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In-domain Control of Partial Differential Equations

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In-domain Control of Partial Differential Equations

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DEDICATION

I dedicate this thesis to my beloved and kind parents.

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RÉSUMÉ

Cette thèse porte sur la commande des systèmes à dimension infinie décrit par les équations aux dérivées partielles (EDP). La commande d'EDP peut être divisée approximativement en deux catégories en fonction de l'emplacement des actionneurs: la commande à la frontière, où les actionnements sont appliqués à la frontière des systèmes d'EDP, et la commande dans le domaine, où les actionneurs pénètrent à l'intérieur du domaine des systèmes d'EDP. Dans cette thèse, nous étudierons la commande dans le domaine de l'équation d'Euler-Bernoulli, de l'équation de Fisher, l'équation de Chafee-Infante et de l'équation de Burgers. L'équation d'Euler-Bernoulli est un modèle classique d'EDP linéaire décrivant la flexion pure des structures flexibles. L'équation de Fisher et l'équation de Chafee-Infante sont des EDP paraboliques semi-linéaires, qui peuvent être utilisées pour modéliser certains phénomènes physiques, chimiques ou biologiques. L'équation de Burgers peut être considérée comme une simplification d'équations de Navier-Stokes en mécanique des fluides, en dynamique des gaz, en fluidité de la circulation, etc. Ces systèmes jouent des rôles très importants en mathématiques, en physique et dans d'autres domaines.

Dans cette thèse, de nouvelles méthodes qui se basent sur la dynamique des zéros et le compensateur dynamique ont été développées pour la conception et l'implémentation de lois de commande pour la commande des EDP avec des actionnements dans le domaine. Tout d'abord, nous étudions le contrôle de l'équation d'Euler-Bernoulli avec plusieurs actionneurs internes. L'inverse de la dynamique des zéros a été utilisé dans la conception de la loi de commande, ce qui permet de suivre la trajectoire prescrit souhaitée. Afin de concevoir la trajectoire souhaitée, la fonction de Green est utilisée pour déterminer la commande statique. La planification de mouvement est générée par des contrôleurs dynamiques basés sur la méthode de platitude différentielle. Pour les équations paraboliques non linéaires, la dynamique des zéros est régie par une EDP non linéaire. Par conséquent, nous avons recours à la méthode de décomposition d'Adomian (ADM) pour générer la commande dynamique afin de suivre les références désirées. Dans le cas de l'équation de Burgers, un compensateur dynamique a été utilisé. Pour obtenir la stabilité globale de l'équation de Burgers contrôlée, une rétroaction non linéaire a été appliquée à la frontière. La méthode d'ADM et la platitude ont été utilisées dans l'implémentation du compensateur dynamique.

ABSTRACT

This thesis addresses in-domain control of partial differential equation (PDE) systems. PDE control can in general be classified into two categories according to the location of the actuators: boundary control, where the actuators are assigned to the boundary of the PDE systems, and in-domain control, where the actuation penetrates inside the domain of the PDE systems. This thesis investigates the in-domain control of some well-known PDEs, including the Euler-Bernoulli equation, the Fisher's equation, the Chafee-Infante equation, and Burgers' equation. Euler-Bernoulli equation is a classical linear PDE used to describe the pure bending of flexible structures. Fisher's equation and the Chafee-Infante equation are semi-linear parabolic PDEs that can be used to model physical, chemical, and biological phenomena. Burgers' equation can be viewed as simplified Navier-Stokes equations in lower dimensions in applied mathematics, and it has been widely adopted in fluid mechanics, gas dynamics, traffic flow modeling, etc. These PDE systems play important roles in mathematics, physics, and other fields.

In this work, in-domain control of linear and semi-linear parabolic equations are treated based on dynamic compensators. First, we consider the in-domain control of an Euler-Bernoulli equation with multiple internal actuators. The method of zero dynamics inverse is adopted to derive the in-domain control to allow an asymptotic tracking of the prescribed desired outputs. A linear proportional boundary feedback control is employed to stabilize the Euler-Bernoulli equation around its zero dynamics. To design the desired trajectory, the Green's function is employed to determine the static control, and then motion planning is generated by dynamic control based on differential flatness. For the semi-linear parabolic equations, zero dynamics are governed by nonlinear PDEs. Therefore, the implementation of the in-domain control of linear PDEs cannot be directly applied. We resort then to the Adomian decomposition method (ADM) to implement the dynamic control in order to track the desired set-points. Finally, the in-domain control of a Burgers' equation is addressed based on dynamic compensator. A nonlinear boundary feedback control is used to achieve the global stability of the controlled Burgers' equation, and the ADM as well as the flatness are used in the implementation of the proposed in-domain control scheme.

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LIST OF SYMBOLS AND ACRONYMS

A	Unbounded closed linear operator
A^*	Adjoint operator of A
$D(A)$	Domain of the operator A
$R(A)$	Range of the operator A
e^{-At}	C_0-semigroup generated by $-A$
$\mathbf{R}(\lambda, A)$	Resolvent of the operator A
ADM	Adomian Decomposition Method
DPS	Distributed Parameter Systems
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
Ω	Open subset of $\mathbb{R}^n$
$\bar{\Omega}$	Closure of a set Ω
$\partial\Omega$	Boundary of the set Ω
$L^p(\Omega)$	Space of Lebesgue measurable functions with $\int_\Omega \lvert f(x)\rvert^p \mathrm{d}x < \infty$
$H^1(\Omega)$	Space of all elements of $L^2(0,1)$ having the first-order generalized derivative, that is 2th-power summable on Ω
$H^{-1}(\Omega)$	Dual space of $H^1(0,1)$
$L^1(\Omega)$	Space of integrable functions on Ω.
$L^1_{loc}(\Omega)$	Space of functions locally integrable in Ω
$C^m(\Omega)$	Space of functions with continuous derivatives of order k
$C_c^\infty(\Omega)$	Space of infinitely differentiable functions with compact support in Ω
$H_0^1(\Omega)$	Closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$
$\lVert \cdot \rVert_{L^2}$	Norm of the space $L^2(\Omega)$
$\lVert \cdot \rVert_{H^1}$	Norm of the Sobolev space $H^1(\Omega)$
$\lVert \cdot \rVert_\infty$	Norm of the essentially bounded measurable functions
$\langle , \rangle$	Inner product of the space $L^2(\Omega)$
X	Banach space with norm $\lVert \cdot \rVert$
$L(X)$	Space of bounded linear operators form X to X
$L^p(0,T;X)$	Space of all strongly measurable function with $\int_0^T \lVert u(t)\rVert^p \mathrm{d}t < \infty$
$L^\infty([0,T];X)$	Space of all strongly measurable function with $\operatorname{ess\,sup}_{0\le t\le T}\lVert u(t)\rVert < \infty$
$C([0,T];X)$	Space of all continuous functions with $\max_{0\le t\le T} \lVert u(t)\rVert < \infty$
$\mathbb{N}^*$	Set of natural numbers except for zero
$\mathbb{R}$	Set of real numbers

$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{C}$	Set of complex numbers
$\mathbf{Re}\ z$	Real part of a complex variable z
$(0,T)\times\partial\Omega$	Lateral surface of a cylinder
$(0,T)\times\Omega$	Cylinder
$\triangleq$	defined as
w_t	Derivative of $w(x,t)$ with respect to t
w_x	Derivative of $w(x,t)$ with respect to x
ξ_{x^-}	Left derivative of $\xi(x,t)$ with respect to x
ξ_{x^+}	Right derivative of $\xi(x,t)$ with respect to x
$[\xi]_{x=x_i}$	Difference between the right limit and the left limit of $\xi(x,t)$ at x_i
δ_i	Dirac delta function supported at the point x_i
$G(x,y)$	Green's function

CHAPTER 1 INTRODUCTION

1.1 General Context of PDE Control

Control theory is an indispensable branch of engineering and mathematics that addresses the behaviours of dynamical systems. It has been applied to huge number of real systems, such as electrical mechanical systems, communications and computing systems, aerospace systems, medical and biological systems, social and economic systems, etc. Feedback control is one of the most essential concepts, which plays an important role in assuring the system stability and achieving the desired performance. The basis scheme of feedback control is illustrated in Fig.1.1. The measurement of the output of the system is captured by the sensors and then, is fed to the controller. In order to compensate for the regulation error between the desired performance and the actual outputs, the resulting feedback controllers can adjust control inputs according to the control objective.

