COALGEBRAS ON MEASURABLE SPACES

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Submitted to the faculty of the Graduate School

in partial fulfillment of the requirements

for the degree

Doctor of Philosophy

in the Department of Mathematics,

Indiana University

August 2005

Accepted by the Graduate Faculty, Indiana University, in partial fulfillment of the requirements of the degree of Doctor of Philosophy.

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To Ana

Acknowledgements

There are many people without whose help I could not have completed this dissertation. These few words here reflect but a small part of the gratitude I own all of them.

My advisor Larry Moss supported me every step of the way. Since my arrival to Bloomington he shared generously his time, knowledge, insight and passion for new challenges. He encouraged me to pursue my ideas and helped me to bring them to fruition. I had the privilege of closely collaborating with him, learning first hand from his expertise.

The members of my Research Committee, Esfandiar Haghverdi, David McCarty and Alberto Torchinsky were also my professors. They always found time to extend their friendship and talk about the bigger picture of the mathematics we discussed. I greatly appreciate and will always remember these discussions.

The faculty of the Departments of Mathematics, Computer Science and Philosophy enriched and stimulated my intellectual life. I'm particularly grateful to Jon Barwise, Grahame Bennet, Michael Dunn, Douglas Hofstadter, Daniel Leivant and Slawomir Solecki.

The staff of the Department of Mathematics was exceptionally helpful all along. Misty Cummings was always a step ahead of my needs, so I never had to worry about any paperwork.

My fellow students and friends in the Departments of Mathematics, Computer Science and

Astronomy, who were always there with me to share the good times, bad times, homework and office spaces: Stefano Borgo, Sofía Brenes Barahona, Anastasiya Chagrova, Fulya Erdinc, Jim Hendrickson, Stella Kafka, Tei Laine, Francisco Lara-Dammer, Hayoung Lee, Maricarmen Martínez, Lei Qian, Thomas Reichherzer, Alek Slominski, Nik Swoboda and Alejandro Valerio. With Saleh Aliyari, Seunghwan Lee and Fernando Tohmé I also shared particularly fruitful discussions.

In Bloomington I found many friends: Sarah Ellis, Julie England, Kate Holden, Jeff and Dru Jarret, Waki Murayama, Gregory Rawlins, Steve Simms, Raja Thiagarajan and the rest of the SFDG. All of them made living here a daily joy.

My friends and family in Bahía Blanca, Argentina, never failed to encourage and support me. I hope they know how much that meant to me.

No words will ever be enough to thank my wife Ana Maguitman for all she does and is.

A teaching assistantship from the Department of Mathematics and a Fulbright fellowship provided financial support during the past six years at Indiana University. I thank these institutions for the trust they put in me.

Abstract

Given an endofunctor T in a category $\mathcal{C}$, a coalgebra is a pair (X, c) consisting of an object X and a morphism $c : X \to T(X)$. X is called the *carrier* and the morphism c is called the *structure map* of the T-coalgebra.

The theory of coalgebras has been found to abstract common features of different areas like computer program semantics, modal logic, automata, non-wellfounded sets, etc. Most of the work on concrete examples, however, has been limited to the category Set. The work developed in this dissertation is concerned with the category Meas of measurable spaces and measurable functions.

Coalgebras of measurable spaces are of interest as a formalization of Markov Chains and can also be used to model probabilistic reasoning. We discuss some general facts related to the most interesting functor in Meas, Δ, that assigns to each measurable space, the space of all probability measures on it. We show that this functor does not preserve weak pullbacks or ω^{op}-limits, conditions assumed in many theorems about coalgebras. The main result will be two constructions of final coalgebras for many interesting functors in Meas. The first construction (joint work with L. Moss), is based on a modal language that lets us build formulas that describe the elements of the final coalgebra. The second method makes use of a subset of the projective limit of the final sequence for the functor in question. That is, the sequence $1 \leftarrow T1 \leftarrow T^2 1 \leftarrow \ldots$ obtained by iteratively applying

the functor to the terminal element 1 of the category. Since these methods seem to be new, we also show how to use them in the category Set, where they provide some insight on how the structure map of the final coalgebra works.

We show as an application how to construct universal Type Spaces, an object of interest in Game Theory and Economics. We also compare our method with previously existing constructions.

Contents

1

Introduction

The main subject of this dissertation is the theory of coalgebras on measurable spaces. Coalgebras are simply defined as pairs (X, c) consisting of an object in a category $\mathcal{C}$ and a morphism $c : X \to FX$ where F is an endofunctor of the category $\mathcal{C}$. For some time now, the standard introduction to coalgebras have been the tutorials by Gumm [Gum99] and Rutten [Rut00]. In those works, numerous examples of applications of the abstract theory are shown, including non-wellfounded sets (see also Aczel [Acz88] and Turi and Rutten [TR98]), infinite data structures like streams and trees, different kinds of automata (including non-deterministic ones), labeled transition systems, graphs, Kripke models or frames and even some dynamical systems. All these examples are realised as coalgebras for functors in the category Set of sets and the usual set functions. The existence of notions like probabilistic automata and transition systems, continuous time systems and Markov chains, make it desirable to extend the theory of coalgebras to the category Meas of measurable spaces and measurable functions, where in particular, probability measures can be studied. Probabilities come into the picture through the analysis of the endofunctor Δ in Meas. For a given measurable space X, ΔX is the space of all the probability measures that can be defined over X.

The first explorations of coalgebras on some measure spaces [RdV99] were led by the observation that bisimulation, as defined for probabilistic systems by Larsen and Skou [LS91] was a particular case of the general coalgebraic concept due to Aczel and Mendler, [AM89] (see also [Mos99]). These studies were restricted to specific measure spaces in categories of metric and ultrametric spaces [RdV99]. Probabilistic transition systems have been studied by many computer scientists, see for example the papers by Desharnais et al., [BDEP97], [DEP02], [DGJP03], and also by van Breugel et al., [vBSW02], [vBMOW03], [vBMOW05]. For a survey on the different functors studied, see Bartels, Sokolova and de Vink, [BSdV03].

One application has been the guiding example for the research presented here. This is the definition of *type spaces* from game theory applied to economics. In particular, the problem of finding the *universal type space*, which turns out to be a final coalgebra (this is, a final object in the category of coalgebras for the functor that represent type spaces). Here the probabilities are used to model subjective beliefs of players in a game. That these beliefs can be modeled with probability measures is called the *Bayesian approach*. This branch of studies is independent from the work in computer science, but as we'll see, both of these are encompassed by the class of functors we deal with in this thesis.

Final coalgebras for a wide class of endofunctors in Meas are obtained through two related methods. The first one was developed jointly with Lawrence S. Moss, and makes use of a modal language $\mathcal{L}(T)$ defined in terms of the functor T. This language proves to be expressive enough to let us build a coalgebra whose elements are certain sets of its formulas. Modal languages like these had been studied by Jacobs [Jac01], Rößiger [Röß99, Röß00, Röß01] and Kurz [Kur01]. No logical system is proposed for the languages introduced. Instead, theories are built by collecting sets of formulas satisfied in some model.

In the second construction we use the *final sequence* for the functor. This is the sequence obtained by applying the functor to the final object in the category, and iterating the procedure ω

many times. We then take the projective limit and identify a subspace that will be the carrier of the final coalgebra. The final sequence has been used before to obtain final coalgebras, and in some cases its projective limit is the final coalgebra itself [SP82], but when the functors involved do not satisfy the condition of being ω^{op}-continuous, more work needs to be done, as is the case here. For the category Set, the final sequence has been studied by Worrell [Wor99] and Kurz and Pattinson [KP02]. This thesis is the first study of the final sequence in Meas.

Chapter 2 introduces the general theory of coalgebras, and the construction of final coalgebras in Set using the final sequence. An example in which this construction fails is included and is original to this dissertation.

In Chapter 3 we include the basic measure theory that will be needed to prove the main results. Lemma 3.6 is new, and allows us to build the languages in the Chapter 4 without using negation.

Chapter 4 presents the results that already appeared in [MV04] and [MV05].

Part of the results of Chapter 5 will appear in [Vig05].

Chapter 6 shows how to adapt the construction in Chapter 5 to the category Set. These results are announced in [Vig05] but their proof appears here for the first time.

Chapter 7 expands on the treatment of type spaces sketched in [MV04], using the results from chapters 4 and 5 to construct the *universal type space*. We also offer a review of the bibliography on the topic, trying to clarify the relationship between different formalizations of type space that have been proposed.

Finally, in Chapter 8 we recapitulate the work and point out directions for further research and some open problems.

2

Coalgebras and Limits

In this chapter we introduce coalgebras and some facts of category theory that will be helpful in organizing the material that follows. We also present a well known method for constructing final coalgebras, and an example where this method fails, motivating the new method from chapters 4, 5 and 6.

2.1 Coalgebras

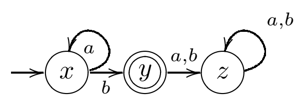
Definition 2.1. Given a category $\mathcal{C}$, and an endofunctor $T : \mathcal{C} \to \mathcal{C}$, a *$T$-coalgebra* or *coalgebra* is a pair (X, c) where X is an object of $\mathcal{C}$ and c is morphism from X to $T(X)$.

Example 2.1. Consider the category Set of sets and set functions, and let the functor T be given by $T(X) = A \times X$ for a fixed set A. If $f : X \to Y$, $T(f) = 1_A \times f$ is the function from $A \times X$ to $A \times Y$ that sends an element $\langle a, x\rangle$ to $\langle a, f(x)\rangle$. Then a coalgebra is a set X, together with a function $c : X \to A \times X$ that assigns to each element in X a pair in $A \times X$. For example, if $X = \{x, y\}$ and $A = \{a, b\}$ then the map $c(x) = \langle a, y\rangle, c(y) = \langle b, x\rangle$ makes X into a coalgebra.

Example 2.2. Automata can be modeled as coalgebras for the functor $TX = X^A \times 2 \times 2$, where

X is the set of states of the automaton, A is the alphabet over which the automaton operates and $2 = \{0, 1\}$ is used to mark which states are initial or final.

As a particular example, consider the automaton below, which recognizes the regular language $\{a^n b | n \geq 0\}$.



We can regard it as the coalgebra with $X = \{x, y, z\}, A = \{a, b\}$ and $c : X \to X^A \times 2 \times 2$ given by:

$$c(x) = \langle (a \mapsto x, b \mapsto y), 1, 0\rangle$$

$$c(y) = \langle (a \mapsto z, b \mapsto z), 0, 1\rangle$$

$$c(z) = \langle (a \mapsto z, b \mapsto z), 0, 0\rangle$$

So this indicates, for example, that x is an initial state, y is final, and z is neither initial nor final.

Example 2.3. Let $\mathcal{D}$ be the functor in Set that assigns to each set X, the set of all functions $\mu : X \to [0, 1]$ such that μ has finite support and $\sum_{x \in X} \mu(x) = 1$. Given a function $f : X \to Y$, and $\mu \in \mathcal{D}X$, $(\mathcal{D}f)(\mu)(y) = \sum_{f(x)=y} \mu(x)$ if $y \in f(X)$, and 0 otherwise. Then a coalgebra for $\mathcal{D}$ can be seen as a model for (discrete time, finitely branching) Markov Chains, and variations of this lead to diverse kinds of *probabilistic transition systems*. See for example, [LS91, Mos99, BSdV03].

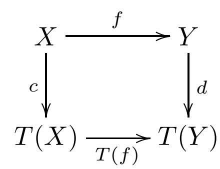
For $\mu \in \mathcal{D}X$, we denote with $Supp\mu$ the set of $x \in X$ such that $\mu(x) > 0$. Notice that if $f : X \to Y$ and $\mu \in \mathcal{D}X$, then

$$|Supp((\mathcal{D}f)\mu)| \leq |Supp(\mu)|. \tag{2.1}$$

Example 2.4. Given a set E, we can consider for each set X the set of all functions from E to

X, denoted as X^E. If $f : X \to Y$, then we can define $f^E : X^E \to Y^E$ by $f^E(\alpha) = f \circ \alpha$ for every $\alpha \in X^E$. Thus $\cdot^E$ is a functor in Set.

Definition 2.2. Given two coalgebras (X, c) and (Y, d), a T-*coalgebra morphism* is a morphism $f : X \to Y$ such that $Tf \circ c = d \circ f$, i.e., the following diagram commutes:



The notion of T-coalgebra morphism agrees with the previously existing notions of morphisms for automata and probability transition systems. That is, the theory of coalgebras captures nicely the notion of morphism in many different examples, including the automata from Example 2.2.

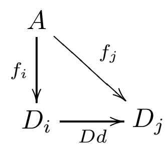
The collection of all coalgebras for a functor T on $\mathcal{C}$, together with the T-coalgebra morphisms form a category, usually denoted as $\mathcal{C}_T$.

2.2 Limits

Definition 2.3. (following [AHS90])

- A *diagram* in a category $\mathcal{C}$ is a functor $D : \mathcal{I} \to \mathcal{C}$. The domain, $\mathcal{I}$ is called the *scheme* of the diagram. For each object i of $\mathcal{I}$, we denote $D(i)$ with D_i.
- A *source* in $\mathcal{C}$ is a pair $(A, (f_i)_{i \in I})$ consisting of an object A in $\mathcal{C}$ and a family of morphisms $f_i : A \to A_i$ indexed by some class I. In particular, we are going to consider I to be the class of objects in the category $\mathcal{I}$. A is called the *domain of the source* and the family $(A_i)_{i \in I}$ is called the *codomain of the source*.

- A $\mathcal{C}$ source $(A \xrightarrow{f_i} D_i)$ is said to be *natural* for D provided that for each $\mathcal{I}$-morphism $i \xrightarrow{d} j$ the triangle in $\mathcal{C}$



commutes. Natural sources are also called *cones*.

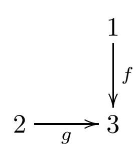
- A *limit* of D is a natural source $(L \xrightarrow{l_i} D_i)_{i \in I}$ for D with the universal property that each natural source $(A \xrightarrow{f_i} D_i)_{i \in I}$ for D uniquely factors through it; i.e., for every such source there exists a unique morphism $f : A \to L$ with $f_i = l_i \circ f$ for each $i \in I$. This limit is also called *projective limit* or *inverse limit*.

- A *weak limit* of D is a natural source $(L \xrightarrow{l_i} D_i)_{i \in I}$ for D with the property that each natural source $(A \xrightarrow{f_i} D_i)_{i \in I}$ for D factors through it; i.e., for every such source there exists a (not necessarily unique) morphism $f : A \to L$ with $f_i = l_i \circ f$ for each $i \in I$.

- We say that a functor T *preserves* a limit $(L \xrightarrow{l_i} D_i)_{i \in I}$ if $(TL \xrightarrow{Tl_i} TD_i)_{i \in I}$ is also a limit.

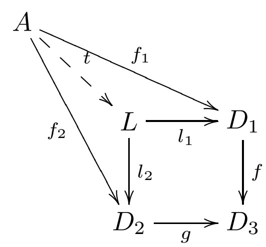
We will use two particular cases of limits: pullbacks and ω^{op}-limits.

Example 2.5. *Pullbacks* are limits of diagrams:



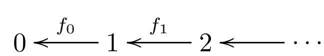
Thus a pullback is an object L together with morphisms $l_1 : L \to D_1$ and $l_2 : L \to D_2$ such that $f \circ l_1 = g \circ l_2$ (notice that $l_3 : L \to D_3$ is usually ignored in the definition of pullback, because it's determined by $l_3 = f \circ l_1 = g \circ l_2$), and for any object A and morphisms $f_i : A \to D_i$, such that

$f \circ f_1 = g \circ f_2$ there exists a unique morphism $t : A \to L$ such that $l_i \circ t = f_i$ for $i = 1, 2$.



In the category Set, pullbacks exists and are constructed as follows: Let $L = \{\langle d_1, d_2 \rangle \in D_1 \times D_2 : (Df)(d_1) = (Dg)(d_2)\}$. Then L, with the projections $\pi_1 : L \to D_1$ and $\pi_2 : L \to D_2$, is a pullback.

Example 2.6. ω^{op}*-limits* have as scheme the category obtained from the poset of the natural numbers with the reverse order:



We will call the diagrams of this scheme ω^{op}*-sequences*, and denote them by the sequence of objects and morphisms $(D_n, Df_n)_{n \in \omega}$ of which they are composed.

The limits of ω^{op} sequences in Set can be constructed by taking the subset P of the product of all the D_n's:

$$P = \{(d_n)_n \in \prod_{n \geq 0} D_n : (Df_n)(d_{n+1}) = d_n\} \tag{2.2}$$

Then the restrictions to P of the projections $\pi_n : \prod_m D_m \to D_n$ make P the limit of the sequence.

2.3 Bisimulations

Definition 2.4. For an endofunctor T in Set, a *bisimulation* between two T-coalgebras (X, c) and (Y, d) is a binary relation $R \subseteq X \times Y$ such that there exists a morphism $b : R \to T(R)$ that makes the two projections, π_X and π_Y into T-coalgebra morphisms.

$$\begin{array}{ccccc} X & \xleftarrow{\pi_X} & R & \xrightarrow{\pi_Y} & Y \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle d} \\ T(X) & \xleftarrow{T\pi_X} & T(R) & \xrightarrow{T\pi_Y} & T(Y) \end{array}$$

Definition 2.5. An endofunctor T *preserves weak pullbacks* if it transforms weak pullbacks into weak pullbacks.

Example 2.7. Many of the usually considered endofunctors in Set preserve weak pullbacks. For example, the constant functor for any fixed set A, the exponential functor from Example 2.1, and the power set functor. If two functors U and V preserve weack pullbacks, then so does their composition $U \circ V$ and their product $U \times V$ (see [Gum99]).

On the other hand, not all functors in Set preserve weak pullbacks. The functor $(\cdot)^3_2$ defined by

$$X^3_2 = \{\langle x, y, z\rangle | |\{x, y, z\}| \leq 2\}$$

and

$$f^3_2(\langle x, y, z\rangle) = \langle f(x), f(y), f(z)\rangle$$

does not (see [AM89]).

Definition 2.6. A functor T is said to *weakly preserve pullbacks* if it transforms pullbacks into weak pullbacks.

In a category with pullbacks, preserving weak pullbacks and weakly preserving pullbacks are equivalent [Gum01]. The importance of this condition in the theory of coalgebra is explained by the following facts (among others).

Theorem 2.1. *[Rut00] Let* $T : \mathsf{Set} \to \mathsf{Set}$ *be a functor that preserves weak pullbacks; and let* $f : (X, c) \to (Z, e)$ *and* $g : (Y, d) \to (Z, e)$ *be* T*-coalgebra morphisms. Then the pullback* (P, k, l) *of* f *and* g *in* Set *is a bisimulation on* X *and* Y*.*

Theorem 2.2. *[Rut00] If the functor* T *preserves weak pullbacks, and* X, Y *and* Z *are* T*-coalgebras, then the composition* $R \circ Q$ *of two bisimulations* $R \subseteq X \times Y$ *and* $Q \subseteq Y \times Z$ *is a bisimulation between* X *and* Z*.*

Definition 2.7. A *final* (or *terminal*) object in a category $\mathcal{C}$ is an object 1 such that for every object A in $\mathcal{C}$ there exists a unique morphism $!_A : A \to 1$.

It follows fom the definition that all terminal objects in a category are isomorphic: if 1 and $1'$ are terminal, then there are unique morphisms $!_{1'} : 1' \to 1$ and $!'_1 : 1 \to 1'$. Their composition $!_{1'} \circ !'_1$ must be $!_1$, which must also be the identity on 1. In a similar fashion, $!'_1 \circ !_{1'}$ is the identity on $1'$, so 1 and $1'$ are isomorphic.

Example 2.8. In the category Set, any singleton is a final object. Since all terminal objects are isomorphic, we can select an arbitrary singleton $1 = \{*\}$ and call it *the* terminal object of Set.

Final objects in the category of coalgebras for a functor T are called final coalgebras, and they have some remarkable properties that make them interesting.

Theorem 2.3. *(Lambek's Lemma, [Lam68]) If* (Z, e) *is a final coalgebra, then* e *is an isomorphism between* Z *and* TZ*.*

Theorem 2.4. *(Coinduction proof principle) [Rut00] If there exists a bisimulation* $R \subseteq Z \times Z$ *in a final coalgebra* Z*for a functor on* Set*, and a pair* $\langle z, z' \rangle \in R$*, then* $z = z'$*.*

2.4 Final Sequences and Final Coalgebras

The main result in this section is Theorem 2.5, due to Smyth and Plotkin, [SP82], which provides a construction of final coalgebras, but we also present an original example showing that the construction cannot always be applied.

Definition 2.8. The *final sequence* for a functor T in a category with final object 1 is the one obtained by applying the functor T to 1. Since 1 is final, there is a unique map $!_{T1} : T1 \to 1$, and we can apply T to this map, obtaining an infinite chain:

$$1 \xleftarrow{!} T1 \xleftarrow{T!} T^2 1 \xleftarrow{T^2!} T^3 1 \longleftarrow \cdots$$

This is a particular case of an ω^{op}-sequence.

Given a T-coalgebra (X, c), there is a way of regarding X as a natural source $(X \xrightarrow{h_n^c} T^n 1)_{n\in\omega}$ mapping X to the final sequence. Let h_0^c be the unique map $!_X : X \to 1$,

$$h_0^c = !_X. \tag{2.3}$$

Given $h_n^c : X \to T^n 1$, let

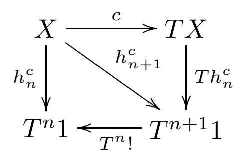
$$h_{n+1}^c = Th_n^c \circ c. \tag{2.4}$$

Lemma 2.1. *Given a T-coalgebra (X, c), for all $n \geq 0, h_n^c = T^n! \circ h_{n+1}^c$.*

Proof. Easy to prove by induction on n: for $n = 0, h_0^c = !_X = !_X \circ h_1^c$ and if $h_{n-1}^c = T^{n-1}! \circ h_n^c$, then $h_n^c = Th_{n-1}^c \circ c = T(T^{n-1}! \circ h_n^c) \circ c = T^n! \circ Th_n^c \circ c = T^n! \circ h_{n+1}^c$. ☐

In the case of the category Set, the Lemma above says that for every element x of a T-coalgebra

(X, c), $h^c(x)$ is in the projective limit P defined as in (2.2).



Let $h^c : X \to P$ be the function defined by

$$\pi_n(h^c(x)) = h^c_n(x) \tag{2.5}$$

for all $n \geq 0$. For a category in general, h^c is the mediating morphism given by the condition on P being a limit.

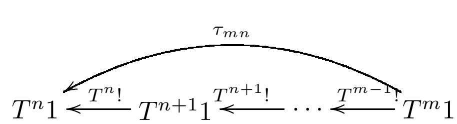
Lemma 2.2. *Given a T-coalgebra morphism $f : (X, c) \to (Y, d)$, for all $n \geq 0, h^d_n \circ f = h^c_n$ and therefore, $h^d \circ f = h^c$.*

Proof. We prove by induction over n that $\pi_n h^d f = \pi_n h^c$, this is, $h^d_n \circ f = h^c_n$:

For $0, h^d_0 f = !_Y f = !_X = h^c_0$. Assuming that $h^d_n f = h^c_n$, we get that $h^d_{n+1} f = Th^d_n \circ d \circ f = Th^d_n \circ Tf \circ c = T(h^d_n \circ f) \circ c = Th^c_n \circ c = h^c_{n+1}$. $\square$

We will ignore the superindex of h whenever doing so does not lead to any confusion.

