)_+$.

Moreover every S in \mathbb{D} is of the form \mathfrak{A}_+ for some \mathfrak{A} in \mathbb{D}^* , as $S \simeq (S_+^*)$ and \mathbb{D} is closed under isomorphism. Therefore a class \mathbb{D} of sv-frames is a co-variety if and only if \mathbb{D}^* is a variety, if and only if there is a modal logic Σ such that

$$\mathbb{D}^* = \{\mathfrak{A} \mid \mathfrak{A} \models \Sigma\}$$

if and only if

$$\mathbb{D} = \{S \mid S \Vdash_{\text{sv}} \Sigma\}.$$

The semantics equivalence lemma and yet another application of the fact justifies the last equivalence that any sv-frame is isomorphic to the ultrafilter frame of some modal algebra and any modal algebra is the modal algebra of clopens of some sv-frames justifies that last equivalence. ⊢

2.10 Descriptive General Frames

In 2.5 we cited The Stone Representation Theorem in order to define sv-frames that are the main object of study here. We also mentioned that this definition is not essentially new. A well known part of the literature on Modal Logic is the discussion of structures called Descriptive General Frames.

Definition 2.10.1. (Descriptive General Frames) *A Descriptive General Frame, a DGF in short, is a structure $D = \langle W, R, A \rangle$, where $F = \langle W, R \rangle$ is a kripke frame, and $A \subseteq P(W)$ is a set of subsets of W with the following properties*

1. *A is closed under the boolean operations of intersection, union, forming complements, and R^{-1} , where $R^{-1}[B] = \{x \mid \exists y \in B \text{ s.t. } Rxy\}$.*
2. *If $x \neq y$ are nodes of D then there is $P \in A$ such that $x \in P$ and $y \notin P$.*
3. *For any node $y \notin R[x]$ then there is $P \in A$ such that $y \in P$ and $x \notin R^{-1}[P]$.*

Definition 2.10.2. (Category Of Descriptive General Frames) *DGF is the category with descriptive general frames as objects, and bounded frame morphisms that also reflect admissible sets as morphisms. That is, given DGF's $F = \langle W_0, R_0, A_0 \rangle$, $G = \langle W_1, R_1, A_1 \rangle$, and a bounded frame morphism*

$$f : F \longrightarrow G$$

is a morphism in DGF provided that

$$f^{-1}[X] \in A_0 \quad \text{if} \quad X \in A_1.$$

Our definition of an sv-frame as a relational structure as opposed to the coalgebraic definition should make it obvious that $\text{Clg}(\mathbf{V})$ is isomorphic to DGF which is in turn dually equivalent to MA, and so the two categories are semantically identical as classes of structures providing a semantics for modal logics. Therefore using sv-frames to study the semantics of modal logic is not an essentially new idea. It is well known that properties of admissible subsets of a DGF are equivalent to conditions making the topology generated by the collection of admissible sets a stone topology.

As mentioned in [GMM], It was S.Kripke who pointed out that the dual equivalence in case of Kripke frames and modal algebras fails, because a modal algebra is in general only a subalgebra of the complex algebra of its ultrafilter frame. In proving categorical duality of $\text{Clg}(\mathbf{V})$ and MA. One avoids this problem, by building the corresponding modal algebra of a frame on a different boolean algebra rather than the power set of the underlying frame, i.e. the boolean algebra of clopen subsets of the universe of the frame.

The discussion above makes it clear that our choice of sv-frames is basically a matter of notational convenience. We can easily refer to validity over an sv-frame under all valuations in contrast to validity under a smaller class of sv-valuations and this makes it quite easy to state results whose statements are otherwise fairly complicated. As said before one can also prove the duality between Clg and MA directly without introducing descriptive general frames. We chose to demonstrate this fact through

a duality for DGF's, for historical reasons not for elegance. DGF is the category of choice for many modal logicians.

2.11 Canonicity

Proposition 2.3 singles out an algebra for a modal logic Σ whose theory is exactly the set of consequences of Σ . There is no frame that plays the same role for Kripke frames. A frame that corresponds to the Lindenbaum algebra can always be built using the standard constructions; the so called Henkin Style constructions. The outcome however, fails to validate Σ in general. A specific valuation on this frame, which makes a modal, proves a completeness result over models, not frames. We are interested in those modal logics that end up being valid on their *canonical* frames. One can say that these logics are those that are complete in a very strong sense.

Definition 2.11.1. (Canonical Frames and Canonical Models) *The canonical frame of a modal logic Σ is a frame whose nodes are maximal consistent theories of Σ and whose accessibility relation R_\diamond is defined by*

$$R_\diamond \Gamma_0 \Gamma_1 \text{ iff } (\diamond \phi \in \Gamma_0 \text{ if } \phi \in \Gamma_1)$$

and is denoted by \mathfrak{F}^Σ .

The canonical model of a modal logic Σ is the model based on the canonical frame of Σ with a valuation μ_c , called the canonical valuation and defined by

$$\mu_c(p) = \hat{p} = \{\Gamma \mid p \in \Gamma\}.$$

We use the notation \mathfrak{M}^Σ to refer to the canonical model of Σ as in [BRV].

Note that maximal consistent theories of a logic Σ are the ultrafilters of L_Σ the Lindenbaum algebra of Σ . The accessibility relation on the canonical frame of Σ coincides with the relation one gets by following the construction of the ultrafilter frame of L_Σ . So $(L_\Sigma)_+ = \mathfrak{F}^\Sigma$. As we have seen before, any ultrafilter frame is an sv-frame by The Stone representation theorem. In particular \mathfrak{F}^Σ is an sv-frame.

The following is the completeness result for models.

Fact 2.3. *Given any modal logic Σ*

$$\mathfrak{M}^\Sigma \Vdash \phi \quad \text{iff} \quad \vdash_\Sigma \phi.$$

C.f. [BRV].

Definition 2.11.2. (Canonical Logics) *A modal logic is canonical if and only if it is valid on its canonical frame.*

Some varieties of modal algebras are closed under the operation of forming canonical extensions. we defined this operator on page 24 here.

Definition 2.11.3. (Canonical Variety) *A variety V_Σ is called a canonical variety if it is closed under forming canonical extensions.*

It is not hard to see that algebraic canonicity is stronger than logical canonicity. We will prove this shortly, but first let us define the dual notion first.

Algebraic canonicity is stronger than logical canonicity in a strong sense. In fact, one can easily show a modal logic Σ is canonical if the canonical extension of a single modal algebra belongs to the variety defined by Σ . Not surprisingly that particular algebra is the algebra that determines the variety.

Proposition 2.15. *A modal logic Σ is canonical iff $\mathfrak{Cm} L_\Sigma \in V_\Sigma$.*

Proof: Since by definition

$$(L_\Sigma)_+ \simeq \mathfrak{F}^\Sigma,$$

by applying the semantics equivalence lemma we have

$$L_\Sigma \models \Sigma \quad \text{iff} \quad \mathfrak{F}^\Sigma \Vdash_{sv} \Sigma.$$

On the other hand as in previous proposition for any \mathfrak{A}

$$(\mathfrak{Cm} \mathfrak{A})_+ \simeq ue(\mathfrak{A}_+).$$

Hence letting $\mathfrak{A} = L_\Sigma$,

$$\mathfrak{Cm} L_\Sigma \models \Sigma \quad \text{iff} \quad ue \mathfrak{F}^\Sigma \Vdash_{sv} \Sigma.$$

However,

$$\mathfrak{F}^\Sigma \Vdash \Sigma \quad \text{iff} \quad ue \mathfrak{F}^\Sigma \Vdash_{sv} \Sigma \quad (\text{proposition 2.11}).$$

Therefore

$$\mathfrak{Cm} L_{\Sigma} \models \Sigma \quad \text{iff} \quad \mathfrak{F}^{\Sigma} \Vdash \Sigma.$$

Finally, Σ is valid on \mathfrak{F}^{Σ} iff Σ is a canonical modal logic, by definition, and $\mathfrak{Cm} L_{\Sigma} \in V_{\Sigma}$ iff Σ is valid on $\mathfrak{Cm} L_{\Sigma}$, also by definition. This means

$$\mathfrak{Cm} L_{\Sigma} \in V_{\Sigma} \quad \text{iff} \quad \Sigma \text{ is a canonical modal logic.}$$

which completes the proof of the proposition. \dashv

2.12 Canonicity in Dual Form

We will now formulate canonicity of varieties and modal logics in terms of classes of sv-frames. We need a definition first.

Definition 2.12.1. (\mathfrak{ue} -Closure) *A class of K of Kripke frames is called \mathfrak{ue} -closed provided that is closed under ultrafilter extension. Naturally, being closed under ultrafilter extensions means:*

$$\text{If } F \in \mathbb{K} \text{ then } \mathfrak{ue} F \in \mathbb{K}.$$

Similarly a class \mathbb{C} of sv-frames is \mathfrak{ue} -closed whenever

$$S \in \mathbb{C} \text{ implies } \mathfrak{ue} S \in \mathbb{C}.$$

The following observations are aimed at streamlining the general discussion on the connection of Frames and algebras.

Proposition 2.16. *The following are implied by the duality between SV and MA.*

1. *A class \mathbb{D} of sv-frames is \mathfrak{ue} -closed iff \mathbb{D}^* is a closed under canonical extensions.*
2. *A covariety SV^Σ is \mathfrak{ue} -closed iff V_Σ is a canonical variety.*

Proof This is almost obvious by the definition of the dual operators on classes of algebras and sv-frames. The second statement is a special case of the first one. Nevertheless we provide a direct proof for the second statement emphasizing semantically

equivalent statements that we use frequently and without notice.

For a modal logic Σ V_Σ is a canonical variety provided that

$$\mathfrak{A} \in V_\Sigma \text{ implies } \mathfrak{Cm} \mathfrak{A} \in V_\Sigma ,$$

for any modal algebra \mathfrak{A} . Equivalently, V_Σ is canonical provided that for any \mathfrak{A}

$$\mathfrak{Cm} \mathfrak{A} \models \Sigma \text{ if } \mathfrak{A} \models \Sigma .$$

On the other hand any modal algebra is a subalgebra of its canonical extension, by proposition 2.2. Therefore it is always true (even if Σ is not canonical) that

$$\mathfrak{A} \models \Sigma \text{ if } \mathfrak{Cm} \mathfrak{A} \models \Sigma .$$

Combining the two statements we conclude that V_Σ is canonical if and only if for any modal algebra \mathfrak{A} ,

$$\mathfrak{A} \models \Sigma \text{ iff } \mathfrak{Cm} \mathfrak{A} \models \Sigma .$$

On the other hand by the Semantics Equivalence Lemma

$$\mathfrak{A} \models \Sigma \text{ iff } \mathfrak{A}_+ \Vdash_{sv} \Sigma ,$$

and

$$\mathfrak{Cm} \mathfrak{A} \models \Sigma \text{ iff } ue \mathfrak{A}_+ \Vdash_{sv} \Sigma ,$$

which is true because

$$(\mathfrak{Cm} \mathfrak{A})_+ \simeq ue (\mathfrak{A}_+) .$$

Therefore V_Σ is canonical if and only if for any modal algebra \mathfrak{A} ,

$$\mathfrak{A}_+ \Vdash_{sv} \Sigma \quad \text{iff} \quad u\epsilon \mathfrak{A}_+ \Vdash_{sv} \Sigma.$$

Since for any modal algebra \mathfrak{A} , $\mathfrak{A} \simeq S^*$, for some sv-frame S , and for any sv-frame S , $S \simeq (S^*)_+$, it follows that V_Σ is canonical if and only if

$$SV^\Sigma = \{S \mid S \Vdash_{sv} \Sigma\} \text{ is } u\epsilon\text{-closed.}$$

⊢

The dual of proposition 2.15 is almost clear by definition.

Proposition 2.17. *A modal logic Σ is canonical if and only if $u\epsilon \mathfrak{F}^\Sigma \Vdash_{sv} \Sigma$. Equivalently*

$$\Sigma \text{ is canonical} \quad \text{iff} \quad u\epsilon \mathfrak{F}^\Sigma \in SV^\Sigma.$$

Proof By proposition 2.11 and the definition of a co-variety

$$\mathfrak{F}^\Sigma \Vdash \Sigma \quad \text{iff} \quad u\epsilon \mathfrak{F}^\Sigma \Vdash_{sv} \Sigma \text{ (proposition 2.11)} \quad \text{iff} \quad u\epsilon \mathfrak{F}^\Sigma \in SV^\Sigma.$$

Here the duality plays an implicit role. That is in the proof of proposition 2.11. ⊢

2.13 The Countable Canonicity Conjecture

Later we will address certain connections between canonicity of modal logics and of varieties they generate. The question of equivalence of these two forms of canonicity is open. One direction is of course proved, as seen above. The reverse direction which is called countable canonicity conjecture is the part which remains open. Our objective is to formulate the conjecture and to show how it could be approached in the context of *sv*-frames.

The Countable Canonicity Conjecture

The countable canonicity conjecture, CCC, surmises the following.

The variety of modal algebras defined by any canonical modal logic is canonical.

So, the conjecture claims that for a canonical modal logic Σ the variety V_Σ of all modal algebras over which Σ is valid, is closed under canonical extension of modal algebras. The "countable" in the name of the conjecture refers to the fact that we only consider countable languages. If arbitrarily large sets of propositional variables are allowed in forming modal languages one can state and prove a canonicity theorem so to say. We will not discuss this issue here. We are interested in canonicity of modal logics and varieties when the modal language under discussion is countable.

By 2.16 above, CCC is equivalent to ue -closure of SV^Σ for any canonical modal logic Σ . That is, given a variety V_Σ , V_Σ is canonical provided that the class SV^Σ of all sv-frames that sv-validate Σ is ue -closed. However to prove CCC one only needs to prove the closure for one operation for either type of classes (i.e. modal algebras or sv-frames).

Proposition 2.18. *A variety V_Σ is canonical if and only if for any arbitrary index set I*

$$\mathfrak{Cm} \prod_I L_\Sigma \in V_\Sigma$$

Dually, a co-variety of sv-frames is ue -closed if and only if

$$ue \bigoplus_I^{sv} \mathfrak{F}^\Sigma \in SV^\Sigma,$$

where I is any index set.