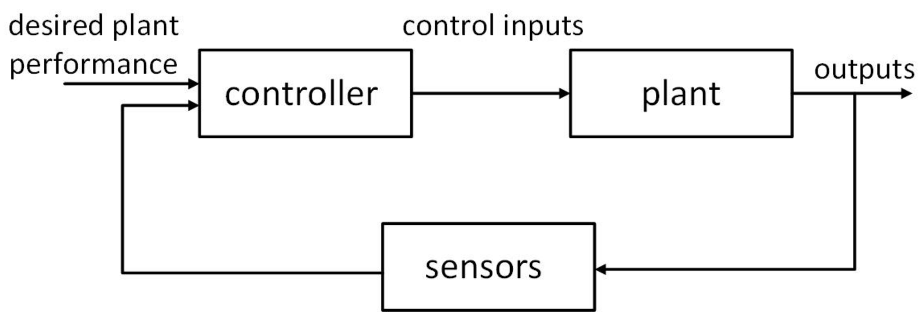

Figure 1.1 Block diagram of feedback control

Partial differential equation (PDE) is one of mainstream fields of mathematics. PDEs can be utilized to describe different physical, chemical and biological phenomenons. This work considered some PDEs that have been used in modeling of a wide variety of scientific and engineering systems. One of the typical application is the deformable mirrors in adaptive optic system in astronomical telescopes, where such devises are used to eliminate the distortion of light due to earth atmosphere turbulence [25]. A simplified model of deformable micro-mirrors in one dimensional space is a micro-beam governed as Euler-Bernoulli equation (see Fig.1.2). Another well-studied class of PDEs is the parabolic PDE. One of the typical paraboic PDE is Fisher's equation that can be used to predict the evolution of the population of advantageous genes through a geographic region [57]. Due to the impact of human activities, climate prediction becomes more and more challenging. Chaffee-Infante equation is other typical parabolic PDE that can be used to reconstruct the model of climate dynamics in order to enhance the prediction accuracy [96]. Burgers' equation considered in this work is widely addressed in the literature. Burgers' equation is indeed a simplified Navier-Stokes equation in low dimensional spaces, which is used in the analysis of fluid turbulence. It combines the effects of the nonlinear advection and the diffusion, and can be applied to other fluid-flow inspired applications, such as traffic flow [26].

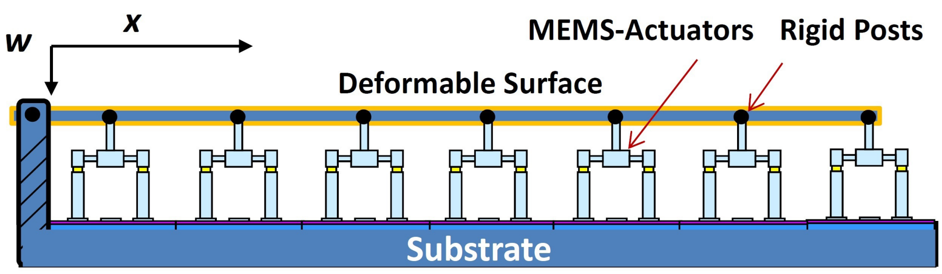

Figure 1.2 Deformable micro-beam [14]

Based on the dimension of the spaces on which the control systems are defined, control systems can be classified into two categories: finite-dimensional systems, given by ordinary differential equations, and infinite-dimensional systems, governed by abstract differential equations. Control theory for finite-dimensional systems has been considered as a classical framework, that has successfully been applied to deal with many control problems [54, 76]. A variety of methods have been developed for finite-dimensional systems, such as root locus design, frequency response methods, optimal control, backstepping control, feedback linearization, sliding mode control, Lyapunov method, etc. Control theory for finite-dimensional systems plays a crucial role in a wide range of applications, from aerospace, computer networks, and mechanical systems to economical and social systems. However, with the evolution of complexity inherent in the industrial and scientific dynamic models, classical finite-dimensional control cannot meet the growing demands of applications in biology, climate modelling, ecosystems, neural networks, telecommunication networks, etc. To bridge the gap between practical applications and control theory, infinite-dimensional systems control has been developed to solve the more complicated control problems, including time-delay systems control and partial differential equation (PDE) control [51]. There has been considerable interest recently in developing new control techniques to solve the control of infinite dimensional systems, also called distributed parameter systems (DPS), described by PDEs [88]. Note that the solutions to PDE systems evolve in an infinite-dimensional Hilbert space, the so-called Sobolev space [2]. Due to the nature of infinite-dimension space, classical finite-dimensional control theory cannot be directly extended to PDE control.

From the perspective of actuator locations, PDE control generally consists of boundary control and in-domain control. Many studies have focused on boundary control design of PDEs [66, 82, 110]. Boundary control plays an important role in stabilizing as well as tracking desired signals and rejection of disturbances. In the control of PDEs, it is still possible to leverage the techniques developed for finite-dimensional systems by discretizing the original infinite-dimensional system. With this approach, called early-lumping, the feedback controller design is based on a finite-dimensional approximation of the original PDE system. A major concern of early-lumping is that it may lead to the well-known spillover phenomenon due to the residual modes [17]. For this reason, a great effort on PDE control is devoted to the development of control system design based on the original infinite-dimensional model. This approach is called late-lumping. One of the well-known methods for PDE control is the semigroup theory, which is a classical mathematical theory and has been exploited for PDE control systems [51, 131]. The most promising advantage of the semigroup method is that it converts PDE control systems to an analogous form of finite-dimensional control systems, for which the solutions can be expressed as an extension of the exponential matrices of the finite-dimensional linear systems. This conversion allows for PDE systems to be transformed into abstract operator ordinary differential equations (ODEs) that in turn make it possible to leverage techniques for finite-dimensional control design to help solve PDE control problems, such as pole-assignment design and optimal control [52]. The semigroup method has demonstrated its effectiveness and its capacity in PDE control, especially in the well-posedness and stability analysis of PDE control systems [23, 131].

The Lyapunov direct method is a very useful tool in the stability analysis of linear and nonlinear systems [48,49]. In general, Lyapunov functions can also be used as energy functions for PDE systems. Another concept, named differential flatness, from the geometrical property of finite-dimensional control systems, has also been successful adopted to some specific linear PDE systems [100], whose variables and inputs can be represented in terms of flat outputs and their time derivatives, such as heat equation, wave equation, open channel flow, Euler-Bernoulli equation, etc.

On the other hand, studies on in-domain control of PDE systems are scattered in the literature. For an in-domain control problem, the actuation penetrates inside the domain of the PDE system or is distributed over the domain. In-domain control can be utilized to design PDE systems with prescribed performance and to achieve the steady-state responses to desired interior inputs. The method considered in this thesis is mainly based on the zero-dynamics inverse design that is developed in a series of articles from Byrns and his collaborators [28, 31, 33–35, 39]. A major contribution of this research lies in the design and implementation of dynamic compensator for in-domain control by combining zero dynamics with flatness and introducing a semi-analytical method, called the Adomian decomposition method, for nonlinear PDEs. A possible application of the considered problems is network congestion control, where the in-domain control can be designed to modify buffer queues or switches in order to avoid the overflowing of resources. A brief literature review on some methods for PDE systems analysis and control related to the present work is given below.

1.2 Boundary Control

Boundary controller synthesis, combined with the need to feasibly implement control systems in real applications, has motivated extensive research efforts on PDE control systems that comprise linear PDE system and nonlinear PDE systems [20, 61, 83, 130]. Boundary control requires actuators and sensors located on the boundary, which is easily implemented in practice [90]. Indeed, the sensors can convert physical signals, such as temperature, force, pressure, position, etc., into electrical signals, such as voltage or current, that are much more convenient to manipulate with computer hardware and software. Some control methodologies that have been successfully employed in PDE boundary control are described below.

1.2.1 Boundary feedback control

Feedback control is the fundamental basis in control theory, and many advances have been made in both theoretical and practical areas [60, 123, 126, 128]. Feedback control can be employed to stabilize infinite-dimensional systems to compensate for unwanted disturbances. It also plays a critical role for other control problems, such as output regulation, tracking control, etc.

Feedback control is particularly important for distributed parameter systems. There are at least two basic approaches to the use of feedback control for the stabilization of DPS. One way is to apply early-lumped ordinary differential equations (ODE) to approximate the original PDEs, simplifying the control procedure and facilitating the analysis of the DPS [18, 19, 45]. However, to avoid the appearance of spill-over and performance degradation, the approach of early-lumping could result in high-dimensional feedback control structures and complicated computational efforts. To compensate for model imperfection and perturbations, feedback control laws are applied directly to DPS to locally stabilize DPS [21, 36]. Meanwhile, it is difficulty or almost impossible to achieve a global stability by only using linear feedback control. Consequently, nonlinear feedback control is developed for DPS to establish global stability and improve the regulation performance [9, 78].

