An Annotated Bibliography of Array Studies

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I have recently completed an extensive investigation into the use of arrays as compositional units. Particular emphasis has been placed on twelve-tone aggregate-forming arrays (also known as Combination Matrices or CM's) in which the columns as well as the rows have been ordered,[1] much attention having been paid to CM's in which the rows and columns are ordered as transforms of the **same** set (so-called self-deriving arrays). In the course of my research, I have compiled a comprehensive if not exactly encyclopedic survey of the currently available literature on arrays and/or ordering problems. While none of the cited articles requires extensive technical knowledge, most of them are likely to seem more palatable to the reader who is comfortable with twelve-tone jargon (of the level of complexity that I have been using in this paper) and with the occasional presentation of theorems in the form of mathematical equations.

Most of these articles are of moderate to high quality. A handful are less well presented and are included because they deal with rarely discussed issues.

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[1] This phenomenom is known as Multiple order function (MOF) since every pitch functions as a given order-element of two or more set statements.

Finally, it should be noted that arrays and their concomitant ordering problems are not widely documented.[2] There are many articles, not listed here, however, which devote a paragraph or so to array-oriented problems. Unless otherwise noted, the articles in this bibliography either contain substantial amounts of array-relevant material or discuss in detail material which itself is of relevance, if only in a less direct manner.

1.A. Arrays

1.A.1 Specific Discussions.

Babbitt, Milton. "Twelve-tone Rhythmic Structure and the Electronic Medium." Perspectives of New Music, vol. 1, no. 1: 49-79. Reprinted in Boretz, Benjamin and Edward T. Cone (eds.). Perspectives on Contemporary Music Theory (New York: W.W. Norton, 1972): 148-179.

Fundamentally, this is the initial presentation of Babbitt's time-point system. This system was developed in part as a response to certain rhythmic implications of arrays, however, and the article contains some thoughts on these implications.

Babbitt, Milton. "Since Schoenberg." Perspectives of New Music, vol. 12 (double issue): 3-28.

A wide-ranging survey of some of the more sophisticated developments in twelve-tone technique since the death of Schoenberg. Included are discussions of transformational criteria in generalized aggregate structures (vis-a-vis the effect of CM-inversion on the internal relationships of the CM) and a description of some of the features of an array from Babbitt's Partitions (the latter is not identified as such).

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[2]With the coming of Milton Babbitt's seventieth birthday, a number of articles on his recent music and on array composition in general can be expected.

Bazelow, Alexander R. and Frank Brickle. "A Partition Problem Posed by Milton Babbitt." Perspectives of New Music, vol. 14, no. 2/ vol. 15, no. 1 (double issue): 280-293.

An attempt to formulate some solutions to the problems of generalizing the structural constraints on all-partition arrays. The authors present an algorithm for generating rows which can create 4-voice CM's capable of any partition scheme. (Unfortunately, the resultant rows are highly redundant.) The constraints so-defined are sufficient to produce all-partition arrays and are, in fact, necessary to produce the type of "any-partition" CM they describe, but they are by no means necessary for the construction of an all-partition array.

Bazelow, Alaxander R. and Frank Brickle. "A Combinatorial Problem in Music Theory--Babbitt's Partition Problem (I)." In Gerwitz, Allan and Louis V. Quintas (eds.). Second International Conference on Combinatorial Mathematics (New York: New York Academy of Sciences, 1979): 47-63.

A more technical discussion of the partition problem described in the author's earlier Perspectives on New Music article (q.v. above).

Brickle, Frank. "The Various Rhythms of Three Passages." Ph.D. dissertation, Princeton University, 1980.

Brickle takes as a point of departure a single 4x48 (unordered) array. After analyzing the basic properties of the set and of the array, he proceeds to produce three brief compositions (for solo flute, for string orchestra and for clarinet, viola and piano) based on that array. A large part of the essay is a critical evaluation of those compositions both in terms of effective realizations of the array and as coherent pieces of music.

Howe, Hubert. "Multi-dimensional Arrays." Ph.D. dissertation, Princeton University, 1972.