For any $n < m$, let $\tau_{mn} : T^m1 \to T^n1$ be defined by $\tau_{mn} = T^n! \circ T^{n+1}! \circ \ldots \circ T^{m-1}!$. Also let $\tau_{mm} = 1_{T^m1}$. We have



Theorem 2.5. *[SP82] Let $\mathcal{C}$ be a category with final object 1, and T an endofunctor of $\mathcal{C}$. Suppose that P is the limit of the final sequence and that T preserves the limit. Then the final T-coalgebra exists and is (P, ρ), where $\rho : P \to TP$ is the mediating morphism obtained by TP being a limit.*

Proof. Let (X, c) be an arbitrary T-coalgebra, and assume there exists a T-coalgebra morphism $f : (X, c) \to (P, \rho)$. By the proof of Lemma 2.2, $h_n^\rho \circ f = h_c^c$ for all $n \geq 0$. Since P is a limit, if such f exists, it must be unique. To prove its existence, it's enough to notice that h^c gives a natural source over the final sequence, so there must be a mediating morphism $f : X \to P$.

To prove that f is indeed a T-coalgebra morphism, note that $h_n^\rho = \pi_n$, and for all $n \geq 0$ we have that $T\pi_n \circ \rho \circ f = Th_n^\rho \circ \rho \circ f = h_{n+1}^\rho \circ f = h_{n+1}^c = Th_n^c \circ c = T(h_n^\rho \circ f) \circ c = T\pi_n \circ Tf \circ c$, so $\rho \circ f = Tf \circ c$. □

The functors that preserve the ω^{op}-limits as in the Theorem above are called ω^{op}-continuous. Not all the functors in Set are ω^{op}-continuous. Here are two important examples:

Example 2.9. ([Wor99]) Let $\mathcal{P}$ be the (covariant) finite powerset functor. Let P be again the limit of the final sequence, and let $A_n = \mathcal{P}^n 1$. There is a unique morphism $g : \mathcal{P}P \to P$ such that for all $n \geq 1, \pi_n g = \mathcal{P}\pi_{n-1}$. If g were an isomorphism, then for some $S \in \mathcal{P}P, g(S) = \langle A_n \rangle$. This sequence of elements is in P since $A_n \in A_{n+1}$ and $\mathcal{P}^n!(A_n) = A_{n-1}$. But now we have that $\pi_n g(S) \pi_n g(S) = A_{n-1}$ so $|S| \geq |\pi_n g(S)| = |A_{n-1}|$ for all $n \geq 1$, which contradicts the fact that S is finite.

Example 2.10. Consider the functor $\mathcal{D}$ on Set, the two element set $A = \{a, b\}$ and the ω^{op}-sequence

$$1 \xleftarrow{k_0} A \xleftarrow{k_1} A^2 \xleftarrow{k_2} A^3 \longleftarrow \cdots$$

where $k_n : A_{n+1} \to A_n$ is the projection of the first n components. Let $\mu_n \in \mathcal{D}A^n$ be the constant function $\frac{1}{2^n}$. It's clear that for all $n \geq 0, (\mathcal{D}k_{n-1})\mu_n = \mu_{n-1}$. The limit of the chain is A^ω, and letting $\pi_n : A^\omega \to A^n$ be the projection of the first n components, it's clear that there is no $\mu \in \mathcal{D}A^\omega$ such that for all $n \geq 0, (\mathcal{D}\pi_n)\mu = \mu_n$.

The example above shows that $\mathcal{D}$ does not preserve ω^{op}-sequences in general. But it could be the case that it preserves the limit of the final sequence, which is all that's required for applying Theorem 2.5. In fact, this is true, but not very interesting, since $\mathcal{D}1$ consists of a single probability distribution (the one that assigns 1 to the single element in the set $1 = \{*\}$). However, if we are interested in more complex functors, like the ones used to model labeled transition systems, we can adapt the counterexample to the final sequence of the functor, as in the following example.

Example 2.11. Let $A = \{a, b\}$ and let $S = A^\omega$, the product of countably many copies of A. Let T be the functor $TX = \mathcal{D}(S \times X)$. Next we define on S the measures ν_n by

$$\nu_n(s) = \begin{cases} \frac{1}{2^n} & \text{if } s_{n+i} = a \text{ for all } i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Now let κ_n be the product measure on S^n

$$\kappa_n = \nu_n \times \nu_n \times \ldots \times \nu_n. \tag{2.6}$$

Note that the support of κ_n has $(2^n)^n$ elements.

The next step of this construction is to embed S^n in $S \times T^{n-1}1$ for $n \geq 1$. Some notation first: given a set X and $x \in X$, $\delta_x \in \mathcal{D}X$ is the probability distribution:

$$\delta_x(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

We can consider δ as a natural transformation from the identity functor Id to $\mathcal{D}$. It is easy to see it is an injective one, that is, for every set $X, \delta_X : X \to \mathcal{D}X$ is injective. Furthermore, δ is the unit of the monad for $\mathcal{D}$, with multiplication $\gamma_X : \mathcal{D}\mathcal{D}X \to \mathcal{D}X$ given by

$$\gamma_X(\mu)(x) = \sum_{\nu \in \mathcal{D}X} \nu(x)\mu(\nu)$$

for all $\mu \in \mathcal{D}\mathcal{D}X$ and $x \in X$. This is a particular case of what's known as the Giry monad (see [Gir82] and Lemma 3.2 of this dissertation).

Now we define the maps $a_n : S^n \to S \times T^{n-1}1$ for $n \geq 1$ as follows: $a_1 : S \to S \times 1 = S \times T^0 1$ is

$$a_1(s) = \langle s, * \rangle \tag{2.7}$$

$$a_{n+1}(s_0, s_1, \ldots, s_n) = \langle s_0, \delta_{a_n(s_1,\ldots,s_n)} \rangle \tag{2.8}$$

In other words, a_{n+1} is the map:

$$S^{n+1} = S \times S^n \xrightarrow{1_S \times a_n} S \times (S \times T^{n-1}1) \xrightarrow{1_S \times \delta_{S \times T^{n-1}}} S \times \mathcal{D}(S \times T^{n-1}) = S \times T^n 1$$

The maps a_n allow us to define

$$\gamma_n = (\mathcal{D}a_n)\kappa_n \tag{2.9}$$

Since $\mathcal{D}a_n : \mathcal{D}S^n \to \mathcal{D}(S \times T^{n-1})1 = T^n 1, \gamma_n \in T^n 1$. Furthermore, since δ is injective, it's easy to check that so is a_n for all n. As a consequence,

$$|Supp(\gamma_n)| = |Supp(\kappa_n)| = (2^n)^n = 2^{(n^2)} \tag{2.10}$$

Now let $k_n : S^{n+1} \to S^n$ be the map sending $(s_0, s_1, \ldots, s_n)$ to $(s_0, \ldots, s_{n-1})$ and consider the following diagram for $n \geq 1$:

$$\begin{array}{ccccccc} \cdots & \xleftarrow{k_{n-1}} & S^n & \xleftarrow{k_n} & S^{n+1} & \longleftarrow & \cdots \\ & & \downarrow{\scriptstyle a_n} & & \downarrow{\scriptstyle a_{n+1}} & & \\ \cdots & \longleftarrow & S \times T^{n-1}1 & \xleftarrow[1_S \times T^{n-1}!]{} & S \times T^n 1 & \longleftarrow & \cdots \end{array} \tag{2.11}$$

We prove by induction that the square commutes. For $n = 1$, we have that $a_1(k_1(s_0, s_1)) = a_1(s_0) = \langle s_0, * \rangle$, while $1_S \times !(a_2(s_0, s_1)) = 1_S \times ! \langle s_0, \delta_{\langle s_1, * \rangle} \rangle = \langle s_0, * \rangle$, so

$$\begin{array}{ccc} S & \xleftarrow{k_1} & S^2 \\ \downarrow{\scriptstyle a_1} & & \downarrow{\scriptstyle a_2} \\ S \times T^0 1 & \xleftarrow[1_S \times !]{} & S \times T^1 1 \end{array}$$

commutes.

For the inductive step, note that the square in (2.11) can be rewritten as follows, using the observation that $k_n = 1_S \times k_{n-1}$:

$$\begin{array}{ccc} S \times S^{n-1} & \xleftarrow{1_S \times k_{n-1}} & S \times S^n \\ {\scriptstyle 1_S \times a_{n-1}}\downarrow & & \downarrow{\scriptstyle 1_S \times a_n} \\ S \times (S \times T^{n-2} 1) & \xleftarrow{1_S \times (1_S \times T^{n-2}!)} & S \times (S \times T^{n-1} 1) \\ {\scriptstyle 1_S \times \delta_{S \times T^{n-2} 1}}\downarrow & & \downarrow{\scriptstyle 1_S \times \delta_{S \times T^{n-1} 1}} \\ S \times (S \times T^{n-1} 1) & \xleftarrow[1_S \times T^{n-1}!]{} & S \times T^n 1 \end{array}$$

So the problem reduces to proving that the following diagram commutes:

$$\begin{array}{ccc}
S^{n-1} & \xleftarrow{k_{n-1}} & S^{n} \\
{\scriptstyle a_{n-1}}\Big\downarrow & & \Big\downarrow{\scriptstyle a_n} \\
S \times T^{n-2}1 & \xleftarrow{1_S \times T^{n-2}!} & S \times T^{n-1}1 \\
{\scriptstyle \delta_{S\times T^{n-2}1}}\Big\downarrow & & \Big\downarrow{\scriptstyle \delta_{S\times T^{n-1}1}} \\
T^{n-1}1 & \xleftarrow[T^{n-1}!]{} & T^{n}1
\end{array}$$

But in this diagram the upper square commutes by inductive hypothesis, and the lower one because of the naturality of the natural transformation δ, and using the fact that $\mathcal{D}(1_S \times T^{n-2}!) = T^{n-1}!$.

Now that we know that (2.11) commutes, we can apply the functor $\mathcal{D}$ and get that:

$$\begin{array}{ccccccc}
\cdots & \xleftarrow{\mathcal{D}k_{n-1}} & \mathcal{D}S^{n} & \xleftarrow{\mathcal{D}k_n} & \mathcal{D}S^{n+1} & \longleftarrow & \cdots \\
 & & \Big\downarrow{\scriptstyle \mathcal{D}a_n} & & \Big\downarrow{\scriptstyle \mathcal{D}a_{n+1}} & & \\
\cdots & \longleftarrow & T^{n}1 & \xleftarrow[T^{n}!]{} & T^{n+1}1 & \longleftarrow & \cdots
\end{array} \qquad (2.12)$$

also commutes. Therefore:

$$\begin{array}{rcll}
T^n!\gamma_{n+1} & = & T^n!(\mathcal{D}a_{n+1})\kappa_{n+1} & \text{by(2.9)} \\
 & = & \mathcal{D}((1_S \times T^{n-1}!)a_n)\kappa_{n+1} & \\
 & = & \mathcal{D}(a_{n-1}k_n)\kappa_{n+1} & \text{by (2.12)} \\
 & = & (\mathcal{D}a_{n-1})(\mathcal{D}k_n)\kappa_{n+1} & \\
 & = & (\mathcal{D}a_{n-1})\kappa_n & \\
 & = & \gamma_n & \text{by(2.9)}
\end{array}$$

To summarize: we have $\gamma_n \in T^n1$ with $T^n!\gamma_{n+1} = \gamma_n$ for all $n \geq 0$, so the sequence $\Gamma =$

$(\gamma_0, \gamma_1, \ldots)$ is in the limit P of the final sequence. If we assume that T preserves this limit, then there exists $\rho : P \to TP$ so that for all $n \geq 0, T\pi_n \circ \rho = \pi_{n+1}$. Then, $\rho(\Gamma) \in TP = \mathcal{D}(S \times P)$, and therefore $\rho(\Gamma)$ has finite support. On the other hand, by equations (2.1) and (2.10), $2^{(n^2)} \leq |Supp(\rho(\Gamma))|$ for all n, contradiction.

We will present in Chapter 6 a construction that allows us to find the final coalgebra for this functor T.

3

Measurable Spaces

3.1 Measure Theory

Definition 3.1. A *ring* (or *boolean ring*) of sets is a non empty class $\mathcal{B}$ of sets such that if $E \in \mathcal{B}$ and $F \in \mathcal{B}$ then $E \cup F \in \mathcal{B}$ and $E \setminus F \in \mathcal{B}$. If furthermore, the class $\mathcal{B}$ is closed under countable unions, then it's a *σ-ring*.

An *algebra* (or *boolean algebra*) of subsets of a set M is a nonempty class $\mathcal{B}$ of subsets of M such that if $E \in \mathcal{B}$ and $F \in \mathcal{B}$ then $E \cup F \in \mathcal{B}$ and $E^{\mathsf{c}} \in \mathcal{B}$, where E^{c} denotes the complement of E with respect to M. It follows that $M \in \mathcal{B}$. It is also a *σ-algebra* if it is closed under countable unions.

Definition 3.2. A *measurable space* is a pair (M, Σ) where M is a set and Σ is a σ-algebra of subsets of M. The subsets of M which are in Σ are called *measurable sets*.

A function between measurable spaces $f : (M, \Sigma) \to (M', \Sigma')$ is said to be *measurable* if for every $E \in \Sigma', f^{-1}(E) \in \Sigma$. We will denote with Meas the category of measurable spaces and measurable functions.

Definition 3.3. A *measure* on a measurable space (M, Σ) is a map $\mu : \Sigma \to [0, \infty]$ such that

$\mu(\emptyset) = 0$ and $\mu(\bigcup_n E_n) = \sum_n \mu(E_n)$ for any pairwise disjoint family $\{E_n\} \subseteq \Sigma$. If $\mu(M) = 1$, then μ is called a *probability measure.*

Definition 3.4. A *family of generators* $\mathcal{F}$ of a σ-algebra Σ is a family of subsets such that the smallest σ-algebra including $\mathcal{F}$ is Σ. This is denoted by $\sigma(\mathcal{F}) = \Sigma$.

We will consider the endofunctor Δ in Meas that assigns to each measurable space M, the set ΔM of all the probability measures over M, endowed with the σ-algebra Σ_Δ generated by the sets of the form $\beta^p(E)$ where $p \in [0,1], E$ is a measurable subset of M and

$$\beta^p(E) = \{\mu \in \Delta M : \mu(E) \geq p\}. \tag{3.1}$$

The same σ-algebra can be generated by considering only the sets $\beta^p(E)$ for rational values of p.

If $f : (M, \Sigma) \to (N, \Sigma')$ is measurable, we define $\Delta f : \Delta M \to \Delta N$ as follows: for $\mu \in \Delta(M)$ and $E \in \Sigma'$,

$$(\Delta f)(\mu)(E) = \mu(f^{-1}(A)).$$

As an easy consequence of the definition, we have that

Lemma 3.1. *If $f : M \to M'$ is a measurable function, then for all measurable $E \subseteq M'$ and $p \in [0,1]$,*

$$\beta^p(f^{-1}(E)) = (\Delta f)^{-1}(\beta^p(E))$$

Proof. Both sets are $\{\mu \in \Delta M \mid \mu(f^{-1}(E)) \geq p\}$. $\square$

From Lemma 3.1, it follows that Δf is measurable, and it's easy to check now that Δ is a functor on Meas.

Definition 3.5. Given an element m in a measurable space M, $\delta_m \in \Delta M$ is the Dirac measure supported at m. It assigns 1 to every subset of M containing m, and 0 to the rest.

For each measurable space M, the map $\delta_M : M \to \Delta M$ maps $m \in M$ to $\delta_M(m) = \delta_m \in \Delta M$. This application δ_M is measurable: for every $E \in \Sigma$. If $p \in (0, 1]$,

$$\begin{aligned} \delta^{-1}(\beta^p(E)) &= \{x \in X : \delta_x \in \beta^p(E)\} \\ &= \{x \in X : \delta_x(E) \geq p\} \\ &= \{x \in X : x \in E\} = E \end{aligned}$$

and if $p = 0$ we have $\delta^{-1}\beta^0(E) = X$.

Integration. In a measurable space X it is usual to consider the real-valued functions χ_E for measurable subsets E. $\chi_E(x) = 1$ if $x \in E$, and 0 otherwise. A *simple function* is a function $f = \sum_{i=1}^n \alpha_i \chi_{E_i}$ where $E_1, \ldots, E_n$ are disjoint measurable sets and $\alpha_i \neq 0$ for $i = 1, \ldots, n$. Given a measure μ on X, if each E_i has finite measure (as is the case when μ is a probability measure), then the simple function f is said to be *integrable* and $\int_X f d\mu$ is defined to be $\sum_{i=1}^n \alpha_i \mu(E_i)$. The *distance* between two simple integrable functions f and g is $\int_X |f - g| d\mu$. More in general, if f is a measurable real-valued function, and there exists a sequence of simple integrable functions f_n that converge in measure to f (that is, $\lim_n \mu(\{x||f_n(x) - f(x)| > \epsilon\}) = 0$ for all $\epsilon > 0$) such that the distance between f_n and f_m tends to 0 when m and n tend to infinity, then f is said to be integrable and its integral is defined to be the limit of the integrals of the simple functions f_n (see [Hal50], Chapter V for more specific details).

The Giry Monad. $\delta : Id \to \Delta$ is a natural transformation. A second natural transformation $\gamma : \Delta\Delta \to \Delta$ given by

$$\gamma_M(\mu)(E) = \int_{\nu \in \Delta(M)} \nu(E)\ d\mu,$$

gives us the following:

Lemma 3.2 (Giry [Gir82]). *(Δ, δ, γ) is a monad on* Meas*.*

Products and coproducts. Products and coproducts exist in Meas, and they can be constructed in a similar fashion to the corresponding constructions in Set (furthermore, Meas is a complete category, [Gir82]). The σ-algebra of the product is the one generated by the "rectangles" formed by taking the cartesian product of measurable sets, while the σ-algebra of the coproduct is formed by taking (disjoint) unions of measurable sets in each of the summands.

If $M \times N$ is a product of measurables spaces and μ is a probability measure on it, $(\Delta\pi_M)\mu$ is a measure on M and it's called the *marginal* of μ over M. Of course, $(\Delta\pi_N)\mu$ is the marginal of μ over N. If on the other hand, $\mu \in \Delta M$ and $\nu \in \Delta N$, it is possible to define a probability measure $\mu \times \nu \in \Delta(M \times N)$ by taking, for measurable subsets E and F in M and N, respectively

$$\mu \times \nu(E \times F) = \mu(E)\nu(F)$$

this definition is then extended to all the measurable subsets of $M \times N$. It is clear that the marginals of $\mu \times \nu$ are μ and ν.

If $(M_i, \Sigma_i)_{i \in I}$ is a family of measurable spaces, then $(M, \Sigma) = \prod_i (M_i, \Sigma_i)$ is also a measurable space, where Σ is taken to be the σ-algebra generated by the union of all the $(\pi_i)^{-1}(\Sigma_i)$, for all the projections $\pi_i : M \to M_i$.

3.2 Weak Pullbacks and the Functor Δ

A natural question that arises in this context is whether the functor Δ preserves weak pullbacks. Here we answer negatively the equivalent question of whether Δ weakly preserves pullbacks.

Pullbacks in Meas are constructed in a similar way as in Set. Given measurable functions $f : X \to Z$ and $g : Y \to Z$, one takes the set $P = \{\langle x, y\rangle : f(x) = g(y)\}$ endowed with the smallest σ-algebra that makes the projections measurable.

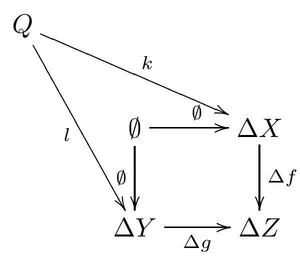
Let's recall that on the real line, the *Borel sets* are defined to be those in the σ-algebra generated by the collection of all the open intervals (a, b) with $a < b \in \mathbb{R}$. The length of an interval gives us a set function $\lambda((a, b)) = b - a$, which can be extended to the completion of the Borel sets of $\mathbb{R}$ (this is, all the unions of Borel sets with subsets of sets of measure zero). This extension is called the *Lebesgue measure*, and the sets in the completion are *Lebesgue measurable*. Its restriction to the interval $[0, 1]$ yields a probability measure over that same interval.

Now let $X = Y = Z$ be the interval $[0, 1]$ on the reals. Let $f : X \to Z$ and $g : Y \to Z$ be the maps:

$$f(x) = \begin{cases} x & \text{if } x \neq 1/2 \\ 0 & \text{if } x = 1/2 \end{cases}$$

and $g(y) = 1 - y$. It is easy to check that f and g are measurable functions and $\{\langle x, y\rangle | f(x) = g(y)\} = \emptyset$.

Now consider $Q = \{q\}$, and functions $k : Q \to \Delta X, l : Q \to \Delta Y$ given by $k(q) = l(q)$ equal to the Lebesgue measure over $[0, 1]$. $(\Delta f)k(q) = (\Delta g)l(q)$ is again the Lebesgue measure, so the diagram commutes, but there is no measurable function $j : Q \to \emptyset$.



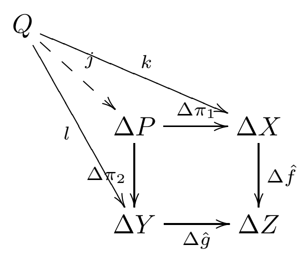
This counterexample can be "fixed" by changing f to be the identity on $[0, 1]$. Could this be the case for all counterexamples? That is, could this be a pathological case arising from the fact that two measurable functions can be essentially the same but differ on a set of measure zero? In other words, is there a pair of functions $\hat{f}, \hat{g}$ so that $f = \hat{f}$ and $g = \hat{g}$ almost everywhere, but yet $\hat{f}$ and $\hat{g}$ have a non-empty pullback that can be weakly preserved by Δ? The answer is again negative as the following construction shows.

We can assume that $\hat{f} = 1_{[0,1]}$ and $\hat{g} = f$ on a set E of measure zero, with $\hat{g} = g$ everywhere else. So the pullback of $\hat{f}$ and $\hat{g}$ is $P = \{\langle x, x\rangle : x \in E\}$. For $q \in [0, 1]$, consider the functions $c_q : [0, 1] \to \mathbb{R}$ defined as follows:

$$c_q(x) = \begin{cases} 1 + q & \text{if } x \in [0, 1/4) \cup (3/4, 1] \\ 1 - q & \text{if } x \in [1/4, 3/4] \end{cases}$$

The functions c_q are naturally associated with a measure in $\Delta[0, 1]$, by taking $\int_F c_q(x)dx$ where dx represents the usual Lebesgue measure over the reals and F is the measurable set for which we are calculating the probability.

Now let $Q = [0, 1]$ and $k(q) = c_q = l(q)$. It is clear that $\Delta 1_{[0,1]} k(q) = c_q = \Delta\hat{g}l(q)$. Furthermore, the functions k and l are measurable. Assume that there is a measurable function $j : Q \to \Delta P$ such that the following diagram commutes



Then, when we calculate $k(q)(E)$ we get: $\int_E c_q(x)dx = 0$, while on the other hand $\Delta\pi_1 j(q)(E) = j(q)(\pi_1^{-1}(E)) = j(q)(P) = 1$, contradiction.

We conclude that ΔP is not a weak pullback for the functions $\Delta\hat{f}$ and $\Delta\hat{g}$, and therefore Δ does not weakly preserve pullbacks.