1.2.2 Backstepping control

Backstepping control has been employed extensively in nonlinear finite-dimensional systems. It has proven its ability to deal with nonlinear strict-feedback system under a lower triangular structure [73]. However, the classical backstepping approach cannot be applied straightforwardly to infinite-dimensional systems, because it is almost impossible to reformulate the original infinite-dimensional systems in the lower triangular form using linear finite-dimensional transformations. To extend the classical backstepping method to infinite-dimensional systems, one would have to resort to infinite dimensional transformations instead of linear finite-dimensional transformations, which can reformulate the original PDE systems into asymptotically stable systems. There are several transformation operators that can transform original PDE control systems into asymptotically stable target systems, including the invertible Volterra integral transformation and the Fredholm-type transformation [10,82]. The process of backstepping control can eliminate the unstable term of the original PDE systems and brings all the undesired terms into the boundary, thereby using boundary control schemes to stabilize the original PDE systems, which facilitates the control design of PDE systems [83]. Backstepping control was successfully applied to a variety of PDE systems, such as heat equations [20,91], wave equations [79,125], and Burgers' equations [92]. In addition, backstepping control involves the computation of the integration of the state variable with respect to the spatial domain, and thus the implementation of backstepping-based controllers requires access to the total state variables of the PDE systems, which is almost impossible for PDEs systems because of the infinite dimensional nature. Consequently, observers are required for backstepping control to make the resulting schemes implementable [81,124].

1.2.3 Differential flatness

The technique of differential flatness has been extensively exploited for finite-dimensional systems, such as DC motor dynamics [115], underactuated crane systems [144], nonlinear chemical reaction [117], etc. Differential flatness states that the state variables and the control inputs of a system can be expressed in terms of the flat outputs and their derivatives, which can greatly facilitate control design. Meanwhile, differential flatness can also be adapted to DPS [99,100,102,121]. Due to the infinite-dimensional nature of DPS, its variables and inputs can be parametrized by infinite power series, whose coefficients depend on the time-derivatives of flat outputs. Thus, it is inevitable to truncate differential flatness-based controllers using the approximated partial sums of the power series in practical implementations [87]. In addition, the choice of flat outputs is crucial for flatness-based controllers, as they should allow for the convergence and regularity of the infinite power series. The Gevrey function is an ideal candidate for flat outputs, as can be differentiated infinitely times, decays very fast and smoothly connects the set-points [121]. Differential flatness-based control design for PDEs has been developed to track prescribed trajectories and stabilize the trajectories of PDEs around the desired reference profiles via motion planning [95, 99, 100, 102, 120]. Controllability of PDEs is treated in [97, 136], where differential flatness is used to generate feasible trajectories to achieve the target outputs. A further generalization of semi- and quasi-linear parabolic PDE systems is considered in [94, 95, 103], where differential flatness is exploited to generate the desired trajectories to track the reference signals.

1.2.4 Zero dynamics inverse

The concept of zero dynamics arises from the notion of left or right half plane zeros and zeros at infinity. The notion of zero dynamics was first introduced to solve the problem of pole-placement for finite-dimensional linear systems and was then adapted to finite-dimensional nonlinear systems. This concept implies that the dynamics of linear or nonlinear systems comply with the constraint that the output is identically zero. Using input-output linearization, the dynamics of a finite-dimensional nonlinear system can be split into an external part (input-output dynamics) and an internal part (internal dynamics). Generally speaking, when the outputs of a nonlinear system are set, the zero dynamics are simultaneously determined, which are not accessible from the input-output dynamics [76]. This observation indicates that the choice of outputs is crucial for input-output linearization, as it determines whether the linearization of a nonlinear system will track the desired prescribed outputs or not while achieving the global stability for the entire system. A number of zero dynamics-based solutions for the control of nonlinear systems can be found in, e.g., [24, 39, 63, 104].

In the literature, tremendous efforts have been devoted to extending the zero dynamics of finite-dimensional systems to infinite-dimensional systems [37, 38, 74, 77], including pole-placement, zero-pole dynamics, and zero dynamics. Recently, zero dynamics inverse design has been developed for PDE systems to track desired profiles and reject disturbances. Compared to the original PDEs, the zero dynamics of the resulting PDEs are that whose boundary complies with the constraints, bringing the regulation error down to zero. One of the advantages of zero dynamics inverse is that the zero dynamics are much easier to be dealt with than the controlled PDE systems, and the state variables of the zero dynamics are available, allowing access to their estimates in both quantitative and qualitative fashion. When the zero dynamics are determined, the tracking problems of PDEs are reduced to force the original PDEs to converge to their zero dynamics via some boundary control schemes. The zero dynamics inverse approach has been applied to linear parabolic PDE systems to regulate the tracking error for set-point problems and harmonic tracking problems [31, 35]. The work in [37] extends the notion of zero dynamics for a boundary controlled Burgers' equation to derive the tracking control scheme, which can guarantee the semi-globally exponential stabilization of the Burgers' equation and secure the regulation error to decay to zero as time varying. In [34], the authors develop a feedback control law combined with zero dynamics inverse for a Kuramoto-Sivashinsky equation to solve tracking and disturbance rejection problem. Nevertheless, the implementation of zero dynamics-based dynamic compensators remains a challenging issue as it requires to solve online the zero dynamics that are also a PDE.

1.3 In-domain Control

In-domain control has been gaining increasing attention and plays a more and more critical role in PDE control. Regulating the interior points to track the desired reference signals for in-domain controlled PDE systems is still a challenging problem. This is due to the fact that the actuation mechanisms for distributed control are not feasible for most of the real-life applications. Whereas, point-wise in-domain control usually leads to unbounded input operators in the Banach space L^2 or H^1, which causes enormous difficulties in analyzing the properties of the solutions to PDEs, such as well-posedness, regularity, etc. One of the intuitive approaches to deal with the in-domain control of PDEs is to convert the in-domain control into boundary control, thereby applying the boundary control schemes developed for boundary control to derive the in-domain controllers. However, due to the effect of in-domain inputs on PDE systems, this approach is not applicable directly to PDEs with point-wise in-domain actuations. Some efforts have been investigating coping with the in-domain control of PDE systems by using differential flatness [119, 139, 145]. As linear PDE systems feature the superposition principle, it allows decomposing multiple in-domain controlled linear PDEs into a setting with parallel connected subsystems. A linear PDE with multiple in-domain control can thus be split into several linear sub-PDEs, each of which comprises only one in-domain control, Laplace transform can be applied to derive the in-domain control. This method has been demonstrated effectively for heat equation [145] and Euler-Bernoulli equation [139].

However, for nonlinear PDE systems, the above approach cannot be applied as the superposition property does not hold anymore. Indeed, it is almost impossible for nonlinear PDEs to obtain the explicit form of solutions. The mathematical tools for obtaining solutions to linear PDEs, such as Laplace transform and Fourier transform, cannot be utilized directly to nonlinear PDEs. New methods are needed for the design of in-domain control for nonlinear PDE systems. Recently, the geometric theory was introduced as a method to address PDE control problems, as it can capture the essence of the controllability and observability of nonlinear PDE systems [70, 75]. The zero dynamics inverse approach is an application of the geometric theory for PDE control and is used to simplify the control design. However, the nonlinearity of zero dynamics makes it difficult to implement the controllers developed by zero dynamics inverse. Static control is a simple method for implementing in-domain control based on zero dynamics for nonlinear PDE systems, as proposed in [11–13].

The drawback of static control is that it may result in unwanted oscillations and degrade the dynamic performance of the controlled PDE systems. To overcome this problem, dynamic control can be applied to PDEs with in-domain actuations, which can significantly enhance the performance of controlled systems. However, there are few solutions that allow solving the in-domain control problems of PDE systems using dynamic control. A dynamic in-domain control scheme capable of tracking desired set-points for a class of semi-linear parabolic equations with multiple in-domain actuations is proposed in [140], where the implementation of in-domain control based on zero dynamics utilizes the Adomian decomposition method (ADM), and linear feedback boundary control is used to guarantee the asymptotical stability of the semi-parabolic equations around their zero dynamics. In-domain control of Burgers' equation is treated using a dynamic compensator, which generates the in-domain control, and the implementation of the resulting in-domain controllers uses the combination of ADM and differential flatness in [141]. This method will be presented in detail in this thesis.

1.4 Objective and Contributions

In-domain control is of theoretical and practical importance in both academia and industry. This thesis investigates the set-point regulation problems of PDE systems with multiple point-wise in-domain actuations. The subject of PDE control involves a variety of different linear and nonlinear PDEs, such as parabolic equations, hyperbolic equations, nonlinear elliptic equations, etc. Every PDE system characterizes its unique properties, such as existence, regularity, and stability. Thus, it is impossible to develop a general theory to systematically solve PDE control problems. For this reason, we limit our scope in this work for addressing control problems of some classical PDE models: Euler-Bernoulli equation and semi-linear parabolic equations, including Fisher's equation, the Chaffee-Infante equation, and Burgers' equation. These PDEs models are instrumental in and applicable to different problems in theory and practice, as they can describe a wide range of phenomena and capture the dominant properties. The Euler-Bernoulli equation can be used to explain how a flexible structure behaves under axial forces and bending under the assumption that deformed angles are small [22]. Fisher's equation is a semi-linear PDE, which can represent the propagation of a virus mutation over a long time series [129]. Burgers' equation can capture the features of turbulent fluid flow and roughly simulate the behavior of Navier-Stokes equations [108].