Howe's principal topic is a refinement of the array technique presented in Winham (q.v., below). Like Winham, these are not twelve-tone arrays nor are they easily extensible to twelve-tone arrays and thus this dissertation has only tangential relevance to my own work. However, a motivating force for Howe throughout "Arrays" is the projection of multiple levels of pitch structure. The opening sections, in fact, include some fairly careful consideration of how many levels may be reasonably projected by a given passage (this topic recurs in various guises throughout the paper).

Mead, Andrew W. "Detail and the Array in Milton Babbitt's My Compliments to Roger." <u>Music Theory Spectrum</u>, vol. 5 (1983): 89-109.

Despite its relative brevity, this is one of the most thorough analyses of a Babbitt work I have yet encountered. In addition to providing a detailed structural description of the work (including the relationship of rhythmic/metric structure to the PC array--a relationship which is generally ignored by analysts), Mead goes on to examine some of the ways in which Babbitt achieves musical continuity in his realization of the array.

Mead, Andrew W. "Some Recent Developments in the Music of Milton Babbitt." <u>Musical Quarterly</u>, vol 70, no. 3 (1984): 310-331.

In many respects aimed at an audience relatively unfamiliar with twelve-tone music, this article is nonetheless valuable for its summation of recent developments in Babbitt's array-manipulative techniques (some of which I have discussed in other contexts above). The principal topics of discussion are the all-partition array, weighted aggregates, all-trichordal rows and what Mead calls the "superarray."[3] Mead mentions many recent pieces of Babbitt in passing but does spend a fair amount of time discussing the large-scale forms of Arie da Capo and Paraphrases.

Morris, Robert. "Combinatoriality without the Aggregate." Perspectives of New Music, vol. 21 (1982/83 (double issue)): 431-486.

Morris here develops a theory of CM structure which replaces the notion of the aggregate with that of "norms," i.e. set-classes which are created by the union of any two adjacent CM positions (vertical or horizontal). The "rows" so created are called "chains" and the arrays formed by their union, "CM's." These CM's are subject to most of the standard array-manipulative techniques (wholesale TnI, swapping, folding, etc.). In addition, because individual positions are not internally ordered, various extensions to these operations become possible. The lack of order also allows the introduction of some other operators which are not generally possible in twelve-tone CM's. Most notable of these are the re-positioning of entire columns of the CM and the rotation of the entire CM.

There are some ideas of potential interest here, even if they are only vaguely relevant to twelve-tone CM's. The most obvious of these is the use of norms to construct rows which are saturated with a given set-class (norm). Because norms are, by definition, crucial to the generation of chains and CM's, there is extensive discussion of their deployment. The construction of a saturated series is simply a specific instance of the general cases covered in this discussion.

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3 Briefly, the simultaneous statement of two or more all partition arrays.

Starr, Daniel and Robert Morris. "A General Theory of Combinatoriality and the Aggregate." Perspectives of New Music, vol. 16, nos. 1 & 2 (1977/78 (double issue)): 3-35 and 50-84, respectively.

Probably the single most comprehensive (and comprehensible) presentation of twelve-tone combinatorial techniques currently available. A must for anyone interested in these techniques and particularly recommended to the reader who is tired of endless pages of often superfluous mathematical proofs.

Winham, Godfrey. "Composition with Arrays." Ph.D. dissertation, Princeton University, 1964. Reprinted in Perspectives of New Music, vol. 9, no. 1 (1970: 43-67 and again in Boretz and Cone, Contemporary Theory, pp. 261-285.

A brief exposition of Winham's compositional techniques which deal with transformations of two-dimensional, non-twelve-tone arrays. In Part I of the paper, Winham formulates a "system" of arrays which is general enough "to encompass any passage of music." A second part presents detailed commentary on the use of the system in one of Winham's compositions. Many suggested types of array usage (falling outside the actual definition of the system) are described here. These usages were later incorporated into a more restricted definition of the system of arrays by Howe (q.v. above).

1.A.2. Relevant analyses. In addition to the Mead article cited above, virtually any minimally competent analysis of the music of Babbitt or later Martino is likely to have at least some interesting observations on the use of arrays as large scale structural units. Those listed below are cited without commentary but it should be noted that their quality varies.