3.3 ω^{op}-limits and the Functor Δ

To present an example where Δ does not preserve an ω^{op}-limit, we need first to introduce some more definitions from measure theory. In the first place, ω^{op} limits can be constructed in Meas as follows. Suppose we have an ω^{op}-sequence $(A_n, k_n)_{n\in\omega}$. Just as in Set, we take

$$P = \{(a_n) \in \prod_{n\geq 0} A_n : k_n(a_{n+1}) = a_n\}$$

Now, if Σ_n is the σ-algebra on A_n, notice that $\pi_n^{-1}(\Sigma_n)$ is a σ-algebra on P and $\pi_n^{-1}(\Sigma_n) \subseteq \pi_{n+1}^{-1}(\Sigma_{n+1})$ (this follows from $k_n \circ \pi_{n+1} = \pi_n$). The σ-algebra we will use on P is

$$\Sigma = \sigma(\bigcup_{n\geq 0} \pi_n^{-1}(\Sigma_n)).$$

Definition 3.6. ([Hal50], p. 42) A class of sets $\mathcal{H}$ is *hereditary* if $E \in \mathcal{H}$ and $F \subseteq E$ implies $F \in \mathcal{H}$.

The least hereditary family containing a ring $\mathcal{R}$ will be denoted with $\mathcal{H}(\mathcal{R})$.

An *outer measure* is an extended real-valued, non negative, monotone and countably subadditive set function μ^* defined on an hereditary σ-ring $\mathcal{H}$ and such that $\mu^*(\emptyset) = 0$.

Extended real-valued functions can take the value $+\infty$, while countably subadditive means that if a sequence $\{E_i\}$ of sets in $\mathcal{H}$ whose union is also in $\mathcal{H}$, we have

$$\mu^*(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$$

Theorem 3.1. *[Hal50] If μ is a measure on a ring $\mathcal{R}$, and if for every $E \in \mathcal{H}(\mathcal{R})$,*

$$\mu^*(E) = \inf\{\sum_{n=1}^{\infty} \mu(E_n) : E_n \in \mathcal{R}, n = 1, 2, \ldots; E \subseteq \bigcup_{n=1}^{\infty} E_n\}$$

then μ^* is an extension of μ to an outer measure on $\mathcal{H}(\mathcal{R})$); if μ is finite, so is μ^*.

The outer measure μ^* may be described as the lower bound of sums of the type $\sum_{n=1}^{\infty} \mu(E_n)$ where $\{E_n\}$ is a sequence of sets in $\mathcal{R}$ whose union contains E. μ^* is called the outer measure *induced* by μ. In particular, if μ is a measure defined on a σ-algebra Σ, then

$$\mu^*(E) = \inf\{\mu(F) : E \subseteq F \text{ and } F \in \Sigma\}$$

Analogously, the *inner measure* μ_* *induced by* μ can be defined, for every $E \in \mathcal{H}(\mathcal{R})$ as

$$\mu_*(E) = \sup\{\mu(F) : F \subseteq E \text{ and } F \in \Sigma\}.$$

Definition 3.7. A subset X_0 of a measure space (X, Σ, μ) is *thick* if for every measurable set E $\mu_*(E - X_0) = 0$. This is the same as saying that $\mu_*(X - X_0) = 0$. If μ is totally finite (i.e., $\mu(X) < \infty$;

all probability measures are totally finite), then X_0 is thick if and only if $\mu^*(X_0) = \mu(X)$.

In the case of the unit interval with the Lebesgue measure on it, thick just means a set with outer Lebesgue measure equal to 1. Note that the outer measure λ^* induced by λ is defined for every subset of the reals, while λ itself is not. On the Lebesgue measurable sets, λ^* and λ agree, so λ is a *regular* measure (as every measure over a metric space is; see Definition 3.11 and the comments before Theorem 3.4).

Theorem 3.2. *([Hal50], p. 75) If X_0 is a thick subset of a measure space (X, Σ, μ), $\Sigma_0 = \Sigma \cap X_0$ and for $E \in \Sigma, \mu_0(E \cap X_0) = \mu(E)$, then (X_0, Σ_0, μ_0) is a measure space.*

Lemma 3.3. *Let λ be the Lebesgue measure on the interval $[0, 1]$ of the reals. There exist sets $X_0 \supset X_1 \supset X_2 \supset \ldots$ in $[0, 1]$ with $\lambda^*(X_k) = 1$ for all k and $\bigcap_k X_k = \emptyset$.*

The proof of this Lemma makes use of sets like the non-measurable one constructed by Vitali, and is sketcked in [Dud89] Exercise 2, p. 81.

Example 3.1. We consider (X_n, Σ_n, μ_n) to be defined as in the Theorem 3.2. Let $i_n : X_{n+1} \hookrightarrow X_n$ be the inclusion map, and $(Y_n, \Gamma_n) = \prod_{k=0}^{n}(X_k, \Sigma_k)$.

Notice that for all $n \geq 0$, $(\Delta i_n)\mu_{n+1} = \mu_n$. Indeed, if $E_n \in \Sigma_n$, then there is a Lebesgue measurable set E such that $E_n = E \cap X_n$. Then,

$$\begin{aligned}
(\Delta i_n)(\mu_{n+1})(E_n) &= \mu_{n+1}(i_n)^{-1}(E_n) \\
&= \mu_{n+1}(i_n)^{-1}(E \cap X_n) \\
&= \mu_{n+1}(E \cap X_{n-1}) \\
&= \mu(E) \\
&= \mu_n(E_n)
\end{aligned}$$

Let $s_n : X_n \to Y_n$ be the application $x \mapsto (x, \ldots, x)$. To see that s_n is measurable, consider the sets

$$D_n = \{y \in Y_n : y_0 = y_1 = \ldots = y_n\}$$

D_n is measurable in Y_n:

$$I_k = \bigcup_{j=1}^{k} (\prod_{i=0}^{n} (\left[\frac{j-1}{k}, \frac{j}{k}\right] \cap X_i))$$

is a series of "cubes" containing D_n and $D_n = \bigcap_{k=1}^{\infty} I_k$.

Now for any $E \in \Gamma_n$,

$$\begin{aligned} s_n^{-1}(E) &= \{x \in X_n : (x, \ldots, x) \in E\} \\ &= E \cap D_n \end{aligned}$$

and this is a measurable subset of Y_n.

Let $k_n : Y_{n+1} \to Y_n$ be the measurable map sending $(y_0, y_1, \ldots, y_n, y_{n+1})$ to $(y_0, y_1, \ldots, y_n)$, and let P be the projective limit of the chain formed by the spaces Y_n and the maps $k_n, n \geq 0$, with $\pi_n : P \to Y_n$ as before.

It's easy to check that for all $n \geq 0$, the following diagram commutes:

$$\begin{array}{ccc} X_{n+1} & \xrightarrow{s_{n+1}} & Y_{n+1} \\ {\scriptstyle i_n}\big\downarrow & & \big\downarrow{\scriptstyle k_n} \\ X_n & \xrightarrow{s_n} & Y_n \end{array}$$

Let $\nu_n = (\Delta s_n)\mu_n \in \Delta Y_n$. We check that for all $n \geq 0$:

$$\begin{aligned}(\Delta k_n)(\nu_{n+1}) &= (\Delta k_n)((\Delta s_{n+1})\mu_{n+1}) \\ &= (\Delta s_n \circ i_n)\mu_{n+1} \\ &= (\Delta s_n)\mu_n \\ &= \nu_n\end{aligned}$$

Suppose toward a contradiction that Δ preserved limits of ω^{op}-chains, so that ΔP is the limit of $\Delta Y_0 \leftarrow \Delta Y_1 \leftarrow \cdots$. Then there would be some $\nu \in \Delta P$ such that for all n, $(\Delta\pi_n)(\nu) = \nu_n$. For each n, let

$$C_n = \pi_n^{-1}(s_n(X_n)) = \pi_n^{-1}(D_n).$$

This is the set of elements of X whose first n coordinates are equal. $\bigcap_n C_n = \emptyset$, since $x \in \bigcap_n C_n$ iff for all $n, x_n \in D_n$ iff $x_0 \in X_n$ for all n, but we know that $\bigcap_n X_n = \emptyset$.

Moreover, each set C_n is measurable, being the inverse images under π_n of D_n. But note that

$$\begin{aligned}\nu(C_n) &= \nu(\pi_n^{-1}(s_n(X_n)) \\ &= (\Delta\pi_n)\nu(s_n(X_n)) \\ &= \nu_n(s_n(X_n)) \\ &= (\Delta s_n)\mu_n(s_n(X_n)) \\ &= \mu_n((s_n)^{-1}s_n(X_n)) \\ &= \mu_n(X_n) \\ &= 1\end{aligned}$$

So $\nu(C_n) = 1$ for all n, yet $\bigcap_n C_n = \emptyset$. This contradiction shows that ν cannot exist.

3.4 Kolmogorov's consistency Theorem

We have seen that the functor Δ does not preserve ω^{op}-limits in general, but under certain extra topological assumptions, however, functors closely related to Δ *do* preserve the ω^{op}-limits. This is usually known as Kolmogorov's consistency theorem.

Definition 3.8. Given a topological space $(X, \mathcal{T})$, a *Borel measure* over X is a measure over the measurable space $(X, \sigma(\mathcal{T}))$.

Whenever $X = (X, \mathcal{T})$ is a topological space, the σ-algebra we will consider over it is $\sigma(\mathcal{T})$. The sets in this σ-algebra are called the *Borel* sets of X, and ΔX will be the space of all Borel probability measures over X.

Definition 3.9. A set F is the *support* of a probability μ on a topological space (denoted $F =$ *Supp*(μ)) if F is the smallest closed set whose complement has measure 0.

Note that the support of a measure μ need not exist.

Let's recall that a *Polish space* is a separable and completely metrizable topological space. That is, it admits a countable dense set and a metric compatible with the topology such that the space is complete under that metric. The usual examples of Polish spaces are the real line and the Cantor space. Any metric space that is not complete, like the rational numbers, fails to be Polish.

Definition 3.10. A measurable space (X, Σ) is a *standard Borel space* if there is a Polish topology $\mathcal{T}$ on X such that $\sigma(\mathcal{T}) = \Sigma$.

If X is a standard Borel space, so is ΔX, and the projective limit of standard Borel spaces is also a standad Borel space.

Theorem 3.3. *(see [Par67], [Kec95]) Let (X_n, Σ_n) be a sequence of standard Borel measurable spaces and $f_n : X_{n+1} \to X_n$ a surjective measurable map for $n \geq 0$. If μ_n is a probability measure* *on* X_n *such that* $\mu_{n+1} f_n^{-1} = \mu_n$ *for all* $n \geq 0$*, then there is a unique probability measure* μ *on* (X, Σ) *the projective limit of the* (X_n, Σ_n) *such that for all* $n \geq 0$*,* $\mu \pi_n^{-1} = \mu_n$*.*

In terms of our previous notation, $(\mu_n)_{n \in \omega}$ is an element of the projective limit of the sequence formed by the spaces ΔX_n and the functions Δf_n. Then there exists $\mu \in \Delta P$, where P is the projective limit of the sequence of spaces X_n and functions f_n, such that $(\Delta \pi_n)\mu = \mu_n$. This establishes an isomorphism between the projective limit of the spaces ΔX_n and ΔP. In other words, we get the following Corollary:

Corollary 3.1. *In the category of standard Borel spaces and measurable functions, the functor* Δ *is* ω^{op}*-continuous.*

There are other sets of conditions under which Kolmogorov's theorem holds. To point them out we need first to recall a few definitions.

Definition 3.11. A Borel measure μ is said to be *regular* if for any μ-measurable set E,

$$\begin{aligned} \mu(E) &= \inf\{\mu(O) : O \text{ is open and } E \subseteq O\} \\ &= \sup\{\mu(C) : C \text{ is compact and } E \supseteq C\}. \end{aligned}$$

We will denote with $\Delta_r(X)$ the set of all the regular Borel measures over a topological space X.

Definition 3.12. Given a topological space X, $C(X)$ is the normed space of all continuous real-valued functions on X, with the supremum norm.

$C_b(X)$ is the space of all the *bounded* continuous real-valued functions.

It is well known that $C_b(X)$ is a *Banach space*, i.e., a normed linear space which is complete.

Definition 3.13. The *dual* of a Banach space X is the space X' of all the continuous linear functions from X to the reals.

For a compact space X, $C(X) = C_b(X)$, since all the real-valued functions over X are bounded. If μ is a probability measure on X, we can consider μ as an element of $C'(X)$, the dual of $C(X)$ as follows: for each $f \in C(X)$, let $L_\mu(f) = \int_X f(x)d\mu$. Then $L_\mu : C(X) \to \mathbb{R}$.

The *weak-** topology on $C'(X)$ is defined as the smallest topology such that for each $f \in C(X)$ the map $L \mapsto L(f)$ where L ranges on $C'(X)$ is continuous.

When restricted to the space ΔX, the weak-* topology has as subbasis the sets:

$$U_{\mu_0,\epsilon,f} = \{\mu \in \Delta X : |\int f d\mu - \int f d\mu_0| < \epsilon\}$$

If X is a Polish space, or a Compact Hausdorff space, this is actually the same topology as the one generated by the sets $\beta^p(E)$ for Borel sets E and $p \in [0,1]$ (see [Bla51]). In general, though, the weak-* topology is weaker.

If a topological space X is Hausdorff, then $\Delta_r(X)$, endowed with the weak-* topology is also a Hausdorff space. And if X is compact, so is $\Delta_r(X)$ (Alaoglu's Theorem. See, e.g., [Hei93], [Kec95]). Furthermore, if X is metric, every probability measure on X is regular, so $\Delta_r X = \Delta X$ (Ulam's theorem, see [Dud89]).

Theorem 3.4. *(see [Hei93], also [Mèt63], Theorem III.3.2) Let X_n be a sequence of Hausdorff topological spaces and $f_n : X_{n+1} \to X_n$ a surjective continuous map for $n \geq 0$. If μ_n is a regular Borel probability measure on X_n such that $\mu_{n+1} f_n^{-1} = \mu_n$ for all $n \geq 0$, then there is a unique regular Borel probability measure μ on the projective limit of the X_n such that for all $n \geq 0$, $\mu\pi_n^{-1} = \mu_n$.*

Corollary 3.2. *The functor Δ_r is ω^{op}-continuous in the category* Haus *of Hausdorff topological spaces and continuous functions between them.*

The theorems 3.3 and 3.4 have been exploited to obtain some of the results mentioned in Chapter

7, and can be easily extended to yield final coalgebras for the class of measure polynomial functors defined in Chapter 4, via Theorem 2.5.

3.5 A Lemma on generators of σ-algebras

Definition 3.14. A π-*system* is a family of subsets of a set M which is closed under finite intersections. A λ-*system* is a family including M and closed under complements and countable intersections.

Theorem 3.5. *(Dynkin's π-λ Theorem, [Dyn61]) If $\mathcal{A}$ is a π-system and $\mathcal{L}$ is a λ-system, then $\mathcal{A} \subseteq \mathcal{L}$ implies $\sigma(\mathcal{A}) \subseteq \mathcal{L}$.*

The following Lemma about probability measures is a consequence of Dynkin's π-λ Theorem and will be used later.

Lemma 3.4. *[Dyn61] Suppose that μ_1, μ_2 are probability measures on $\sigma(\mathcal{F})$, where $\mathcal{F}$ is a π-system. If μ_1 and μ_2 agree on $\mathcal{F}$, then they agree on $\sigma(\mathcal{F})$.*

Lemma 3.5. *(Heifetz and Samet, [HS98]) Let $\mathcal{F}$ be a boolean algebra of sets that generates the σ-algebra Σ on a measurable space (M, Σ). Then the σ-algebra $\mathcal{F}_\Delta$ generated by the family of sets*

$$\{\beta^p(E) : E \in \mathcal{F} \text{ and } p \in [0,1]\}$$

is the same as Σ_Δ, the σ-algebra generated by the sets $\beta^p(E)$ with $E \in \Sigma$.

We have strengthened this Lemma in the following way:

Lemma 3.6. *Let $\mathcal{F}$ be a π-system of sets that generates the σ-algebra Σ on a measurable space M. Then the σ-algebra $\mathcal{F}_\Delta$ is the same as Σ_Δ.*

To prove it, we first need some facts about real numbers.

Definition 3.15. Given a real number x, the *floor* of x is the greatest integer that is less or equal than x. We denote it with $\lfloor x \rfloor$.

It follows from the definition that

$$\lfloor x \rfloor - 1 = \lfloor x - 1 \rfloor \leq x - 1 < \lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$$

Given $p \in [0, 1]$ and $n \in \mathbb{N}$, $k = \lfloor np \rfloor$ is such that

$$\frac{k}{n} \leq p < \frac{k+1}{n}$$

Furthermore, from

$$\frac{np - 1}{n} < \frac{\lfloor np \rfloor}{n} \leq \frac{np}{n}$$

it follows that

$$\lim_{n \to \infty} \frac{\lfloor np \rfloor}{n} = p$$

The following easy facts will also be used in the proof of Lemma 3.6:

- $p \leq q$ iff for all $n \geq 1$, $p < q + \frac{1}{n}$.
- If for all $n, a_n \geq 0$ then $\sum_{n=1}^{\infty} a_n \geq p$ iff for all $m \geq 1$, $\exists k (\sum_{n=1}^{k} a_n \geq p - \frac{1}{m})$.

Proof. (of Lemma 3.6) : Let $\mathcal{F}' = \bigcap_p (\beta^p)^{-1}(\mathcal{F}_\Delta)$. That is,

$$\mathcal{F}' = \{F \in \Sigma : \forall p \in [0, 1](\beta^p(F) \in \mathcal{F}_\Delta)\}.$$

So for all p, $\beta^p(\mathcal{F}') \subseteq \mathcal{F}_\Delta$. Clearly $\mathcal{F} \subseteq \mathcal{F}'$. We show below that $\mathcal{F}'$ is a λ-system. Therefore by Dynkin's Theorem, we have $\Sigma = \sigma(\mathcal{F}) \subseteq \mathcal{F}'$. It follows that for all p, $\beta^p(\Sigma) \subset \mathcal{F}_\Delta$. So $\Sigma_\Delta = \sigma(\bigcup_p \beta^p(\Sigma)) \subset \mathcal{F}_\Delta$, as desired.

- Since $\beta^p(M) = \Delta M \in \mathcal{F}_\Delta$ for all $p \in [0,1]$, M is in $\mathcal{F}'$.

- If we assume that $E \in \mathcal{F}'$, then for all $q \in [0,1], \beta^q(E) \in \mathcal{F}_\Delta$. In particular, for a fixed $p \in [0,1]$ and all $n \geq \frac{1}{p}, \beta^{(1-p)+\frac{1}{n}}(E) \in \mathcal{F}_\Delta$. But then $\beta^p(E^\mathsf{c}) = \bigcap_n (\beta^{(1-p)+\frac{1}{n}}(E))^\mathsf{c} \in \mathcal{F}_\Delta$ so $E^\mathsf{c} \in \mathcal{F}'$.

  With some more detail:

$$\begin{aligned}
\beta^p(E^\mathsf{c}) &= \{\mu \in \Delta X : \mu(E^\mathsf{c}) \geq p\} \\
&= \{\mu \in \Delta X : 1 - \mu(E) \geq p\} \\
&= \{\mu \in \Delta X : \mu(E) \leq 1 - p\} \\
&= \{\mu \in \Delta X : \forall n \geq \tfrac{1}{p}\ \mu(E) < (1-p) + \tfrac{1}{n}\} \\
&= \textstyle\bigcap_{n \geq \frac{1}{p}} (\beta^{1-p+\frac{1}{n}}(E))^\mathsf{c}
\end{aligned}$$

- Now, if $\{E_n\}$ is a sequence of disjoint sets from $\mathcal{F}'$, then we need to prove that $\bigcup_n E_n \in \mathcal{F}'$.

  First we prove that if A, B are disjoint sets from $\mathcal{F}'$, then $A \cup B \in \mathcal{F}'$.

  Claim: Let $I_n = \{\langle l, m \rangle : l, m \in \mathbb{N}, l, m \leq n \text{ and } l + m \geq \lfloor np \rfloor - 2\}$. Then

$$\beta^p(A \cup B) = \bigcap_n \bigcup_{I_n} (\beta^{\frac{l}{n}}(A) \cap \beta^{\frac{m}{n}}(B)).$$

  Assume that $\mu \in \beta^p(A \cup B)$. Then $\mu(A \cup B) = \mu(A) + \mu(B) \geq p$ Let $\mu(A) = a; \mu(B) = b$ and $r = a + b$. Fix a natural number n. Let $k_a = \lfloor na \rfloor \leq n, k_b = \lfloor nb \rfloor \leq n$. Then we have that $k_a + k_b > na - 1 + nb - 1 = nr - 2 \geq \lfloor nr - 2 \rfloor = \lfloor nr \rfloor - 2 \geq \lfloor np \rfloor - 2$, so $\langle k_a, k_b \rangle \in I_n$. Since $k_a/n \leq a$, $\mu \in \beta^{k_a/n}(A)$, and in a similar fashion, $\mu \in \beta^{k_b/n}(B)$. This proves that

$\mu \in \beta^{\frac{l}{n}}(A) \cap \beta^{\frac{m}{n}}(B)$ for some $\langle l, m\rangle \in I_n$.

Now assume that for all n, $\mu \in \bigcup_{I_n}(\beta^{\frac{l}{n}}(A) \cap \beta^{\frac{m}{n}}(B))$. Then for all n and some l, m, $\mu(A \cup B) = \mu(A) + \mu(B) \geq \frac{l+m}{n} \geq \frac{\lfloor np \rfloor - 2}{n}$. Since this last one converges to p when $n \to \infty$, we conclude $\mu(A \cup B) \geq p$, i.e., $\mu \in \beta^p(A \cup B)$.

Since $A, B \in \mathcal{F}'$, then for all m, l, n, $\beta^{\frac{l}{n}}(A) \in \mathcal{F}_\Delta$ and $\beta^{\frac{m}{n}}(B) \in \mathcal{F}_\Delta$, and so are the countable intersections and unions. Therefore then $A \cup B \in \mathcal{F}'$.

It follows that any finite union of sets in $\mathcal{F}'$ is also in $\mathcal{F}'$, and this is what we use below.

Now consider the sequence of disjoint sets $\{E_n\}$. We claim that

$$\beta^p(\bigcup_n E_n) = \bigcap_m \bigcup_n \beta^{p-\frac{1}{m}}(\bigcup_{k=1}^{n} E_k)$$

Consider the following equivalent statements:

$$\mu \in \beta^p(\bigcup_n E_n)$$

$$\mu(\bigcup_n E_n) = \sum_n \mu(E_n) \geq p$$

$$\forall m \geq \frac{1}{p}\ \exists n_m\ \mu(\bigcup_{k=1}^{n_m} E_k) \geq p - \frac{1}{m}$$

$$\forall m \geq \frac{1}{p}\ \exists n_m\ \mu \in \beta^{p-\frac{1}{m}}(\bigcup_{k=1}^{n_m} E_k)$$

$$\forall m \geq \frac{1}{p}\ \mu \in \bigcup_n \beta^{p-\frac{1}{m}}(\bigcup_{k=1}^{n} E_k)$$

$$\mu \in \bigcap_{m \geq \frac{1}{p}} \bigcup_n \beta^{p-\frac{1}{m}}(\bigcup_{k=1}^{n} E_k)$$

□

This Lemma will be important in the proof of Lemma 4.6 in the next chapter.

4

Modal Languages and Satisfied Theories

4.1 Modal Languages for Measure Polynomial Functors

Definition 4.1. The class of *measure polynomial functors* is the smallest class of functors on Meas containing the identity Id, the constant functor M for each measurable space M, and closed under binary products, coproducts and applications of the functor Δ.

The main goal of this chapter is to provide a construction of final coalgebras for all the measure polynomial functors in the category Meas. The method of work is related to the construction of canonical models in modal logic, but also presents significant differences. We begin the construction by defining the *ingredients* of such functors, which will be used to index the formulas in a modal language $\mathcal{L}(T)$.