We propose using zero dynamics and dynamic compensators in the control of nonlinear PDEs with multiple in-domain inputs to produce the desired in-domain control signals. However, zero dynamics (or dynamic compensators) are governed by in-homogeneous nonlinear parabolic equations, which cannot be dealt with using existing tools for obtaining their exact solutions. Hence, we resort to numerical solutions to approximate the exact solutions to the zero dynamics or dynamic compensators. As the ADM has been proved as an effective and efficient numerical method to approximate exact solutions to linear and nonlinear PDEs, it is leveraged in the present work for this purpose. One of the advantages of the ADM is that it can reduce the computational complexity while it does not sacrifice the accuracy of the numerical solution [3, 113]. Another advantage of this method is that, unlike other classical numerical methods, there is no need to discretize or linearize the system [5]. Truncating the solution until the final computational implementation is all that is required. In summary, the developed approaches in this thesis belong to the category of late-lumping.

The major contributions of this thesis are listed as follows:

1 Deformation control of an Euler-Bernoulli equation with interior actuations is developed by using zero dynamics inverse technique combined with differential flatness. The well-posedness and stability of the controlled Euler-Bernoulli equation are established based on the semigroup theory. An efficient control algorithm for the implementation of zero dynamics with multiple in-domain control inputs is devised using differential flatness. Furthermore, the Green's function of the static equation of the Euler-Bernoulli equation is employed in motion planning. The approach developed in this work presents a systematic design procedure and is applicable to a wide range of linear PDEs with point-wise actuations.

   The results of this contribution are published in the 55th IEEE Conference on Decision and Control in 2016 [139].

2 The problem of asymptotic output regulation of a class of semi-linear parabolic equations with in-domain control is treated using the zero dynamics inverse technique along with the Adomian decomposition method. Zero dynamics inverse design allows generating the desired in-domain control inputs, which can track the prescribed desired set-points. Differential flatness and the Adomian decomposition method are employed to implement the in-domain control based on zero dynamics inverse design. Finally,

Chaffee-Infante equation and Fisher's equation are used to illustrate the effectiveness of our proposed control scheme through both theoretical analysis and numerical simulations. This work contributed to extending the method of flatness-based trajectory planning to nonlinear PDEs with multiple point-wise in-domain control.

This contribution is published in the International Journal of Robust and Nonlinear Control [140].

3 Due to the fact that the nonlinear term of a Burgers' equation is not smooth, the control design procedure of Fisher's equation and Chaffee-Infante equation cannot be directly applied to Burgers' equation. To extend the proposed control scheme, a dynamic compensator is introduced to generate the desired in-domain control, which can force the controlled Burgers' equation to track the desired set-point signals. A nonlinear feedback boundary control is used to globally stabilize the controlled Burgers' equation around the trajectory of the dynamic compensator. The resulting in-domain controllers can be implemented based on the ADM.

The results of this work have been reported in a paper submitted to the International Journal of Control [141].

1.5 Dissertation Organization

The rest of this thesis is organized as follows:

Chapter 2 outlines the background knowledge of PDE control theory required for this thesis. It starts by presenting the Sobolev space, which is the fundamental tool of PDE analysis and is crucial for the stability analysis of PDE systems. As semigroups theory is a classical tool for assessing the well-posedness and stability of linear and nonlinear PDEs, some basic notions and results related to this tool are provided. Differential flatness is then presented, which can generate explicit trajectories of linear PDE systems. The Adomian decomposition method is an important numerical method to obtain the approximate solutions of linear and nonlinear PDEs, which is used intensively in this thesis.

Chapter 3 is devoted to studying the in-domain control of an Euler-Bernoulli equation. The problem is solved using differential flatness to complete the in-domain control design, which only relies on the so-called flat outputs. The Gevery function is chosen as the flat outputs in order to guarantee the convergence of the solution that is under an infinite series. To facilitate the motion planning, a static control algorithm based on the Green's function is proposed to generate the static control inputs. The simulation results confirmed the effectiveness of the proposed method.

Chapter 4 presents the in-domain control of a class of semi-linear parabolic equations, including Fisher's equation and Chaffee-Infante equation. The aim is to derive in-domain controllers to regulate the multiple interior outputs of a class of semi-linear parabolic equations to track the desired set-points. A linear boundary feedback control is used to allow the semi-linear parabolic equations to approach their zero dynamics asymptotically. A zero dynamics-based dynamic in-domain control is implemented by means of the Adomian decomposition method and flatness. The simulation results are carried out to validate the proposed approach.

In-domain control of a Burgers' equation under nonlinear boundary feedback control is presented in Chapter 5. A dynamic compensator is introduced to produce the in-domain control, which can drive the in-domain control inputs to track the desired outputs. A Lyapunov stability analysis confirms that nonlinear boundary control enables the Burgers' equation to converge to the dynamic compensator. The ADM is used in the implementation of the resulting in-domain control scheme. The simulation results confirmed the validity of the proposed control scheme.

Chapter 6 discusses the general concepts of this thesis. It covers the main ideas on how to deal with in-domain control of PDE systems as well as the most promising trends of this subject.

Chapter 7 summarizes the work reported in this thesis and discusses the research direction related to this work.

CHAPTER 2 BASIC NOTIONS AND TOOLS FOR PDE CONTROL

This chapter introduces some basic notions and mathematical tools, including Sobolev spaces, their properties, semigroup theory, basic knowledge of elliptic equations and linear parabolic equations, and the Adomian decomposition method, which are used throughout the thesis.

2.1 Sobolev Spaces and PDE

The theory of Sobolev spaces is a standard tool for the study of partial differential equations, which provides a systematic methodology to address different problems in this field. In the thesis, Sobolev inequalities are utilized to estimate H^1- and L^2-norms of solutions to PDE systems while dealing with stability and tracking problems. Before introducing the Sobolev space, we first give the definition of L^p space and Hölder inequality. Assume that Ω is an open set in $\mathbb{R}^n$, and $\bar{\Omega}$ is the closure of the set Ω. We denote by $L^1(\Omega)$ the space of integrable functions on Ω and by $L^1_{loc}(\Omega)$ the space of functions locally integrable in Ω. Let $C^k(\Omega)$ denote the space of functions with continuous derivatives of order k, and $C_c^\infty(\Omega)$ denote the space of infinitely differentiable functions with compact support in Ω [56]. Some notations for derivative are listed as follows, which will be used in the upcoming development.

Notation for derivatives [56]. Assume $u : \Omega \to \mathbb{R}, x \in \Omega$.

(i) $\dfrac{\partial u}{\partial x_i} = \lim_{h\to 0} \dfrac{u(x + he_i) - u(x)}{h}$, provide the limit exists, where e_i represents the vector with a 1 in the ith coordinate and 0's elswhere in $\mathbb{R}^n$.

(ii) We usually write u_{x_i} for $\dfrac{\partial u}{\partial x_i}$

(iii) Similarly, $\dfrac{\partial^2 u}{\partial x_i \partial x_j} = u_{x_i x_j}$, $\dfrac{\partial^3 u}{\partial x_i \partial x_j \partial x_k} = u_{x_i x_j x_k}$, etc.

(iv) Multi-index notation:
(a) A vector of the form $\alpha = (\alpha_1, \ldots, \alpha_n)$, where each component α_i is nonnegative integer, is called a multi-index of order

$$|\alpha| = \alpha_1 + \ldots + \alpha_n. \tag{2.1}$$

(b) Given a multi-index α, define

$$D^\alpha u(x) := \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha^n} u. \tag{2.2}$$

Definition 2.1. *[65] Let $p \in \mathbb{R}$ with $1 < p < \infty$. We set*

$$L^p(\Omega) = \Big\{ f : \Omega \to \mathbb{R}; f \text{ is measurable and } |f|^p \in L^1(\Omega) \Big\} \tag{2.3}$$

with

$$\|f\|_{L^P} = \left[\int_\Omega |f(x)|^p \mathrm{d}x \right]^{1/p}. \tag{2.4}$$

In particular, the space L^∞ is defined as follows.

Definition 2.2. *[65] We set*

$$L^\infty(\Omega) = \left\{ \begin{matrix} f : \Omega \to \mathbb{R} : f \text{ is measurable and there is a constant } C \\ \text{such that } |f(x)| \leq C \quad \text{a.e. on } \Omega \end{matrix} \right\} \tag{2.5}$$

with

$$\|f\|_{L^\infty} = \inf \{ C : |f(x)| \leq C \text{ a.e. on } \Omega \}. \tag{2.6}$$

Let $1 \leq p \leq \infty$. We denote by p' the conjugate exponent, i.e.

$$\frac{1}{p} + \frac{1}{p'} = 1. \tag{2.7}$$

Theorem 2.1. *[65] Assume that $f \in L^p$ and $g \in L^{p'}$ with $1 < p < \infty$. Then $fg \in L^1$ and*

$$\int_\Omega |fg| \mathrm{d}x \leq \|f\|_{L^p} \|g\|_{L^{p'}}. \tag{2.8}$$

In most cases, it is difficult to study the classical solutions to PDEs because of the lack of knowledge on the explicit form of the solutions to PDEs. Moreover, although functional analysis is a fundamental mathematical tool for PDEs, it cannot directly be employed to deal with classical solutions. Therefore, one needs to resort to studying weak solutions of PDEs, which makes the theory of functional analysis an efficient tool in the study of PDE problems. To study weak solutions, weak derivatives should be introduced.