Arnold, Stephen and Graham Hair. "An Introduction and a Study: String Quartet No. 3." Perspectives of New Music, vol. 14, no. 2/vol. 15, no. 1 (double issue): 155-186.

Babbitt, Milton. [On Relata I] in Hines, Robert Stephen (ed.). The Orchestral Composer's Point of View (Norman: University of Oklahoma Press, 1970). Reprinted in Perspectives of New Music, vol. 9, no. 1 (1970), pp. 1-22.

Boretz, Benjamin. "Babbitt, Milton." in Vinton, John (ed.) Dictionary of Contemporary Music (New York: E. P. Dutton & Co., 1971).

Fennelly, Brian. "DONALD MARTINO: Parasonatina al'Dodecaphonia (1964)." Perspectives of New Music, vol. 8, no. 1 (1969): 133-135.

Mead, Andrew W. "Detail and the Array in Milton Babbitt's My Compliments to Roger." See entry above.

Rahn, John. "How do you Du (by Milton Babbitt)?" Perspectives of New Music, vol. 14, no. 2/vol. 15, no. 1 (double issue): 61-80.

Rothstein, William. "Linear Structure in the Twelve-tone System: An Analysis of Donald Martino's Pianississimo." Journal of Music Theory, vol. 24, no. 2 (1980): 129-165.

Weinberg, Henry. "Donald Martino: Trio (1959)." Perspectives of New Music, vol. 2, no. 1 (1963): 82-90.

Zuckermann, Mark. "On Milton Babbitt's String Quartet No. 2." Perspectives of New Music, vol. 14, no. 2/vol. 15, no. 1 (double issue): 85-110. Excerpted from Zuckermann, Mark. "Derivation as an Articulation of Set Structure: A Study of the first Ninety-Two measures of Milton Babbitt's String Quartet No. 2. Ph.D. dissertation, Princeton University, 1976.

1.B. Ordering

To the best of my knowledge, there has been relatively little research done on the problems of constructing ordered CM's. Only a very small portion of this deals with self-deriving rows and much of what has been done has not been published. None of this work has been at all exhaustive. Much of it has the further drawback of dealing only cursorily with the actual structural constraints which the MOF property places on the set itself. Consequently, most of the articles listed in this section are here because they include information that is fundamental, if only in a roundabout way, to the study of ordering relations in CM's.

1.B.1. Generalized Ordering Concerns.

Babbitt, Milton. [Letter to the editor]. See Serial Forum, below.

Browne, Richmond. "Re the Babbitt/Mallalieu Fully Cyclically Permutational Row." See Serial Forum, below.

Lewin, David. "On Certain Techniques of Reordering in Serial Music." *Journal of Music Theory*, vol. 10, no. 2 (1966): 276-287.

Lewin here formalizes the operation of taking every nth PC of a row,[4] enumerating the various

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[4]This is equivalent to multiplying each order number of the original set by n(mod 13), and thus bears a similar relation to the M (cycle of fourths transform) operator as R does to I in that the first member of each pair operates on order numbers and the second performs the same arithmetic on PC numbers.

interactions of different values of n. This procedure has only a tenuous connection with the present investigation but if some theory of invariance could be developed for the operation, it might serve as a basis for generating one class of self-deriving row. Lewin also makes passing reference to the "Mallalieu row" discussed in the Serial Forum cited below.

Lewin, David. [Letter to the Editor]. See Serial Forum, below.

Lewin, David. "On Partial Ordering." *Perspectives of New Music*, vol. 14, no.2/vol. 15, no. 1 (double issue):252-257.

A brief discussion of the order constraints of a given partition with emphasis on how to determine the order constraints which two rows (or two columns) have in common.

Martino, Donald. "The Source Set and its Aggregate Formations." *Journal of Music Theory*, vol. 5, no. 2 (1961): 224-273. Addendum in *Journal of Music Theory*, vol.6, no. 2 (1962): 322-323.