Definition 4.2. Given a measure polynomial functor T we define the set of *ingredients of* T, $\mathsf{Ing}(T)$ as follows: $\mathsf{Ing}(Id) = \{Id\}$; $\mathsf{Ing}M = \{M, Id\}$; $\mathsf{Ing}(U \times V) = \{U \times V\} \cup \mathsf{Ing}(U) \cup \mathsf{Ing}(V)$; $\mathsf{Ing}(U + V) = \{U + V\} \cup \mathsf{Ing}(U) \cup \mathsf{Ing}(V)$ and $\mathsf{Ing}(\Delta S) = \{\Delta S\} \cup \mathsf{Ing}(S)$. All the ingredients of T are measure polynomial functors, and for every maesure polynomial functor $T, \mathsf{Ing}(T)$ is a finite set that includes T.

The ingredients of a measure polynomial functor are defined in terms of the syntax with which the polynomial is presented.

Example 4.1. Let $T = Id + \Delta(Id \times M)$ for some fixed measurable space M. Then

$$\mathsf{Ing}(T) = \{Id, M, Id \times M, \Delta(Id \times M), T\}$$

Definition 4.3. For each measure polynomial functor T, we are going to define a language $\mathcal{L}(T)$. Each formula in $\mathcal{L}(T)$ is associated to one of the ingredients S of T. We denote this by $\varphi : S$, where S is an ingredient of T. Then the formulas are formed as follows:

- For each $S \in \mathsf{Ing}(T), \mathsf{true}_S : S$ is a formula.
- If M is a measurable space which is an ingredient of T and A is a measurable subset of M, or $A = \{m\}$ for some $m \in M$, then $A : S$ is a formula. We call these formulas *formulas of constant sort.*
- If $\varphi : S$ and $\psi : S$ are formulas, then $\varphi \wedge \psi : S$ is also a formula.
- If $U \times V \in \mathsf{Ing}(T)$, and $\varphi : U$ and $\psi : V$ are formulas, then $\langle \varphi, \psi \rangle_{U \times V} : U \times V$ is also a formula.
- If $U + V \in \mathsf{Ing}(T)$ and $\varphi : U$ is a formula, then $\mathsf{inl}_{\ U+V}\varphi : U + V$ is also a formula.
- If $U + V \in \mathsf{Ing}(T)$ and $\varphi : V$ is a formula, then $\mathsf{inr}_{\ U+V}\varphi : U + V$ is also a formula.
- If $\Delta S \in \mathsf{Ing}(T), p \in [0, 1]$ and $\varphi : S$ is a formula such that all its subformulas of constant sort are measurable sets in the appropriate spaces, then $\beta^p\varphi : \Delta S$ is a formula.
- If $\varphi : T$ is a formula, then $[\mathsf{next}]\varphi : Id$ is a formula.

We will indicate with $\varphi :: S$ that φ is a formula of sort S such that all its subformulas of constant sort are measurable sets. Note that $\varphi :: S$ implies $\varphi : S$.

Example 4.2. Let $T = Id + \Delta(Id \times M)$ as in the Example 4.1. Here are some examples of formulas and their sorts:

$$\begin{array}{ccc} \mathsf{true}_{Id} : Id & A : M & \langle \mathsf{true}_{Id}, A\rangle : Id \times M \\ \beta^p \langle \mathsf{true}_{Id}, A\rangle : \Delta(Id \times M) & \mathsf{inl}\ \mathsf{true}_{Id} : T & \mathsf{inr}\ \beta^p \langle \mathsf{true}_{Id}, A\rangle : T \\ [\mathsf{next}]\mathsf{inl}\ \mathsf{true}_{Id} : Id & [\mathsf{next}]\mathsf{inr}\ \beta^p \langle \mathsf{true}_{Id}, A\rangle : Id & \end{array}$$

Languages like $\mathcal{L}(T)$ defined here appeared in papers by Rößiger, [Röß99, Röß01], Kurz, [Kur01], and Jacobs [Jac01] on coalgebraic generalizations of modal logic. Those languages had semantics in coalgebras in the category Set. For our work on Meas here, the precedent is the paper by Heifetz and Samet [HS98], where they develop a language for the specific case of type spaces, which we will analize in Chapter 7. It should also be compared to the language introduced by Larsen and Skou in [LS91] to test for bisimulation in probabilistic systems.

Definition 4.4. Given a measure polynomial functor T and a T-coalgebra (X, c), the formulas of the language $\mathcal{L}(T)$ of each sort S will be interpreted in $S(X)$ as follows:

$$\begin{array}{rcl@{\qquad}rcl} [\![\mathsf{true}_S]\!]^c_S &=& SX & [\![\mathsf{inl}\ \varphi]\!]^c_{U+V} &=& \mathsf{inl}\ ([\![\varphi]\!]^c_U) \\ [\![A]\!]^c_M &=& A & [\![\mathsf{inr}\ \varphi]\!]^c_{U+V} &=& \mathsf{inr}\ ([\![\varphi]\!]^c_V) \\ [\![\varphi \wedge \psi]\!]^c_S &=& [\![\varphi]\!]^c_S \cap [\![\psi]\!]^c_S & [\![\beta^p \varphi]\!]^c_{\Delta S} &=& \beta^p([\![\varphi]\!]^c_S) \\ [\![\langle \varphi, \psi\rangle]\!]^c_{U\times V} &=& [\![\varphi]\!]^c_U \times [\![\psi]\!]^c_V & [\![[\mathsf{next}]\varphi]\!]^c_{Id} &=& c^{-1}([\![\varphi]\!]^c_T) \end{array}$$

Here A is any measurable subset or singleton of M, a constant functor which is an ingredient of T.

We say that a point $x \in SX$ *satisfies* the formula $\varphi : S$ if $x \in [\![\varphi]\!]^c_S$. Notice that if $\varphi :: S$, then $[\![\varphi]\!]_S$ is measurable.

Example 4.3. Let's make Example 4.2 where $T = Id + \Delta(Id \times M)$ somewhat more concrete by letting M be the interval $[0,1]$ of the reals, and $A = (1/2, 2/3]$. Let $X = \{x, y, z\}$ and let

$$c(x) = \mathsf{inl}\ (y)$$

$$c(y) = \mathsf{inr}\ (\delta_z \times \mu)$$

$$c(z) = \mathsf{inr}\ ((\frac{1}{3}\delta_x \times \lambda) + (\frac{2}{3}\delta_y \times \lambda))$$

where λ is the Lebesgue measure over $[0,1]$.

$$[\![\mathsf{true}_{Id}]\!]^c_{Id} = X$$

$$[\![(1/2, 2/3]]\!]^c_{[0,1]} = (1/2, 2/3]$$

$$[\![\langle \mathsf{true}_{Id}, (1/2, 2/3]\rangle]\!]^c_{Id\times[0,1]} = X \times (1/2, 2/3]$$

$$[\![\beta^{1/6}\langle \mathsf{true}_{Id}, (1/2, 2/3]\rangle]\!]^c_{\Delta(Id\times[0,1])} = \{\mu \in \Delta(X \times [0,1]) : \mu(X \times (1/2, 2/3]) \geq 1/6\}$$

$$[\![\mathsf{inl}\ \mathsf{true}_{Id}]\!]^c_T = \mathsf{inl}\ X$$

$$[\![\mathsf{inr}\ \beta^{1/6}\langle \mathsf{true}_{Id}, (1/2, 2/3]\rangle]\!]^c_T = \mathsf{inr}\ \{\mu \in \Delta(X \times [0,1]) : \mu(X \times (1/2, 2/3]) \geq 1/6\}$$

$$[\![[\mathsf{next}]\mathsf{inl}\ \mathsf{true}_{Id}]\!]^c_{Id} = \{x\}$$

$$[\![[\mathsf{next}]\mathsf{inr}\ \beta^{1/6}\langle \mathsf{true}_{Id}, (1/2, 2/3]\rangle]\!]^c_{Id} = \{y, z\}$$

In contrast,

$$[\![[\mathsf{next}]\mathsf{inr}\ \beta^{1/2}\langle \mathsf{true}_{Id}, (1/2, 2/3]\rangle]\!]^c_{Id} = \emptyset$$

Lemma 4.1. *Coalgebra morphisms reflect the semantics. In other words, if $f : (X, c) \to (Y, b)$ is a*

T-coalgebra morphism, then for every $\varphi \in \mathcal{L}(T)$, if $\varphi : S$ then $(Sf)^{-1}([\![\varphi]\!]^b_S) = [\![\varphi]\!]^c_S$.

Proof. We prove this by induction on the formulas of $\mathcal{L}(T)$. If φ is true_S, then $(Sf)^{-1}([\![\varphi]\!]^b_S) = (Sf)^{-1}SY = SX = [\![\varphi]\!]^c_S$. Similarly, for $A : M$ for some constant sort M, $(Mf)^{-1}([\![A]\!]^b_S) = (1_M)^{-1}A = A = [\![A]\!]^c_S$.

Since inverse images preserve intersections, the Lemma is trivial for the case of conjunctions. For products, we have to use the inductive hypothesis:

$$\begin{aligned}
((U \times V)f)^{-1}([\![\langle\varphi,\psi\rangle]\!]^b_{U\times V}) &= (Uf \times Vf)^{-1}([\![\varphi]\!]^b_U \times [\![\psi]\!]^b_V) \\
&= (Uf)^{-1}([\![\varphi]\!]^b_U) \times (Vf)^{-1}([\![\psi]\!]^b_V) \\
&= [\![\varphi]\!]^c_U \times [\![\psi]\!]^c_V \\
&= [\![\langle\varphi,\psi\rangle]\!]^c_{U\times V}
\end{aligned}$$

For sums, we have:

$$\begin{aligned}
((U + V)f)^{-1}([\![\mathsf{inl}\ \varphi]\!]^b_{U+V}) &= (Uf + Vf)^{-1}(\mathsf{inl}\ [\![\varphi]\!]^b_U \\
&= \mathsf{inl}\ (Uf)^{-1}([\![\varphi]\!]^b_U) \\
&= \mathsf{inl}\ [\![\varphi]\!]^c_U) \\
&= [\![\mathsf{inl}\ \varphi]\!]^c_{U+V}
\end{aligned}$$

and similarly for V and the inclusion to the right inr .

For formulas of sort ΔS and of the form $\beta^p\varphi$, we also use Lemma 3.1

$$\begin{aligned}
(\Delta S f)^{-1}(\llbracket \beta^p \varphi \rrbracket^c_{\Delta S}) &= (\Delta S f)^{-1}(\beta^p \llbracket \varphi \rrbracket^c_S) \\
&= \beta^p((Sf)^{-1}(\llbracket \varphi \rrbracket^c_S)) \\
&= \beta^p \llbracket \varphi \rrbracket^b_S \\
&= \llbracket \beta^p \varphi \rrbracket^b_{\Delta S}
\end{aligned}$$

Finally, for formulas of the form $[\mathsf{next}]\varphi$ with $\varphi : T$,

$$\begin{aligned}
f^{-1}(\llbracket [\mathsf{next}]\varphi \rrbracket^b_{Id}) &= f^{-1}(b^{-1}(\llbracket \varphi \rrbracket^c_T)) \\
&= c^{-1}((Tf)^{-1}(\llbracket \varphi \rrbracket^c_T)) \\
&= c^{-1}(\llbracket \varphi \rrbracket^c_T) \\
&= \llbracket [\mathsf{next}]\varphi \rrbracket^c_{Id}
\end{aligned}$$

Here we used that $b \circ f = Tf \circ c$, because f is a T-coalgebra morphism. □

4.2 The Space of Satisfied Theories

We will use the formulas in $\mathcal{L}(T)$ to describe the elements of coalgebras and other sets SX. We do this by defining for each T-coalgebra (X, c) and each $x \in SX$ where $S \in \mathsf{Ing}(T)$,

$$d^c_S(x) = \{\varphi : S | x \in \llbracket \varphi \rrbracket^c_S\} \tag{4.1}$$

The set $d^c_S(x)$ will be called the *theory satisfied* by point $x \in SX$. We call the maps d^c_S the *description maps*. As a consequence of Lemma 4.1, we have:

Corollary 4.1. *If $f : (Xc) \to (Y, b)$ is a morphism of coalgebras, then for every $S \in \mathsf{Ing}(T)$,*

$d^b_S \circ Sf = d^c_S$. *That is, coalgebra morphisms preserve description maps.*

Definition 4.5. For each of the ingredients S of a polynomial functor T, we define the set S^* as follows:

$$S^* = \{d^c_S(x) \mid \text{ for some coalgebra } (X,c), x \in SX\}. \tag{4.2}$$

Notice that S^* is a collection of subsets of $\mathcal{L}(T)$, and therefore a set. For each formula φ of sort S, we let

$$|\varphi|_S = \{s \in S^* \mid \varphi \in s\}. \tag{4.3}$$

Now we can endow S^* with the σ-algebra generated by the sets $|\varphi|_S$ for all $\varphi :: S$. Since $|\varphi| \cap |\psi| = |\varphi \wedge \psi|$, we have in the sets $|\varphi|$ a π-system of generators of the σ-algebra on S^*.

Now that we defined S^* as a measurable space, we can ask the question if the function $d^c_S : SX \to S^*$ is measurable.

Lemma 4.2. *For all* (X,c) *and all* $S \in \mathsf{Ing}(T)$*:*

1. *For all* $\varphi : S$, $[\![\varphi]\!]^c_S = (d^c_S)^{-1}(|\varphi|)$.

2. $d^c_S : SX \to S^*$ *is measurable.*

Proof. The first part follows immediately from the definition, while the second follows from the first. Recall that to prove that a function is measurable, it is enough to prove that for a family of generators, the inverse images under the function are measurable sets. So, it's enough to recall that for all $\varphi :: S$, $[\![\varphi]\!]^c_S$ is measurable. □

Our purpose is to prove that Id^* is the carrier of the final coalgebra for a measure polynomial functor. The space Id^* is similar to the *canonical model* often used in Modal Logic (see, for example, [BdRV01]). The difference is that we don't use a deductive system of inference to determine the sets of formulas that make up the model, but use instead all the sets of formulas that are realised in *some* model.

The idea is that each point s in Id^* is a set of formulas and these formulas will contain the information of what $c^*(s)$ will be, if c^* is the structure map of the final coalgebra. Each $s \in Id^*$ is the description of some element x in a coalgebra (X, c). The description of $c(x)$ will be already contained in s. The following Lemma makes this formal.

Lemma 4.3. *There is a measurable map $g : Id^* \to T^*$ such that for every T-coalgebra (X, c), the following diagram commutes,*

$$\begin{array}{ccc} X & \xrightarrow{c} & TX \\ {\scriptstyle d^c_{Id}}\downarrow & & \downarrow{\scriptstyle d^c_T} \\ Id^* & \xrightarrow[g]{} & T^* \end{array} \tag{4.4}$$

and for all $\varphi : T$, $g^{-1}|\varphi|_T = |[\mathsf{next}]\varphi|_{Id}$.

Proof. We define g by

$$g(s) = \{\varphi : T \mid [\mathsf{next}]\varphi \in s\}. \tag{4.5}$$

Assume that (X, c) is a coalgebra with $x \in X$ such that $s = d^c_{Id}(x)$. Then $c(x) \in TX$, and

$$\begin{aligned} g^{-1}(d^c_{Id}(x)) &= g^{-1}(\{\varphi : Id \mid x \in [\![\varphi]\!]_{Id}\}) \\ &= \{\psi : T \mid [\mathsf{next}]\psi \in \{\varphi : Id \mid x \in [\![\varphi]\!]_{Id}\}\} \\ &= \{\psi : T \mid x \in [\![[\mathsf{next}]\psi]\!]_{Id}\} \\ &= \{\psi : T \mid c(x) \in [\![\psi]\!]_T\} \\ &= d^c_T(c(x)) \end{aligned}$$

Now take $s \in g^{-1}(|\varphi|_S)$. Then $g(s) \in |\varphi|$, which is equivalent to saying that $\varphi \in g(s)$, or that

$[\mathsf{next}]\varphi \in s$. This proves the second statement, and when applied to formulas $\varphi :: T$, it proves the measurability of g. □

4.3 From S^* to $S(Id^*)$

Now we have the map $g : Id^* \to T^*$ and for every coalgebra, the map $d^c : X \to Id^*$ makes the diagram in (4.4) commute. But to get the final coalgebra, we need a map $c^* : Id^* \to T(Id^*)$. We'll achieve this by finding maps r_S connecting S^* with $S(Id^*)$ for each ingredient of T. As a preliminary step, we need some measurable maps introduced in the following three Lemmas.

Lemma 4.4. *Let $U \times V \in \mathsf{Ing}(T)$. There is a measurable map $\langle \pi_1, \pi_2 \rangle : (U \times V)^* \to U^* \times V^*$ such that for every coalgebra (X, c), $\langle \pi_1, \pi_2 \rangle \circ d^c_{U \times V} = d^c_U \times d^c_V$ that is, the following diagram commutes:*

$$\begin{array}{ccc} (U \times V)X & & \\ \downarrow{\scriptstyle d^c_{U\times V}} & \searrow{\scriptstyle d^c_U \times d^c_V} & \\ (U \times V)^* & \xrightarrow[\langle \pi_1, \pi_2 \rangle]{} & U^* \times V^* \end{array} \tag{4.6}$$

Furthermore, if $\varphi : U$ and $\psi : V$, $\langle \pi_1, \pi_2 \rangle^{-1}(|\varphi| \times |\psi|) = |\langle \varphi, \psi \rangle|$.

Proof. We define $\pi_1 : (U \times V)^* \to U^*$ by

$$\pi_1(s) = \{\varphi : U \mid \langle \varphi, \mathsf{true}_V \rangle \in s\}.$$

$\pi_2 : (U \times V)^* \to V^*$ is defined in a similar way. First we check that for every $s \in (U \times V)^*, \pi_1(s) \in U^*$. We know there must exist some (X, c) such that for some $x \in (U \times V)X, d^c_{U\times V}(x) = s$. But then $x = \langle u, v \rangle$ with $u \in UX$ and $v \in VX$. Now it's easy to check that $\pi_1(s) = \{d^c_U(u) : \varphi \in \pi_1(s)\}$ iff $\langle \varphi, \mathsf{true}_V \rangle \in s = d^c_{U\times V}(\langle u, v \rangle)$ iff $\langle u, v \rangle \in [\![\langle \varphi, \mathsf{true}_V \rangle]\!]^c_{U\times V}$ iff $u \in [\![\varphi]\!]^c_U$ iff $\varphi \in d^c_U(u)$. In the same manner, one can check that $\pi_2(s) = d^c_V(v)$ and therefore $\pi_2(s) \in V^*$.

To verify that $\langle \pi_1, \pi_2 \rangle^{-1}(|\varphi| \times |\psi|) = |\langle \varphi, \psi \rangle|$, consider the following equivalent statements:

$$s \in |\langle \varphi, \psi \rangle|_{U \times V}$$

$$\langle \varphi, \psi \rangle \in s$$

$$\langle \varphi, \mathsf{true}_V \rangle, \langle \mathsf{true}_U, \psi \rangle \ \in s$$

$$\varphi \in \pi_1(s) \ \text{ and } \psi \in \pi_2(s)$$

$$\pi_1(s) \in |\varphi|_U \text{ and } \pi_2(s) \in |\psi|_V$$

$$\langle \pi_1, \pi_2 \rangle(s) \in |\varphi|_U \times |\psi|_V$$

The measurability of $\langle \pi_1, \pi_2 \rangle$ follows from the equation above and the fact that the σ-algebra on $U^* \times V^*$ is generated by the sets $|\varphi|_U \times |\psi|_V$, with $\varphi :: U$ and $\psi :: V$. □

Lemma 4.5. *Let $U + V \in \mathsf{lng}(T)$. There is a measurable map $\alpha : (U+V)^* \to U^* + V^*$ such that for every coalgebra (X, c), $\alpha \circ d^c_{U+V} = d^c_U + d^c_V$, and if $\varphi : U$ and $\psi : V$, then $\alpha^{-1}(\mathsf{inl}\ _{U+V}|\varphi|_U) = |\mathsf{inl}\ \varphi|_{U+V}$ and $\alpha^{-1}(\mathsf{inr}\ _{U+V}|\psi|_V) = |\mathsf{inr}\ \psi|_{U+V}$.*

Proof. We define $\alpha : (U+V)^* \to U^* + V^*$ by

$$\alpha(s) = \begin{cases} \mathsf{inl}\ _{U^*+V^*}(\{\varphi : U \mid \mathsf{inl}\ \varphi \in s\}), & \text{if } \mathsf{inl}\ \mathsf{true}_U \in s \\ \mathsf{inr}\ _{U^*+V^*}(\{\varphi : V \mid \mathsf{inr}\ \varphi \in s\}), & \text{if } \mathsf{inr}\ \mathsf{true}_V \in s \end{cases} \tag{4.7}$$

We check that for every $s \in (U+V)^*, \alpha(s) \in U^* + V^*$. We know as before that there must exist some (X, c) such that for some $x \in (U+V)X, d^c_{U+V}(x) = s$. But then $x = \mathsf{inl}\ u$ with $u \in UX$ or $x = \mathsf{inr}\ v$ with $v \in VX$, and only one of these is valid. Let's assume that $x = \mathsf{inl}\ u$ (the other case is handled in the same way). In this case we have that $u \in UX = [\![\mathsf{true}_U]\!]^c_U$ and therefore $\mathsf{inl}\ \mathsf{true}_U \in s$. So $\alpha(s) = \mathsf{inl}\ _{U^*+V^*}(\{\varphi : U \mid \mathsf{inl}\ \varphi \in s\})$. We claim that $\{\varphi : U \mid \mathsf{inl}\ \varphi \in s\} = d^c_U(u)$. To verify this, it is enough to note that $\mathsf{inl}\ \varphi \in s = d^c_{U+V}(\mathsf{inl}\ u)$ iff $\mathsf{inl}\ u \in [\![\mathsf{inl}\ \varphi]\!] = \mathsf{inl}\ [\![\varphi]\!]$ iff $u \in [\![\varphi]\!]$.

Now $(\alpha \circ d^c_{U+V} \circ \mathsf{inl}\ )(u) = (\alpha \circ d^c_{U+V})(x) = \mathsf{inl}\ d^c_U(u)$, and similarly for v, which proves that $\alpha \circ d^c_{U+V} = d^c_U + d^c_V$,

Finally, to prove the last assertion, and that α is measurable, consider the following equivalent statements:

$$s \in |\mathsf{inl}\ \varphi|$$

$$\mathsf{inl}\ \mathsf{true}_U \in s \text{ and } \mathsf{inl}\ \varphi \in s$$

$$\mathsf{inl}\ \mathsf{true}_U \in s \text{ and } \varphi \in \{\psi : U \mid \mathsf{inl}\ \psi \in s\}$$

$$\mathsf{inl}\ \mathsf{true}_U \in s \text{ and } \{\psi : U \mid \mathsf{inl}\ \psi \in s\} \in |\varphi|$$

$$\alpha(s) \in \mathsf{inl}\ _{U^*+V^*}(|\varphi|)$$

□

Lemma 4.6. *Let $\Delta S \in \mathsf{lng}(T)$. There is a measurable map $\epsilon : (\Delta S)^* \to \Delta(S^*)$ such that for any coalgebra (X, c), $\epsilon \circ d^c_\Delta S = \Delta d^c_S$, and for all $\varphi :: S$, $\epsilon^{-1}(\beta^p(|\varphi|)) = |\beta^p\, \varphi|$.*

Proof. For a given $s \in (\Delta S)^*$, fix (X, c) and $\mu \in \Delta S X$ such that $d^c_{\Delta S}(\mu) = s$. We define

$$\epsilon(s) = (\Delta d^c_S)\mu. \tag{4.8}$$

This definition depends on (X, c) and μ, but notice that we get a probability measure on S^*.