Proof By the Birkhoff Variety Theorem and its dual for sv-frames, any algebra in V_Σ (sv-frame in SV^Σ) is in $HSP(L_\Sigma, (GQC(\mathfrak{F}^\Sigma)))$. Since both ue and \mathfrak{Cm} preserve injective and surjective morphisms, if $\mathfrak{Cm}\mathfrak{A}$ is in V_Σ so is $\mathfrak{Cm}\mathfrak{B}$ where \mathfrak{B} is a subalgebra or a quotient of \mathfrak{A} . Similarly, if $ue S$ is in SV^Σ so is $ue S'$ where S' is a generated sv-subframe or an sv-quotient of S . Therefore V_Σ (ue -closed-ness of V_Σ) rests on the case of products (sv-coproducts). \dashv

In the rest of our work here we freely use duality. When a fact transfers from sv-frames to modal algebras and vice versa we simply do that by saying "by duality".

We will return to the question of canonicity after exploring some properties of sv-frames and proving some results that hold for ultrafilter extensions in particular.

Proposition 2.19. *Suppose \mathbb{C} and \mathbb{D} are dually equivalent categories. That is, there are contravariant functors F , and G such that*

$$F : \mathbb{C} \rightarrow \mathbb{D} \quad \text{and} \quad G : \mathbb{D} \rightarrow \mathbb{C}$$

such that for all objects C in \mathbb{C} , and D in \mathbb{D}

$$F(G(D)) \simeq D \quad \text{and} \quad G(F(C)) \simeq C.$$

Let L be a limit of a cone $\{C_i\}_{i \in I}$ in \mathbb{C} then $F(L)$ is the colimit of the dual diagram (sink) of objects $\{F(C_i)\}_{i \in I}$ in \mathbb{D} and vice versa.

Proof The proof is just a routine inspection of the diagrams for limits and colimits. The image of a cone under a contravariant functor is a sink and vice versa. Uniqueness of the maps into (out of) a limit (colimit) is also preserved, since each object in either one of the two categories is isomorphic to image of an object in the other category under the corresponding functor. ⊣

Remark This proposition has numerous applications. For instance an initial object

turns into a final object in the dual category, and vice versa. What is of particular interest to us is the correspondence between the product in the category of modal algebras and the coproduct in the dual category of sv-frames. We will see the application of this fact later.

In the particular duality between MA and SV there are particular properties that we are going to use. The following is a list of properties that are essential for establishing the results in the next section. Our reference is [BRV]. In the rest of our discussion we shall use the following notations. But first let us introduce a definition.

The following properties are either obvious or have been introduced before. Here we list them for future reference.

1. If S is an sv-frame S^* is the modal algebra of clopens of S .
2. If \mathfrak{A} is a modal algebra \mathfrak{A}_+ is the ultrafilter frame of \mathfrak{A} , which is an sv-frame.
3. If $f : S_0 \rightarrow S_1$ is a sv-frame morphism $f^* : S_1^* \rightarrow S_0^*$ is the dual modal homomorphism between the corresponding modal algebras.
4. If $f : \mathfrak{A} \rightarrow \mathfrak{B}$ is a modal homomorphism, $f_+ : \mathfrak{B}_+ \rightarrow \mathfrak{A}_+$ is the corresponding sv-frame morphism.
5. If $f : F_0 \rightarrow F_1$ is a frame morphism $f^+ : F_1^+ \rightarrow F_0^+$ is the dual modal homo-

morphism between the corresponding modal algebras. Following the notation $\mathcal{Cm} \mathfrak{A} = (\mathfrak{A}_+)^+$ and $\mathfrak{ue} F = (F^+)_+$ we also adopt the following notations.

6. If $f : F_0 \rightarrow F_1$ is a frame morphism $\mathfrak{ue} f : \mathfrak{ue} F_0 \rightarrow \mathfrak{ue} F_1$ is the map $\mathfrak{ue} f = (f^+)_+$.
7. If $f : \mathfrak{A} \rightarrow \mathfrak{B}$ is a modal homomorphism, $\mathcal{Cm} f$ is the modal homomorphism $(f_+)^+$.

Fact 2.4. *The following hold for all modal algebras and all sv-frames. C.f. [BRV] for details.*

1. *If $f : S_0 \rightarrow S_1$ is an injective (surjective) sv-frame morphism $f^* : S_1^* \rightarrow S_0^*$ is a surjective (injective) modal homomorphism between the corresponding modal algebras.*
2. *If $f : \mathfrak{A} \rightarrow \mathfrak{B}$ is an injective (surjective) modal homomorphism, $f_+ : \mathfrak{B}_+ \rightarrow \mathfrak{A}_+$ is a surjective (injective) sv-frame morphism. It follows immediately that the following also hold.*
3. *If $f : S_0 \rightarrow S_1$ is an injective (surjective) sv-frame morphism $\mathfrak{ue} f : \mathfrak{ue} S_0 \rightarrow \mathfrak{ue} S_1$ is also an injective (surjective) frame morphism.*
4. *If $f : \mathfrak{A} \rightarrow \mathfrak{B}$ is an injective (surjective) homomorphism, $\mathcal{Cm} f : \mathcal{Cm} \mathfrak{A} \rightarrow \mathcal{Cm} \mathfrak{B}$ is also an injective (surjective) homomorphism.*

3 Theory Maps and Their Properties

3.1 Classification of sv-Theory Maps

The following facts are easy consequences of the definition of an sv-frame.

1. Any closed generated subframe of an sv-frame S , is a generated sv-subframe of S .
2. Any generated sv-subframe of S is closed.
3. The image of an sv-frame S under an sv-morphism into an sv-frame S' is a generated sv-subframe of S' .
4. The inverse image of an sv-frame S' under an sv-frame morphism from S into S' is a generated sv-subframe of S .

At this point we are in a position to prove a theorem classifying the set of bounded frame morphisms from an sv-frame \mathfrak{F}^k . This allows us to translate statements about validity of modal logics on sv-frames into statements about the range of continuous bounded frame morphisms from those sv-frames. We will elaborate on this point after proving the theorem.

Theorem 3.1. (sv-Theory Classification theorem) *If S is an sv-frame and f a function*

$$f : S \longrightarrow \mathfrak{F}^k$$

93

then

f is an sv-morphism iff $f = T_\mu$ for some $\mu \in SVal(S)$.

Proof Let μ be an sv-valuation on S and T_μ the theory map it defines. We refer to the accessibility relations on S and \mathfrak{F}^k , by R and R_\diamond , respectively.

We claim that T_μ is a continuous bounded frame morphism. Firstly, by definition

$$T_\mu^{-1}[\hat{p}] = \mu(p),$$

where \hat{p} is a basis element of the topology on \mathfrak{F}^k . Since μ is an sv-valuation $T_\mu^{-1}[\hat{p}]$ is a clopen subset of S , and T_μ is continuous.

It is obvious that T_μ preserves the relation on S . To prove that the μ -theory map is a bounded morphism, assume $a \in |S|$, and there is a maximal consistent set of formulae, say Γ_1 , in \mathfrak{F}^Σ such that:

$$R_\diamond T_\mu(a) \Gamma_1$$

Let $\Gamma_0 = T_\mu(a)$. For any ϕ :

$$\diamond \phi \in \Gamma_0 \text{ if } \phi \in \Gamma_1 \quad (\text{by the definition of } R_\diamond)$$

Let $\{\phi_i\}_{i \in \mathbb{N}}$ be an enumeration of formulae in Γ_1 , and define a subset Γ_2 of Γ_1 , as follows:

$$\Gamma_2 = \{\psi_i\}_{i \in \mathbb{N}} = \left\{ \bigwedge_{j=1}^{j=i} \phi_j \right\}_{i \in \mathbb{N}}.$$

Since Γ_1 is maximally consistent it is obvious that Γ_2 is a subset of Γ_1 which is logically equivalent to Γ_1 . By the definition of R_\diamond $\Diamond\psi_i \in \Gamma_0$, if $\psi_i \in \Gamma_2$. Since $\Gamma_0 = T_\mu(a)$, for any $i \in \mathbb{N}$ there is $b \in |S|$ such that:

$$Rab \text{ and } S, \mu, b \Vdash \psi_i$$

For any i let

$$B_i = \{b \mid S, \mu, b \Vdash \psi_i\} \cap R[a].$$

or

$$B_i = R[a] \cap \mu(\psi_i).$$

Obviously for any i , $\mu(\psi_i)$ is closed (clopen). Therefore B_i is closed for all i 's. It is also obvious by the definition of ψ_i that $B_j \subseteq B_i$ if $i \leq j$.

Finally, since

$$S, \mu, a \Vdash \Diamond\psi_i \quad \text{for any } \psi_i \text{ in } \Gamma_2,$$

it follows, from the definition of truth, that B_i is nonempty for all i .

Hence the sequence of sets

$$\{B_i\}_{i \in \mathbb{N}}$$

is a nested sequence of nonempty closed sets with finite intersection property, in a compact Hausdorff space. Therefore

$$\bigcap_{i \in \mathbb{N}} B_i \neq \emptyset.$$

By definition, this implies the existence of a node $b \in |S|$, accessible from a , such that for any $\psi_i \in \Gamma_2$ (hence for any $\psi_i \in \Gamma_1$), we have:

$$S, \mu, b \Vdash \psi_i .$$

That is,

$$\Gamma_1 \subseteq T(S, \mu, b) .$$

Because Γ_1 is maximally consistent it follows that

$$T(S, \mu, b) = \Gamma_1 ,$$

or

$$T_\mu(b) = \Gamma_1 .$$

The theory map T_μ is therefore shown to be a continuous bounded frame morphism.

Conversely, assume f is an sv-morphism

$$f : S \longrightarrow \mathfrak{F}^k .$$

We define μ , an sv-valuation on S , by

$$\mu(p) = f^{-1}[\hat{p}] ,$$

and prove that f is the theory map defined by μ .

Since f is continuous and \hat{p} is a clopen basis of \mathfrak{F}^K , μ is an sv-valuation. We use induction to show that for any $a \in |S|$ and any modal formula ϕ

$$\phi \in T_\mu(a) \quad \text{iff} \quad \phi \in f(a) .$$

The statement is obvious for atomic formulae by definition. The inductive steps for $\neg\phi$ and $\phi \wedge \psi$ assuming the statement holds for ϕ and ψ are readily verified. So assume the statement holds for any $b \in |S|$ and ϕ . That is, assume for any a in S

$$\phi \in T_\mu(a) \quad \text{iff} \quad \phi \in f(a) .$$

Under this assumption we prove that for an arbitrary node a in S

$$\Diamond\phi \in T_\mu(a) \quad \text{iff} \quad \Diamond\phi \in f(a) .$$

Assume

$$\Diamond\phi \in T_\mu(a) ,$$

or equivalently

$$S, \mu, a \Vdash \Diamond\phi .$$

By the definition of truth

$$\exists b R a b \quad \text{and} \quad S, \mu, b \Vdash \phi .$$

or equivalently,

$$\exists b R a b \quad \text{and} \quad \phi \in T_\mu(b) .$$

By induction hypothesis this is equivalent to

$$\exists b R a b \quad \text{and} \quad \phi \in f(b) ,$$

which implies

$$\Diamond\phi \in f(a) .$$

This is true because f preserves R and consequently by the definition of R_\Diamond

$$\text{if } \phi \in f(b) \text{ then } \Diamond\phi \in f(a) .$$

Conversely, assume $\Diamond\phi \in f(a)$. By the definition of R_\Diamond

$$\exists \Gamma \ R_\Diamond f(a)\Gamma \quad \text{and} \quad \phi \in \Gamma .$$

Since f is bounded, there is some b in S such that $\Gamma = f(b)$. Hence ϕ belongs to $f(b)$, which implies ϕ belongs to $T_\mu(b)$, by induction hypothesis. So far we know that b is a node accessible from a and $\phi \in T_\mu(b)$, or equivalently $S, \mu, b \Vdash \phi$.

Therefore the definition of the truth implies

$$S, \mu, a \Vdash \Diamond\phi ,$$

or equivalently,

$$\Diamond\phi \in T_\mu(a) ,$$

as required. ⊢

Next we consider some corollaries of the Classification Theorem above. Starting with a functional description of validity.

Corollary 3.2. *For any sv-frame S and any modal logic Σ , we have*

$$S \Vdash_{\text{sv}} \Sigma \text{ iff for any sv-theory map } T_\mu, T_\mu[S] \subseteq \mathfrak{F}^\Sigma.$$

This actually holds at the level of sv-valuations and individual nodes. That is, for any μ in $\text{SVal}(S)$ and any a in $|S|$

$$S, \mu, a \Vdash \Sigma \text{ iff } T_\mu(a) \in \mathfrak{F}^\Sigma.$$

Proof: Note that for any modal formula ϕ

$$T_\mu(a) \in \hat{\phi} \text{ iff } \phi \in T_\mu(a) \text{ iff } S, \mu, a \Vdash \phi,$$

and that

$$\phi \in \mathfrak{F}^\Sigma \text{ iff } \mathfrak{F}^\Sigma \subseteq \hat{\phi}, \text{ and } \neg\phi \cap \mathfrak{F}^\Sigma = \emptyset.$$

So it is obvious that for any ϕ in Σ

$$\text{if } S, \mu, a \Vdash \neg\phi \text{ then } T_\mu(a) \in \neg\hat{\phi}.$$

Since \mathfrak{F}^Σ is a closed subset of the compact Hausdorff space \mathfrak{F}^κ ,

$$T_\mu(a) \in \mathfrak{F}^\Sigma \text{ provided that for all } \phi, \hat{\phi} \cap \mathfrak{F}^\Sigma \neq \emptyset \text{ if } T_\mu(a) \in \hat{\phi}.$$

This proves the claim for individual nodes and sv-valuations. The claim on validity follows by definition. \dashv

Corollary 3.3. (sv-Coproduct Decomposition) *Any sv-frame S is sv-equivalent to an sv-coproduct of a family*

$$\{S_\mu\}_{\mu \in \text{SVal}(S)}$$

where each S_μ is a quotient sv-frame of S . Moreover there is a largest generated subframe of \mathfrak{F}^k determined by a modal logic Σ such that each S_μ is a generated sv-subframe of \mathfrak{F}^Σ .