Martino examines and summarizes all possible ways to create equally partitioned aggregates (i.e., 6^2, 4^3, 3^4 and 6^2) from arbitrary source sets and extrapolates to present some data on the formation of asymmetrical (e.g. 5 7) partitions. If any aspect of symmetric combinatoriality can be generalized, it is summarized amongst the plethora of tables given in this article. The discussions of "trichordal/tetrachordal intersections" and "harmonic hexachords" show a desire to relate certain linear formations to significant "vertical" events and to set forth criteria for determining what vertical events will result from various classes of combination.

Morris, Robert. "More on 0,1,4,2,9,5,11,3,8,10,7,6." See Serial Forum, below.

Rothgeb, John. "Some Ordering Relations in Twelve-Tone Music." Journal of Music Theory, vol. 11, No. 2 (1967): 176-197. Adapted from Yale University Master's degree thesis, 1965.

This bears a somewhat cursory relation to the MOF problem. Rothgeb is concerned with order constraint intersections of different forms of a given set. Because the calculation of these intersections deals with the behavior of ordered cycles under different operations, Rothgeb's study may be viewed as preliminary ground work for the Morris article cited above and, in a more general way, for the constructive algorithm I have developed elsewhere.[5]

Serial Forum: "On Maximally Scrambled Twelve-tone Sets." In Theory Only, vol. 2, no. 5 (1976): 35 and vol. 2, no. 7 (1976): 8-20.

A series of brief letters and articles by Milton Babbitt, Richmond Browne, David Lewin and Robert Morris discussing various aspects of the row 014295B38A76 (dubbed "the Mallalieu row" after its discoverer, Pohlman Mallalieu). This row exhibits a remarkable degree of self-embedding: if every nth PC is taken (n=1....11, counting an extra space every time you "go around the end"), the result is always a transposition of the original row. This set cannot, strictly speaking be the basis of a self-deriving CM, however, because the residual PC's from this operation are not transforms of the row. For instance, taking every other (every "2'th") PC yields Example 1.1, the upper row of which is T1 of the original set but the bottom row of which is not related except as a rotation (by six places) of the set (i.e., the bottom row is also T1 of the original set but with the hexachords swapped). If n is greater than 2, even this "self-derivation-to-within-rotation" is not possible unless the row is polyphonized into n voices.

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[5]David Kowalski, "An Algorithm and a Computer Program for the Calculation of Self-Deriving Arrays."

Example 1.1 Every "2'th" PC from the Mallalieu row.

T1	1 2 5 3 A 6	0 4 9 B 8 7
[rotation6(P)]	0 4 9 B 8 7	1 2 5 3 A 6
	P	P

Starr, Daniel. "Sets, Invariances and Partitions." Journal of Music Theory, vol. 22, no. 1 (1978): 1-42.

The principal aim of this article is to ". . . develop a calculus of unordered pitch class sets."[6] Somewhat akin to Rothgeb here, Starr devotes a good deal of this paper to the behavior of cycles under the various TTO's. Properties of order are not generally discussed, but the possibility of ordering certain resultant combinations into a meaningful sequence is touched on. While self-derivation is not referred to, the discussion of operational partitions and of the "strength lattice"[7] play a crucial role in the full understanding of multi-dimensional cyclically-generated CM's.

1.B.2 Self-derivation.

Batstone, Philip. "Multiple Order Functions in Twelve-Tone Music." Ph.D. dissertation, Princeton University, 1965. Reprinted in Perspectives of New Music, vol. 10, no. 2 (1972) and vol. 11, no. 1 (1972): 60-72 and 92-111, respectively.

Batstone views the MOF issue in a slightly different light than I do. Whether for some personal/aesthetic reason or due to lack of

[6]Starr, "Sets," p. 1.

[7]A tabular ranking of all the TTO's according to the number and size of the cycles created by their corresponding operational partition.

awareness of the alternatives, he deals only with arrays in which all the rows consist of identical set forms but staggered so as to produce at least one aggregate which may be ordered as a transform of the set (see Example 1.2). [8] Note that contrapuntally created aggregates are not consistently formed.

Example 1.2

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P                        0 5 4             7 2 3        6 (BA189)
P (054723) 6 B A               1 8 9
P   (054)        7 2 3                6 B A       1 8 9
                 |--------------------------|
                 T7I
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Batstone does generate a formula for producing such occurrences. The formula allows one to specify the number of rows taking part in the combination, but otherwise, it is a sort of "black box" affair with no clear relationship between input data and the actual sets output. Thus it is of limited use as a practical hand-algorithm. He never mentions that multiple order functions are possible given the simultaneous use of two or more different forms of a set.