Definition 2.3. *[56] Suppose $u, v \in L^1_{loc}(\Omega)$ and α is a multi-index. We say that v is the α-th weak partial derivative of u, denoted by*

$$D^\alpha u = v, \tag{2.9}$$

provided

$$\int_\Omega u D^\alpha \phi \mathrm{d}x = (-1)^\alpha \int_\Omega v \phi \mathrm{d}x, \text{ for all } \phi \in C_c^\infty(\Omega). \tag{2.10}$$

Lemma 2.1. *[56] A weak α-th partial differential derivative of $u \in L^1_{loc}(\Omega)$, if it exists, is uniquely defined up to a set of measure zero.*

In this context, we will introduce some elementary properties of Sobolev spaces. Let $\Omega \subset \mathbb{R}^n$ be an open set and $p \in \mathbb{R}$ with $1 \leq p \leq \infty$. The Sobolev space $W^{k,p}(\Omega)$ consists in all locally summable functions $u : \Omega \to \mathbb{R}$ such that for each multi-index α with $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(\Omega)$. Specifically, the space $W^{m,p}$ is defined as follows.

Definition 2.4. *[65] The Sobolev space $W^{m,p}(\Omega)$ is defined to be*

$$W^{m,p}(\Omega) = \left\{ \begin{matrix} u \in L^p(\Omega); \forall \alpha \text{ with } |\alpha| \leq m, \ \exists g_\alpha \in L^p(\Omega) \text{ such that} \\ \int_\Omega u D^\alpha \varphi \mathrm{d}x = (-1)^\alpha \int_\Omega g_\alpha \varphi \mathrm{d}x, \ \forall \varphi \in C_c^\infty(\Omega) \end{matrix} \right\}. \tag{2.11}$$

If $p = 2$, $H^k(\Omega) = W^{k,2}(\Omega), k = 0, 1, \ldots, \infty$. We denote by $H_0^1(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$ and by $H^{-1}(\Omega)$ the dual space to $H_0^1(\Omega)$. Note that the space $W^{1,p}$ is equipped with the norm

$$\|u\|_{W^{1,p}} = \|u\|_{L^p} + \|Du\|_{L^p}. \tag{2.12}$$

The following inequality is called the interpolation inequality of Gagliardo-Nirenberg. It is a very powerful tool for establishing the estimates of higher power norm in H^1 and L^2 spaces.

Theorem 2.2. *[65] For any $w \in H^1(0,1)$ and $2 \leq q \leq \infty$, there exists a constant $C > 0$ such that*

$$\|w\|_{L^q} \leq C \|w\|_{H^1}^{\theta} \|w\|_{L^2}^{1-\theta}, \tag{2.13}$$

where $\theta = 1/2 - 1/q$ and C is a constant independent of w.

2.1.1 Second-order elliptic equations

In this section, we introduce the weak solution to a linear elliptic PDE and the properties of its solution. Consider the boundary-value problem of a second-order elliptic equation described by

$$\begin{cases} Lu = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.14}$$

where the function $u : \Omega \to \mathbb{R}$ is the state of the system, Ω is an open bounded subset of $\mathbb{R}^n$, $f : \Omega \to \mathbb{R}$, and L represents a second-order partial differential operator of the form:

$$Lu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a^{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} b^1(x)\frac{\partial u}{\partial x_i} + c(x)u. \tag{2.15}$$

We assume the symmetry condition

$$a^{ij} = a^{ji}, \quad i, j = 1, \ldots, n. \tag{2.16}$$

Definition 2.5. *[56] The partial differential operator L defined in* (2.15) *is uniformly elliptic if there exists a constant $\theta > 0$ such that*

$$\sum_{i,j=1}^{n} a^{ij}\xi_i\xi_j \geq \theta|\xi|^2 \tag{2.17}$$

a.e. $\forall x \in U$ and all $\xi \in \mathbb{R}^n$.

We proceed to formulate the week solution to the second-order elliptic equation (2.14).

Definition 2.6. *[56] We say that $u \in H_0^1(\Omega)$ is a weak solution to the elliptic equation* (2.14) *if*

$$\int_\Omega \sum_{i,j=1}^{n} a^{ij}u_{x_i}v_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} v + cuv\mathrm{d}x = \int_\Omega fv\mathrm{d}x \tag{2.18}$$

for any $v \in H_0^1(\Omega)$.

The existence of a weak solution to the elliptic equation (2.14) is an application of the Fredholm theory. Before giving the existence theory, the formal adjoint of L is introduced.

Definition 2.7. *[56] The operator L^*, the formal adjoint of L, is given by*

$$L^*v := -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \quad a^{ij}\frac{\partial v}{\partial x_j}\right) - \sum_{i=1}^{n} b^i \frac{\partial v}{\partial x_i} + \left(c - \sum_{i=1}^{n} b^1_{x_i}\right) v, \tag{2.19}$$

provided $v : \Omega \to \mathbb{R}$, $b^i \in C^1(\bar{\Omega}), i = 1, \ldots, n$.

Theorem 2.3. *[56]*

(i) One of the following statements holds:

(α) for each $f \in L^2(\Omega)$ there exists a unique weak solution to the boundary-value problem (2.14)*;*

(β) there exists a weak solution $u \not\equiv 0$ to the homogeneous problem:

$$\begin{cases} Lu = 0, & \text{in } \Omega, \\ \ \ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.20}$$

(ii) Furthermore, if assertion (β) *holds, then the dimension of the subspace* $N \subset H_0^1(\Omega)$ *of* (2.20) *is finite and equals to the dimension of the subspace* $N^* \subset H_0^1(\Omega)$ *of the weak solutions of*

$$\begin{cases} L^* v = 0, & \text{in } \Omega, \\ \quad v = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.21}$$

(iii) Finally, the boundary-value problem (2.14) *has a weak solution if and only if*

$$\int_\Omega f v \mathrm{d}x = 0 \quad \text{for all } v \in N^*. \tag{2.22}$$

Introducing of the regularity of domain boundaries allows us to study the smoothness of the weak solution up to the boundary. First, $B(x, r) = \{v \in \mathbb{R}^n : |x - v| < r\}$, where $x \in \mathbb{R}^n, r > 0$. Specifically, the smoothness of domain boundaries is defined below.

Definition 2.8. *[56] The boundary* $\partial\Omega$ *is* C^k *if for each point* $x^0 \in \partial\Omega$*, there exist* $r > 0$ *and a* C^k *function* $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ *such that, upon relabeling and reorienting the coordinates axes if necessary, we have*

$$\Omega \cap B(x^0, r) = \{x \in B(x^0, r) | x_n > \gamma(x_1, \ldots, x_{n-1})\}. \tag{2.23}$$

Likewise, $\partial\Omega$ *is* C^∞ *if* $\partial\Omega$ *is of* C^k *for* $k = 1, 2, \ldots,$ *and* $\partial\Omega$ *is analytical if the mapping* γ *is analytic.*

The following theorem provides some insight on the smoothness effect on the elliptic operator for the second-order elliptic equation.

Theorem 2.4. *[56] Assume that* $a^{ij} \in C^1(\bar{\Omega}), b^i, c \in L^\infty(\Omega), i, j = 1, \ldots, n$ *and* $f \in L^2(\Omega)$*. Suppose that* $u \in H_0^1(\Omega)$ *is a weak solution to the elliptic boundary-value problem* (2.14)*. Assume furthermore that* ∂U *is of* C^2*. Then* $u \in H^2(\Omega)$*, and we have the estimate*

$$\|u\|_{H^2(\Omega)} \leq C \left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right), \tag{2.24}$$

where $C > 0$ *is a constant, depending only on* U *and the coefficients of* L*.*

Note that although the nonhomogeneous term f and the initial value of u only belong to $L^2(\Omega)$, the weak solution u belongs to $H^2(\Omega)$, which implies that the elliptic operator can improve the regularity of the weak solution to make it more smooth than the nonhomogeneous term and the initial data.