Proof: Suppose $T_{sv}(S) = \Sigma$. This implies for all a in S and all μ in $SVal(S)$

$$T_\mu(a) \subseteq \mathfrak{F}^\Sigma,$$

by corollary 3.2 above. Since for each sv-valuation of S , T_μ is continuous and bounded $T_\mu[S]$ is a closed generated subframe of \mathfrak{F}^Σ which implies

$$T_\mu[S] \text{ is a generated sv-subframe of } \mathfrak{F}^\Sigma.$$

On the other hand

$$T_\mu(S) = T_{\bar{\mu}}(S_\mu),$$

where

$$\bar{\mu}(p) = \{x \in S_\mu \mid x = T_\mu(a) \text{ and } a \in \mu(p)\} = T_\mu[\mu(p)].$$

This proves

$$T_{sv}(S) = \bigcap_{\mu \in SVal(S)} T_\mu(S) = \bigcap_{\mu \in SVal(S)} T_{\bar{\mu}}(S_\mu) = T_{sv}\left(\bigoplus_{SVal(S)}^{sv} S_\mu\right),$$

or

$$S \equiv \bigoplus_{SVal(S)}^{sv} S_\mu.$$

Obviously each S_μ is a quotient of S . ⊢

Remark The result we just proved is not the dual of the well-known algebraic fact that

any algebra is a subalgebra of a product of its subdirectly irreducible subalgebras. We do not associate any specific properties to the quotients but each quotient is defined in an almost explicit manner. In contrast there is no claim in this corollary about S being a quotient of the given sv-coproduct.

Proposition 3.4. *Given theory maps T_μ and T_ν , induced by sv-valuation μ and ν , respectively the following is obviously true*

$$T_\mu = T_\nu \quad \text{iff} \quad T_\mu^{-1}[\mu_c(p)] = T_\nu^{-1}[\nu_c(p)], \quad \text{for any atomic proposition } p.$$

Proof Note that both the domain and the range of an sv-theory map are compact Hausdorff spaces. Hence any two continuous functions between the corresponding Stone spaces are equal if their inverse images of all clopens are equal. This is true, because any point is the intersection of all clopens that include it. The proposition above simply states that once bounded-ness is assumed equality of inverse images over a subset of the clopen basis of \mathfrak{F}^Σ results in equality of sv-theory maps. This in turn is true as these maps are theory maps. Any two valuations that agree on all propositions are equal. Propositions correspond to a subset of all clopens. That is, those that are inverse images of a clopen of the form $\mu_c(p)$, where p is an atomic proposition. ⊣

3.2 Hennessy-Milner Property and sv-Frames

As mentioned before we mainly deal with Kripke frames, not models. Yet it is worth taking a detour to see a special property of sv-models, models that are obtained from an sv-frame by fixing an sv-valuation.

Definition 3.2.1. (sv-Models) A model $\langle S, \mu \rangle$, is called an sv-model provided that S is an sv-frame and $\mu \in \text{SVal}(S)$.

Definition 3.2.2. (Hennessy-Milner Property) A class M of models has the Hennessy-Milner property provided that for all \mathfrak{M} , and \mathfrak{N} , in M

$$\forall w \in |\mathfrak{M}| \forall u \in |\mathfrak{N}| \quad [T(\mathfrak{M}, w) = T(\mathfrak{N}, u) \quad \text{iff} \quad \mathfrak{M}, w \leftrightarrow \mathfrak{N}, u].$$

Proposition 3.5. The class of sv-models has the Hennessy-Milner property.

The idea of the proof is almost exactly the same as the one used in (3.1). Using compactness of the underlying frame we find an appropriate node to assign to a given node of a either one of the two models.

Proof Assume \mathfrak{M} and \mathfrak{N} are two sv-models, and define a binary relation $B \subseteq |\mathfrak{M}| \times |\mathfrak{N}|$ by

$$B = \{(x, y) \mid T(\mathfrak{M}, x) = T(\mathfrak{N}, y)\}.$$

We prove that B is a bisimulation relating any two nodes of \mathfrak{M} and \mathfrak{N} that have the same theory. Suppose w and u , are given such that

$$T(\mathfrak{M}, w) = T(\mathfrak{N}, u).$$

By the definition of B , $\langle w, u \rangle \in B$. We need to show that these two nodes are bisimilar.

Obviously w and u agree on all propositional variables. By symmetry of the definition, we just need to check the bisimilarity condition in one direction. So suppose $R_0 ww'$, for $w' \in |\mathfrak{M}|$. We show that there is $u' \in |\mathfrak{N}|$, such that $R_1 uy$. Here R_0 and R_1 are the accessibility relations of \mathfrak{M} and \mathfrak{N} , respectively.

Let $\{\phi_i\}_{i \in \mathbb{N}}$ be the set of formulae in $T(\mathfrak{M}, w')$. So

$$\forall i \in \mathbb{N} \quad \mathfrak{M}, w \Vdash \Diamond \phi_i.$$

This is equivalent to

$$\forall i \in \mathbb{N} \quad \mathfrak{N}, u \Vdash \Diamond \phi_i.$$

So if we let

$$A_i = \{v \in |\mathfrak{N}| \mid (R_1 uv) \ \& \ (\mathfrak{N}, v \Vdash \bigwedge_{j \leq i} \Diamond \phi_j)\},$$

the set $\{A_i\}_{i \in \mathbb{N}}$ is a nested sequence of compact sets in a compact Hausdorff space, with finite intersection property (exactly as in the Embedding lemma). Therefore there is a node u' in \mathfrak{N} such that

$$u' \in \bigcap_{i \in \mathbb{N}} A_i.$$

This implies

$$T(\mathfrak{M}, w') = T(\mathfrak{N}, u') \text{ and hence } R_1 u u'.$$

Therefore

$$\exists u' R_1 u u' \text{ and } (w', u') \in B,$$

as required. \dashv

We are going to prove a corollary to the proposition we just proved. The proof is quite straightforward but we need some definitions.

In proving semantics equivalence lemma (lemma 2.9) we defined an sv-valuation μ_+ of \mathfrak{A}_+ for any valuation on the modal algebra \mathfrak{A} . On the other hand for any a in \mathfrak{A} the basis element in the Stone topology on \mathfrak{A}_+ that (uniquely) corresponds to a was denoted by $\langle a \rangle$, c.f. 2.5. Therefore given a Kripke frame F , a valuation $\mu \in \text{Val}(F)$, and a modal formula ϕ , $\mu(\phi)$ is an element of the modal algebra of subsets of F . The basis element in the Stone topology of $(F^+)_+ = \mathfrak{u} \mathfrak{e} F$ that (uniquely) corresponds to $\mu(\phi)$ is denoted by $\langle \mu(\phi) \rangle$. This justifies the following definition.

Definition 3.2.3. *Given a valuation μ on a Kripke frame F the is unique sv-valuation on $\mathfrak{u} \mathfrak{e} F$ that corresponds to μ is denoted by $\langle \mu \rangle$.*

Our earlier discussions makes it clear that $\langle \mu \rangle$ satisfies the following condition for any

modal formula.

$$\langle \mu \rangle(\phi) = \langle \mu(\phi) \rangle = \{U \in \mathfrak{u}\mathfrak{e} F \mid \mu(\phi) \in U\}.$$

In fact by inspection of the definition of ultrafilter extension it is obvious that any sv-valuation on $\mathfrak{u}\mathfrak{e} F$ is of this form.

Note that the semantics equivalence lemma was stated for a valuation μ on a modal algebra \mathfrak{A} hence denoting the basis element generated by $\mu(\phi)$ by $\mu_+(\phi)$. Here, in contrast, μ is a valuation on F . So in a way we could have used $(\mu^+)_+$ or even $\mathfrak{u}\mathfrak{e} \mu$ to denote the unique sv-valuation that corresponds to μ . However the notation we have chosen is more faithful to the notation used in The Stone Representation Theorem.

We can now state and prove the following corollary.

Corollary 3.6. *Given Kripke frames F , $\mu \in \text{Val}(F)$, $a \in |F|$, F' , $\mu' \in \text{Val}(F')$, and $a' \in |F'|$, we have*

$$\mathfrak{u}\mathfrak{e} F, \langle \mu \rangle, \uparrow a \iff \mathfrak{u}\mathfrak{e} F', \langle \mu' \rangle, \uparrow a'$$

iff

$$T_\mu(F, a) = T_{\mu'}(F', a').$$

Proof By the definition of $\langle \mu(\phi) \rangle$, for any a in F

$$a \in \mu(\phi) \iff \mu(\phi) \in \uparrow a \iff \uparrow a \in \langle \mu(\phi) \rangle = \langle \mu \rangle(\phi).$$

Hence

$$F, \mu, a \Vdash \phi \quad \text{iff} \quad \mathfrak{u}e F, \langle \mu \rangle, \uparrow a \Vdash \phi,$$

or equivalently

$$T_\mu(F, a) = T_{\langle \mu \rangle}(\mathfrak{u}e F, \uparrow a).$$

Similarly

$$T_{\mu'}(F', a') = T_{\langle \mu' \rangle}(\mathfrak{u}e F, \uparrow a').$$

On the other hand since $(\mathfrak{u}e F, \langle \mu \rangle)$ and $(\mathfrak{u}e F', \langle \mu' \rangle)$ are sv-models proposition 3.5 above implies

$$\mathfrak{u}e F, \langle \mu \rangle, \uparrow a \Leftrightarrow \mathfrak{u}e F', \langle \mu' \rangle, \uparrow a'$$

iff

$$T_{\langle \mu \rangle}(\mathfrak{u}e F, \uparrow a) = T_{\langle \mu' \rangle}(\mathfrak{u}e F', \uparrow a').$$

Combining the equivalences above establishes the result. \dashv

Corollary 3.7. *For any Kripke frame F and any sv-valuation μ on $\mathfrak{u}e F$*

$$\mathfrak{u}e F, \mu \Vdash_{sv} \phi \quad \text{iff} \quad \forall a \in F \quad \mathfrak{u}e F, \mu, \uparrow a \Vdash_{sv} \phi.$$

Consequently

$$\mathfrak{u}e F, \Vdash_{sv} \phi \quad \text{iff} \quad \forall a \in F \quad \mathfrak{u}e F, \uparrow a \Vdash_{sv} \phi.$$

In other words to determine whether a modal formula ϕ is valid under an sv-valuation μ , or to check whether ϕ is sv-valid on $\mathfrak{u}e F$, we need only check those nodes in $\mathfrak{u}e F$ that

are principal ultrafilters of subsets of $|F|$.

Proof Note that the sv -valuation μ on $ue F$ is of the form $\mu = \langle \nu \rangle$ where ν is a valuation on F . However as in the proposition we just proved

$$ue F, \langle \nu \rangle \Vdash_{sv} \phi \quad \text{iff} \quad F, \nu \Vdash \phi,$$

On the other hand, again by the proposition,

$$F, \nu \Vdash \phi \quad \text{iff} \quad \forall a \in |F| \quad F, \nu, a \Vdash \phi \quad \text{iff} \quad \forall a \in |F| \quad ue F, \langle \nu \rangle, \langle a \rangle \Vdash \phi.$$

⊢

4 Properties of SV-Frames and Ultrafilter Extension

4.1 Natural Injections and Surjections

Definition 4.1.1. (Weak Morphism) *A weak morphism between Kripke frames*

$$F = \langle F, R_F \rangle \quad \text{and} \quad G = \langle G, R_G \rangle ,$$

is a map, $f : F \rightarrow G$, for which the following properties holds, for any two nodes x and y of f .

$$f(x) R_G f(y) \quad \text{if} \quad x R_F y .$$

Differently put, a weak morphism is one that preserves the accessibility relation, or is a relational homomorphism.

Definition 4.1.2. (Natural Injection) *For any Kripke frame $F = \langle F, R \rangle$, the map*

$$\eta : F \rightarrow \mathfrak{u}F$$

is defined by

$$\eta(x) = \{A \subseteq F \mid x \in A\} = x \uparrow .$$

That is, $\eta(x)$ is the principal ultrafilter generated by x .

Given any Kripke frame F , the natural injection $\eta : F \rightarrow \mathfrak{u}F$, is easily seen to be a weak morphism, but not a bounded morphism in general. However the natural injection η reflects the relation as well as preserving it. That is,

$$x R y \quad \text{iff} \quad x \uparrow R^{\mathfrak{u}F} y \uparrow .$$

This can be seen by unfolding the definition

$$\begin{aligned}
& x \uparrow R^{ue} y \uparrow \\
& \text{iff} \\
& \forall A \in y \uparrow \text{ (i.e. } y \in A) \{z \mid R[z] \cap A \neq \emptyset\} \in x \uparrow \text{ (i.e. } x \in \{z \mid R[z] \cap A \neq \emptyset\}) \\
& \text{iff} \\
& \forall A \in y \uparrow R[x] \cap A \neq \emptyset \\
& \text{iff} \\
& \forall A (R[x] \cap A \neq \emptyset \text{ if } y \in A) \\
& \text{iff} \\
& x R y .
\end{aligned}$$

Given any map f between Kripke frames, one can define the map $ue f$, exactly as defined for bounded morphisms. That is, by taking the double dual of f . The following lemma helps proves property of this map.

Lemma 4.1. *For any function $f : X \longrightarrow Y$, the function $f^+ : P(Y) \longrightarrow P(X)$ defined by*

$$f^+(B) = f^{-1}[B], \text{ where } B \subseteq Y,$$

is a homomorphism of boolean algebras.

Proof The inverse image of a function preserves union, intersection, and complements of sets. The inverse image of Y is X and the inverse image of the empty set is empty.

⊢

The Stone representation theorem for boolean algebras applied to the inverse function in the previous lemma implies the following.

Corollary 4.2. *Given a function $f : X \rightarrow Y$, and its corresponding boolean homomorphism $f^+ : P(Y) \rightarrow P(X)$ as in the lemma above, the function $ue f : ue X \rightarrow ue Y$, defined by*

$$ue f(U) = \{B \subseteq Y \mid f^+[B] \in U\},$$

is a morphism of the category of Stone spaces.