Let $\mathcal{F}$ be the family of the sets of the form $|\varphi|$ for $\varphi :: S$. For each set in $\mathcal{F}$ we get that:

$$\begin{aligned} ((\Delta d^c_S)\mu)(|\varphi|) &= \mu((d^c_S)^{-1}(|\varphi|)) && \text{by the definition of } \Delta \text{ as a functor} \\ &= \mu([\![\varphi]\!]^c_S) && \text{by Lemma 4.2} \\ &= \max\{p \mid \beta^p\varphi \in s\} \end{aligned}$$

The last equality deserves a little explanation. If we let $q = \mu[\![\varphi]\!]^c_S$, then for all $p \leq q, \mu \in \beta^p\varphi$, and this proves that $\max\{p \mid \beta^p\varphi \in s\} \leq q$. But also $q \in \{p \mid \beta^p\varphi \in s\}$, so the equality follows.

Since the expression '$\max\{p \mid \beta^p\varphi \in s\}$' depends only on s, for the sets in $\mathcal{F}$ we have proved that the definition is in fact independent of (X, c) and μ. But $\mathcal{F}$ is a π-system of generators of the σ-algebra on S^*. By Lemma 3.4, it follows that $\epsilon(s)$ is also well defined, and it follows directly from (4.8) that $\epsilon \circ d^c_\Delta S = \Delta d^c_S$.

Finally, we need to prove that ϵ is measurable. By Lemma 3.6, the sets E of the form $\beta^p|\varphi|$ with $\varphi :: S$ or finite intersections of them, $\epsilon^{-1}(E)$ are enough to generate the σ-algebra of measurable subsets of $\Delta(S^*)$. These are easier to handle than the sets $\beta^p(E)$ for just any measurable subset E.

So for these sets we have:

$$
\begin{aligned}
\epsilon^{-1}(\beta^p|\varphi|) &= \{s \in (\Delta S)^* | \epsilon(s) \in \beta^p(|\varphi|)\} \\
&= \{s \in (\Delta S)^* | \epsilon(s)(|\varphi|) \geq p\} \\
&= \{s \in (\Delta S)^* | \beta^p\varphi \in s\} \qquad (*) \\
&= |\beta^p\varphi|
\end{aligned}
$$

The second case (finite intersections of sets of this kind), follows easily from the fact that inverse images preserve intersections.

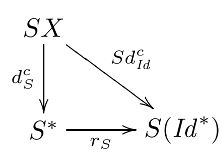
The line marked above with (*) also needs to be explained. The argument is similar to one we had before in this proof. Suppose that $\epsilon(s)(|\varphi|) = q \geq p$. Then $\beta^q\varphi \in s$, but also $\beta^p\varphi \in s$ for all $p \leq q$, since all theories of all points must have this monotonicity property. In the other direction, if $\beta^p\varphi \in s$, then the largest q such that $\beta^\varphi \in s$ is at least p, and $\epsilon(s)(|\varphi|)$ is the maximum of those, so $\epsilon(s)(|\varphi|) \geq p$. □

Before we proceed, we need to define some natural sets $\hat{\varphi}$ that act as interpretations of the formulas of $\mathcal{L}(T)$ in the sets $S(Id^*)$, as we'll see shortly.

Definition 4.6. For the special case of the functor Id, which is always an ingredient of T, $Id(Id^*) = Id^*$, so we can take for all $\varphi : Id, \hat{\varphi} = |\varphi|$. For all S, $\widehat{\mathsf{true}_S} = S(Id^*)$. For $A : M$, $\widehat{A} = A$. $\widehat{\varphi \wedge \psi} = \widehat{\varphi} \cap \widehat{\psi}$. For $\langle \varphi, \psi \rangle : U \times V$, $\widehat{\langle \varphi, \psi \rangle} = \widehat{\varphi} \times \widehat{\psi}$. For $\varphi : U$, $\widehat{\mathsf{inl}\ \varphi} = \mathsf{inl}\ (\widehat{\varphi})$; and $\widehat{\mathsf{inr}\ \varphi} = \mathsf{inr}\ (\widehat{\varphi})$. For $\varphi :: S$, $\widehat{\beta^p \varphi} = \beta^p \widehat{\varphi}$.

Recall that our goal was to establish a connection between T^* and $T(Id^*)$. To do this we need a slightly stronger result, that presents the connection between S^* and $S(Id^*)$ for all the ingredients S of T.

Lemma 4.7. *There is a family of measurable maps $r_S : S^* \to S(Id^*)$ indexed by the ingredients of T such that for all coalgebras (X, c), the diagram below commutes:*



(4.9)

and for all $\varphi : S$, $r_S^{-1}(\widehat{\varphi}) = |\varphi|$.

Proof. For $S = Id$, it is enough to let r_{Id} be the identity on Id^* and then the result is immediate.

For a constant functor M, notice that when we defined $\mathcal{L}(T)$ we included each singleton $\{m\}$ as a formula of sort M. This way, for each $s \in M^*$, there is a unique $m \in M$ such that $\{m\} \in s$. So we let $r_M(s) = m$. Then for $m \in M = M(X), r_M(d^c_M(m)) = m = 1_M(m) = M(d^c_{Id}(m))$, so (4.9) holds. For any $A : M$, if $s = d^c_M(m)$ as before, then $s \in |A|$ iff $A \in s$ iff $r_M(s) = m \in A = \hat{A}$, so $r_M^{-1}(\hat{A}) = |A|$. As in all the following cases, it is trivial to check the equality holds also for conjunctions of formulas, so we won't mention it any more.

For a product functor $U \times V$, we use define $r_{U \times V} = (r_U \times r_V) \circ \langle \pi_1, \pi_2 \rangle$.

$$
\begin{array}{ccccc}
 & & UX \times VX & & \\
 & \swarrow^{d^c_{U\times V}} & \downarrow{\scriptstyle d^c_U \times d^c_V} & \searrow^{(U\times V)d^c_{Id}} & \\
(U \times V)^* & \xrightarrow{\ \epsilon\ } & U^* \times V^* & \xrightarrow{\ r_U \times r_V\ } & U(Id^*) \times V(Id^*)
\end{array}
\tag{4.10}
$$

The triangle on the left commutes because of Lemma 4.4, and the one on the right by induction hypothesis, and this proves (4.9) for this case.

The second part is easy to check:

$$
\begin{aligned}
r^{-1}_{U\times V}(\widehat{\langle\varphi,\psi\rangle}) &= \langle\pi_1,\pi_2\rangle^{-1}((r_U \times r_V)^{-1}(\widehat{\varphi}\times\widehat{\psi})) \\
&= \langle\pi_1,\pi_2\rangle^{-1}(|\varphi| \times |\psi|) && \text{by induction hypothesis} \\
&= |\langle\varphi,\psi\rangle| && \text{by Lemma 4.4}
\end{aligned}
$$

For coproducts, we take $r_{U+V} = (r_U + r_V) \circ \alpha$. Using the induction hypothesis and Lemma 4.5, we get (4.9). The second part is also similar:

$$
\begin{aligned}
r^{-1}_{U+V}(\widehat{\mathsf{inl}\ \varphi}) &= \alpha^{-1}((r_U + r_V)^{-1}(\mathsf{inl}\ \widehat{\varphi})) \\
&= \alpha^{-1}\mathsf{inl}\ r_U(\hat{\varphi}) \\
&= \alpha^{-1}\mathsf{inl}\ |\varphi| && \text{by induction hypothesis} \\
&= |\mathsf{inl}\ \varphi| && \text{by Lemma 4.5}
\end{aligned}
$$

Similar calculations are valid for the case of ΔS: using Lemma 4.6 one gets (4.9) and for the second part:

$$
\begin{aligned}
r^{-1}_{\Delta S}(\widehat{\beta^p\varphi}) &= \epsilon^{-1}(\Delta r_S)^{-1}\beta^p(\widehat{\varphi}) \\
&= \epsilon^{-1}\beta^p(r_S)^{-1}(\widehat{\varphi}) && \text{by Lemma 3.1} \\
&= \epsilon^{-1}\beta^p(|\varphi|) && \text{by induction hypothesis} \\
&= |\beta^p\varphi| && \text{by Lemma 4.6}
\end{aligned}
$$

□

4.4 Final Coalgebras

Now we can define the coalgebra (Id^*, c^*) by letting c^* be the composition:

$$r_T \circ g : \; Id^* \to T^* \to T(Id^*) \tag{4.11}$$

where g is the function from Lemma 4.3.

Lemma 4.8. *For each T-coalgebra (X, c), d^c_{Id} is a morphism of coalgebras.*

Proof. Consider

$$\begin{array}{ccccc} X & \xrightarrow{c} & TX & & \\ \downarrow{\scriptstyle d^c_{Id}} & & \downarrow{\scriptstyle d^c_T} & \searrow^{T d^c_{Id}} & \\ Id^* & \xrightarrow[g]{} & T^* & \xrightarrow[r_T]{} & T(Id^*) \end{array}$$

The square commutes as we proved in Lemma 4.3, and the triangle is a special case of equation (4.9) in Lemma 4.7. □

In order to prove that (Id^*, c^*) is the final coalgebra for T, it remains to be proved that d^c_{Id} is the only morphism from (X, c) to (Id^*, c^*). For the proof we need the following two Lemmas.

Lemma 4.9 (Truth Lemma). *For all $\varphi : S$, $[\![\varphi]\!]^{c^*}_S = \widehat{\varphi}$.*

Proof. We prove this by induction on φ. The first base case is $\mathsf{true}_S : S$. Here $[\![\mathsf{true}]\!]^{c^*}_S = S(Id^*) = \widehat{\mathsf{true}}$.

The other base case is $A : M$, where M is a constant functor in $\mathsf{lng}(T)$. Now we have that $[\![A]\!]^{c^*}_S = A = \widehat{A}$.

The cases for the formulas of the form $\varphi \wedge \psi, \langle \varphi, \psi \rangle, \mathsf{inl}\ \varphi, \mathsf{inr}\ \psi$ are trivial from the definition of $\hat{\varphi}$.

For the formulas $[\mathsf{next}]\varphi : Id$ however, a bit more of calculation is required:

$$\begin{aligned}
[\![[\mathsf{next}]\varphi]\!]^{c^*}_{Id} &= (c^*)^{-1}([\![\varphi]\!]^{c^*}_{T}) \\
&= (c^*)^{-1}(\widehat{\varphi}) && \text{by induction hypothesis} \\
&= g^{-1}(r_T^{-1}(\widehat{\varphi})) && \text{by the definition of } c^* \text{ in (4.11)} \\
&= g^{-1}(|\varphi|_T) && \text{by Lemma 4.7} \\
&= |[\mathsf{next}]\varphi|_{Id} && \text{by Lemma 4.3} \\
&= \widehat{[\mathsf{next}]\varphi}
\end{aligned}$$

□

Lemma 4.10. *The description function $d^{c^*}_{Id} : Id^* \to Id^*$ is the identity 1_{Id^*}*

Proof. Recall that for every $\varphi : Id$, by the previous Lemma, $[\![\varphi]\!]^{c^*}_{Id} = \hat{\varphi} = |\varphi|$. Then, for all $s \in Id^*$,

$$d^{c^*}_{Id}(s) = \{\varphi : Id \mid s \in [\![\varphi]\!]^{c^*}_{Id}\} = \{\varphi : Id \mid s \in |\varphi|\} = s.$$

□

Theorem 4.1. *(Id^*, c^*) is the final T-coalgebra.*

Proof. Given an arbitrary T-coalgebra (X, c), we have seen in Lemma 4.8 that d^c_{Id} is a coalgebra morphism. If $f : (X, c) \to (Id^*, c^*)$ is another one, then by Lemma 4.1 and Lemma 4.10, $f = d^{c^*}_{Id} \circ f = d^c_{Id}$. □

5

Final Coalgebras Found in Final Sequences

In this chapter we will give a different construction of the final coalgebras for measure polynomial functors. This time we will use the final sequence of a functor T instead of the language $\mathcal{L}(T)$. The resulting construction is of course isomorphic to the one presented in Chapter 4. To emphasize the parallel between the constructions, we re-use the names of the functions $g, \langle \pi_1, \pi_2 \rangle, \alpha$ and ϵ. They will now have different definitions, but they will play similar roles in the structure of the proof.

5.1 The Space of Realised Points in the Final Sequence

For each ingredient S of T we will consider the ω^{op}-sequence $(ST^n1, ST^n!)_{n\in\omega}$

$$S1 \xleftarrow{S!} ST1 \xleftarrow{ST!} ST^2 1 \xleftarrow{ST^2!} ST^3 1 \longleftarrow \cdots$$

Let P_S be the projective limit of this chain, and let $\pi_S^n : P_S \to ST^n 1$ be the corresponding projections. To determine an element z of P_S it will be enough to know its components, that is, $\pi_S^n z$ for all $n \geq 0$.

For the particular case of $S = Id$ we will denote with π_n the projections, as we did in Example 2.6. It is worth noting that unless the functor S is ω^{op}-continuous, π_S^n won't in general be equal, or even isomorphic, to $S\pi_n$. For each coalgebra (X, c), recall the maps $h_n^c : X \to T^n 1$ defined in Section 2.4 by setting $h_0^c = !_X$ the unique function from X to 1 and $h_{n+1}^c = (Th_n^c) \circ c$. The natural source $Sh_n^c : SX \to ST^n 1$ induces a mapping $h_S^c : SX \to P_S$ such that

$$\pi_S^n h_S^c = Sh_n^c. \tag{5.1}$$

We will sometimes omit the superscript c when this is not a cause of confusion.

Definition 5.1. Let Z_S be the set of all $z \in P_S$ such that for some coalgebra (X, c), and some $x \in SX, h_S^c(x) = z$. It follows that the functions h_S^c map SX to Z_S.

Thus Z_S is the analog of the space S^* in the previous chapter. It is the collection of all sequences in the projective limit that are realized in some coalgebra. In the particular case of $S = Id$, we will denote Z_{Id} just by Z, and π_{Id}^n by π_n.

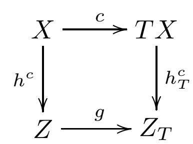
The sets Z_S inherit the measurable space structure (i.e., the σ-algebra) from the projective limit P_S. Furthermore, this σ-algebra is generated by the family $\mathcal{F}$ of sets of the form $(\pi_S^n)^{-1}(E_n)$ with E_n measurable in $ST^n 1$.

Lemma 5.1. *The family $\mathcal{F}$ of sets of the form $(\pi_S^n)^{-1}(E_n)$ with E_n measurable in $ST^n 1$ is a π-system.*

Proof. Suppose that $E = (\pi_S^n)^{-1}(E_n)$ and $F = (\pi_S^k)^{-1}(E_k)$, that is, E and F are in $\mathcal{F}$. We can assume also that $n \leq k$. Then $E \cap F = (\pi_S^n)^{-1}(E_n) \cap (\pi_S^k)^{-1}(E_k) = (\pi_S^k)^{-1}(\tau_{kn}^{-1} E_n \cap E_k)$. Since τ_{kn} is measurable, $\tau_{kn}^{-1} E_n \cap E_k$ is a measurable subset of $ST^k 1$ and $E \cap F \in \mathcal{F}$. $\square$

The measurable space Z will be the carrier of the final coalgebra for T. To define the structure map on Z, we will proceed in two stages. First, we define a map $g : Z \to Z_T$, and then we find a map r_T from Z_T to $T(Z)$ that will establish the result. The first part is taken care of in the following Lemma:

Lemma 5.2. *There exists a measurable map $g : Z \to Z_T$, such that for all coalgebras (X, c) and $x \in X, gh^c = h^c_T c$.*



Proof. We define $g : Z \to P_T$ by $\pi^n_T g = \pi_{n+1}$ for all $n \geq 0$. We need to prove that $\pi^n_T gh = \pi^n_T h_T c$.

$$\begin{aligned} \pi^n_T h_T c &= (Th_n)c \\ &= h_{n+1} \\ &= \pi_{n+1} h \\ &= \pi^n_T gh \qquad \text{by the definition of } g \end{aligned}$$

This proves that $gh = h_T c$ and also that $g : Z \to Z_T$.

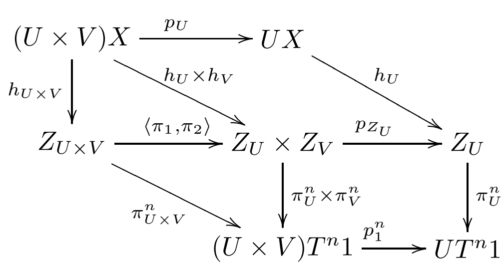
Next we prove that g is measurable. Consider a measurable set $E_n \subseteq TT^n 1$; since the sets of the form $E = (\pi^n_T)^{-1}(E_n)$ generate the σ-algebra on Z_T, it will be enough to prove that $g^{-1}(E)$ is measurable for all such sets E. But $g^{-1}(E) = g^{-1}(\pi^n_T)^{-1}(E_n) = (\pi^n_T \circ g)^{-1}(E_n) = (\pi^{n+1}_T)^{-1}(E_n)$, which we know to be measurable. □

5.2 From Z_S to $S(Z)$

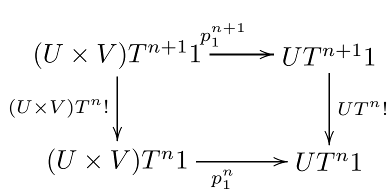
To find the map $r_T : Z_T \to T(Z)$ satisfying the appropriate conditions, we need to work making reference to the structure of T. First, we introduce some auxiliary results.

Lemma 5.3. *If $(U \times V) \in \mathsf{Ing}(T)$, there is a measurable map $\langle \pi_1, \pi_2 \rangle : Z_{U\times V} \to Z_U \times Z_V$ so that for every T-coalgebra (X, c), $\langle \pi_1, \pi_2 \rangle h^c_{U\times V} = h^c_U \times h^c_V$ and for every $n \geq 0, (\pi^n_U \times \pi^n_V)\langle \pi_1, \pi_2 \rangle = \pi^n_{U\times V}$.*

Proof. Let $p^n_1 : (U \times V)T^n 1 \to UT^n 1$, $p^n_2 : (U \times V)T^n 1 \to VT^n 1$, $p_U : (U \times V)X \to UX, p_V : (U \times V)X \to VX, p_{Z_U} : Z_U \times Z_V \to Z_U, p_{Z_V} : Z_U \times Z_V \to Z_V$ be the natural projections. The following diagram will be helpful throughout the proof:



We start by defining $\pi_1 : Z_{U\times V} \to P_U$ through $\pi^n_U \pi_1 = p_1 \pi^n_{U\times V}$. One needs to check that the sequence thus obtained is actually in P_U, i.e., for every $n \geq 0, UT^n!p^{n+1}_1 \pi^{n+1}_{U\times V} = p^n_1 \pi^n_{U\times V}$. The following diagram commutes:



Furthermore, for any $z \in P_{U\times V}$ (and therefore also for $z \in Z_{U\times V}$) and $n \geq 0$, we have that $(U \times V)T^n!\pi^{n+1}z = \pi^n_{U\times V}z$. So we can calculate $UT^n!p^{n+1}_1 \pi^{n+1}_{U\times V} = p^n_1(U \times V)T^n!\pi^{n+1}_{U\times V} = p^n_1 \pi^n_{U\times V}$.

From the definition it follows that $(\pi_U^n \times \pi_V^n)\langle \pi_1, \pi_2 \rangle = \pi_{U\times V}^n$.

To prove that $\langle \pi_1, \pi_2\rangle h_{U\times V}^c = h_U \times h_V = \langle h_U p_U, h_V p_V\rangle$, we need to show that $\pi_1 h_{U\times V} = h_U p_U$ (and the corresponding equation for V). This will be proved once we prove that for all n, $\pi_U^n \pi_1 h_{U\times V} = \pi_U^n h_U p_U$. In fact:

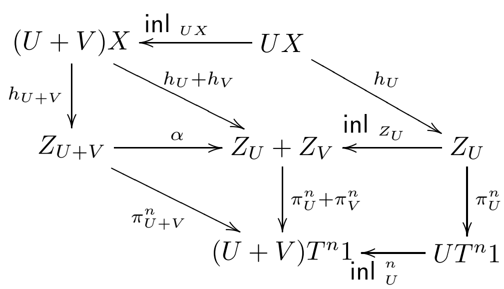
$$\pi_U^n \pi_1 h_{U\times V} = p_1^n \pi_{U\times V}^n h_{U\times V} = p_1^n (U\times V) h_n = p_1^n (U h_n \times V h_n) = U h_n p_U = \pi_U^n h_U p_U.$$

The above equation also proves that π_1 maps $Z_{U\times V}$ to Z_U.

To prove that $\langle \pi_1, \pi_2\rangle$ is measurable, it's enough to prove each of the components is measurable. We do it for π_1. Let E, E_n be as in the proof of Lemma 5.2. Then $\pi_1^{-1}(E) = \pi_1^{-1}(\pi_U^n)^{-1}(E_n) = (\pi_U^n \pi_1)^{-1}(E_n) = (p_1 \pi_{U\times V}^n)^{-1}(E_n)$, which is measurable. □

Lemma 5.4. *If $(U+V) \in \mathsf{lng}(T)$, there is a measurable map $\alpha : Z_{U+V} \to Z_U + Z_V$ so that for every T-coalgebra (X, c), $\alpha h_{U+V}^c = h_U^c + h_V^c$, and for all $n \geq 0$, $(\pi_U^n + \pi_V^n)\alpha = \pi_{U+V}^n$.*

Proof.



Since for each $z \in Z_{U+V}, z = h_{U+V}^c(x)$ for some $x \in (U+V)X$ for some coalgebra (X, c), we define α by $\alpha h_{U+V}(x) = (h_U^c + h_V^c)(x)$.

For every $n \geq 0$ we have:

$$\begin{aligned}
\pi^n_{U+V} h_{U+V} &= (U+V)h_n \\
&= Uh_n + Vh_n \\
&= \pi^n_U h_U + \pi^n_V h_V \qquad \text{by}(5.1) \\
&= (\pi^n_U + \pi^n_V)(h_U + h_V)
\end{aligned}$$

Now we can check α is well-defined. If (Y,d) is another coalgebra and $y \in (U+V)Y$ is such that $h^c_{U+V}(x) = h^d_{U+V}(y)$, then for all $n \geq 0$, $(\pi^n_U + \pi^n_V)(h^c_U + h^c_V)x = (\pi^n_U + \pi^n_V)(h^d_U + h^d_V)y$. So $(h^c_U + h^c_V)x = (h^d_U + h^d_V)y$, i.e., $\alpha h^c_{U+V}(x) = \alpha h^d_{U+V}(y)$.

It also follows from the computation above that $(\pi^n_U + \pi^n_V)\alpha h_{U+V} = (\pi^n_U + \pi^n_V)(h_U + h_V) = \pi^n_{U+V} h_{U+V}$.

To prove that α is measurable, consider $E = \mathsf{inl}_{\,Z_U}(\pi^n_U)^{-1}(E_n)$, with E_n a measurable subset of UT^n1. Then $\alpha^{-1}(E) = \alpha^{-1}(\mathsf{inl}_{\,Z_U}(\pi^n_U)^{-1}(E_n)) = \alpha^{-1}(\pi^n_U + \pi^n_V)^{-1}\mathsf{inl}_{\,U}^{\,n}(E_n) = ((\pi^n_U + \pi^n_V)\alpha)^{-1}\mathsf{inl}_{\,U}^{\,n}(E_n) = (\pi^n_{U+V})^{-1}\mathsf{inl}_{\,U}^{\,n}(E_n)$ is a measurable set. Here we used the fact that $\mathsf{inl}_{\,Z_U}(\pi^n_U)^{-1}(E_n) = (\pi^n_U + \pi^n_V)^{-1}\mathsf{inl}_{\,U}^{\,n}(E_n)$, which is easy to verify. □

Lemma 5.5. *Let ΔS be an ingredient of T. Then there exists a measurable function $\epsilon : Z_{\Delta S} \to \Delta Z_S$ so that for every T-coalgebra (X,c) $\epsilon h^c_{\Delta S} = \Delta h^c_S$ and for every $n \geq 0, (\Delta\pi^n_S)\epsilon = \pi^n_{\Delta S}$.*

Proof. To define $\epsilon(z)$ for a given $z \in Z_{\Delta S}$, we start by doing it for the family $\mathcal{F}$ of sets of the form $E = (\pi^n_S)^{-1}(E_n)$ with E_n measurable in ST^n1.