2.1.2 Linear parabolic equations

Next, we introduce some energy inequalities, and notions on the existence and the regularity of a parabolic equation. Assume that Ω is an open bounded subset of $\mathbb{R}^n$ and $\Omega_T = \Omega \times (0, T]$. Consider the following parabolic equation:

$$\begin{cases} u_t + Lu = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \times [0, T], \\ u = u_0, & \text{on } \Omega \times \{t = 0\}, \end{cases} \tag{2.25}$$

where the function $u : \Omega \times [0, T] \to \mathbb{R}$ is the state of the system, $f : \Omega \to \mathbb{R}$ and $u_0 : \Omega \to \mathbb{R}$ are given, and $u : \bar{\Omega}_T \to \mathbb{R}$ is unknown. The operator L can be expressed as

$$Lu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left(a^{ij}(x,t) \frac{\partial u}{\partial x_i} \right) + \sum_{i=1}^{n} b^i(x,t) \frac{\partial u}{\partial x_i} + c(x,t)u \tag{2.26}$$

for given coefficients a^{ij}, b^i, c, and

$$\sum_{i,j=1}^{n} a^{i,j}(x,t) \xi_i \xi_j \geq \theta |\xi|^2 \tag{2.27}$$

where θ is a constant and $(x, t) \in \Omega_T$, $\xi \in \mathbb{R}^n$. To construct the weak solutions to linear parabolic equations, we need to define some other Sobolev spaces. Let X denote a real Banach space equipped with norm $\| \cdot \|$.

Definition 2.9. *[56] The space $L^p(0, T; X)$ consists of all strongly measurable functions $u : [0, T] \to X$ with*

(i)

$$\|u\|_{L^p(0,T;X)} := \left(\int_0^T \|u(t)\|^p \mathrm{d}t \right)^{1/p} < \infty \tag{2.28}$$

for $1 \leq p < \infty$ and

(ii)

$$\|u\|_{L^\infty(0,T;X)} := ess \sup\nolimits_{0 \leq t \leq T} \|u(t)\| < \infty. \tag{2.29}$$

Definition 2.10. *[56] The space $C([0, T]; X)$ comprises all continuous functions $u : [0, T] \to$*

X with

$$\|u\|_{C([0,T];X)} := \max_{0\le t\le T} \|u(t)\| < \infty. \tag{2.30}$$

Definition 2.11. *[56] Let $u \in L^1(0,T;X)$. We say that $v \in L^1(0,T;X)$ is the weak derivative of u, written as*

$$u' = v, \tag{2.31}$$

provided

$$\int_0^T \phi'(t)u(t)\mathrm{d}t = -\int_0^T \phi(t)v(t)\mathrm{d}t \tag{2.32}$$

for all $\phi \in C_c^\infty(0,T)$.

Next, we consider below the weak solution to a parabolic equation.

Definition 2.12. *[56] We say that a function*

$$u \in L^2(0,t;H_0^1(\Omega)), \;\; \textit{with } u' \in L^2(0,T;H^{-1}(\Omega)), \tag{2.33}$$

is a weak solution to the parabolic boundary-value problem (2.25) provided

$$\begin{aligned}&\langle u', v\rangle + \int_\Omega \sum_{i,j}^{n} a^{i,j}(\cdot,t)u_{x_i}v_{x_j} + \sum_{i=1}^{n} b^i(\cdot,t)u_{x_i}v + c(\cdot,t)uv\mathrm{d}x = (f,v),\\ &\textit{for } v \in H_0^1(\Omega) \;\; \textit{and a.e. } 0\le t\le T\end{aligned} \tag{2.34}$$

and

$$u(0) = u_0, \tag{2.35}$$

where the pairing $(,)$ denotes the inner product in $L^2(\Omega)$.

The following theorem provides an *a priori* bound of the solution to a parabolic equation, which makes it possible to analyse the stability of the linear parabolic equation without having an explicit form of the solution. We assume the coefficient a^{ij}, b^i, c $(i,j = 1,\dots,n)$ are smooth on $\bar{\Omega}$ and do not depend on t.

Theorem 2.5. *[56] Assume $u_0 \in H_0^1(\Omega)$, $f \in L^2(0,T;L^2(\Omega))$. Suppose that $u \in L^2(0,T;H_0^1(\Omega))$,* *with* $u' \in L^2(0,T;H^{-1}(\Omega))$, *is the weak solution to the parabolic equation* (2.25). *Then*

$$u \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega)), u' \in L^2(0,t;L^2(\Omega)), \tag{2.36}$$

and we have the estimate

$$\operatorname{ess\,sup}_{0\leq t\leq T}\|u(t)\|_{H_0^1(\Omega)} + \|u\|_{L^2(0,T;H_0^2(\Omega))} + \|u'\|_{L^2(0,T;L^2(\Omega))} \leq C\left(\|f\|_{L^2(0,t;L^2(\Omega))} + \|u_0\|_{H_0^1(\Omega)}\right), \tag{2.37}$$

where the constant C *depends only on* Ω, T *and the coefficients of* L.

Theorem 2.6. *[56] Assume that*

$$u_0 \in H^{2m+1}(\Omega),\ \frac{\mathrm{d}^k f}{\mathrm{d}t^k} \in L^2(0,T;H^{2m-2k}(\Omega)), \quad k = 0,\dots,m. \tag{2.38}$$

We have the estimate

$$\sum_{k=0}^{m+1}\left\|\frac{\mathrm{d}^k u}{\mathrm{d}t^k}\right\|_{L^2(0,T;H^{2m+2-2k}(\Omega))} \leq C\left(\sum_{k=0}^{m}\left\|\frac{\mathrm{d}^k f}{\mathrm{d}t^k}\right\|_{L^2(0,T;H^{2m-2k}(\Omega))} + \|u_0\|_{H^{2m+1}(\Omega)}\right), \tag{2.39}$$

where $C > 0$ *is the constant, depending only on* m, Ω, T *and the coefficients of* L.

Theorem 2.7. *[56] Assume that*

$$u_0 \in C^\infty(\bar{\Omega}),\ f \in C^\infty(\bar{\Omega}_T), \tag{2.40}$$

and the m*-th order compatibility conditions hold for* $m \geq 0$. *Then the parabolic boundary value problem* (2.25) *has a unique solution*

$$u \in C^\infty(\bar{\Omega}_T). \tag{2.41}$$

2.2 Theory of Semigroups

The semigroup theory is a theory that can be used for the analysis of both finite dimensional and infinite dimensional problems. From the practical and theoretical point of view, it has been proved to be a useful tool for PDE control theory, especially for linear PDE systems, such as parabolic equations, wave equations, Euler-Bernoulli equations, etc. We introduce first some basic notations. Let X denote a complex Hilbert space equipped with the norm $\|\cdot\|$. We denote by $\langle x, y\rangle$ the scalar product of two elements x and y in X. We denote the space of bounded linear operators from X to X by $L(X)$. I represents the identity operator from X to X. **Re** z denotes the real part of the complex number z. The linear operator A is a linear map $A : D(A) \subset X \to X$ defined on a domain $D(A)$ that is a linear subspace of X. $R(A)$ denote the range of the operator A. A linear operator is said to be closed if its graph

$$G(A) = \{(x, Ax) : x \in D(A)\} \tag{2.42}$$

is closed in the product space $X \times X$. The resolvent set $\rho(A)$ is the set of all complex numbers λ such that $\lambda I - A$ has a bounded inverse, namely,

$$\mathbf{R}(\lambda, A) \stackrel{\text{def}}{=} (\lambda I - A)^{-1}. \tag{2.43}$$

Definition 2.13. *[51] The adjoint operator A^* of the operatorA is the linear operator*

$$\begin{aligned} A^* :&D(A^*) \subset X \to X \\ &x \to A^* x, \end{aligned} \tag{2.44}$$

where $D(A^*)$ is the set of all y such that the linear map

$$\begin{aligned} D(A) &\subset X \to X \\ &x \to \langle Ax, y\rangle \end{aligned} \tag{2.45}$$

is continuous, a.e., and there exists $C > 0$ depending on y such that

$$|\langle Ax, y\rangle| \leq C\|x\|, \ \forall x \in D(A). \tag{2.46}$$

Definition 2.14. *[51] A family $e^{At} = \{e^{At}\}_{t\geq 0}$ of operators in $L(X)$ is a strongly continuous semigroup, also denoted as C_0-semigroup, on X if*

1. *$e^{A0} = I$,*
2. *$e^{A(t+\tau)} = e^{At}e^{A\tau}$ for every $t, \tau \geq 0$, (semigroup property)*
3. *$\lim_{t\to 0, t>0} e^{At}z = z$ for all $z \in X$. (strong continuity)*

The infinitesimal generator A of this semigroup in X is defined by

$$D(A) = \left\{ x \in X : \text{ such that the } \lim_{t\to 0^+} \frac{e^{At}x - x}{t} \text{ exists} \right\}, \tag{2.47}$$

and

$$Ax = \lim_{t\to 0^+} \frac{e^{At}x - x}{t}, \quad x \in D(A). \tag{2.48}$$

Theorem 2.8. *$D(A)$ is dense in X [51].*

The following Hille-Yosida-Miyadera-Feller-Phillips theorem provides a necessary and sufficient condition to determine whether a linear operator A is the infinitesimal generator of a strongly continuous semigroup or not.