Proof As mentioned above, this lemma is a direct consequence of the Stone representation theorem. Here is a direct proof. Let B be a subset of Y , and U an ultrafilter of subsets of X . By the definition of $ue f$

$$B \in ue f(U) \quad \text{iff} \quad f^{-1}[B] \in U.$$

Hence if $\langle B \rangle$ is an arbitrary clopen of $ue Y$ and $A = f^{-1}[B]$ we have

$$(ue f)^{-1}[\langle B \rangle] = \langle f^{-1}[B] \rangle = \langle A \rangle,$$

and $\langle A \rangle$, is a clopen of $ue X$ as required.

⊢

Lemma 4.3. *Given Kripke frames X and Y , natural injections η and η' , and a function, $f : X \longrightarrow Y$, the following diagram commutes.*

$$\begin{array}{ccc} X & \xrightarrow{\eta} & \mathfrak{u}\mathfrak{e} X \\ f \downarrow & & \downarrow \mathfrak{u}\mathfrak{e} f \\ Y & \xrightarrow{\eta'} & \mathfrak{u}\mathfrak{e} Y \end{array}$$

Proof Let x be an arbitrary element of X and B a subset of Y . By definition we have

$$B \in \mathfrak{u}\mathfrak{e} f(\eta(x)) \quad \text{iff} \quad f^{-1}[B] \in \eta(x) \quad \text{iff} \quad x \in f^{-1}[B] \quad \text{iff} \quad f(x) \in B \quad \text{iff} \quad B \in \eta'(f(x)).$$

This establishes the result. ⊢

Lemma 4.4. *Let $F = \langle W_0, R_0 \rangle$ and $G = \langle W_1, R_1 \rangle$ be Kripke frames. If $f : F \longrightarrow G$, is a weak morphism of Kripke frames, the map $\mathfrak{u}\mathfrak{e} f : \mathfrak{u}\mathfrak{e} F \longrightarrow \mathfrak{u}\mathfrak{e} G$, is a continuous weak morphism between the corresponding ultrafilter extensions (equipped with Stone topology).*

Proof: Continuity of $\mathfrak{u}\mathfrak{e} f$ follows from the Stone Representation Theorem, as mentioned in the corollary above. To show that $\mathfrak{u}\mathfrak{e} f$ is a weak morphism, let P and Q ultrafilters such that

$$U R_0^{\mathfrak{u}\mathfrak{e}} V.$$

Let $P = \mathfrak{u}\mathfrak{e} f(U)$ and $Q = \mathfrak{u}\mathfrak{e} f(V)$. We will prove

$$P R_1^{\mathfrak{u}\mathfrak{e}} Q.$$

To that effect, we must show that for any B in Q the set

$$m_1(B) = \{x \mid R_1[x] \cap B \neq \emptyset\},$$

is in P . Let B be in Q , and let $A = f^{-1}[B]$. A is in U by the definition of $u \epsilon f$. $B' = f[A]$ is in P as well, since $f^{-1}[B'] = A$.

Obviously

$$m_1(B') \subseteq m_1(B)$$

and as f preserves the relation

$$m_1(f[A]) \subseteq f[m_0(A)] .$$

Since A is in V

$$m_0(A) \in U .$$

Since

$$m_0[A] \subseteq f^{-1}[f[m_0(A)]] ,$$

we conclude

$$f^{-1}[f[m_0(A)]] \in U .$$

Hence

$$f[m_0(A)] \in P .$$

This in turn implies $m_1(B) \in P$, as

$$f[m_0(A)] = m_1(B') \subseteq m_1(B) ,$$

which is enough to establish the result as we discussed above. \dashv

Remark If the map f in the previous lemma is a bounded frame morphism then $u \epsilon f$ is easily seen to be a bounded frame morphism. However the reverse is true under

special conditions. For example for sv-frames, if $u \vDash f$ is a bounded frame morphism, and f is continuous under the Stone topology on corresponding frames, then f must be bounded as well. The reverse statement of the lemma for weak morphisms is easy to prove without any extra assumptions. That is, if $u \vDash f$ is a weak morphism between ultrafilter extensions of two Kripke frames, f is a weak morphism between those frames.

4.2 sv-Frames as Modal Frames

Next, we consider structures called "Normal modal frames", in [GMM]. As we will see below sv-frames could be seen as a special case of a wider class of frames.

Definition 4.2.1. ([Normal] Modal Frame) A (normal) modal frame \mathfrak{S} , is a structure $\mathfrak{S} = \langle S, R, A \rangle$, where $\langle S, R \rangle$ is a Kripke frame and $A \subseteq \mathcal{P}(S)$ is closed under boolean operations, and also the operation $\langle R \rangle$ which is defined, for each $X \subseteq S$ by

$$\langle R \rangle X = \{y \in S \mid y R x \text{ for some } x \in X\}.$$

The following definition is a special case of a definition in [GBQC], where frames with several relations are considered.

Definition 4.2.2. (Topological τ -structure) Given a modal frame $\mathfrak{S} = \langle S, R, A \rangle$, the set A is a basis for a topology τ . \mathfrak{S} is called a topological τ -structure provided that the following two conditions are satisfied.

1. For any open set U in τ the sets $\langle R \rangle(U)$ and $[R](U) (= (\langle R \rangle(U^c))^c)$ are open.

2. For any node $x \in S$ the set $R^{-1}(x) = \{y \in S \mid y R x\}$ is a closed subset of S .

Any sv-frame $\mathfrak{S} = \langle S, R \rangle$ can be considered as a modal frame, Where B is the set of clopens of S , the basis of the Stone topology of \mathfrak{S} . That is,

$$\langle S, R, B \rangle$$

satisfies the definition of normal modal frame. This is true by the equivalence between DGF and SV, as discussed in 78.

Proposition 4.5. *An sv-frame $\mathfrak{S} = \langle S, R \rangle$, considered as a modal frame*

$$\mathfrak{S} = \langle S, R, B \rangle \text{ (} B \text{ is the set of clopens of } \tau \text{),}$$

is a topological τ -structure, where τ is the Stone topology of \mathfrak{S} .

Proof As seen on page 49 above, following [KKV] one can consider an sv-frame \mathfrak{S} as a coalgebra of the Vietoris functor on Stone, where $\mathfrak{S} = \langle S, R, B \rangle$ corresponds to the coalgebra

$$\gamma : S \longrightarrow VS,$$

where

$$\gamma(x) = R[x] = \{y \mid y R x\}.$$

As discussed in [KKV] the basis of the Stone topology on $\mathbb{V}S$ is the following collection of subsets of S :

$$B(\tau) = \{F \subseteq X \mid F \cap U \neq \emptyset\}_{U \in \tau} \cup \{F \subseteq X \mid F \subseteq U\}_{U \in \tau}.$$

For any open subset U of S , We can write

$$\begin{aligned} \langle R \rangle(U) &= \{x \mid \gamma(x) \cap U \neq \emptyset\} \\ &= \{x \mid \gamma(x) \in \{K \mid K \text{ is a closed subset of } S \text{ and } K \cap U \neq \emptyset\}\} \\ &= \gamma^{-1}[\{K \mid K \text{ is a closed subset of } S \text{ and } K \cap U \neq \emptyset\}]. \end{aligned}$$

Since the set

$$\{K \mid K \text{ is a closed subset of } S \text{ and } K \cap U \neq \emptyset\},$$

is a basic open set in the Vietoris topology on $\mathbb{V}S$, and because γ is continuous, $\langle R \rangle U$ is open.

In case of $[R]U$ we can write

$$[R]U = \{x \mid R[x] \cap U^c \neq \emptyset\}^c = \{x \mid R[x] \subseteq U\}.$$

This in turn can be written as

$$\{x \mid \gamma(x) \in \{K \mid K \subseteq S \text{ is closed and } K \subseteq U\}\} = \gamma^{-1}\{K \mid K \subseteq S \text{ is closed and } K \subseteq U\}.$$

Again by continuity of γ , $[R]U$, being the inverse image of a basic open subset of $\mathbb{V}S$, is open.

Finally, because

$$\begin{aligned} R^{-1}[x] &= \{y \mid R[y] \cap \{x\} \neq \emptyset\} \\ &= \{y \mid R[y] \subseteq \{x\}^c\}^c, \end{aligned}$$

and because $\{x\}$ is closed in S , it becomes obvious that $R^{-1}[x]$ is closed. \dashv

In the previous subsections, we considered a natural injection from a frame to its ultrafilter extension, sending each node to the principal ultrafilter it generates. This injection is not a frame morphism in general, because it might not be bounded. However as mentioned in [BRV], this injection preserves and reflects the relations of the corresponding frames.

Definition 4.2.3. Let $\mathfrak{F} = \langle F, R \rangle$, be a compact Hausdorff topological τ -structure. The natural injection of \mathfrak{F}

$$\eta : \mathfrak{F} \longrightarrow \mathbf{ue} \mathfrak{F},$$

has a continuous bounded inverse

$$\epsilon : \mathbf{ue} \mathfrak{F} \longrightarrow \mathfrak{F},$$

which we call the natural surjection onto \mathfrak{F} .

The natural surjection above is defined by sending an ultrafilter of subsets of a frame to the intersection of the closure of all of its elements.

Alternatively, we can define this natural surjection (on sv-frames) by

$$\epsilon(P) = u$$

iff

$$O \in P \text{ for any clopen } O \text{ of } \mathfrak{F} \text{ that includes } u$$

It is easily verified that ϵ has the properties specified in the definition above.

Remark: The natural surjection above is the dual of the Jónsson-Tarski embedding of boolean modal algebras.

Based on the duality between modal algebras and sv-frames we can have an alternative view of the natural surjection. Recall that the Jónsson-Tarski embedding for modal algebras

$$J : \mathfrak{A} \rightarrow \mathfrak{Cm} \mathfrak{A},$$

is defined by

$$J(a) = \langle a \rangle, \text{ where } \langle a \rangle = \{U \in \mathfrak{A}_+ \mid a \in U\},$$

where \mathfrak{A}_+ is the ultrafilter (sv) frame of \mathfrak{A} .

Given sv-frames

$$\mathfrak{F} = \langle F, R_F \rangle \text{ and } \mathfrak{G} = \langle G, R_G \rangle = ue \mathfrak{F},$$

if $\text{Clp}(\mathfrak{F}) \simeq \mathfrak{A}$, and $\text{UF}(\mathfrak{A}) \simeq \mathfrak{F}$, then $\mathfrak{G} \simeq \text{UF}(\mathfrak{Cm} \mathfrak{A})$, by the the duality between MA and SV.

By the discussions above it is now clear that the surjection defined in the following definition is a well-defined surjective sv-frame morphism.

Definition 4.2.4. *Given an sv-frame \mathfrak{F} the frame morphism and \mathfrak{A} the modal algebra of clopen subsets of \mathfrak{F} , we define*

$$\epsilon_{\mathfrak{F}} : \text{ue}\mathfrak{F} \twoheadrightarrow \mathfrak{F} ,$$

by

$$\epsilon_{\mathfrak{F}}(P) = J^+(P) = \{a \in \mathfrak{A} \mid \langle a \rangle \in P\} \text{ where } \langle a \rangle = \{u \in \mathfrak{A}_+ \mid a \in u\} ,$$

Proposition 4.6. *For an sv-frame \mathfrak{F} , the natural surjection defined in 4.2.4 above, is an inverse to the natural injection η , defined in 4.1.2.*

Proof: we have

$$\begin{aligned} \eta(\epsilon_{\mathfrak{F}}(U)) &= \{a \in \mathfrak{A} \mid \langle a \rangle \in U\} \\ &= \{a \in \mathfrak{A} \mid \{V \mid a \in V\} \in U\} \\ &= \{a \in \mathfrak{A} \mid a \in U\} = U \end{aligned}$$

(Note that clopens of \mathfrak{F} are of the form $\langle a \rangle$ for $a \in \mathfrak{A}$).

This establishes that $\epsilon_{\mathfrak{F}}$ is an inverse to the natural injection η . ⊢

Lemma 4.7. *Given compact Hausdorff topological frames*

$$\mathfrak{F} = \langle F, R_F \rangle \quad \text{and} \quad \mathfrak{G} = \langle G, R_G \rangle ,$$

and a continuous morphism, $f : \mathfrak{F} \rightarrow \mathfrak{G}$. The following diagram commutes.

$$\begin{array}{ccc} u\epsilon \mathfrak{F} & \xrightarrow{u\epsilon f} & u\epsilon \mathfrak{G} \\ \epsilon_{\mathfrak{F}} \downarrow & & \downarrow \epsilon_{\mathfrak{G}} \\ \mathfrak{F} & \xrightarrow{f} & \mathfrak{G} \end{array}$$

Proof: Let u be in $|\mathfrak{F}|$, P in $u\epsilon \mathfrak{F}$ such that $\epsilon_{\mathfrak{F}}(P) = u$, and $w = f(u)$, and let O be any open neighborhood of w in \mathfrak{G} . $w \in O$ implies

$$u \in f^{-1}[O] ,$$

and so

$$f^{-1}[O] \in P ,$$

as $f^{-1}[O]$ is an open neighborhood of u . Hence

$$O \in u\epsilon f(P) .$$

This in turn implies

$$\epsilon_{\mathfrak{G}}(u\epsilon f(P)) = w ,$$

since any arbitrary neighborhood of w is proved to be in $u\epsilon f(P)$. ⊢

Proposition 4.8. *Let S be any frame. $\eta[S]$ is a dense subset of $u\epsilon S$ with Stone topology.*

Proof Any nonempty clopen of $\mathfrak{u}\mathfrak{e} S$ can be written as

$$\langle A \rangle = \{P \in \mathfrak{u}\mathfrak{e} S \mid A \in P\},$$

where A is a nonempty subset of $|S|$. If u is an element of A

$$A \in u \uparrow,$$

and hence

$$\eta(u) = u \uparrow \in \langle A \rangle.$$

Therefore

$$\eta[S],$$

the image of S under η intersects any arbitrary clopen of $\mathfrak{u}\mathfrak{e} S$ (is a dense subset of $\mathfrak{u}\mathfrak{e} S$). ⊢

In topological terminology the lemma above shows that $\mathfrak{u}\mathfrak{e} X$ is a compactification of X , where X is given the discrete topology.