Given $z \in Z_{\Delta S}$, we define

$$\epsilon(z)(E) = \pi^n_{\Delta S}(z)(E_n)$$

It's worth remarking this definition just depends on z and not on any μ such that $h(\mu) = z$. To check that it does not depend on the selection of n, consider (X,c) and $\mu \in \Delta SX$ so that $h_{\Delta S}\mu = z$.

$$\begin{aligned}
\epsilon(z)(E) &= \pi^n_{\Delta S}(z)(E_n) \\
&= \pi^n_{\Delta S}(h_{\Delta S}(\mu))(E_n) \\
&= (\Delta S h_n)(\mu)(E_n) \qquad \text{by (5.1)} \\
&= \mu(S h_n)^{-1}(E_n) \\
&= \mu(h_S)^{-1}(\pi^n_S)^{-1}(E_n) \quad \text{by (5.1)} \\
&= \mu(h_S)^{-1}(E) \\
&= (\Delta h_S)(\mu)(E)
\end{aligned}$$

The above equation not only proves the independence of the definition from the selection of n, but also that $\epsilon h_{\Delta S}\mu(E) = \Delta h_S \mu(E)$ for every $E \in \mathcal{F}$ and $\mu \in \Delta SX$. Now we extend the definition of $\epsilon(z)$ to every measurable subset F of Z_S by letting $\epsilon(z)(F) = \Delta h_S \mu(F)$. We need to check that the probability measure $\epsilon(z)$ is well defined. If $z = h^c_{\Delta S}(\mu) = h^d_{\Delta S}(y)$, then we know that $\epsilon h^c_{\Delta S}(\mu)$ and $\epsilon h^d_{\Delta S}(y)$ agree on all the elements of the family $\mathcal{F}$. By Lemma 5.1, $\mathcal{F}$ is a π-system, then by Lemma 3.4, the measures must agree on all measurable subsets.

To prove that ϵ is measurable, first notice that for any measurable subset E_n of ST^n1, $(\Delta\pi^n_S\epsilon)(z)(E_n) = \epsilon(z)(\pi^n_S)^{-1}(E_n) = \pi^n_{\Delta S}(z)(E_n)$. By Lemma 3.5 it will be enough to prove that for a measurable subset $E_n \subseteq ST^n1, \epsilon^{-1}\beta^p(\pi^n_S)^{-1}(E)$ is measurable.

$$\begin{aligned}
\epsilon(\beta^p(\pi^n_S)^{-1}(E_n)) &= \epsilon^{-1}(\Delta\pi^n_S)^{-1}\beta^p(E_n) \\
&= (\Delta\pi^n_S\epsilon)^{-1}\beta^p(E_n) \\
&= (\pi^n_{\Delta S})^{-1}\beta^p(E_n)
\end{aligned}$$

We know the set in the last line to be measurable. We used Lemma 3.1 in the first line of the equation. □

Lemma 5.6. *There exists a measurable function $r_T : Z_T \to T(Z)$ so that for every (X, c), $r_T \circ h^c =$*

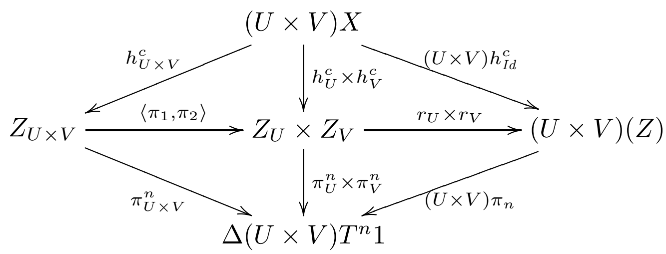
Th^c, and for every $n \geq 0, T\pi_n \circ r_T = \pi^n_T$.

Proof. We will prove this by induction over the ingredients of T. This is, if $S \in \mathsf{Ing}(T)$, then there exists a measurable map $r_S : Z_S \to S(Z)$ such that $r_S h_S = Sh$ and for all $n \geq 0, S\pi_n \circ r_S = \pi^n_S$.

For $S = Id$, $r_{Id} = 1_Z$ is measurable and trivially satisfies the conditions.

For $S = M$, a constant functor, we let $r_M = \pi^0_M : Z_M \to M = M(1)$. Then $r_M h_M = \pi^0_M \langle 1_M \rangle_{n \geq 0} = 1_M = M(h)$, and $M\pi_n r_M = 1_M \pi^0_M = \pi^n_M$ for all $n \geq 0$.

Products We define $r_{U\times V}$ as $(r_U \times r_V) \circ \langle \pi_1, \pi_2 \rangle$.

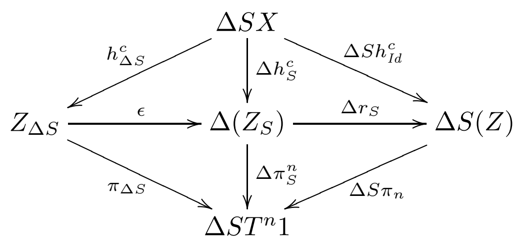


(5.2)

The triangles on the left commute by Lemma 5.3, and the ones on the right by the induction hypothesis. Hence the diagram commutes.

Coproducts We take r_{U+V} to be $(r_U + r_V) \circ \alpha$. We use the diagram from (5.2), replacing $U \times V$ with $U + V$, and Lemma 5.3 with Lemma 5.4.

Probability measures We define $r_{\Delta S}$ as $\Delta r_S \circ \epsilon$.



(5.3)

The triangles on the left commute by Lemma 5.5, and the ones on the right by the induction hypothesis. Hence the diagram commutes. □

5.3 Final Coalgebras

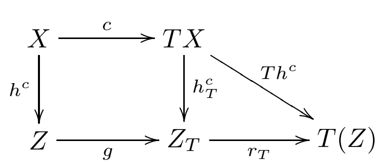
Now we are ready to define $\gamma : Z \to T(Z)$ as

$$r_T \circ g : \ Z \to Z_T \to T(Z) \tag{5.4}$$

We shall show that (Z, γ) is a final T-coalgebra.

Lemma 5.7. *For each coalgebra (X, c), h^c is a morphism of coalgebras.*

Proof. Consider the diagram:



The square commutes by Lemma 5.2, and the triangle by Lemma 5.6. □

Lemma 5.8. $h^\gamma = 1_Z$.

Proof. It will be enough to prove that for each $n \geq 0$, $\pi_n h^\gamma = \pi_n$. For $n = 0$, we have that $\pi_0 h^\gamma = h_0 =! = \pi_0 1_Z$. Now assume that $h_n^\gamma = \pi_n h^\gamma = \pi_n$. Then,

$$\begin{aligned}
\pi_{n+1} h^\gamma &= h_{n+1}^\gamma \\
&= T h_n^\gamma \circ \gamma \\
&= T\pi_n \circ \gamma && \text{by inductive hypothesis} \\
&= T\pi_n \circ r_T \circ g && \text{by the definition of } \gamma \\
&= \pi_T^n \circ g && \text{by Lemma 5.6} \\
&= \pi_{n+1} && \text{by the definition of } g
\end{aligned}$$

□

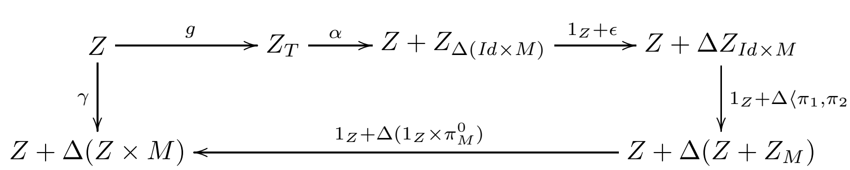
Theorem 5.1. *(Z, γ) is a final coalgebra of T.*

Proof. Let (X, c) be a T-coalgebra. By Lemma 5.7, h^c is a coalgebra morphism. For the uniqueness, suppose that f is any morphism.

By Lemma 2.2, $h^\gamma \circ f = h^c$. But by Lemma 5.8, $h^\gamma = 1_Z$, so $f = h^\gamma \circ f = h^\gamma$.

□

Example 5.1. Returning to the functor T of example 4.1, now we have that the structure map γ for Z is



$$\gamma = (1_Z + \Delta((1_Z \times \pi_M^0)\langle \pi_1, \pi_2 \rangle)\epsilon)\alpha g$$

Here we see how we use the different functions we defined in this Chapter to build the map γ for this specific functor T. Since we have given the definition of each of the intermediate maps, we see here that this method yields some more information on the structure map of the final coalgebra of a measure polynomial functor.

6

Probabilistic Kripke Polynomial Functors in Set

The work from the previous Chapters can also be carried out in Set, for probabilistic Kripke polynomial functors as introduced in [MV04]. These are functors built from the identity functor, constant functors for fixed sets, the finite, covariant *finite* power set functor denoted by $\mathcal{P}$, functions from a fixed set E, denoted by $\cdot^E$ (see Example 2.1), and the discrete measure functor $\mathcal{D}$, that assigns to a set X the set of all functions $\mu : A \to [0,1]$ with finite support and such that $\sum_{a \in A} \mu(a) = 1$ (Example 2.3).

The set $\mathsf{Ing}(T)$ is defined similarly as done for measure polynomial functors, omitting the clause for Δ and adding $\mathsf{Ing}(\mathcal{P}S) = \{\mathcal{P}S\} \cup \mathsf{Ing}(S); \mathsf{Ing}(S^E) = \{S^E\} \cup \mathsf{Ing}(S); \mathsf{Ing}(\mathcal{D}S) = \{\mathcal{D}S\} \cup \mathsf{Ing}(S)$.

In the paper [MV05], a construction of final coalgebras for probabilistic Kripke polynomial functors in Set using the method from Chapter 4 is given. In this Chapter we will do the same but using the final sequence as in Chapter 5.

Recall that in Set, the projective limit of an ω^{op}-sequence $(D_n, f_n)_{n \in \omega}$ is constructed by taking the set

$$P = \{(d_n)_n \in \prod_{n \geq 0} D_n : f_n(d_{n+1}) = d_n\}$$

and the projections $\pi_n : P \to D_n$. We will use the following fact about finite subsets of such limit P:

Lemma 6.1. *Suppose $x \in \mathcal{P}P$ and $y \in P$ satisfies $\pi_n(y) \in (\mathcal{P}\pi_n)(x)$ for all $n \geq 0$. Then $y \in x$.*

Proof. Let's write $x = \{y_1, \ldots, y_r\}$, and assume the claim is not true. Then, for every $i, 1 \leq i \leq r, y \neq y_i$ so

$$\exists n_i \ \pi_{n_i}(y) \neq \pi_{n_i}(y_i). \tag{6.1}$$

Let $K = \max\{n_1, \ldots, n_r\}$. By hypothesis, $\pi_K(y) \in (\mathcal{P}\pi_K)(x)$, so for some $j, 1 \leq j \leq r$,

$$\pi_K(y) = \pi_K(y_j).$$

Now let $\tau_{mn} : D_m \to D_n$ be the composition $f_n \circ f_{n+1} \circ \ldots \circ f_{m+1}$ and therefore for all $n \leq m, \tau_{mn}\pi_m = \pi_n$. We have that

$$\pi_{n_j}(y) = \tau_{Kn_j}\pi_K(y) = \tau_{Kn_j}\pi_K(y_j) = \pi_{n_j}(y_j),$$

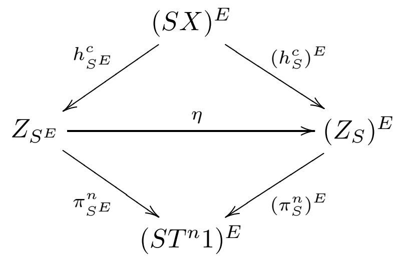
a contradiction with (6.1). □

Following the structure of the proof of Theorem 5.1, three new connecting maps are required, and we present them in the following three Lemmas.

Lemma 6.2. *Let $S^E \in \mathsf{Ing}(T)$. There is a function $\eta : Z_{S^E} \to (Z_S)^E$ such that*

1. *For all T-coalgebras (X, c), $\eta \circ h^c_{S^E} = (h^c_S)^E$.*

2. *For all* $n \geq 0, z \in Z_{S^E}$ *and* $e \in E$, $\pi^n_S(\eta(z)(e)) = \pi^n_{S^E}(z)(e)$.



Proof. We define $\eta : Z_{S^E} \to (P_S)^E$ by

$$\pi^n_S(\eta(z)(e)) = \pi^n_{S^E}(z)(e) \tag{6.2}$$

Since $z \in Z_{S^E}$, for every $n \geq 0$ we have $(ST^n!)^E \pi^{n+1}_{S^E}(z) = \pi^n_{S^E}(z)$, that is to say that $(ST^n!) \circ \pi^{n+1}_{S^E}(z) = \pi^n_{S^E}(z)$. So the sequence in the right hand of (6.2) is actually in $(P_S)^E$.

From the definition of η the second condition of the Lemma is immediately satisfied. We also note that it makes η well-defined.

For all $n \geq 0, (\pi^n_S)^E(\eta \circ h^c_{S^E}) = \pi^n_S \circ \eta \circ h^c_{S^E} = \pi^n_{S^E} \circ h^c_{S^E} = (Sh^c_n)^E = (\pi^n_S \circ h^c_S)^E = (\pi^n_S)^E(h^c_S)^E$. Since they agree on all the projections, it follows that $\eta \circ h^c_{S^E} = (h^c_S)^E$, and also proves that the image of η is in $(Z_S)^E$. □

Lemma 6.3. *Let* $\mathcal{P}S \in \mathsf{Ing}(T)$. *There is a function* $\zeta : Z_{\mathcal{P}S} \to \mathcal{P}(Z_S)$ *such that*

1. *For all* T*-coalgebras* (X, c), $\zeta \circ h^c_{\mathcal{P}S} = \mathcal{P}h^c_S$.

2. *For all* $n \geq 0$, $\mathcal{P}\pi^n_S\zeta = \pi^n_{\mathcal{P}S}$.

Proof. Let $z \in Z_{\mathcal{P}S}$. We set

$$\zeta(z) = \{u \in Z_S | \forall n \geq 0 \ \pi_S^n(u) \in \pi_{\mathcal{P}S}^n(z)\} \tag{6.3}$$

Given $x \in \mathcal{P}SX$ such that $z = h_{\mathcal{P}S}(x)$,

$$\begin{aligned}
\zeta(z) &= \zeta h_{\mathcal{P}S}(x) \\
&= \{u \in Z_S | \forall n \geq 0, \ \pi_S^n(u) \in \pi_{\mathcal{P}S}^n h_{\mathcal{P}S}(x)\} \\
&= \{u \in Z_S | \forall n \geq 0, \ \pi_S^n(u) \in \mathcal{P}Sh_n(x)\} \\
&= \{u \in Z_S | \forall n \geq 0, \ \pi_S^n(u) \in \mathcal{P}(\pi_S^n h_S)(x)\} \\
&= \{u \in Z_S | \forall n \geq 0, \ \pi_S^n(u) \in \mathcal{P}(\pi_S^n)\mathcal{P}(h_S)(x)\}
\end{aligned}$$

Using the Lemma 6.1, we get that the condition defining the set in the last line is equivalent to $u \in \mathcal{P}h_S(x)$, so $\zeta \circ h_{\mathcal{P}S} = \mathcal{P}(h_S)$, and we also get that $\zeta(z)$ is a finite set for all $z \in Z_{\mathcal{P}S}$.

To prove the second part, we use the fact that every $z \in Z_{\mathcal{P}S}$ is $z = h_{\mathcal{P}S}^c(x)$ for some coalgebra (X, c) and $x \in \mathcal{P}SX$. So for all $n \geq 0$,

$$\begin{aligned}
(\mathcal{P}\pi_S^n) \circ \zeta(h_{\mathcal{P}S}(x)) &= (\mathcal{P}\pi_S^n)(\mathcal{P}h_S)(x) && \text{by the first part of this Lemma} \\
&= (\mathcal{P}Sh_n)(x) && \text{by (5.1)} \\
&= \pi_{\mathcal{P}S}^n h_{\mathcal{P}S}(x) && \text{again by (5.1)} \\
&= \pi_{\mathcal{P}S}^n(z)
\end{aligned}$$

□

Lemma 6.4. *Let $\mathcal{D}S \in \mathsf{Ing}(T)$. There is a function $\theta : Z_{\mathcal{D}S} \to \mathcal{D}(Z_S)$ such that*

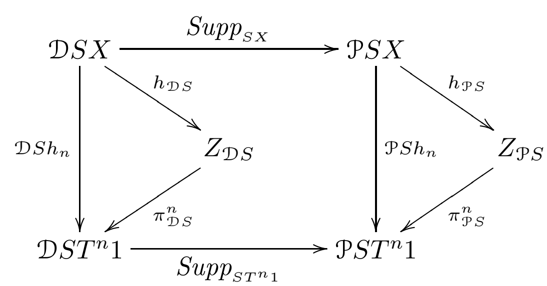
1. *For all T-coalgebras (X, c), $\theta \circ h_{\mathcal{D}S}^c = \mathcal{D}h_S^c$.*

2. For $n \geq 0$, $\mathcal{D}\pi^n_S\theta = \pi^n_{\mathcal{D}S}$

Proof. Given $z \in Z_{\mathcal{D}S}$, let $Q_z = \{u \in Z_S | \forall n \geq 0\ \pi^n_S(u) \in Supp\ \pi^n_{\mathcal{D}S}(z)\}$. (This will be the support of $\theta(z)$).

Claim: Q_z is a finite set. Every $z \in Z_{\mathcal{D}S}$ is $h_{\mathcal{D}S}(\mu)$ for some $mu \in \mathcal{D}SX$. We will use as an auxiliar construction $\mathcal{P}SX$, the finite powerset of SX, and $Z_{\mathcal{P}S}$, together with the maps $h_{\mathcal{P}S}$ and $\pi^n_{\mathcal{P}S}$, although $\mathcal{P}S$ may not be an actual ingredient of T.

For each set X, $Supp_X : \mathcal{D}X \to \mathcal{P}X$ is the application that sends each $\mu \in \mathcal{D}X$ to the finite set $\{x \in X : \mu(x) > 0\}$. It is easy to check that *Supp* is a natural transformation. Now consider the diagram:



The square commutes because of the naturality of *Supp*, and the two triangles commute because of (5.1). Now we calculate:

$$
\begin{aligned}
Q_{h_{\mathcal{D}S}}(x) = &= \{u \in Z_S | \forall n \geq 0\ \pi^n_S(u) \in Supp_{ST^n1}\pi^n_{\mathcal{D}S}h_{\mathcal{D}S}\mu\} \\
&= \{u \in Z_S | \forall n \geq 0\ \pi^n_S(u) \in (\mathcal{P}Sh_n)Supp_X\mu\} \\
&= \{u \in Z_S | \forall n \geq 0\ \pi^n_{\mathcal{P}S}(u) \in \pi^n_{\mathcal{P}S}h_{\mathcal{P}S}Supp_X\mu\} \\
&= \{u \in Z_S | u \in h_{\mathcal{P}S}Supp_X\mu\} \qquad \text{by Lemma 6.1} \\
&= h_{\mathcal{P}S}Supp_{SX}\mu
\end{aligned}
$$

To define $\theta : Z_{\mathcal{D}S} \to \mathcal{D}(Z_S)$ one must first consider for each $z \in Z_{\mathcal{D}S}$ the set Q_z. Since this is equal to $h_{\mathcal{P}S}Supp_{SX}\mu$, it is finite, so there's a number n such that if $u, u' \in Q_z$, and $u \neq u'$, then

$\pi^n(u) \neq \pi^n(u')$. Let N be the first such number. Then

$$\theta(z)(u) = \begin{cases} \pi^N_{\mathcal{D}S}(z)(\pi^N_S(u)) & \text{if } u \in Q_z \\ 0 & \text{otherwise.} \end{cases}$$

If $u \in Q_z$, then

$$\begin{aligned} \theta(h_{\mathcal{D}S}(x))(u) &= (\pi^N_{\mathcal{D}S} h_{\mathcal{D}S}(x))(\pi^N_S(u)) \\ &= \mathcal{D}Sh_N(x)(\pi^N_S(u)) \\ &= x(Sh_N)^{-1}(\pi^N_S(u)) \end{aligned}$$

On the other hand, $(\mathcal{D}h_S)(x)(u) = x(h_S)^{-1}(u)$. So if we prove that $(Sh_N)^{-1}(\pi^N_S(u)) = h_S^{-1}(u)$ we'll have proved that $\theta h_{\mathcal{D}S} = \mathcal{D}h_S$. As a consequence, this will also prove that for all $z \in Z_{\mathcal{D}S}$, $\theta(z)$ is a discrete measure.

If $s \in h_S^{-1}(u)$, then $h_S(s) = u$ and therefore $\pi^N_S h_S(s) = Sh_N(s) = \pi^N_S(u)$, i.e. $s \in (Sh_N)^{-1}(\pi^N_S(u))$.

Now we'll assume that $\pi^N_S(h_S)(s) = Sh_N(s) = \pi^N_S(u)$, and we'll prove that $h_S(s) = u$. This will be accomplished by showing that for all $n, \pi^n_S h_S(s) = \pi^n_S(u)$. We have two cases:

Case 1: $n < N$. From $Sh_N(s) = \pi^N_S(u)$ it follows that

$$\pi^{N-1}_S h_S(u) = Sh_{N-1}(s) = S(T^{N-1}! \circ h_N)(s) = ST^{N-1}!\pi^N_S(u) = \pi^{N-1}_S(u).$$

by induction, one can prove the same for all $n \leq N$.

Case 2: $n \geq N$. We have chosen N so that if $u, u' \in Q_z$ are different, then $\pi^N_S(u) \neq \pi^N_S(u')$. but it follows that for all $n \geq N, \pi^n_S(u) \neq \pi^n_S(u')$. Otherwise, we'd have $\pi^N_S(u) = S\tau_{nN}\pi^n_S(u) = S\tau_{nN}\pi^n_S(u') = \pi^N_S(u')$, contradiction.

Then, if for some $n \geq N$ we had $\pi_S^n h_S(s) \neq \pi_S^n(u)$, then $h_S(s) \neq u$, so $\pi_S^N(h_S(s)) \neq \pi_S^N(u)$, contradicting the hypothesis. This concludes the proof of part 1.

Finally, to prove the second part of the Lemma, let $v \in SF^n 1$.

$$\begin{aligned} \mathcal{D}\pi_S^n \theta(z)(v) &= \theta(z)(\pi_S^n)^{-1}(v) \\ &= \pi_{\mathcal{D}S}^n(z)(\pi_S^n[(\pi_S^n)^{-1}(v)]) \\ &= \pi_{\mathcal{D}S}^n(z)(v) \end{aligned}$$

□

Lemma 6.5. *There is a family of maps $r_S : Z_S \to S(Z)$ indexed by the ingredients of T such that the following hold:*

1. *For all T-coalgebras (X, c) and all $S \in \mathsf{Ing}(T)$, $r_S \circ h_S^c = Sh^c$.*
2. *For all $n \geq 0$, $S\pi_n r_S = \pi_S^n$.*

Proof. By induction on the ingredients S of T. The base cases and the induction steps for products, and coproducts are the same as in Lemma 5.6.