Theorem 2.9. *[51] Let $A : D(A) \subset X \to X$ be a linear operator. Then the following statements are equivalent:*

(i) *$D(A)$ is dense in X, there exist real numbers $M > 0$ and $\omega \in \mathbb{R}$ such that $\rho(A) \supset \{\lambda \in \mathbb{C} : \mathbf{Re}\ \lambda > \omega\}$, and the following inequalities hold:*

$$\|\mathbf{R}^k(\lambda, A)\| \leq M(\mathbf{Re}\ \lambda - \omega)^{-k}, \quad k \in \mathbb{N}, \forall \lambda, Re\lambda > \omega. \tag{2.49}$$

(ii) *A is the infinitesimal generator of a strongly continuous semigroup e^{At}, and there exist real numbers $\omega \in \mathbb{R}$ and $M > 0$ such that*

$$\|e^{At}\| \leq Me^{\omega t}, \quad \forall t \geq 0. \tag{2.50}$$

Definition 2.15. *e^{At} is a contraction semigroup if it is a C_0-semigroup that satisfies an estimate $\|e^{At}\| \leq 1$ for all $t \geq 0$.*

The following theorem provides a simpler criterion for a closed, densely defined operator to be the infinitesimal generator of a contraction semigroup.

Theorem 2.10. *[51] A closed, densely defined operator in a Hilbert space is the infinitesimal generator of a C_0-semigroup satisfying $\|T(t)\| \leq e^{\omega t}$ if the following conditions hold:*

$$\mathbf{Re}\ (\langle Az, z\rangle) \leq \omega\|z\|^2 \quad \text{for } z \in D(A); \tag{2.51}$$
$$\mathbf{Re}\ (\langle A^*z, z\rangle) \leq \omega\|z\|^2 \quad \text{for } z \in D(A^*). \tag{2.52}$$

Definition 2.16. *[70] A is a sectorial operator if it is a closed densely defined operator such that, for some ϕ in $(0, \frac{\pi}{2})$ and some $M \geq 1$ and real a, the sector*

$$S_{a,\phi} = \left\{\lambda \;\middle|\; \phi \leq |arg(\lambda - a)| \leq \pi,\ \lambda \neq a\right\} \tag{2.53}$$

is in the resolvent set of A and

$$\|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda - a|} \quad \text{for} \quad \forall\ \lambda \in S_{a,\phi}. \tag{2.54}$$

Definition 2.17. *[70] Suppose that A is a sectorial operator and* $\mathbf{Re}\ \sigma > 0$*. Then for any* $\alpha > 0$

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-At} \mathrm{d}t \tag{2.55}$$

defines A^α as the inverse of $A^{-\alpha}$ $(\alpha > 0)$, $D(A^\alpha) = R(A^{-\alpha})$, and A^0 is equal to identity on X.

Theorem 2.11. *[70] Suppose that A is sectorial and* $\mathbf{Re}\ \sigma(A) > \delta > 0$*. For $\alpha \geq 0$, there exists C_α such that*

$$\|A^\alpha e^{-At}\| \leq C_\alpha t^{-\alpha} e^{-\delta t}, \quad t > 0. \tag{2.56}$$

Example 2.1. *[51] Consider the operator $A = -\alpha \dfrac{\mathrm{d}^2}{\mathrm{d}x^2}$, whose domain can be defined as*

$$D(A) = \Big\{ v \in H^2(0,1) : v(0) = 0,\ v(1) = 0 \Big\}. \tag{2.57}$$

The eigenvalues and eigenfunctions of A are then given by

$$_i(x) = \sqrt{2}\sin(i\pi x), \quad \lambda_i = \alpha(i\pi)^2, \quad \text{for } i \in \mathbb{N}^*, \tag{2.58}$$

which consist of the complete orthonormal vectors in $L^2(0,1)$. By Hill-Yosida's theorem, $-A$ is the infinitesimal generator of the semigroup e^{-At}. Furmore, we have

$$\|e^{-At}\varphi\|_{L^2} \leq e^{-Lt}\|\varphi\|_{L^2}, \tag{2.59}$$

where $L = \alpha(\pi)^2$. We define the fractional powers of A by $A^s, s \in \mathbb{R}$,

$$D(A^s) = \left\{ \varphi \in L^2(0,1);\ \sum_{i=1}^\infty \lambda_i^{2s} |\langle \psi_i, \varphi \rangle|^2 < \infty \right\}, \tag{2.60}$$

and for $\varphi \in D(A^s)$

$$A^s \varphi = \sum_{i=1}^\infty \lambda^s \langle \psi_i, \varphi \rangle \psi_i. \tag{2.61}$$

The following inequality holds:

$$\|A^s e^{-At}\varphi\|_{L^2} \leq \kappa t^{-s} e^{-Lt} \|\varphi\|_{L^2}, \quad t, s > 0. \tag{2.62}$$

It is easy to see that $D(A) \subset H_0^1(0,1) = \{\varphi \in H^1(0,1); \varphi(0) = \varphi(1) = 0\}$, which is equipped *with the norm*

$$\|\varphi\|_{H_0^1} = \|\varphi_x\|_{L^2}. \tag{2.63}$$

Note that the norm $\|\varphi\|_{H_0^1}$ *of the space* $H_0^1(0,1)$ *is equivalent to the norm* $\|\varphi\|_{H^1}$ *[56].*

Lemma 2.2. *Let* $\varphi \in H_0^1(0,1)$*. Then*

$$\|\varphi\|_{H_0^1} = \|A^{1/2}\varphi\|_{L^2}. \tag{2.64}$$

Furthermore, the following inequality holds:

$$\|\varphi\|_{L^2}^2 \leq \frac{\alpha}{\lambda_1}\|\varphi\|_{H_0^1}^2. \tag{2.65}$$

Proof. For each $\varphi \in H_0^1(0,1)$, we have $\varphi = \sum_{i=1}^{\infty}\langle\varphi, \ _i\rangle\psi_i$. Based on the definition of the power of the operator A, we have

$$\begin{aligned}
\|\varphi\|_{H_0^1}^2 &= \int_0^1 \varphi_x^2 \mathrm{d}x = -\int_0^1 \varphi_{xx}\varphi \mathrm{d}x = \alpha^{-1}\int_0^1 A\varphi\varphi \mathrm{d}x \\
&= \alpha^{-1}\sum_{i=1}^{\infty}\lambda_i\langle\varphi,\psi_i\rangle\int_0^1 \ _i\varphi \mathrm{d}x = \alpha^{-1}\sum_{i=1}^{\infty}\lambda_1|\langle\varphi,\psi_i\rangle|^2 \\
&= \alpha^{-1}\|A^{1/2}\varphi\|_{L^2}^2,
\end{aligned} \tag{2.66}$$

which implies the first claim of the lemma. Then, using the above result, it yields

$$\|\varphi\|_{L^2} = \sum_{i=1}^{\infty}|\langle\varphi,\psi_i\rangle|^2 \leq \sum_{i=1}^{\infty}\frac{\lambda_i}{\lambda_1}|\langle\varphi,\psi_i\rangle|^2 \leq \lambda_i^{-1}\|A^{1/2}\varphi\|_{L^2}^2 \leq \frac{\alpha}{\lambda_1}\|\varphi\|_{H_0^1}, \tag{2.67}$$

which leads to the desired inequality. □

Next, we introduce the Lumer-Phillips theorem, which is about the existence of semigroup under some conditions.

Definition 2.18. *[51] The operator A is dissipative if*

$$\mathbf{Re}\ \langle Ax, x\rangle \leq 0, \quad \forall x \in D(A). \tag{2.68}$$

We can now sate the well-known Lumer-Phillips theorem.

Theorem 2.12. *[51] Assume that A is densely defined and closed. If both A and A^* are* *dissipative, then for every $x_0 \in D(A^*)$, there exists a unique*

$$x \in C^1([0,+\infty);H) \cap C^0([0,+\infty);D(A)) \tag{2.69}$$

such that

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = Ax, \ t \in (0,+\infty), \\ x(0) = x_0. \end{cases} \tag{2.70}$$

Furthermore,

$$\|x(t)\| \leq \|x_0\|, \ \forall t \in [0,+\infty). \tag{2.71}$$

Theorem 2.13. *[51] Assume that A is densely defined and closed. If both A and A^* are dissipative, then for every $x_0 \in D(A)$, for every $T \in (0,+\infty)$, and for every $f \in C^1((0,T),X)$, there exists a unique*

$$x \in C^1([0,T);H) \cap C^0([0,T);D(A)) \tag{2.72}$$

such that

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = Ax + f(t), \ t \in (0,T), \\ x(0) = x_0. \end{cases} \tag{2.73}$$

Furthermore,

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)} f(\tau)\mathrm{d}\tau, \ \forall t \in [0,T). \tag{2.74}$$

2.3 Differential Flatness

Differential flatness has been proven to be a powerful tool for the control of linear and nonlinear finite-dimensional systems. Differential flatness states basically that the state variables and the control inputs of a system can be parametrized in terms of basic outputs and their derivatives up to certain orders. With this method, motion planning for tracking control involves only differential and algebraic computations without integration. It means that trajectory generation of dynamical systems can be reduced to that of algebraic systems, which could considerably simplify the design process and implementation of the control scheme. The definition of finite dimensional flat systems is given below.