Remark Corollary 3.7 can now be proved in a much simpler way by the density of the image of η . Given a Kripke frame F , and an sv -valuation μ on $\mathfrak{u}\mathfrak{e} F$, by the definition of truth:

$$\exists U \in \mathfrak{u}\mathfrak{e} F \text{ such that } \mathfrak{u}\mathfrak{e} F, \mu, U \Vdash_{sv} \phi$$

iff

$$\mu(\phi) \neq \emptyset.$$

Because $\mu(\phi)$ is a clopen set and η has a dense image, the latter statement is equivalent to

$$\exists a \in F \quad \text{such that} \quad \eta(a) \in \mu(\phi),$$

hence equivalent to

$$\exists a \in F \quad \text{such that} \quad u \vDash F, \mu, \uparrow a \Vdash_{sv} \phi.$$

⊢

commented

$$u \vDash F, \mu, x \uparrow \Vdash \phi,$$

then

$$u \vDash F, \mu \Vdash \phi.$$

4.2.1 Two Observations

Here we consider two properties of the ultrafilter extensions. One applies to Kripke frames in general and the other is a property of Compact Hausdorff modal frames (sv-frames for our purposes).

Proposition 4.9. *Ultrafilter extension, as a functor on KF, is injective on objects.*

Proof Let F and G be Kripke frames such that $u \vDash F \simeq u \vDash G$. Since ultrafilter extension of any Kripke frame is an sv-frame, since $u \vDash = ((\)_+)^+$ and since SV is dually equivalent,

it is obvious that

$$F^+ \simeq G^+.$$

So assume that

$$h : F^+ \longrightarrow G^+,$$

is an isomorphism of modal algebras. By definition

$$|F^+| = P(|F|) \quad \text{and} \quad |G^+| = P(|G|).$$

Since h is an isomorphism the image of an atom in F^+ is an atom in G^+ . Atoms of power set algebras are obviously singletons. So define

$$h^- : F \longrightarrow G,$$

by

$$h^-(a) = b \quad \text{iff} \quad h(\{a\}) = \{b\}.$$

Obviously this is a well defined bijection between F and G . Surjectivity of h^{-1} implies that to prove that h^{-1} is a bounded frame morphism and therefore an isomorphism of Kripke frames we need only show

$$x R_F y \quad \text{iff} \quad h(x) R_G h(y).$$

Moreover by symmetry the proof is complete once we show that h^- preserves the relation on F . To this end assume x and y are nodes of F such that $x R_F y$. We should prove

$$h^-(x) R_G h^-(y).$$

note that by the definition of h^- it is obvious that

$$h(\{a\}) = \{h^-(a)\}.$$

Moreover if f_F denotes the modal operator on F^+ (similarly f_G for G^+)

$$x R_F y$$

iff

$$\{x\} \subseteq f_F(\{y\}) \quad \text{iff} \quad h(\{x\}) \subseteq h(f_F(\{y\})) \quad \text{iff} \quad \{h^-(x)\} \subseteq f_G(h(\{y\})) = f_G\{h^-(y)\}$$

iff

$$h^-(x) R_G h^-(y).$$

This completes the proof. ⊢

Our next observation justifies our lax attitude towards specifying the exact Stone topology on an sv-frame.

Proposition 4.10. *Let \mathfrak{F} be an compact Hausdorff modal frame. The quotient topology induced by the natural surjection*

$$\epsilon : ue \mathfrak{F} \longrightarrow \mathfrak{F},$$

is the same as the topology of \mathfrak{F} .

Proof This follows from a simple fact of point set topology. That is, any surjective continuous map which is also a closed (or open) map is a quotient map. C.f. [MUT].

Here is a short proof.

Let

$$f : X \longrightarrow Y$$

be a continuous surjective and closed map. Let $U = C^c$ be a subset of Y such that.

$f^{-1}[U]$ is an open subset of X . Therefore

$$(f^{-1}[U])^c = f^{-1}[U^c]$$

is closed. Since f is surjective

$$f[f^{-1}[U^c]] = U^c = C.$$

Since f is a closed map C is closed, hence U is open. By continuity of f it is obvious that $f^{-1}[U]$ is open for any open $U \subseteq Y$. Hence U is open in Y if and only if $f^{-1}[U]$ is open in X . f is therefore a quotient map. In case of the natural surjection continuity is obvious. Since closed subsets of compact spaces are compact, and continuous images of compact sets are compact (closed) under ϵ we conclude that ϵ is a closed map as well. ⊢

4.3 Coproducts in the KF and SV

Comparing Coproducts We are going to discuss the coproducts in the categories of sv-frames in contrast to that of category of Kripke frames we use the notations in a sloppy way for notational simplicity. For example

$$\coprod_i S_i,$$

is the coproduct, in KF, of the underlying kripke frames of the family of sv-frames

$$\{S_i\}_{i \in I}.$$

A more careful notation would be

$$\coprod_i |S_i|,$$

where for each i

$$|S_i| = \langle W_i, R_i \rangle$$

is the corresponding sv-frame. Each S_i comes with its own Stone topology, which we conveniently forget when we look at it as a Kripke frame. Alternatively we could look at any such sv-frame as having an attached set of subsets, the basis of the stone topology of the frame.

$$S_i = \langle W_i, R_i, A_i \rangle.$$

This amounts to considering an sv-frame as a Descriptive general frame. A notion that we will elaborate on below. As mentioned earlier the duality between descriptive general frames and modal algebras together with the obvious isomorphism between the

category of descriptive general frames and sv-frames proves the duality $SV \simeq MA^{\text{op}}$.

In our discussion on canonical varieties we mentioned that canonical varieties are always determined by canonical (normal) modal logics. We also considered the dual notion of a variety, a covariety of sv-frames. We then showed that a variety of modal algebras is canonical iff the corresponding covariety of sv-frames is $u\epsilon$ -closed. C.f section 1.7.

The following is an observation that can be used to make a connection between classes of Kripke Frames, on the one hand and varieties of modal algebras and covarieties of sv-frames on the other.

Lemma 4.11. (Coproduct Preservation) *The ultrafilter extension of a coproduct of a family of Kripke frames is isomorphic to the sv-coproduct of ultrafilter extensions of the members of the same family. That is, given a family*

$$\{F_i\}_{i \in I}$$

of Kripke frames

$$\text{ue} \left(\coprod_I F_i \right) \simeq \bigoplus_I^{\text{sv}} \text{ue} F_i .$$

Proof This is a direct corollary of the definitions of a coproduct of frames, an sv-coproduct of sv-frames, a direct product of algebras and the complex modal algebra of a frame.

For any frame F , $(\text{ue } F)^* = F^+$. Also if $F_i = \langle W_i, R_i \rangle$ and $\{F_i\}_{i \in I}$ is a class of frames

$$\left(\coprod_I F_i \right) = \langle \coprod_I W_i, \coprod_I R_i, \rangle .$$

So

$$|(\coprod_I F_i)^+| = P(\coprod_I W_i).$$

Since sequences of subsets of a disjoint family of sets, and the set of subsets of their disjoint union are isomorphic as sets in an obvious way, we conclude

$$|(\coprod_I F_i^+)| \simeq |\prod_I (F_i^+)|.$$

Similar inspection of the operations on the complex algebra of the coproducts, comparing them with the operations of the product of complex algebras of the individual frames, shows that

$$\left(\coprod_I F_i^+\right) \simeq \prod_I (F_i^+).$$

The claim now becomes obvious by the definition of sv-coproduct (note that For all F , $(ue F)^* = F^+$).

$$ue \left(\coprod_I F_i \right) \simeq \left(\left(\coprod_I F_i \right)^+ \right)_+ \simeq \left(\prod_I F_i^+ \right)_+ \simeq \left(\prod_I (ue F_i)^* \right)_+ \simeq \bigoplus_I^{sv} ue F_i .$$

⊢

In our discussion on canonical varieties we mentioned that canonical varieties are always determined by canonical (normal)logics. We also considered the dual notion of a covariety of sv-frames. We then showed that a variety of modal algebras is canonical iff the corresponding covariety of sv-frames is ue-closed. C.f section 1.7.

Finally we showed that for a covariety C of sv-frames with a class D of generators

$$C \text{ is ue - closed} \quad \text{iff} \quad ue \bigoplus_I^{sv} S_i \in C,$$

for any family $\{S_i\}_{i \in I}$ of sv-frames in D .

We need a more detailed analysis of coproducts in the category of Kripke frames and sv-frames. Recall that disjoint unions of Kripke frames are coproducts in KF , as discussed in [BRV]. We defined the notion of an sv-coproduct of sv-frames using duality, C.f. 2.9.5, and verified that it coincides with the categorical notion of a coproduct for the category SV . So what we proved in lemma 4.11 shows that forming ultrafilter extensions of frames, as a functor, preserves coproducts of families of frames. That is,

$$u\epsilon \coprod_I F_i \simeq \bigoplus_I^{sv} u\epsilon F_i .$$

On that note, to be more accurate we should specify the insertion (injection) maps of the coproducts in the lemma 4.11. By inspection of the definitions, it is obvious that

$$\langle \bigoplus_I^{sv} u\epsilon F_i, u\epsilon \alpha_i \rangle_{i \in I} \text{ is the sv-coproduct of the family } \{u\epsilon F_i\}_{i \in I} ,$$

if

$$\langle \coprod_I F_i, \alpha_i \rangle_{i \in I} \text{ is the coproduct of the family } \{F_i\}_{i \in I} .$$

Since the category SV is a subcategory of KF , for any family of sv-frames, we can form two distinct coproducts, one in KF and one in SV . It is readily seen that these two notions are, in fact, distinct, except for finite families of sv-frames. It is known that the KF coproduct of an infinite family of sv-frames is never an sv-frame. C.f. [GMM].

We will contrast these two coproducts, and focus on special families of sv-frames to prove a series of lemmas that we employ to answer the question of countable canonicity.

Notations: For the rest of this chapter all frames are sv-frames, unless specified otherwise. I is an arbitrary infinite index set. Based on the previous comment on coproducts of infinite families, we know that the Kripke coproduct of an infinite family of frames is not an sv-frame, but the ultrafilter extension of any frame is an sv-frame. As one can easily see below, except for (Kripke) coproducts of infinite families, all the other frames formed during our constructions are sv-frames.

We use the following notations

$$\coprod_I F \text{ instead of } \coprod_I F_i \quad \text{and} \quad \bigoplus_I^{sv} S \text{ instead of } \bigoplus_I^{sv} S_i,$$

whenever $F_i \simeq F$ and $S_i \simeq S$, for all $i \in I$. Also, for the rest of this chapter

$$\alpha_i : S_i \rightarrow \coprod_I S_i \quad \text{and} \quad \beta_i : S_i \rightarrow \bigoplus_I^{sv} S_i$$

are the injection maps of the corresponding coproducts.

Finally, whenever a morphism is introduced in a lemma it is the same morphism for the rest of the section, and we refer to it by the same name.

Lemma 4.12. *Suppose $\{S_i\}_I$ is a family of sv-frames. Then there is a unique frame morphism, in KF ,*

$$m : \coprod_I S_i \longrightarrow \bigoplus_I^{sv} S_i.$$

such that for all i the following diagram commutes.

$$\begin{array}{ccc}
\coprod_i S_i & \xrightarrow{m} & \bigoplus_i^{sv} S_i \\
\alpha_i \uparrow & \nearrow \beta_i & \\
S_i & &
\end{array}$$

Proof: Existence and uniqueness of m is obvious by the characteristic property of coproducts. \dashv

Corollary 4.13. *For a family $\{S_i\}_{i \in I}$ of sv-frames the following diagram commutes, and $ue m$ is the unique map with this property.*

$$\begin{array}{ccc}
ue \coprod_i S_i & \xrightarrow{ue m} & ue \bigoplus_i^{sv} S_i \\
ue \alpha_i \uparrow & \nearrow ue \beta_i & \\
ue S_i & &
\end{array} \tag{12}$$

Proof: Since ue is a functor, the commutativity of the diagram is obvious. The uniqueness is a consequence of the fact $\bigoplus_i^{sv} ue S_i$ is the sv-coproduct of $\{ue S_i\}_{i \in I}$. \dashv

Lemma 4.14. *The morphism m in lemma 4.12 is injective.*

Proof: This fact becomes obvious once we verify that images of S_i under injection maps β_i are disjoint for distinct indices. To see this consider the structure

$$\bigoplus_i^{sv} S_i \oplus^{sv} S_{i_0} \oplus^{sv} S_{i_1}$$

and a family of sv-morphisms

$$\{f_i\}_{i \in I},$$

where,

$$f_i = \beta_i, \text{ for } i \neq i_0, i_1,$$

and f_{i_0} and f_{i_1} are identity maps into extra copies of S_{i_0} and S_{i_1} . Note that the sv-coproduct of a finite family of sv-frames is simply their disjoint union. It is easy to see, by uniqueness of coproduct maps, that having an element in the intersection

$$k_{i_0}[S_{i_0}] \cap k_{i_1}[S_{i_1}],$$

leads to a contradiction. Therefore the coproduct map m is an injection, in fact a bounded frame embedding. ⊢

4.4 Characterization of Ultrafilter Extensions

In what follows we will prove a characterization theorem for ultrafilter extensions. The characteristic property of ultrafilter extensions is instrumental in establishing our main result. The subtle fact is that the isomorphism

$$\text{ue} \coprod_I S \simeq \bigoplus_{\text{sv}}^I \text{ue} S,$$

provides us with two different characterizations of

$$\bigoplus_{\text{sv}}^I \text{ue} S,$$

one as a sv-coproduct and the other as an ultrafilter extension of a Kripke frame. The results proved below establish a connection between these two points of view. For a reader familiar with category theory it should be clear that the results in this section could provide an indirect proof for coproduct preservation.

We start by the following theorem which is similar to Theorem 3.1 in [GBQC].