We treat in detail the induction step for functors $\mathcal{P}S$. Let $r_{\mathcal{P}S}$ be $\mathcal{P}r_S \circ \zeta$. Then for part *1* we have that

$$r_{\mathcal{P}S} \circ h_{\mathcal{P}S}^c = \mathcal{P}r_S \circ \zeta \circ h_{\mathcal{P}S}^c = \mathcal{P}r_S \circ \mathcal{P}h_S^c = \mathcal{P}(r_S \circ h_S^c) = \mathcal{P}Sh_{Id}^c.$$

For part *2* we have:

$$\begin{aligned}
\mathcal{P}S\pi_n r_{\mathcal{P}S} &= \mathcal{P}(S\pi_n) \circ (\mathcal{P}r_S) \circ \zeta && \text{by the definition of } r_{\mathcal{P}S} \\
&= \mathcal{P}(S\pi_n \circ r_S) \circ \zeta \\
&= \mathcal{P}(\pi_S^n) \circ \zeta && \text{by inductive hypothesis} \\
&= \pi_{\mathcal{P}S}^n && \text{by Lemma 6.3}
\end{aligned}$$

In a similar manner, we define r_{S^E} by $r_{S^E} = r_S^E \circ \eta$ and $r_{\mathcal{D}S}$ by $r_{\mathcal{D}S} = (\mathcal{D}r_S) \circ \theta$. □

The rest of the results of Chapter 5 are valid in Set, yielding the following Theorem:

Theorem 6.1. *All the probabilistic Kripke polynomial functors have a final coalgebra.*

7

Type Spaces

7.1 Motivation

The theory of type spaces has its origins in game theory. The intuitive idea is that a type describes a player. A player in a game can be optimistic, pessimistic, cautious, daring, suspicious, paranoid, etc. To get a mathematical definition, we need to be clear on which kind of games we are talking about, and then we can proceed to see how we can describe the 'type' of a player. We will not go into too much detail about game theory, but just enough to understand the setting in which type spaces originated.

Definition 7.1. (following [OR94]) An *extensive game with perfect information* $G = (N, A, H, P, (U_n)_{n \in N})$ consists of:

- A set N, the set of players.
- A set A, the set of actions.
- A set H of sequences (finite or infinite) of elements in A that satisfies the following three properties:

– The empty sequence $\emptyset$ is in H.

– If $(a_k)_{k=1,\ldots,K} \in H$ (where K may be infinite) and $L < K$ then $(a_k)_{k=1,\ldots,L} \in H$.

– If an infinite sequence $(a^k)_{k=1}^{\infty}$ is such that $(a^k)_{k=1,\ldots,L} \in H$ for every positive integer L, then $(a^k)_{k=1}^{\infty} \in H$.

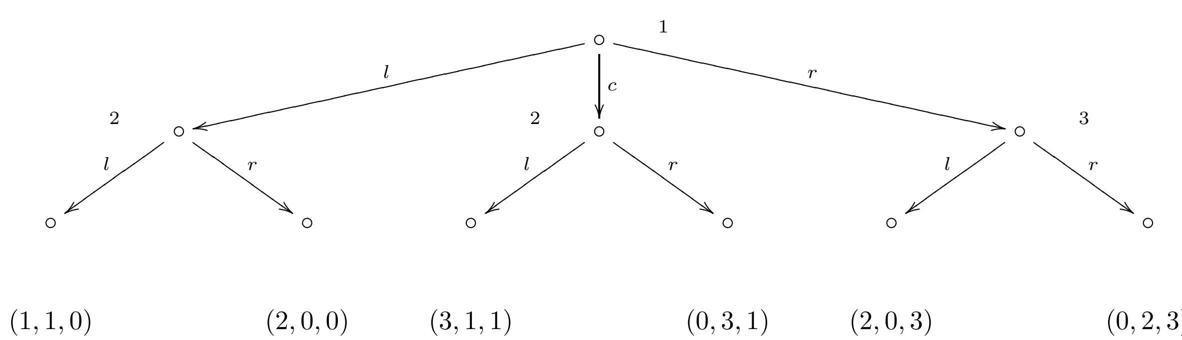
The members of H are called *histories*. A history $(a^k)_{k=1,\ldots,K} \in H$ is *terminal* if it is infinite or there is no a^{K+1} such that $(a^k)_{k=1,\ldots,K+1} \in H$. The set of terminal histories is denoted with Z. The set of actions available after the nonterminal history h is denoted $A(h) = \{a \in A | (h, a) \in H\}$.

- A function $P : H \setminus Z \to N$, that indicates for each non-terminal history in H which one of the players takes an action after the history.

- Functions $U_n : Z \to \mathbb{R}$ for $n \in N$ that give for each terminal history and each player, the *payoff* of that player after that history.

The set H can be seen as a tree with root $\emptyset$, and with its nodes labeled by the function P, and the leaves labeled by the functions U_n. We indicate the elements a^k on the edges of the tree so following a particular branch from the root will give the history that names each node.

Example 7.1.



In the diagram above we have a game where $N = \{1, 2, 3\}; P(\emptyset) = 1$ meaning that player 1 gets to decide the first move in the game, and has three options available: l, c, r (the letters stand for left, center or right, respectively). If player 1 chooses l or c, then player 2 decides what's the next action, and she has options l and r available. If player 1 chooses r instead, it is player 3 who decides what's the final move. Under each terminal node in the tree, a triple indicates the values of the utility functions U_1, U_2 and U_3. So, for example if the history of the game is (c, l), then player 1 gets a payoff of 3, while players 2 and 3 get a payoff of 1 each.

Alternatively, extensive games with complete information can be given by indicating a family of preorders $(\prec_n)_{n \in N}$ that indicate the *preferences* of the players. For our purposes, it will be enough to assume that all players prefer to maximize their payoffs and are indifferent to what other players' payoffs are.

Games with incomplete information are games in which the incompleteness of the information arises in three main ways.

1. The players may not know the *physical outcome function* of the game which specifies the physical outcome produced by each strategy available to the players.
2. The players may not know their own or some other players' *utility functions*, which specify the utility payoff that a given player i derives from every physical outcome.
3. The players may not know their own or some other players' *strategy space*, i.e. the set of all strategies available to various players.

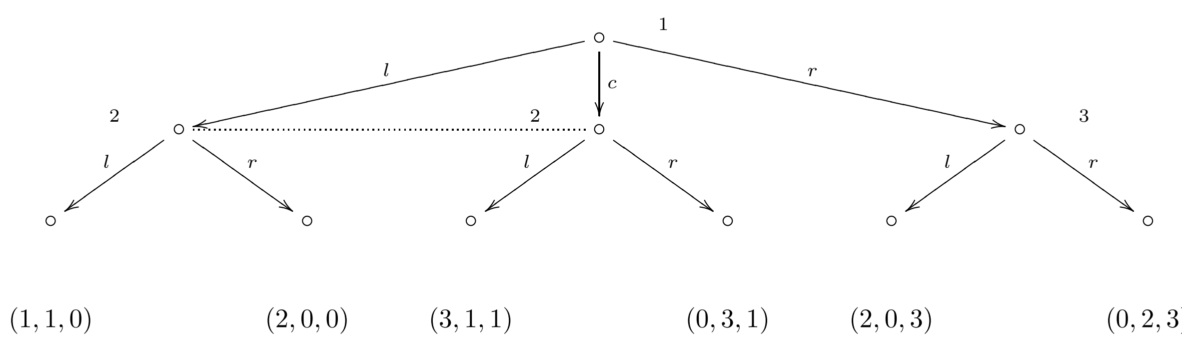
'All other causes of incomplete information can be reduced to these three basic cases– indeed sometimes this can be done in two or more different (but essentially equivalent) ways' [Har67]. The challenge is to be able to take the best possible decisions when these uncertainties are present.

A breakthrough in this field was made in 1967, when a series of papers by John C. Harsanyi, [Har67, Har68a, Har68b] saw print. The idea was to tame the uncertainty by transforming the games with incomplete information into games with complete but *imperfect* information.

Definition 7.2. An extensive game with *imperfect information* is a game $G = (N, A, H, P, U_n, \mathcal{I}_n)$ where N, A, H, P and U_n are as in Definition 7.1, and for each player $n \in N, \mathcal{I}_n$ is a partition on the set $H_n = \{h \in H \setminus Z : P(h) = n\}$ such that for two elements h, h' in the same component of the partition, $A(h) = A(h')$. The equivalence classes in this partitions are called *information sets*.

The idea here is that player n knows in which information set the game currently is, but doesn't know exactly the whole history that has lead the game into that set. Note that the players still have perfect information. They know the payoffs in all the possible outcomes.

Example 7.2.



Now the dotted line indicates that the set $\{l, c\}$ is an information set for player 2. She does not have information about whether player 1 moved to the right or to the center, but she does know what the payoffs will be in each case, and also knows that, since it's her turn, player 1 did not choose r.

If all the information sets contain exactly one node of the tree, we have a game with perfect information. The information sets allow us to represent games in which the players make their moves simultaneously (and thus don't know when making their decision what are the other players' moves), and also to represent situations in which "nature" or "chance" makes a move we cannot predict. This feature will be exploited later.

Some further assumptions are made about the games under study. In the first place, it is assumed that the beliefs the players of the game have can be represented through probability measures (this is called the *Bayesian approach*). It is also assumed that the players are aware of the extent of the knowlege or ignorance of the other players, and that they will always act "rationally", that is, they will take the action that gives them the highest possible expected payoff, based on the information available to them. The notion of rationality is quite hard to formulate and still topic of debate among game theorists (see, e.g., [Bra04]).

In Harsanyi's words, [Har67]:

> It seems to me that the basic reason why the theory of games with incomplete information has made so little progress so far lies in the fact that these games give rise, or at least appear to give rise, to an infinite regress in reciprocal expectations on the part of the players.

The argument is the following: suppose the game has incomplete information and just two players. Player 1 has some beliefs about what are the actual values of the missing information. This is represented as a probability measure over the space of all possible values the unknown could take. Player 1 also knows that player 2 cannot know the actual value and hence resorts to using a probability distribution representing her beliefs as well. In order to take a decision, player 1 then must form some mental model of what player 2's beliefs are. Player 2's beliefs include those that, in turn, player 2 has about player 1's beliefs. This kind of reasoning promptly leads to an infinite regression of unfolding beliefs. Harsanyi calls any model of this kind a *sequential-expectations* model for games with incomplete information.

Harsanyi was concerned with finding ways of analizing these games with incomplete information. The solution he offered involved the construction of a game with complete but imperfect information based on the given one with incomplete information. In the new game, there are new chance moves that are assumed to occur before the two players choose their strategies. In these random moves, the actual payoff of the two players are determined, but being a game with imperfect information, the players only know they are in some information set, and a probability distribution for the random moves (this probability distribution is assumed to be common knowledge to all the players). Using conditional probabilities, they can then derive the different expected values they need to assess the strategies to be taken in the game.

There is an alternative interpretation of the random moves added to the game, which originates the intuition in which we base our model. Instead of assuming that they determine important characteristics of the players (in particular, their payoffs), it could be assumed that the players themselves are being chosen at random from 'certain hypothetical populations containing individuals of different "types", each possible "type" of a player i being characterized by a different attribute vector c_i, i.e., by a different combination of production costs, financial resources, and states of information' [Har67].

It is these populations that we'll call *type spaces*, and their elements will be of course, *types*. While Harsanyi assumes the type space was given, he already suggested they could be constructed from the considerations about beliefs explained above:

> As we have seen, if we use the Bayesian approach, then the sequential-expectations model for any given [incomplete information] game G will have to be analyzed in terms of infinite sequences of higher and higher-order subjective probability distributions, i.e. subjective probability distributions over subjective probability distributions [Har67].

Harsanyi was discouraged from this approach by the technical difficulties it presented:

> Probability distributions over some space of payoff functions or of probability distributions, and more generally probability distributions over function spaces, involve certain mathematical difficulties [...]. However, as Aumann has shown [Aum61] and [Aum64], these mathematical difficulties can be overcome. But even if we succeed in defining the relevant higher order probability distributions in a mathematically admissible way, the fact remains that the resulting model –like *all* models based on the sequential-expectations approach–will be extremely complicated and cumbersome.

The difficulty pointed out by Aumann in [Aum61] is that if X and Y are measurable spaces and we denote by Y^X the set of all measurable functions from X to Y, then there is no natural way of endowing Y^X with a σ-algebra that makes the evaluation function $ev : Y^X \times X \to Y$ given by $ev(f, x) = f(x)$ measurable. Aumann proposes in [Aum64] to choose a single real number that represents a probability distribution. In our approach, the problem is overcome by considering the spaces ΔX instead of looking at all the measurable functions in $[0, 1]^X$ that have integral 1 over X.

So, to formalize the notion of types that Harsanyi had in mind, we want a mathematical object, the *type space*, such that each element or *type* will have associated to it, in a natural way, beliefs (represented by probability distributions) over the states of nature and the types of the other players in the game. In a game with N players, each player will assume one of the types $t \in T$, as if they were roles in a play.

A first approach would be to solve the equation $T \cong \Delta(S \times T)$, where the set T would be the type space and S the *states of nature*. The states of nature are the possible values the unknown variables in the game can take. We want both S and T to be measurable spaces so we can define probability measures on them. Let $m : T \to \Delta(S \times T)$ be the desired isomorphism. Then for each

$t \in T$, $m(t)$ represents the beliefs of a player of type t.

There are some problems with this approach. If the game has N players, then each player i with type t_i should have beliefs about the types of all the other players, so the equation to solve could be

$$T \cong \Delta(S \times T^N). \tag{7.1}$$

Furthermore, we want each type to know what his own type is, so we don't want T to be isomorphic to $\Delta(S \times T^N)$, but to the subset of $\Delta(S \times T^N)$ of probability distributions in which the marginal of each $m(t_i)$ on the i-th copy of T is the distribution δ_{t_i} which has support on the point t_i. Adding this extra condition to the definition would steer us away from the definition of coalgebras on Meas, but we can overcome this difficulty by changing the functor in an appropriate way. The key observation here is that for any product of measurable spaces $A \times B$ and $b_0 \in B$ such that the singleton $\{b_0\}$ is measurable, there is an isomorphism between the spaces $\{\mu \in \Delta(A \times B) : mar_B\mu = \delta_{b_0}\}$ and ΔA.

Recall that a probability measure μ on $A \times B$ induces, via the projections, a measure on each of the factor spaces. These measures are called *marginals*, and denoted by $mar_A\mu = (\Delta\pi_A)\mu = \mu \circ \pi_A^{-1}$; $mar_B\mu = (\Delta\pi_B)\mu = \mu \circ \pi_B^{-1}$.

The following Lemma proves that in the case above, it is enough to know the marginals to determine the measure.

Lemma 7.1. *Let μ be a probability measure on a product measurable space $A \times B$. If $mar_B\mu = \delta_{b_0}$ for some $b_0 \in B$, then $\mu = mar_A\mu \times \delta_{b_0}$.*

Proof. We only need to prove it for rectangles $G \times F$, where G is a measurable subset of A and F is a measurable subset of B.

We want to prove that $\mu(G \times F) = (mar_A\mu)(G) \times \delta_{b_0}(F)$. We have two cases: if $b_0 \notin F$, this reduces to proving that $\mu(G \times F) = 0$, and if $b_0 \in F$, then we want to show that $\mu(G \times F) = mar_A\mu(G) = \mu(\pi_A^{-1}(G)) = \mu(G \times B)$.

Notice first that for $\mu(G \times B) = \mu(\pi_A^{-1}(G)) = mar_A\mu(G) = mar_A\mu(G) \times \delta_{b_0}(B)$. Also $\mu(A \times F) = mar_B\mu(F) = \delta_{b_0}(F) = (mar_A\mu)(A) \times \delta_{b_0}(F)$.

Now we can prove that if $b_0 \notin F$, then $\mu(G \times F) \leq \mu(A \times F) = 0$, and if $b_0 \in F$, then $\mu(G \times F) = \mu(G \times \{b_0\}) + \mu(G \times (F \setminus \{b_0\}) \leq \mu(G \times \{b_0\}) + \mu(A \times (F \setminus \{b_0\})) = \mu(G \times \{b_0\})$. On the other hand, $\mu(G \times B)$ is also equal to $\mu(G \times \{b_0\}) + 0$. $\square$

Note that even though the Lemma requires the singletons in B to be measurable, once we decide to model types using the isomorphism, we can drop the condition. Now we can model the introspection condition by considering equations like

$$T \cong \Delta(S \times T^{N-1}). \tag{7.2}$$

The problem of finding a *universal type space*, that is, a type space containing all the possible types a player could adopt, could be solved by finding the final coalgebra for the functor $F(X) = \Delta(S \times X^{N-1})$. This can be done using Theorem 4.1 or Theorem 5.1. Lambek's Lemma 2.3 provides the isomorphism we are looking for.

But when we look at a single coalgebra for this functor, that is, a measurable map $m : T \to \Delta(S \times T^{N-1})$ we get a somewhat unsatisfactory model. Why should all the players come from the same type space? It would be better to be more general and to assume that there are type spaces $T_1, T_2, \ldots, T_N$ and the type of player i is selected from the corresponding T_i.

Definition 7.3. Let Meas^N be the N-fold product of the category Meas. Each object M in Meas^N is a N-tuple of measurable spaces $(M_1, \ldots, M_N)$, and the morphisms are N-tuples of measurable functions $f_i : M_i \to M_i'$. Let $Proj_i^N : \mathsf{Meas}^N \to \mathsf{Meas}$ be the i-th projection functor.

Definition 7.4. We define then a *type space* for a game with N players over the measurable space S of states of nature, as a coalgebra for the endofunctor in Meas^N given by $T = (T_1, T_2, \ldots, T_N)$ where for $1 \leq i \leq N$,

$$T_i = \Delta(S \times \prod_{j \neq i} Proj_j^N). \tag{7.3}$$

The diagram for a coalgebra (X, m) of this functor is:

$$\begin{array}{ccccccc}
(X_1 & , & X_2 & , & \cdots & , & X_N) \\
\downarrow{\scriptstyle m_1} & & \downarrow{\scriptstyle m_2} & & & & \downarrow{\scriptstyle m_N} \\
(\Delta(S \times \prod_{j \neq 1} X_j) & , & \Delta(S \times \prod_{j \neq 2} X_j) & , & \cdots & , & \Delta(S \times \prod_{j \neq N} X_j))
\end{array}$$

The definition above is a particular case of the more general one that follows.

7.2 Measure polynomial functors in many variables

Definition 7.5. A *measure polynomial functor in many variables* $T : \mathsf{Meas}^N \to \mathsf{Meas}$ is a functor built from the functors $Proj_1^N, \ldots, Proj_N^N$ and constant functors for measurable spaces, using either products, coproducts and Δ. For any natural number N', we can extend the notion of a measure polynomial functor to functors $T = (T_1, \ldots, T_{N'}) : \mathsf{Meas}^N \to \mathsf{Meas}^{N'}$ such that each $T_i, 1 \leq i \leq N'$, is a measure polynomial functor in many variables from Meas^N to Meas as defined above.

Example 7.3. For a fixed measurable space M, consider the polynomial functor in three variables $F : \mathsf{Meas}^3 \to \mathsf{Meas}^2$ given by:

$$F \;=\; (\; \Delta(Proj_1^3 + Proj_2^3) \;,\; ((\Delta Proj_3^3) \times Proj_2^3) + M \;)$$

Definition 7.6. The *ingredients* of a measure polynomial functor in many variables $T = (T_1, T_2, \ldots, T_{N'}) : \mathsf{Meas}^N \to \mathsf{Meas}^{N'}$ are defined by:

- $\mathsf{Ing}(T) = \cup_{i=1}^{N'} \mathsf{Ing}(T_i)$
- $\mathsf{Ing}(Id_i) = \{Proj_j^N : 1 \leq j \leq N\}$
- $\mathsf{Ing}(M) = \{M, Proj_j^N : 1 \leq j \leq N\}$
- $\mathsf{Ing}(U \times V) = \{U \times V\} \cup \mathsf{Ing}(U) \cup \mathsf{Ing}(V)$
- $\mathsf{Ing}(U + V) = \{U + V\} \cup \mathsf{Ing}(U) \cup \mathsf{Ing}(V)$
- $\mathsf{Ing}(\Delta U) = \{\Delta U\} \cup \mathsf{Ing}(U)$

$\mathsf{Ing}(T)$ is a finite set of functors from Meas^N to Meas.

We are going to center our attention on measure polynomial functors in many variables that are endofunctors of the category Meas^N, and the coalgebras for those functors.

Theorem 7.1. *If* $T : \mathsf{Meas}^N \to \mathsf{Meas}^N$ *is a measure polynomial functor in many variables, then it has a final coalgebra.*

Proof. We could prove this theorem using the methods from Chapter 4 or Chapter 5. Using the method of modal languages, we have to start by defining for each $i, 1 \leq i \leq N$ new sorts $Proj_i^N$, the formulas $\mathsf{true}_i : Proj_i^N$ and modal operators $[\mathsf{next}_i]$ so that if $\varphi : T_i$, then $[\mathsf{next}_i]\varphi : Proj_i^N$, with semantics given by

$$[\![\mathsf{true}_i]\!]^m_{Proj_i^N} = X_i,$$

$$[\![[\mathsf{next}_i]\varphi]\!]^m_{Proj_i^N} = m_i^{-1}[\![\varphi]\!]^m_{T_i}$$

for every T-coalgebra (X, m).

For each ingredient S of T, the measurable spaces S^* are defined as in Definition 4.5. Let $d_i^m : X_i \to (Proj_i^N)^*$ be the description map:

$$d_i^m(x_i) = \{\varphi : Proj_i^N | x_i \in [\![\varphi]\!]^m_{Proj_i^N}\}$$

These descriptions maps establish the function $d_{Id}^m : X \to Id^*$, where $Id^* = ((Proj_1^N)^*, \ldots, (Proj_N^N)^*)$. In a similar way, we have the maps $d_S^m : SX \to S^*$ for each ingredient S.

For each i, we let $g_i : (Proj_i^N)^* \to T_i^*$ be defined by

$$g_i(s) = \{\varphi : T_i | [\mathsf{next}_i]\varphi \in s\} \tag{7.4}$$

for every $s \in (Proj_i^N)^*$. Using the same arguments as in Lemma 4.3, we get that for each i the following diagram commutes.

$$\begin{array}{ccc} X_i & \xrightarrow{m_i} & T_i X \\ {\scriptstyle d^m_{Proj_i^N}}\downarrow & & \downarrow{\scriptstyle d^m_{T_i}} \\ Proj_i^* & \xrightarrow[g_i]{} & T_i^* \end{array}$$

Therefore, the diagram

$$\begin{array}{ccc} X & \xrightarrow{m} & TX \\ {\scriptstyle d^m_{Id}}\downarrow & & \downarrow{\scriptstyle d^m_{T}} \\ Id^* & \xrightarrow[g]{} & T^* \end{array}$$

also commutes. Here $T^* = (T_1^*, \ldots, T_N^*), g = (g_1, \ldots, g_N)$ and $d_T^m = (d_1^m, \ldots, d_N^m)$.