Morris, Robert. "On the Generation of Multiple Order Function Rows." _Journal of Music Theory_, vol. 21, no. 2 (1977): 238-263.

Morris provides an effective method of generating sets which may be viewed as either "...the concatenation or the merging of segments"[9] (see Example 1.3). This essentially results in a single column of a self-deriving CM. This column can be concatenated with its retrograde to produce a CM of trivial combinatoriality (see Example 1.3). As formulated by Morris, the algorithm is incapable of generating more than a

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[8]Batstone, "MOF," p. 61 (diss.), p. 111 (PNM, vol. 11, no. 1). This is an integer representation of a musical example. Pitch classes shown in parentheses are not shown in Batstone's example but presumably occur in the piece from which the example is excerpted.

[9]Morris, "Generation," p. 238.

single CM column except in this very restricted way. The basic concepts behind the algorithm are important, however.

Example 1.3: Concatenated versus merged segments.

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                                                X.
P  0 5 B 8 4 2 1 3 A 9 6 7      T7  7 0 6 5 B 9 8 A 3 4 1 2
   X.          Y.
                                    R(Y.)
         (a)                                   (b)

   P    0   5 B   8     4   2 |  1   3 A   9     6   7
   R  7   6     9   A 3   1   | 2  4     8   B 5   0
     T7                          T7R
                          (c)
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Starr, Daniel. "Derivation and Polyphony in Twelve-Tone Music." Ph.D. dissertation, Princeton University, 1980. Reprinted in Perspectives of New Music (forthcoming).

Starr provides us with a very thorough investigation of the vertical/horizontal relationship in general. Taken in conjunction with Starr and Morris, "General Theory" (q.v., above), these articles provide what may be simultaneously the most concise and the most comprehensible summation of twelve-tone polyphony available. Self-deriving CM's are only a small part of the discussion but the author does include an effective if slightly cumbersome method of generating such CM's. Like Morris's method, Starr's is introduced as a means of getting P+R arrays, but unlike Morris, Starr also mentions combinations wherein the row/row relation includes but is not restricted to R. His algorithm is easily modified to search for these other CM's.

One of the appendices to this dissertation is the author's Twelve-Tone System Library (TTSL), a very large library of FORTRAN computer routines for the analysis of twelve-tone rows. This library contains many routines which, although normally used internally by "utility routines" (i.e., user accessible subroutines), are very valuable for examining the accumulation of order constraints when constructing CM's.

Westergaard, Peter. "Towards a Twelve-Tone Polyphony." Perspectives of New Music, vol. 4, no. 2 (1966): 90-112. Reprinted in Boretz and Cone, Contemporary Theory: 238-260.

This article is notable in many respects, not the least of which is some very cogent arguing for the need for self-deriving CM's and for the need, in general, to relate more convincingly the vertical and the horizontal. It does not, however, deal with *why* self-deriving CM's are constructible nor does it go into very much detail regarding the "how" of creating such combinations. Westergaard concerns himself primarily with four, six and twelve voice counterpoint with its concomitant greater flexibility, conceding that since smaller arrays offer fewer partitional alternatives ". . .the entire advantage of the combining method--its freedom--is lost."[10] Alternative means of relating the two dimensions in two voice counterpoint are suggested in an attempt to reach a compromise between the advantages of intervallic correlation provided by MOF CM's and the partitional restrictions which make their employment in sparse counterpoint so awkward.

Wintle, Christopher. "Multiple Order Functions in Twelve-Tone Music, an Informal Addendum." Perspectives of New Music, vol. 12 (1973/74): 86-89.

This very brief essay is of little interest. It consists of little more than an annotated example suitable for appendation to the Balstone article cited above (which not coincidentally had appeared in the preceeding issue of PNM). I mention it here only in the interest of comprehensiveness.

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[10] Westergaard, "Polyphony," p. 110; PNM, p. 258 (reprint).