Theorem 4.15. *Let $\mathfrak{F} = \langle F, R \rangle$ be a Kripke frame. If*

$$\eta : \mathfrak{F} \longrightarrow \text{ue} \mathfrak{F},$$

is the natural injection of \mathfrak{F} , then for any compact Hausdorff frame, $\mathfrak{G} = \langle G, R_g \rangle$, and any function

$$f : \mathfrak{F} \longrightarrow \mathfrak{G}$$

there is a continuous map

$$\hat{f} : ue\mathfrak{F} \longrightarrow \mathfrak{G}$$

such that

$$\hat{f} \circ \eta = f.$$

In other words, given f above, there is a unique map \hat{f} , such that the following diagram commutes.

$$\begin{array}{ccc} \mathfrak{F} & \xrightarrow{\eta} & ue\mathfrak{F} \\ & \searrow f & \downarrow \hat{f} \\ & & \mathfrak{G} \end{array}$$

Moreover

1. If $\mathfrak{F} = \langle F, R \rangle$ is a compact and Hausdorff frame, and f is continuous then

$$\hat{f} = f \circ \epsilon_{\mathfrak{F}}.$$

2. If f is a weak/bounded frame morphism, so is \hat{f} .

Proof: Recall that

$$ue : KF \longrightarrow SV,$$

is a functor preserving injective and surjective frame morphisms.

Consider the continuous map

$$\epsilon_{\mathfrak{G}} : ue\mathfrak{G} \twoheadrightarrow \mathfrak{G},$$

introduced in 4.2.4, which is an inverse to $\eta_{\mathfrak{G}}$. Lemma 4.3 implies commutativity of the square in the following diagram.

$$\begin{array}{ccc}
 \mathfrak{F} & \xrightarrow{\eta_{\mathfrak{F}}} & u\epsilon \mathfrak{F} \\
 f \downarrow & & \downarrow u\epsilon f \\
 \mathfrak{G} & \xrightarrow{\eta_{\mathfrak{G}}} & u\epsilon \mathfrak{G} \\
 \text{id} \downarrow & \swarrow \epsilon_{\mathfrak{G}} & \\
 \mathfrak{G} & &
 \end{array}$$

Hence, letting

$$\hat{f} = \epsilon_{\mathfrak{G}} \circ u\epsilon f,$$

we conclude

$$\hat{f} \circ \eta_{\mathfrak{F}} = f,$$

as required.

To see uniqueness of \hat{f} , assume g is a continuous function such that

$$g \circ \eta_{\mathfrak{F}} = f.$$

Since $\eta_{\mathfrak{F}}$ has a dense image in $|u\epsilon \mathfrak{F}|$ which is a dense subset of $|u\epsilon \mathfrak{G}|$ on which g and \hat{f} agree, $\hat{f} = g$.

If f is continuous the following diagram commutes as in lemma 4.7

$$\begin{array}{ccc}
 \mathfrak{F} & \xleftarrow{\epsilon_{\mathfrak{F}}} & u\epsilon \mathfrak{F} \\
 f \downarrow & & \downarrow u\epsilon f \\
 \mathfrak{G} & \xleftarrow{\epsilon_{\mathfrak{G}}} & \mathfrak{G}
 \end{array}$$

Hence

$$\hat{f} = \epsilon_{\mathfrak{G}} \circ u\epsilon f = f \circ \epsilon_{\mathfrak{F}},$$

satisfies the condition of the lemma.

Finally, since ue preserves weak and bounded morphisms, since natural injections (surjections) are weak (bounded) it is obvious that \hat{f} is weak/bounded whenever f is weak/bounded. \dashv

Let us discuss the characteristic property of the ultrafilter extension a little further. For any set X the set of ultrafilters of X , is a compactification of X . In the lemma above the topological aspect of the induced map \hat{f} has nothing to do with Kripke frames. We might very well have defined the induced map in a more general sense. However, the range of a map f must be at least a compact Hausdorff space, otherwise the natural surjection cannot be defined. This is why we have not stated above lemmas more *categorically*, so to say.

We could state and prove the following.

Proposition 4.16. *Given a set X , a Stone space Y , and a function*

$$f : X \longrightarrow Y ,$$

There is a unique continuous function

$$f^* : ue X \longrightarrow Y ,$$

such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\eta} & \mathfrak{u}\mathfrak{e} X \\ & \searrow f & \downarrow f^* \\ & & Y \end{array}$$

The proof is of course identical to that of theorem 4.15. Moreover it is obvious that assignment

$$f \rightsquigarrow f^*,$$

is one-to-one. On the other hand by uniqueness of f^* , we can see that each continuous map from $\mathfrak{u}\mathfrak{e} X$ to Y must be the induced map of some function with domain X . That is, given

$$g : \mathfrak{u}\mathfrak{e} X \longrightarrow Y,$$

to find the unique map that invokes g consider the composition

$$g \circ \eta : X \longrightarrow Y,$$

whose domain is obviously X . So there is a unique continuous map

$$g^* : \mathfrak{u}\mathfrak{e} X \longrightarrow Y,$$

such that

$$g^* \circ \eta = g \circ \eta.$$

Hence $g = (g \circ \eta)^*$.

The proposition we just proved can be restated as

$$\text{Hom}_{\text{Sets}}(X, |Y|) \simeq \text{Hom}_{\text{Stone}}(\mathfrak{u}\mathfrak{e} X, Y).$$

Here $|Y|$ indicates the underlying set of the topological space Y . Writing down the triangular identities we can see that, with η and ϵ , the natural injection and surjection, as the unit and co-unit we have an adjoint situation. That is, the forgetful functor from Stone to Sets is a right adjoint to the ultrafilter functor.

In contrast, since η is not a bounded morphism, hence not in the category of Kripke frames, a similar argument to show ue as a functor

$$ue : KF \longrightarrow SV.$$

has a right adjoint, fails. Had this been the case preservation of coproducts would have been a corollary of this adjunction. This is true because any functor with a right adjoint preserves colimits. There is a way around this problem if one wants to prove preservation of the coproduct via adjunction. However a direct proof of preservation is much easier.

Existence of a unique continuous map with domain f^* given a map f in proposition 4.16 characterizes ultrafilter extension of a set or Kripke frame. The proof is by a Standard uniqueness argument. This is mentioned in [GBQC]. We include a proof for completion.

Theorem 4.17. *For any set X , $ue X$ is the unique compact Hausdorff topological space*

for which there is a function

$$\eta : X \longrightarrow ue X,$$

such that for any compact Hausdorff topological space Y , and any function

$$f : X \longrightarrow Y,$$

There is a unique continuous function

$$f^* : ue X \longrightarrow Y,$$

making the following diagram commute.

$$\begin{array}{ccc} X & \xrightarrow{\eta} & ue X \\ & \searrow f & \downarrow f^* \\ & & Y \end{array}$$

Proof: Let \bar{X} be a compact Hausdorff topological space together with a function

$$\bar{\eta} : X \longrightarrow \bar{X},$$

such that for any compact Hausdorff space Y there is a unique continuous map

$$\bar{f} : \bar{X} \longrightarrow Y,$$

making the following diagram commute

$$\begin{array}{ccc} \bar{X} & \xrightarrow{\eta} & ue \bar{X} \\ & \searrow f & \downarrow f^* \\ & & Y \end{array}$$

The following diagram is also commutative

$$\begin{array}{ccccc} X & \xrightarrow{\eta} & ue X & \xrightarrow{Id} & ue X \\ & \searrow \bar{\eta} & \downarrow \bar{\eta}^* & \nearrow \bar{\eta}^* & \\ & & \bar{X} & & \end{array}$$

Since identity is a continuous map the homeomorphism is established. \dashv

Remark Obviously the existence of a unique weak/bounded frame morphisms g^* given a weak/bounded frame morphism g , the ultrafilter extension of a Kripke (sv-frame). The fact the map invoked by a weak/bounded morphism is a weak/bounded continuous morphism, does not change the proof of uniqueness (up to isomorphism) of the ultrafilter extension of a frame.

Classifying General Theory Maps

We close this section with a classification theorem. A direct proof could be provided however our proof uses the results in this section. Hence the inclusion of the result here.

Theorem 3.1 classifies sv-theory maps, as continuous bounded frame morphisms whose range is included in \mathfrak{F}^Σ . Having proved enough about sv-frames and in particular ultrafilter extensions we can prove a classification theorem for theory maps in general.

Proposition 4.18. *A function*

$$f : F \longrightarrow \mathfrak{F}^k,$$

is a theory map if and only if

$$ue f : ue F \longrightarrow ue \mathfrak{F}^k$$

is bounded.

Proof Note that uef is always continuous. If uef is bounded then the continuous and bounded map $(\epsilon_k \circ uef)$ is an sv -theory map on ueF by 3.1. On the other hand by 4.15 any sv -theory map on ueF is of the form

$$\epsilon_k \circ uef,$$

where

$$f : F \longrightarrow \mathfrak{F}^k.$$

However f cannot be just any function!

In the proof of the corollary 3.6 we showed that

$$T_\mu(F, a) = T_{\langle \mu \rangle}(ueF, \uparrow a).$$

Since $(\epsilon_k \circ uef)$ is an sv -theory map, it must be of the form $\langle \mu \rangle$ for some $\mu \in \text{Val}(F)$.

Therefore by commutativity of the diagram in theorem 4.15 for any a in F

$$f(a) = T_\mu(a).$$

Therefore f is a theory map.

The reverse direction is almost obvious. If $f = T_\mu$ is a theory map for μ in $\text{Val}(F)$ then $T_{\langle \mu \rangle}$ and hence uef are bounded. Again we are using the fact

$$T_\mu(F, a) = T_{\langle \mu \rangle}(ueF, \uparrow a).$$

⊢

4.5 Negative Results

In the earlier parts of this section we proved certain properties of coproducts of sv-frames, and that of ultrafilter extensions. In proposition 2.17 we saw that to prove CCC it is enough to show that for any canonical modal logic Σ , and an arbitrary index I

$$\text{ue} \bigoplus_{I}^{\text{sv}} \mathfrak{F}^{\Sigma}_{\text{sv}} \Vdash \Sigma.$$

On the other hand corollary 4.13, with $S_i = \mathfrak{F}^{\Sigma}$ for all $i \in I$, shows that $\bigoplus_{I}^{\text{sv}} \text{ue} \mathfrak{F}^{\Sigma}$ is embedded in $\text{ue} \bigoplus_{I}^{\text{sv}} \mathfrak{F}^{\Sigma}$. It is natural to ask whether this embedding is an isomorphism. If the answer is positive CCC is resolved. This is true because for any canonical modal logic Σ

$$\text{ue} \mathfrak{F}^{\Sigma}_{\text{sv}} \Vdash \Sigma,$$

and therefore

$$\bigoplus_{I}^{\text{sv}} \text{ue} \mathfrak{F}^{\Sigma}_{\text{sv}} \Vdash \Sigma. \quad (1)$$

Another question; is $\text{ue} \mathfrak{m}$ an isomorphism if it is bijective? The following proposition answers this question positively.

Proposition 4.19. *Any bijective weak, continuous map between sv-frames is an sv-isomorphisms.*

Proof If

$$f : S_0 \longrightarrow S_1$$

is a bijective continuous function, it is a homeomorphism on the underlying Stone spaces. This is true, because any bijective continuous map from a compact space into a Hausdorff space is a homeomorphism. C.f. [MRS] for a proof. On the other hand, any surjective weak morphism is obviously bounded. So both f and its inverse f^{-1} are bounded. Therefore f has an inverse which is both continuous and bounded, which means f is an isomorphism of sv -frames. \dashv

There is another way to look at a potential isomorphism

$$\bigoplus_{\mathbf{I}}^{sv} u\epsilon S_i \simeq u\epsilon \bigoplus_{\mathbf{I}}^{sv} S_i, \quad (2)$$

By our definition of ultrafilter extensions, $u\epsilon$ can be seen as a functor

$$u\epsilon : SV \longrightarrow SV.$$

If $u\epsilon$ has a right adjoint it preserves coproducts on SV , therefore for any sv -frame S ,

$$u\epsilon \bigoplus_{\mathbf{I}}^{sv} S \simeq \bigoplus_{\mathbf{I}}^{sv} u\epsilon S,$$

and by letting $S = \mathfrak{F}^{\Sigma}$ the desired isomorphism is established.

Unfortunately, as the title of the current section suggests, an isomorphism as in (2) cannot exist, as we argue below.

It is easy to see that for any infinite index set I and a family $\{S_i\}_{i \in I}$ of sv-frames

$$\coprod_I S_i,$$

is not an sv-frame, since it is not compact. on the other hand by the observation made in proposition 4.9 if

$$\bigoplus_I^{sv} S \simeq \bigoplus_I^{sv} S,$$

then

$$\coprod_I S \simeq \bigoplus_I^{sv} S.$$

This leads to a contradiction. Note that any Stone space can be seen as an sv-frame where each node is related to itself and nothing else. Let S be any countable Stone space (an example is any countable compact ordinal) and \mathfrak{A} be the modal algebra of clopens of S . The power of \mathfrak{A} over an countably infinite index set I is not countable. However the disjoint union of countably many copies of S is countable. Since

$$\bigoplus_I^{sv} S = (\prod_I \mathfrak{A})_+$$

is uncountable and

$$\coprod_I S,$$

is countable they cannot possibly be isomorphic.

Although this cardinality argument proves the proposition below let us mention that more could be said about how the sv-coproduct of a family of sv-frames compares to their disjoint union. In fact

$$\bigoplus_I^{sv} S \text{ is the Stone-Jech compactification of } \coprod_I S,$$

which is the largest compactification of $\coprod_i S$. It should not then be surprising that the two distinct coproducts fail to be isomorphic as sets, let alone Kripke frames.

We provided a direct proof for injectivity of the map m in lemma 4.12. Topologically speaking m is the embedding map of compactification and hence obviously injective and continuous. So if a copower is not compact the embedding cannot be surjective. However to refute the existence of an isomorphism it is not enough to show that m is not a bijection. One should prove that there are no bijections between a copower of S over an infinite index set and its Stone-Jech compactification. We remain content with a single counterexample for the sake of brevity.

Since the isomorphism (2) above cannot be established we have the following negative result.

Proposition 4.20. *The operator ue as a functor on SV cannot have a right adjoint.*

Proof Obvious by our discussion above. ⊢

5 Canonicity, Definability, and Completeness

5.1 Canonicity Results

In what follows we prove some connections between notions of canonicity and definability, as well as completeness. We will also provide some alternative forms of CCC.

We prove some results in the direction of proving CCC.