Defining r_T as N-tuple of the maps $r_{T_i} : T_i^* \to T_i(Id^*)$, and letting $m^* = r_T \circ g$, we get that $d_{Id}^m : (X, m) \to (Id^*, m^*)$ is a T-coalgebra morphism. Furthermore, using Lemma 4.10 for each component of $d_{Id}^{m^*}$, this function is the identity on Id^*, from which it follows that (Id^*, m^*) is a final coalgebra for T. □

7.3 Universal type spaces

Going back to the type spaces for a game with N players, application of the Theorem above yields a final type space, also known in the literature as *universal type space*. We also get the following Lemma:

Lemma 7.2. *If* $T : \mathsf{Meas}^N \to \mathsf{Meas}^N$ *is the functor given by* $(T_i = \Delta(S \times \prod_{j \neq i} Proj_j^N))_{1 \leq i \leq N}$*, then for each* $i, (Proj_i^N)^*$ *is isomorphic to* $\Delta(S \times \prod_{j \neq i}(Proj_j^N)^*)$ *and all the spaces* $(Proj_i^N)^*, 1 \leq i \leq N$ *are isomorphic.*

Proof. By Lambek's Lemma, 2.3, Id^* is isomorphic to $T(Id^*)$, from which follows that for each $i, (Proj_i^N)^* \cong T_i(Id^*) = \Delta(S \times \prod_{j \neq i}(Proj_j^N)^*)$.

For the second assertion, it is enough to notice the symmetry in the definition of the functor T. The construction of each space $(Proj_i^N)^*$ is the same, up to a reassignment of the indices. □

The fact that all the type spaces in the universal type space for a game with N players are isomorphic justifies naming it *the* universal type space for the game: each one of the players are of one of the types in this universal space.

Example 7.4. Assume that $N = 4$, se take a look at some of the formulas for the functor T and its ingredients.

$$true_j : Proj_j^4, 1 \leq j \leq 4$$

For A measurable in $S, A : S$

$$\langle A, \mathsf{true}_1, \mathsf{true}_3, \mathsf{true}_4 \rangle : S \times \prod_{j \neq 2} Proj_j^4$$

$$\beta^p \langle A, \mathsf{true}_1, \mathsf{true}_3, \mathsf{true}_4 \rangle : T_2$$

$$[\mathsf{next}]_2\beta^p\langle A, \mathsf{true}_1, \mathsf{true}_3, \mathsf{true}_4\rangle : Proj^4_2$$

This last formula expresses that player 2 in the game believes that the event A has probability bigger or equal than p. If we name this formula φ_2, then we can build more complex formulas, like

$$[\mathsf{next}]_1\beta^q\langle A, \varphi_2, \mathsf{true}_3, \mathsf{true}_4\rangle : Proj^4_1.$$

In this way, we can express beliefs about beliefs, and by further nesting formulas, beliefs about beliefs about beliefs, etc.

It is in this kind of formulas that we see the relevance of using coalgebras in dealing with this problem. The repeated application of the structure map allows us to unfold the different levels of beliefs of the players.

7.4 A brief review of the literature on type spaces

There have been several constructions of type spaces and universal type spaces in the literature, each one trying to capture the intuitive idea behind the definition in a slightly different way. Here we review them, as we compare them with the framework we just developed.

Armbruster, Böge and Eisele

In *Bayesian Game Theory* [AB79], W. Armbruster and W. Böge present their approach to the study of games with unknown utility functions, in which the players "will have at least a subjective probability distribution on [the] alternatives". This is called the *Bayesian assumption.* In order to construct "canonical representations for the players' subjective probability measures", the following notion is introduced, and attributed to Böge, in a lecture on game theory given in 1970.

Definition 7.7. Let $S_1^0, \ldots, S_N^0$ be compact Hausdorff spaces. An N-tuple of compact sets and continuous maps $(S_1, \ldots, S_N, \rho_1, \ldots, \rho_N)$ is called an *oracle system* for $S_1^0, \ldots, S_N^0$ if for all i, $\rho_i : S_i \to S_i^0 \times \prod_{j \neq i} \Delta_r(S_j)$.

This is the same as saying that (S, ρ) is a coalgebra for the functor $T = (S_i^0 \times \prod_{j\neq i} \Delta_r(Proj_j^N))_{1 \leq i \leq N}$ in the category CHaus of compact Hausdorff spaces and continuous functions. The underlying assumption here is that each player has a different space of states of nature S_i^0 in which their unknowns lie.

The final coalgebra is constructed by taking the projective limit of the corresponding final sequence. This final coalgebra is called the *canonical oracle system*. Note that not all the components of the functor are the same, so in general the spaces Id_i^* will not be isomorphic to each other as in Lemma 7.2. This is a reasonable assumption, and using Theorem 7.1, one can extend the definition and existence of canonical oracle systems to the general case of measurable spaces.

It is important to note that here appears for the first time a coalgebra (not necessarily the final one) as a model of the beliefs of a player. This transcends the idea of just looking for the space of all possible types, to give more restricted models that can be useful to describe situations in more manageable terms.

W. Böge and Th. Eisele present a slightly different approach in the paper *On Solutions of Bayesian Games*, [BE79]. Here again the topological setting is the category CHaus. The space over which the behavior of the players is selected is similar to the one we proposed in (7.1), but with certain restrictions.

Given a compact space of states of nature R^0, a nonempty subspace $R^1 \subseteq R^0 \times (\Delta_r R^0)^N$ of common a-priori information is selected.

Definition 7.8. A system (R, ρ) with

$$\rho : R \to R^0 \times (\Delta_r R)^N$$

is called a *system of complete reflections over the information set* R^1 if

$$(1_{R^0} \times (\Delta_r(\pi_{R^0} \circ \rho))) \circ \rho \subseteq R^1 \subseteq R^0 \times (\Delta_r R^0)^N. \tag{7.5}$$

$$\begin{array}{ccc} R & \qquad & R \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} \\ R^0 \times (\Delta_r R)^N & & R^0 \times (\Delta_r R)^N \\ \downarrow{\scriptstyle \pi_{R^0}} & & \downarrow{\scriptstyle 1_{R^0} \times (\Delta_r(\pi_{R^0} \circ \rho))^N} \\ R^0 & & R^0 \times (\Delta_r R^0)^N \end{array}$$

The space R^1 has to satisfy a couple of conditions, the first one specifying that each player knows what their beliefs are, and the second one saying that each player will try to maximize their utility function. These requirements preclude the systems of complete reflections from being coalgebras. We have seen before how the first condition, of each player knowing their beliefs, can be dealt with by taking a different functor.

The construction of the final object in the category of systems of complete reflections is done by taking the projective limit, and restricting the spaces so that the image of the map ρ for the final object is contained in R^1. It would be interesting to adapt our methods from Chapters 4 and 5 to include this kind of restrictions.

Mertens and Zamir

The paper *Formulation of Bayesian Analysis for Games with Incomplete Information* by Jean-François Mertens and Shmuel Zamir, [MZ84], is the most often cited one in the literature about type spaces.

Starting from a compact space S called *parameter-space* or set of states of nature, they seek to define a set Y of the "states of the world" in which every point contains all characteristics, beliefs and mutual beliefs of all players. The equations that summarize their goals are:

$$Y = S \times T^N \tag{7.6}$$

$$T = \text{ the set of all probability distributions on } (S \times T^{N-1}) \tag{7.7}$$

These equations are, of course, intended to be solved up to isomorphism. Equation (7.7) is essentially our (7.2). Some of the definitions in this work are interesting and we will analyze them here, trying to understand their motivation and how they are accounted for in our model.

Definition 7.9. [MZ84] Let S be a compact space. An *S-based abstract beliefs space (BL-space)* is an $(N + 3)$ tuple $(C, S, f, (t^i)_{i=1}^N)$ where C is a compact set, f is a continuous mapping $f : C \to S$ and $t^i, i = 1, \ldots, N$, are continuous mappings $t^i : C \to \Delta(C)$ (with respect to the weak-* topology) satisfying:

$$\tilde{c} \in C \text{ and } \tilde{c} \in Supp(t^i(c)) \Rightarrow t^i(\tilde{c}) = t^i(c). \tag{7.8}$$

The condition (7.8) specifies that "a player assigns positive probability (in the discrete case) only to those points in C in which he has the same beliefs. In other words, he is certain of his own beliefs." It can be rewritten as:

$$\tilde{c} \in C \text{ and } \tilde{c} \in Supp(t^i(c)) \Rightarrow \tilde{c} \in (t^i)^{-1}[t^i(c)].$$

Or the following equivalent equations:

$$Supp(t^i(c)) \subseteq (t^i)^{-1}[t^i(c)]$$

$$t^i(c)[(t^i)^{-1}[t^i(c)]] = 1$$

$$(\Delta t^i)t^i(c) = \delta_{t^i(c)}.$$

Thus, even though the first impression could be that Belief spaces are coalgebras for the functor $FX = S \times \Delta X$, we see immediately that we need the function f to have the specific codomain S, and we need many different functions t^i with codomain ΔC.

However, we can see that an adaptation from our definitions yields spaces with the same properties. Furthermore, we can drop the requirements about compactness for the space $S$and continuity for the functions. If (X, m) is a type space for a game over S with N players, as in Definition 7.4, then let

$$C = S \times \prod_{i=1}^{N} X_i$$

$$C_{-i} = S \times \prod_{j \neq i} X_j$$

Let π_i and π_{-i} be the projections from C to X_i and C_{-i}, respectively. Now for all $c \in C$, let $t^i : C \to \Delta C$ be defined by

$$t^i(c) = m_i \pi_i(c) \times \delta_{\pi_i(c)}.$$

Thus $t^i(c) \in \Delta C$. Letting $\pi_S : C \to S$ be the projection, we have that

Proposition 7.1. *$(C, S, \pi_S, (t^i)_{i=1}^N)$ is a BL-space.*

Proof. We only need to check that condition (7.8) is satisfied. Notice that the type spaces of Definition 7.4 are defined for any measurable space S, and the functions m_i need not be continuous, just measurable. Condition (7.8) is stated in terms of the support of the probability measure $t^i(c)$, which does not necessarily exist in the more general case. We will prove the condition $t^i(c)[(t^i)^{-1}[t^i(c)]] = 1$ which is equivalent to (7.8) when the support is defined.

$$\begin{aligned}
t^i(c)[(t^i)^{-1}(t^i(c))] &= t^i(c)[(t^i)^{-1}(m_i\pi_i(c) \times \delta_{\pi(c)})] \\
&= t^i(c)[((m_i \times \delta) \circ \pi_i)^{-1}(m_i\pi_i(c) \times \delta_{\pi_i(c)})] \\
&= t^i(c)[(\pi_i)^{-1}(m_i \times \delta)^{-1}(m_i\pi_i(c) \times \delta_{\pi_i(c)})]
\end{aligned}$$

The set $(m_i \times \delta)^{-1}(m_i\pi_i(c) \times \delta_{\pi_i(c)})$ is not empty, since at least $\pi_i(c)$ is in it. It is also equal to the set $m_i^{-1}m_i\pi_i(c) \cap \delta^{-1}(\delta_{\pi_i(c)}) = m_i^{-1}m_i\pi_i(c) \cap \{\pi_i(c)\}$ so its inverse image under π_i is $C_{-i} \times \{\pi_i(c)\}$. Therefore

$$\begin{aligned}
t^i(c)[(t^i)^{-1}(t^i(c))] &= t^i(c)[C_{-i} \times \{\pi_i(c)\}] \\
&= m_i\pi_i(c)(C_{-i}) \times \delta_{\pi_i(c)}(\pi_i(c)) \\
&= 1
\end{aligned}$$

□

Note that in Mertens and Zamir's approach, the universal type spaces are constructed by constructing first the universal BL-space Y and then taking taking $T = t^i(Y)$, while here we have shown how to construct belief spaces from the type spaces.

Definition 7.10. [MZ84] A *coherent beliefs hierarchy [over S] of level K* ($K = 1, 2, \ldots$) is a sequence $(C_0, C_1, \ldots, C_K)$ where:

1. C_0 is a compact subset of S and for $k = 1, \ldots, K, C_k$ is a compact subset of $C_{k-1} \times [\Delta(C_{k-1})]^N$ (as topological spaces). We denote by ρ_{k-1} and t^i the projections of C_k onto C_{k-1} and the i-th copy of $\Delta(C_{k-1})$ respectively.

$$C_0 \xleftarrow{\rho_0} C_1 \xleftarrow{\rho_1} \cdots \xleftarrow{\rho_{K-1}} C_K$$

2.

$$\rho_{k-1}(C_k) = C_{k-1}; k = 1, \ldots, K$$

3. For all $c_k \in C_k$, let $c_{k-1} = \rho_{k-1}(c_k)$. Then for all i, and $k = 2, \ldots, K$,

H1) the marginal distribution of $t^i(c_k)$ on C_{k-2} is $t^i(c_{k-1})$;

H2) the marginal distribution of $t^i(c_k)$ in the i-th copy of $\Delta(C_{k-2})$ is the unit mass at $t^i(c_{k-1}) = t^i(\rho_{k-1}(c_k))$.

The coherent hierarchies are used to build the universal beliefs space Y. They can be seen as the first K steps in the iteration that leads to the final sequence. The additional conditions we see come from different complications introduced in the construction. Part 2 of the definition states that the projections should be surjective. This condition is necessary here because the spaces C_k are compact subspaces of $C_{k-1} \times (\Delta C_{k-1})^N$ and not that whole space.

Conditions **H1**) and **H2**) of part 3 have the following intuitive meaning:

H1) says that player i's k-level beliefs coincide with his $(k-1)$ level beliefs in whatever concerns hierarchies up to level$(k-2)$. Condition **H2**) says that player i knows his own previous order beliefs. [MZ84]

Under a more technical light, **H1**) can be written as

$$(\Delta\rho_{k-2})t^i(c_k) = t^i(c_{k-1}) = t^i(\rho_{k-1}(c_k)) \tag{7.9}$$

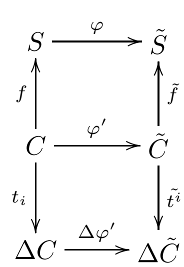
for every $c_k \in C_k$. This condition is saying that c_k is an element of the projective limit of the spaces C_k. The condition **H2**) can be written as: for every $c_k \in C_k$,

$$(\Delta\rho_{k-2})t^i(c_k) = t^i(c_{k-1}) = t^i\rho_{k-1}(c_k). \tag{7.10}$$

There is some abuse of notation here: for each number $k \geq 1$, functions $t^i : C_k \to \Delta C_{k-1}$ are defined, so there is a different function t^i that is applied to c_k and another one that's applied to c_{k-1}, and it should be clear which one is needed in each occurrence of t^i. Having (7.10) is needed in order to obtain (7.8) in the projective limit.

Morphisms between BL-spaces are defined as follows:

Definition 7.11. [MZ84] A *beliefs morphism* (BL-morphism) from a BL-space $(C, S, f, (t^i)_{i=1}^N)$ to a BL-space $(\tilde{C}, \tilde{S}, \tilde{f}, (\tilde{t^i})_{i=1}^N)$ is a pair (φ, φ') where φ' is a continuous mapping from C to $\tilde{C}$ and φ is a continuous mapping of S to $\tilde{S}$ such that for each $i; i = 1, 2, \ldots, n$, the following diagram commutes:



Given a fixed space S of states of nature, the *universal BL-space* is the final object in the category of BL-spaces over S and BL-morphisms. The universal BL-space over a fixed space S is built by taking the projective limit Y of a sequence of coherent beliefs hierarchies:

Let $Y_0 = S; Y_1 = S \times \Delta S \times \ldots \times \Delta S$ and for $k \geq 2$, let $Y_k = \{y_k \in Y_{k-1} \times [\Delta(Y_{k-1})]^N$: **H1)** For all i the marginal distribution of $t^i(y_k)$ on Y_{k-2} is $t^i(y_{k-1})$ and **H2)** the marginal distribution of $t^i(y_k)$ on $\Delta^i(Y_{k-2})$ is the unit mass at $t^i(y_{k-1})\}$.

With this definition, for each value of k, the sequence $(Y_0, \ldots, Y_k)$ is a coherent beliefs hierarchy over S of level k, and it also is the biggest one that can be constructed. All the coherent hierarchies of beliefs can be mapped to the ones constructed above, and all the BL-spaces can be mapped in a unique way to their limit Y.

It is clear from the proof given that the spaces under consideration are assumed to be compact Hausdorff topological spaces. Mertens and Zamir use Riesz's Representation theorem to prove what essentially amounts to Theorem 3.4, but one needs to also assume that the probability measures involved are all regular, as [AB79] and [Hei93] point out.

Heifetz and Samet

Aviad Heifetz and Dov Samet, in their paper *Topology-Free Typology of Beliefs*, [HS98], are the first to solve the problem of finding the universal type space in the general case of measurable spaces. They present two constructions of the space, much in the spirit of the two constructions of final coalgebras for measure polynomial functors presented in Chapters 4 and 5, respectively.

Their methods have provided us with guiding insight for the extensions we presented here. In the language-based construction, we have presented new languages $\mathcal{L}(T)$ based on each measure polynomial functor T. We have also introduced two important refinements:

Their operator $B_i^p(e)$ is used to express that a player i believes that an event represented by e has probability bigger than p. In our formulation, this would be expressed as $[\mathsf{next}]_i \beta^p(e)$, a formula of sort $Proj_i^N$ for the functor T from definition 7.3 that is, the syntactic operator B_i^p has been factored in two parts. This allows us to have more expressive power and describe points of coalgebras that are not of the form $\Delta S(X)$ for some measure polynomial functor S.

Our refinement of their Lemma 3.5 to get Lemma 3.6 allows us to work with languages without negation, by proving that a π-system of generators is enough to generate the σ-algebra on ΔX, and a boolean algebra is not necessary.

On section 5 of the paper, *The Universal Type Space in Terms of Hierarchies*, they describe types by their *histories*, and therefore the notation h we borrowed for the map $h^c : X \to Z$. Here the coalgebraic and categorial machinery we have used allowed us to simplify the construction, providing both more clarity and generality.

Other related work

Among other work related to type spaces, we'd like to mention some in particular.

Spyros Vassilakis, in [Vas91], identifies the final sequence method as the right one to obtain a solution for $X = \Delta(S \times X)$ in the category of Compact Hausdorff spaces. He also suggests further applications in [Vas90].

Brandenburger and Dekel in [BD93] propose a similar construction to that of [MZ84], and explore the relation of the concept of types with the one of common knowledge.

Luc Lismont in [Lis92] and Aviad Heifetz in [Hei96] present models of type spaces in the framework of non-wellfounded set theory, which has a close relation to the theory of coalgebras since its origin (see [Acz88] and [BM96]).

Probabilistic logic applied to type spaces has been studied by Heifetz and Mongin in [HM01], and Meier in [Mei01]. Meier also explored the simpler case of type spaces when the probabilities are given by *finitely additive* measures in [Mei02].

This being just a cursory overview of the literature on this topic, it shows the interest in the problem, and also suggest directions for further development of both the applications and the general theory presented in previous chapters.

8

Conclusions

In this dissertation we have tried to expand the usefulness of the theory of coalgebras, by studying them in a previously unexplored category, that of measurable spaces and functions. The main purpose on doing so, is to be able to express and model problems involving probability measures.

Most of the applications of coalgebras have been done in the category of sets, with a particular emphasis on theoretical computer science. Some examples of this include data structures and automata theory. But these notions have been generalized to include probabilities: in some systems the transitions are not deterministic, but they may follow some probability distribution.

Coalgebras provide a formal framework to think about processes, transitions between states are the essence of the structure maps $c : X \to T(X)$. The particular functor T used in the model determines what may be referred to as “observable behavior” of the system: some new information abou an element $x \in X$ is obtained after we apply the transition c to it. But this element $c(x)$ will also make reference to new elements in X, the new state (or set of states, or probabilities about the states) in the system. To this new state we can apply c again to look further into the behavior of the element x. This idea leads naturally to the quest of somehow collecting all possible behaviors under a certain functor. Finding the final coalgebra for the functor achieves even more than this:

two elements in different coalgebras with the same history of behavior are mapped to the very same *element* in the final coalgebra.

We have presented two constructions of the final coalgebra for measure polynomial functors. These functors comprehend a wide class of useful functors for probabilistic systems. The purpose of giving two constructions is that they illuminate different aspects of the mathematical object under study (i.e. the final coalgebras for the measure polynomial functors).

Using the languages $\mathcal{L}(T)$ from Chapter 4 provides a way of reasoning about the systems. Although an axiomatization of a logic has not been given yet, some of the literature points in that direction (for example, [Mos99], [HM01], [Mei01], [CP04]). One would like to have a deductive method for telling when a set of formulas is actually a realized theory of Id^* without appealing to a model. In other words, the goal would be to produce a complete logic for these languages.

Reasoning about probabilistic systems is specially appealing when we regard the probabilities as subjective beliefs as we've done in Chapter 7. We have used the category Meas^N to model several agents having beliefs about the beliefs of each other, and we were able to model introspection (knowledge of one's own beliefs) in this setting. It will be interesting to take these insights to the category of sets and use them to create modal logics in which to study not only introspection but also the problem of common knowledge.

The second construction offered is simpler, possibly facilitating the use of these coalgebras for a wider audience not interested in logic. In this approach, the idea of describing behaviors is accomplished not by formulas but by trajectories a space that is easy to build just by iterating applications of the functor. Furthermore, the category-theoretic presentation helps clarifying the different assumptions made in the existing solutions for the problem of universal type spaces.

In both constructions the *ingredients* of the functor play an important role. For each one of the ingredients S a space is constructed, S^* in the language driven construction and Z_S in the final sequence approach. We have kept in both approaches the same names for the measurable maps connecting the different spaces so to make the parallel between the constructions clear. The structure map of the final coalgebra is presented factored as a series of maps that have easy and natural definitions. That is not to say that proving they have the required properties is easy. Some technical difficulties need to be overcome, in particular for the map ϵ going from $(\Delta S)^*$ in Lemma 4.6 and from $Z_{\Delta S}$ to ΔZ_S in Lemma 5.5. In particular, it is in the proof of Lemma 4.6 that our Lemma 3.6 lets us do the construction without incorporating negations (they are not needed) to the languages $\mathcal{L}(T)$.

The general intuition on how the two given proofs work is that a given element in the final coalgebra must be the description of a particular element x in some coalgebra (X, c). This element already has a prescribed target $c(x)$, and we can look for the description of $c(x)$ in T^* or Z_T. Having one of the constructions does not automatically yield the other. By the nature of their underlying sets, the proofs of finality require different techniques.

We hope that the tools presented in this dissertation will find their applications both in computer science, economics and any other fields where spaces of probabilities may appear, and need to be reasoned upon.

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Curriculum Vitae

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Education

- 2005. PhD (Mathematics) Indiana University, Bloomington, USA. Advisor: Lawrence S. Moss.
- 2005. Logic Certificate. Logic Program, Indiana University, Bloomington, USA.
- 1999. Magister en Matemática, Universidad Nacional del Sur, Bahía Blanca, Argentina. Advisor: Luiz F. Monteiro.
- 1994. Licenciado en Matemática, Universidad Nacional del Sur, Bahía Blanca, Argentina.

Teaching Experience

- August 1999-July 2005. Associate Instructor. Department of Mathematics, Indiana University, Bloomington.
- July 1998-August 1999. Teaching Assistant. Department of Mathematics, Universidad Nacional del Sur, Argentina.
- March 1993-March 1998. Junior Teaching Assistant. Department of Mathematics, Universidad Nacional del Sur, Argentina.

Professional Service

- Local Student co-chair for NASSLLI 2003, Indiana University, Bloomington.

Honors

- Rothrock Teaching Award for Associate Instructors, Department of Mathematics, Indiana University, April 2004.
- James P. Williams Memorial Award, Department of Mathematics, Indiana University, April 2000.
- Fulbright Fellowship. 1999-2001.
- Fellowship from CONICET (National Research Council, Argentina). 1995-1999.
- “25 de Mayo” Award, (Best GPA among graduates in Exact, Physical and Natural Sciences), Universidad Nacional del Sur, June 1995.

Articles in Journals and Conference Proceedings

- I. Viglizzo. *Final sequences and final coalgebras for measurable spaces.* To appear in the proceedings of CALCO (Conference on Algebra and Coalgebra in Computer Science) 2005, LNCS.
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