In proposition 2.18 we proved that the countable canonicity conjecture is equivalent to the claim that for any canonical modal logic Σ and arbitrary index set I ,

$$ue \bigoplus_I^{sv} \mathfrak{F}^\Sigma \in SV^\Sigma.$$

This is equivalent to

$$ue \bigoplus_I^{sv} \mathfrak{F}^\Sigma \Vdash_{sv} \Sigma \quad \text{or equivalently} \quad K\Sigma \subseteq Tsv(ue \bigoplus_I^{sv} \mathfrak{F}^\Sigma). \quad (1)$$

The second equivalence holds because $Tsv(ue \bigoplus_I^{sv} \mathfrak{F}^\Sigma)$ is closed under logical deduction, by soundness.

We also showed (proposition 2.17) that the canonicity of a modal logic Σ is equivalent to

$$ue \mathfrak{F}^\Sigma \in SV^\Sigma.$$

On the other hand

$$\bigoplus_I^{sv} ue \mathfrak{F}^\Sigma \equiv_{sv} ue \mathfrak{F}^\Sigma.$$

Moreover, if Σ is canonical

$$ue \mathfrak{F}_{sv}^\Sigma \Vdash \phi \quad \text{iff} \quad \mathfrak{F}^\Sigma \Vdash \phi \quad \text{iff} \quad \vdash_\Sigma \phi \quad \text{iff} \quad \phi \in K\Sigma.$$

Therefore

$$Tsv(\bigoplus_I^{sv} ue \mathfrak{F}^\Sigma) = Tsv(ue \mathfrak{F}^\Sigma) = K\Sigma. \quad (2)$$

It is obvious that the theory of a frame is included in the theory of any of its subframes.

by Therefore proposition 4.13 with $S_i = \mathfrak{F}^\Sigma$ implies

$$Tsv(ue \bigoplus_I^{sv} \mathfrak{F}^\Sigma) \subseteq Tsv(\bigoplus_I^{sv} ue \mathfrak{F}^\Sigma).$$

Hence CCC implies

$$ue \bigoplus_I^{sv} \mathfrak{F}^\Sigma \in SV^\Sigma \quad \text{iff} \quad Tsv(ue \bigoplus_I^{sv} \mathfrak{F}^\Sigma) = K\Sigma = Tsv(\bigoplus_I^{sv} ue \mathfrak{F}^\Sigma).$$

Thus we have proved that CCC is equivalent to

$$Tsv(ue \bigoplus_I^{sv} \mathfrak{F}^\Sigma) = Tsv(\bigoplus_I^{sv} ue \mathfrak{F}^\Sigma). \quad (3)$$

This can be reformulated as

$$ue \bigoplus_I^{sv} \mathfrak{F}^\Sigma \equiv_{sv} \bigoplus_I^{sv} ue \mathfrak{F}^\Sigma.$$

However for any modal logic Σ by 2.3, and the semantics equivalence lemma

$$\mathfrak{F}_{sv}^\Sigma \Vdash \phi \quad \text{iff} \quad \vdash_\Sigma \phi.$$

Therefore

$$\bigoplus_I^{\text{sv}} \mathfrak{F}^\Sigma \Vdash_{\text{sv}} \phi \quad \text{iff} \quad \mathfrak{F}^\Sigma \Vdash_{\text{sv}} \phi \quad \text{iff} \quad \vdash_\Sigma \phi.$$

That is,

$$\text{Ts}_{\text{sv}}(\bigoplus_I^{\text{sv}} \mathfrak{F}^\Sigma) = \mathbf{K}\Sigma.$$

Hence by equation (2) above if Σ is a canonical modal logic

$$\text{Ts}_{\text{sv}}(\bigoplus_I^{\text{sv}} \mathfrak{F}^\Sigma) = \text{Ts}_{\text{sv}}(\bigoplus_I^{\text{sv}} \text{ue } \mathfrak{F}^\Sigma) \quad (4)$$

This implies that CCC is also equivalent to

$$\text{Ts}_{\text{sv}}(\bigoplus_I^{\text{sv}} \mathfrak{F}^\Sigma) = \text{Ts}_{\text{sv}}(\text{ue } \bigoplus_I^{\text{sv}} \mathfrak{F}^\Sigma). \quad (5)$$

This can be reformulated as

$$\bigoplus_I^{\text{sv}} \mathfrak{F}^\Sigma \equiv_{\text{sv}} \text{ue } \bigoplus_I^{\text{sv}} \mathfrak{F}^\Sigma.$$

In contrast to

$$\text{Tk}(\bigoplus_I^{\text{sv}} \mathfrak{F}^\Sigma) = \text{Ts}_{\text{sv}}(\text{ue } \bigoplus_I^{\text{sv}} \mathfrak{F}^\Sigma),$$

which holds by 2.11 regardless of canonicity of Σ .

In other words, one of the statement equivalent to CCC is the equation (5) above.

We know that the

$$\text{ue } \bigoplus_I^{\text{sv}} \mathfrak{F}^\Sigma \not\equiv \bigoplus_I^{\text{sv}} \text{ue } \mathfrak{F}^\Sigma,$$

(proposition 2), and we just concluded that the sv-equivalence of the two structures is exactly what is needed to prove CCC. One last note is that by propositions 2.10 and 4.11 the equivalence

$$ue \bigoplus_I^{sv} \mathfrak{F}^\Sigma \equiv_{sv} \bigoplus_I^{sv} ue \mathfrak{F}^\Sigma .$$

holds, if and only if

$$\bigoplus_I^{sv} \mathfrak{F}^\Sigma \equiv \coprod_I \mathfrak{F}^\Sigma . \quad (6)$$

We prove a different equivalence that could be seen as a mid-step for a potential proof of CCC.

Proposition 5.1. *For any canonical modal logic Σ , and any infinite index set I*

$$\bigoplus_I^{sv} ue \mathfrak{F}^\Sigma \equiv \bigoplus_I^{sv} \mathfrak{F}^\Sigma .$$

Proof Note that a rather easy sv-equivalence,

$$\bigoplus_I^{sv} ue \mathfrak{F}^\Sigma \equiv_{sv} \bigoplus_I^{sv} \mathfrak{F}^\Sigma .$$

follows from proposition 2.17. The equivalence we prove here is for all valuations, not just sv-valuations.

Since Σ is canonical $ue \mathfrak{F}^\Sigma \in SV^\Sigma$. Therefore there is an sv-frame S , and an index set I , such that $ue \mathfrak{F}^\Sigma \rightarrowtail S$ and $\bigoplus_I^{sv} \mathfrak{F}^\Sigma \twoheadrightarrow S$. These embedding and surjective maps invoke corresponding coproduct maps

$$\bigoplus_I^{sv} ue \mathfrak{F}^\Sigma \rightarrowtail \bigoplus_I^{sv} S ,$$

and

$$\bigoplus_I^{SV} \mathfrak{F}^\Sigma \twoheadrightarrow \bigoplus_I^{SV} S.$$

The latter statement is true because one can easily deduce

$$\bigoplus_I^{SV} (\bigoplus_I^{SV} \mathfrak{F}^\Sigma) \simeq \bigoplus_I^{SV} \mathfrak{F}^\Sigma,$$

from the one to one correspondence

$$I \times I \simeq I,$$

which holds as I is an infinite set.

On the other hand there is a surjective continuous frame morphism

$$\bigoplus_I^{SV} \epsilon : \bigoplus_I^{SV} \mathfrak{ue} \mathfrak{F}^\Sigma \longrightarrow \bigoplus_I^{SV} \mathfrak{F}^\Sigma.$$

This is the coproduct map for the natural surjection ϵ from $\mathfrak{ue} \mathfrak{F}^\Sigma$ onto \mathfrak{F}^Σ .

Now an inspection of the semantics of the structures above using 2.6 readily implies

that for any ϕ

$$\bigoplus_I^{SV} \mathfrak{ue} \mathfrak{F}^\Sigma \Vdash \phi \quad \text{iff} \quad \bigoplus_I^{SV} \mathfrak{F}^\Sigma \Vdash \phi.$$

This establishes the desired equivalence. \dashv

Remark It is worth mentioning that one can prove an isomorphism result here. However the isomorphism does not hold for arbitrary infinite index sets.

We can now proceed with some results linking modal definability and canonicity.

5.2 Modal definability and Completeness

A Kripke frame, being a relational structure, can be studied from the view point of first order logic. A class of Kripke frames can therefore be definable as a class of first order structures. We will define the notion of modal definability shortly; first order definability is the analogous notion for first order logic which is not part of our discussion per se. However on a few occasions we refer to results related to first order logic as they become relevant to modal definability. We only need some basic facts about first order definable classes. [BRV] contains enough background material on first order definability for our purposes.

Our definitions are very close to the standard definitions, if not exactly the same, c.f. [BRV]. Our terminology is somewhat different.

Definition 5.2.1. (Classes of Frames Defined by a Modal Logic)

1. *The class of Kripke frames defined by a modal logic Γ , is the class of all Kripke frames over which Γ is valid. We denote this class by KF^Γ . In other words*

$$KF^\Gamma = \{F \mid F \models \Gamma\}.$$

2. *The class of sv-frames defined by Γ , is the class of all sv-frames over which Γ is sv-valid. We denote this class by SV^Γ . In other words*

$$SV^\Gamma = \{S \mid S \models_{sv} \Gamma\}.$$

Following our discussion in section 1.8. we have

$$\text{TsV}(\text{SV}^\Gamma) = \mathbf{K}\Gamma.$$

However there is no reason to assume

$$\text{Tk}(\mathbf{K}\mathbf{F}^\Gamma) = \mathbf{K}\Gamma.$$

It is Obvious by soundness that

$$\mathbf{K}\Gamma \subseteq \text{Tk}(\mathbf{K}\mathbf{F}^\Gamma).$$

The equality of these two classes is a property which is called frame completeness.

Definition 5.2.2. *A modal logic Γ is called a frame complete logic provided that there is a nonempty class of Kripke frames such that*

$$\Gamma = \text{Tk}(\mathbb{C}).$$

It is obvious by soundness that a frame complete logic is closed (under logical deduction). It is also obvious by soundness that for any modal logic Γ

$$\mathbf{K}\mathbf{F}^\Gamma \subseteq \mathbf{K}\mathbf{F}^{\mathbf{K}\Gamma}. \quad (1)$$

The fact that equality is not a given in (1) above means that $\mathbf{K}\mathbf{F}^\Gamma$ is not *determined* by Γ , so to say. That is, $\mathbf{K}\mathbf{F}^\Gamma$ is not always the class of Kripke frame that validate Γ , its logical consequences and nothing more. $\mathbf{K}\mathbf{F}^\Gamma$ might very well validate some formulae that are not deducible from Γ . This is usually referred to as (frame) incompleteness

of Kripke semantics. Note that any canonical modal logic Σ is easily seen to be frame complete. This is true because

$$\mathfrak{F}^\Sigma \in \mathbf{KF}^\Sigma,$$

and so

$$\mathbf{Tk}(\mathbf{KF}^\Sigma) \subseteq \mathbf{Tk}(\mathfrak{F}^\Sigma) = \mathbf{K}\Sigma.$$

We will return to frame completeness after introducing a few more notions.

Definition 5.2.3. (Definable Classes of Frames and sv-Frames) *A class \mathbb{C} of (Kripke) frames is Kripke definable iff there is a logic modal Γ such that for any (Kripke) frame F ,*

$$F \in \mathbb{C} \quad \text{iff} \quad F \Vdash \Gamma.$$

A class \mathbb{C} of sv-frames is sv-definable iff there is a modal logic Γ such that for any sv-frame S ,

$$S \in \mathbb{C} \quad \text{iff} \quad S \Vdash_{\text{sv}} \Gamma.$$

Remark Once again our discussion in 1.8. shows that a class \mathbb{C} of sv-frames is definable if and only if it is a covariety. This is in turn equivalent to closure of \mathbb{C} under GQC. As far as we know such a characterization does not exist for definable classes of Kripke frames.

It is natural to ask if any consistent modal logic defines some (nonempty) class of Kripke frames. Since our focus is on modal logics the answer to this questions is pos-

itive.

Fact 5.1. *Any modal logic is either valid on a Kripke frame that consists of a single reflexive node or valid on a frame that consists of a single irreflexive node.*

Proof C.f. [BRV].

⊢

The fact we just mentioned trivially implies that no modal logic (as far as we are concerned!) defines an empty class of Kripke frames.

We have introduced the notion of frame completeness. We will see the justification for this terminology. We should first define completeness, and to do that we need the notion of semantic entailment.

Definition 5.2.4. (Semantic Consequence) *Let ϕ be a modal formula, Γ a set of modal formulae, Σ a modal logic, and \mathbb{C} a class of Kripke frames. We say Γ locally entails ϕ over \mathbb{C} , or ϕ is a local semantic consequence of Γ over \mathbb{C} , provided that for all F in \mathbb{C} , for all μ in $\text{Val}(F)$, and for all a in $|F|$*

if $F, \mu, a \Vdash \Gamma$ then $F, \mu, a \Vdash \phi$.

This is notated by

$\Gamma \Vdash_{\mathbb{C}} \phi.$

Definition 5.2.5. (Completeness) Let \mathbb{C} be a class of structures, i.e. Kripke frames, sv-frames, or models.

1. **(Weak Completeness)** A modal logic Σ is weakly complete with respect to a class of structures, \mathbb{C} , provided that

$$\text{if } \Vdash_{\mathbb{C}} \phi \text{ then } \vdash_{\Sigma} \phi.$$

2. **(Strong Completeness)** A modal logic Σ is strongly complete with respect to \mathbb{C} provided that for any set of formulae Γ and any modal formula ϕ

$$\text{if } \Gamma \Vdash_{\mathbb{C}} \phi \text{ then } \Gamma \vdash_{\Sigma} \phi.$$

Proposition 5.2. If \mathbb{C} is a nonempty class of Kripke frames and Σ a modal logic such that

$$\Sigma = \text{Tk}(\mathbb{C}).$$

then Σ is weakly complete with respect to \mathbb{C} . Obviously, \mathbb{C} is a sound class of frames for Σ , as well.

Proof Any modal formula valid on all frames in \mathbb{C} is, by definition, in the Kripke theory of \mathbb{C} , and hence in Σ . Conversely since Σ is closed under logical deduction, for any ϕ if $\vdash_{\Sigma} \phi$ then ϕ is in Σ . Hence $\Vdash_{\mathbb{C}} \phi$. In other words it is almost obvious that the assumption of this propositions is equivalent to

$$\phi \in \Sigma \text{ iff } \Vdash_{\mathbb{C}} \phi.$$

We can refer to this by saying \mathbb{C} *provides a sound and complete semantics for Σ* .

It should now be obvious that any definable class \mathbb{C} is definable by a frame complete modal logic which is included in any other frame complete logic that defines \mathbb{C} . Therefore we have the following proposition.

Proposition 5.3. *If \mathbb{C} is a class of Kripke frames definable by a modal logic Γ , i.e. for any Kripke frame F*

$$F \in \mathbb{C} \quad \text{iff} \quad F \Vdash \Gamma,$$

then Γ is frame complete.

Proof Obvious by earlier results in the current section. ⊢

Remark An equivalent form of frame completeness is the following.

$$\Sigma = \text{Tk}(\text{KF}^\Sigma)$$

As mentioned above there is no known theorem on definability analogous to the Birkhoff Variety Theorem. That is, a theorem that classifies definable classes of Kripke frames as classes that are closed under a certain set of operations.

In what follows we shall try to point out the importance of the notion of ultrafilter extension as it relates to definability.

The classic theorem on definability, the Goldblatt-Thomason's theorem, under a strong set of assumptions, provides necessary and sufficient conditions for a class of Kripke frames to be definable. To be more specific, Goldblatt-Thomason's theorem starts with a class that is definable by a first order logic, before proceeding to lay out conditions that are equivalent to modal definability of the class. Let us state the theorem first. Note that disjoint union for Kripke frames is obviously the coproduct in the category of Kripke frames. We skip the definition of an ultrapower which is standard in first order logic. Let us just say that the ultrafilter extension of a frame F is the bounded morphic image of some ultrapower of F . Therefore a class that is closed under bounded morphic images and ultrapowers is \mathfrak{ue} -closed. As mentioned before [BRV] contains all the necessary definitions and facts we use here.

Theorem 5.4. (Goldblatt-Thomason) *A class \mathbb{C} of Kripke frames closed under ultrapowers, is modally definable iff \mathbb{C} is closed under forming (generated)subframes, quotients, disjoint unions of frames (coproducts), and reflects ultrafilter extensions (i.e. $F \in \mathbb{C}$ if $\mathfrak{ue} F \in \mathbb{C}$).*

In the standard proof of Goldblatt-Thomason's theorem, it becomes apparent that first order definability of a class \mathbb{C} implies that the variety of modal algebras deter-

mined by \mathbb{C} is a canonical variety. C.f. [BRV]. This in turn implies that \mathbb{C} is \mathfrak{ue} -closed. However canonicity of the variety determined by \mathbb{C} is weaker than first order definability of \mathbb{C} . The reason is not trivial by any means. In fact whether the two concepts were equivalent was an open question resolved not long ago C.f. [GHV]. In any case this shows that the statement of Goldblatt-Thomason's theorem assumes something stronger than canonicity $V_{\mathbb{C}}$ before even stating necessary and sufficient conditions for \mathbb{C} to be modally definable. We should mention that the original proof of Goldblatt-Thomason's theorem is modified to reduce the set of prerequisites for a class whose definability is characterized by the theorem. Yet what is really *needed* for the proof is the \mathfrak{ue} -closure of the class whose definability is under investigation. We will prove a stronger form of Goldblatt-Thomason's theorem. Some definitions first.

Definition 5.2.6. *The variety determined by a class \mathbb{C} of Kripke frames is the variety $V_{\mathbb{C}} = \text{HSP}(\mathbb{C}^+)$, where*

$$\mathbb{C}^+ = \{F^+ \mid F \in \mathbb{C}\}.$$

Definition 5.2.7. *Given a class \mathbb{C} of frames, i.e. Kripke frames or sv-frames, $\mathfrak{ue} \mathbb{C}$ is the following class of sv-frames*

$$\mathfrak{ue} \mathbb{C} = \{S \mid S = \mathfrak{ue} F \text{ for some } F \in \mathbb{C}\}.$$

Proposition 5.5. *If \mathbb{C} is a class of Kripke frames closed under disjoint union and ultra-filter extension, then the variety $V_{\mathbb{C}}$ is canonical. 4.11*

Proof We prove $\langle \text{ue } \mathbb{C} \rangle$ is ue -closed, which is equivalent to canonicity of V_Σ by proposition 2.16.

Since forming ultrafilter extension preserves subframes and quotients, we only need to verify that the following holds.

$$\text{If } S = \bigoplus_{\text{I}}^{\text{sv}} \text{ue } F_i \quad \text{and} \quad \forall i \, F_i \in \mathbb{C}, \quad \text{then } \text{ue } S \in \langle \text{ue } \mathbb{C} \rangle.$$

This is true because, as mentioned before, subframes and quotients are preserved under forming ultrafilter extensions.

Now if

$$\text{ue } S = \text{ue } \left(\bigoplus_{\text{I}}^{\text{sv}} \text{ue } F_i \right),$$

we have

$$\text{ue } S \simeq \text{ue } \left(\text{ue } \prod_{\text{I}} F_i \right) \quad (\text{by 4.11 above}).$$

Since \mathbb{C} is closed under coproduct, and is ue -closed,

$$\text{ue } \left(\text{ue } \prod_{\text{I}} F_i \right) \in \mathbb{C}, \text{ hence } \text{ue } S \in \text{ue } \mathbb{C} \subseteq \langle \text{ue } \mathbb{C} \rangle.$$

⊢

Here is the stronger form of Goldblatt-Thomason's theorem.

Theorem 5.6. *A class \mathbb{C} closed under ultrafilter extensions is modally definable if and only if*

1. If $F \in \mathbb{C}$ and $f : F \rightarrow G$ then $G \in \mathbb{C}$.
2. If $F \in \mathbb{C}$ and $f : F \rightarrowtail G$ then $G \in \mathbb{C}$.
3. If $\{F_i\}_{i \in I}$ is a family of frames in \mathbb{C} , then $\coprod_I F_i$ is in \mathbb{C} .
4. If $u \vDash F \in \mathbb{C}$ then $F \in \mathbb{C}$.

Proof One direction of the implication is obvious. So assume the conditions 1-4 above hold, and let Σ be a modal logic such that $SV^\Sigma = \langle u \vDash \mathbb{C} \rangle$. We prove that \mathbb{C} is defined by Σ .

Obviously for any $F \in \mathbb{C}$,

$$F \Vdash \Sigma, \text{ because } u \vDash_{sv} F \Vdash \Sigma.$$

Conversely, we prove $F \in \mathbb{C}$ whenever $F \Vdash \Sigma$. Let $F \Vdash \Sigma$, and hence $u \vDash_{sv} F \Vdash \Sigma$, by lemma 2.9.

Since $SV^\Sigma = \langle u \vDash \mathbb{C} \rangle$,

$$u \vDash F \in \langle u \vDash \mathbb{C} \rangle.$$

But

$$SV^\Sigma = \langle u \vDash \mathbb{C} \rangle = GQC(u \vDash \mathbb{C}).$$

So $u \vDash F$ belongs to $GQC(u \vDash \mathbb{C})$. Since \mathbb{C} is closed under subframes (injective frame morphisms), and quotients (surjective frame morphisms), we need only consider the

case in which $ue F$ is an sv-coproduct of members of $ue \mathbb{C}$.

So assume

$$ue F = \bigoplus_I^{sv} ue F_i \quad \text{and} \quad F_i \in \mathbb{C} \quad \text{for all } i \text{ in } I.$$

Hence

$$ue F \simeq ue \left(\prod_I ue F_i \right) \quad (\text{by proposition 4.11}).$$

Since \mathbb{C} is closed under coproducts, and is ue -closed, $ue F \in \mathbb{C}$, and because \mathbb{C} reflects ultrafilter extensions, $F \in \mathbb{C}$. ⊢

We could try to prove a superficial characterization theorem. That is, transfer the conditions for definability to a different class. Here is the reason.

Proposition 5.7. *A class \mathbb{C} of Kripke frames is definable if and only if for any frame F*

$$F \in \mathbb{C} \quad \text{iff} \quad ue F \in \langle ue \mathbb{C} \rangle.$$

⊢

One last connection between canonicity and definability is the following proposition. In this proposition by a nontrivial class of frames we mean a nonempty class whose complement is also nonempty.

Proposition 5.8. *Suppose \mathbb{C} is a nontrivial class of Kripke frames definable by Σ . That is,*

$$\emptyset \neq \mathbb{C} = \mathbf{KF}^\Sigma,$$

$$\mathbf{Tk}(\mathbb{C}) = \mathbf{K}\Sigma$$

The theory of \mathbb{C}^c , the complement of \mathbb{C} is a canonical modal logic.

Proof: Let

$$\Gamma = \mathbf{Tk}(\mathbb{C}^\Gamma).$$

$\Sigma \not\subseteq \Gamma$ otherwise any Kripke frame validates Σ and $\mathbb{C} = \mathbf{KF}$.

This implies that \mathfrak{F}^Γ does not belong to \mathbb{C} . To see this, consider \mathfrak{M}^Γ , the canonical model of Γ , whose underlying frame is \mathfrak{F}^Γ . For any modal formula ϕ ,

$$\mathfrak{M}^\Gamma \Vdash \phi \quad \text{iff} \quad \vdash_\Gamma \phi \quad (\text{Fact 2.3}).$$

If

$$\mathfrak{F}^\Gamma \in \mathbb{C}$$

we have

$$\mathfrak{F}^\Gamma \Vdash \Sigma \quad \text{and} \quad \text{hence} \quad \mathfrak{M}^\Gamma \Vdash \Sigma$$

which implies $\Sigma \subseteq \Gamma$. Since $\mathfrak{F}^\Gamma \in \mathbb{C}^c$ we have

$$\mathfrak{F}^\Gamma \Vdash \mathbf{Tk}(\mathbb{C}^c) = \Gamma,$$

that is Γ is a canonical modal logic. ⊢

Conclusion

In the chapters 2 and 4 of this dissertation we provided some reformulations of the CCC using the terminology of sv-frames. In chapter 5 our discussion on logical equivalence of copowers of structures that generate the covariety of sv-frames for a canonical logic, i.e. proposition 5.1 and the argument prior to it suggests an approach for a possible proof of the CCC.

We hope that these results contribute to an eventual answer to the intriguing problem that is the CCC. We should mention that using the theory maps to form yet another point of view for the study of canonicity seems plausible to us. We have not followed this idea here but we hope to formulate a more general notion of the theory map and study its possible consequences.

Some connections between the notions of canonicity and definability have long been known. However obtaining a complete description of this connection seems to remain evasive. We hope that our work could be of use to anyone who might be interested in pursuing research on this subject.

References

[BRV] Modal Logic, Patrick Blackburn, M. de Rijke, Y. Venema, Cambridge Tracts in Theoretical Computer Science 53, Cambridge University Press, 2001.

[GHV] Erdős Graph Resolves Fine's Canonicity Problem. R. Goldblatt, I. Hodkinson, Y. Venema, The Bulletin of Symbolic Logic, vol. 10, no.2, 2004, PP 186-208.

[GMM] Mathematics of modality, R. Goldblatt, vol 43 CSLI Lecture Notes - Center for the Study of Language and Information, Stanford, 1993.

[GQC] Questions of Canonicity, R. Goldblatt, Trends in Logic, 50 Years of Studia Logica, Kluwer Academic Publishers, 2003 PP 93-128.

[KKV] Stone Coalgebras, C. Kupke, A. Kurz, Y. Venema, Theoretical Computer Science Vol. 327, Issues 1-2, 25 October 2004, PP 109-134

[MLC] Categories for Working Mathematician, S. Mac Lane, 2nd. Edition Springer, 1998.

[MUT] Topology, J.R. Munkres, 2nd Edition, Prentice Hall, Inc. 2000.

[JTBAO1] Boolean algebras with operators. I, B. Jónsson and A. Tarski, Amer. J. Math. 73 (1951), PP 891-939.

[JTBAO2] Boolean algebras with operators. II, B. Jónsson and A. Tarski, Amer. J. Math. 74 (1952), PP 127-162.

[BPA] Memoirs of the American Mathematical Society Number 396, W. J. Blok and D. Pigozzi, Algebraizable logics. (1989).

[VBUE] Canonical Modal Logics and Ultrafilter Extensions. Journal of Symbolic Logic 44, J. van Benthem Vol 1, PP 1-8.

[GBFD] First-order definability in modal logic, R. Goldblatt, J. Symbolic Logic 40 (1975) PP 35-40.

[FFM] Some connections between modal and elementary logic, K. Fine, in S. Kanger, ed., Proceedings of Third Scandinavian Logic Symposium, Studies in Logic 82 (North-Holland, Amsterdam, 1975) PP 15-31.

[GBME] Mathematical Modal Logic: A View of its Evolution. R. Goldblatt Journal of Applied Logic, volume 1, nos. 5-6, 2003, PP 309-392.

CV Saleh Aliyari Address: Math Dept. Indiana University

Rawles Hall 120

Bloomington, IN 47405

Phone: (+1) 812 339 2742

E-mail: saliyari@indiana.edu

Education

Ph.D. Mathematics, Indiana University, Mathematics Department, April 2010

Master in Logic, Mathematical Research Institute, Utrecht, The Netherlands, July 1999

B.Sc. Mathematics, Sharif University of Technology, Tehran, Iran, August 1994

Research Interests

Modal and First order Logic

Foundations of Computer Science

Algebraic and Coalgebraic Methods in Mathematical Logic and Computer Science

Coalgebras of Topological and Measurable Spaces

Publications

Weak Extensions of Recursively Enumerable Degrees, with M.Arslanov, Technical Report, Institute for Theoretical Physics and Mathematics (IPM), Tehran, 1996

Accepted for Presentation in Logic Colloquium 1997, Leeds, England

A Brief Survey of Calculus, A Companion book, Indiana University, Independent Studies Program, 2003

In preparation:

Theory Maps in Logic

Ultrafilter Extension and Canonicity, a Topological Approach