Definition 2.19. *[64] The system $\dot{x} = f(x,u), x \in \mathbb{R}^n, u \in \mathbb{R}^m$ is differentially flat, or flat for short, if there exist*

$$h : \mathbb{R}^n \times (\mathbb{R}^m)^{r+1} \to \mathbb{R}^m, \tag{2.75}$$
$$\phi : (\mathbb{R})^r \to \mathbb{R}^n, \tag{2.76}$$
$$: (\mathbb{R}^m)^{r+1} \to \mathbb{R}^m, \tag{2.77}$$

such that

$$y = h(x, u, \dot{u}, \cdots, u^{(r)}), \tag{2.78}$$
$$x = \phi(y, \dot{y}, \cdots, y^{(r-1)}), \tag{2.79}$$
$$u = \psi(y, \dot{y}, \cdots, y^{(r-1)}, y^{(r)}). \tag{2.80}$$

In recent years, differential flatness has been adopted to address control problems of linear and nonlinear PDEs [95, 100, 102, 103]. Although the properties of the flatness for infinite dimensional systems are basically the same for that for finite dimensional systems, its statement is not straightforward. For this reason, we give below an example to illustrate the design process for tracking control of infinite dimensional systems.

Example 2.2. *Consider the in-domain control problem of a heat equation:*

$$\begin{aligned}
&\xi_t - \xi_{xx} = 0, \quad x \in \Omega,\ t > 0,\\
&\xi_x(0,t) = k_1\xi(0,t),\ \xi_x(1,t) = k_2\xi(1,t),\\
&[\xi_x(x,t)]_{x_i} = u_i(t), i = 1, \ldots, n,\\
&\xi(x,0) = 0,
\end{aligned} \tag{2.81}$$

where $\Omega \triangleq (0, x_1) \cup (x_1, x_2) \cup \cdots \cup (x_{n-1}, x_n) \cup (x_n, 1)$, *and* $[v(x)]_{x=x_i} = v(x_i^+) - v(x_i^-)$, *with* $v(x_i^+)$ *and* $v(x_i^-)$ *denoting, respectively, the right and the left limits of* $v(x)$ *at the point* x_i*.* $u_i(t), i = 1, \ldots, n$, *are control inputs located in the domain,* $k_1, k_2 > 0$ *are proportional boundary feedback control gains.*

We first divide the system (2.81) *into* n *parallelly connected systems* $\xi_i(x,t), i = 1, \ldots, n$*:*

$$\begin{aligned}
&\xi_{it} - \xi_{ixx} = 0, \quad x \in (0, x_j) \cup (x_j, 1),\\
&\xi_{ix}(0,t) = k_1\ \xi_i(0,t), \quad \xi_{ix}(1,t) = -k_2\ \xi_i(1,t),\\
&\xi_i(0,t) = 0,\\
&\xi_{ix^+} - \xi_{ix^-} = u_i(t).
\end{aligned} \tag{2.82}$$

We split every system ξ_i into two sub-systems, i.e., for fixed $x_i \in (0,1)$, considering

$$\xi_{it}^{-}(x,t) - \xi_{ixx}^{-}(x,t) = 0, \quad x \in (0, x_i), \tag{2.83}$$

$$\xi_i^{-}(0,t) = 0, \quad \xi_{ix}^{-}(0,t) = k_1 \xi_i^{-}(0,t), \tag{2.84}$$

and

$$\xi_{it}^{+}(x,t) - \xi_{ixx}^{+}(x,t) = 0, \quad x \in (x_i, 1), \tag{2.85}$$

$$\xi_i^{+}(0,t) = 0, \quad \xi_{ix}^{+}(1,t) = -k_2 \xi_i^{+}(1,t), \tag{2.86}$$

with the joint conditions

$$\xi_i^{-}(x_i,t) = \xi_i^{+}(x_i,t), \tag{2.87}$$

$$\xi_{ix}^{+}(x_i,t) - \xi_{ix}^{-}(x_i,t) = v_i. \tag{2.88}$$

Applying the Laplace transform to both sides of (2.85) *with boundary conditions* (2.86) *and* (2.87) *with boundary conditions* (2.88), *it yields*

$$\begin{aligned} &s\widehat{\xi_i^{-}}(x,s) = \widehat{\xi_{ixx}^{-}}(x,s), x \in (0, x_i), \\ &\widehat{\xi_{ix}^{-}} = k_1 \widehat{\xi_i^{-}}(0,s), \end{aligned} \tag{2.89}$$

and

$$\begin{aligned} &s\widehat{\xi_i^{+}}(x,s) = \widehat{\xi_{ixx}^{+}}(x,s), x \in (x_i, 1), \\ &\widehat{\xi_{ix}^{+}}(1,s) = -k_2 \widehat{\xi_i^{+}}(1,s). \end{aligned} \tag{2.90}$$

which admits the solution

$$\begin{aligned} &\widehat{\xi_i^{-}}(x,s) = \hat{C}_1(s)\phi_1(x,s) + \hat{C}_2(s)\phi_2(x,s), \\ &\widehat{\xi_i^{+}}(x,s) = \hat{C}_3(s)\phi_1(\zeta,s) + \hat{C}_4(s)\phi_2(\zeta,s), \end{aligned} \tag{2.91}$$

where

$$\zeta = x - x_i, \quad \phi_1(x,s) = \frac{\sinh(\sqrt{s}x)}{\sqrt{s}}, \quad \phi_2(x,s) = \cosh(\sqrt{s}x). \tag{2.92}$$

Then, deriving $\phi_1(x,s)$ and $\phi_2(x,s)$ with respect to x, we can obtain:

$$\phi_1'(x,s) = \phi_2(x,s), \quad \phi_2'(x,s) = s\phi_1(x,s). \tag{2.93}$$

By the boundary conditions (2.89) and (2.90), we have

$$
\begin{aligned}
&\hat{C}_1(s) = k_1 \hat{C}_2(s), \\
&\hat{C}_3(s)\phi_2(1-x_i,s) + s\hat{C}_4(s)\phi_1(1-x_i,s) = \\
&-k_2\left(\hat{C}_3(s)\phi_1(1-x_i,s) + \hat{C}_4(s)\phi_2(1-x_i,s)\right).
\end{aligned}
\tag{2.94}
$$

Applying the boundary conditions (2.87) and (2.88), we have

$$
\begin{aligned}
&\hat{C}_1(s)\phi_1(x_i,s) + \hat{C}_2(s)\phi_2(x_i,s) = \hat{C}_4(s), \\
&\hat{C}_1(s)\phi_2(x_i,s) + s\hat{C}_2(s)\phi_1(x_i,s) - \hat{C}_3(s) = \hat{v}(s).
\end{aligned}
\tag{2.95}
$$

Substituting (2.94) *into* (2.95) *yields*

$$
\begin{aligned}
&\hat{C}_4(s) = \hat{C}_2(s)\left(k_1\phi_1(x_i,s) + \phi_2(x_i,s)\right), \\
&\hat{C}_3(s)\left(\phi_2(1-x_i,s) + k_2\phi_1(1-x_i,s)\right) = \\
&-\left(k_2\phi_2(1-x_i,s) + s\phi_1(1-x_i,s)\right)\left(k_1\phi_1(x_i,s) + \phi_2(x_i,s)\right)C_2(s).
\end{aligned}
\tag{2.96}
$$

Setting

$$
\begin{aligned}
&C_2(s) = -\left(\phi_2(1-x_i,s) + k_2\phi_1(1-x_i,s)\right)\hat{h}_i(s), \\
&C_3(s) = \left(k_2\phi_2(1-x_i,s) + s\phi_1(1-x_i,s)\right)\left(k_1\phi_1(x_i,s) + \phi_2(x_i,s)\right)\hat{h}_i(s),
\end{aligned}
\tag{2.97}
$$

and substituting the $C_2(s)$ *and* $C_3(s)$ *into* (2.94) *and* (2.96), *respectively, we can obtain* $C_1(s)$ *and* $C_4(s)$, *which leads to*

$$
\hat{u}_i(s) = \left(\sqrt{s}\phi_1(1,s) + k_1k_2\phi_1(1,s) + k_1\phi_2(1,s) + k_2\phi_2(1,s)\right)\hat{h}_i(s). \tag{2.98}
$$

Note that $\hat{h}_i(s) \leftrightarrow h_i(t)$ *is the so-called basic output, or flat output. Recall that*

$$
\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}\ , \quad \cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}\ . \tag{2.99}
$$

The full-state trajectory $\hat{u}_j$ *can be written in the form*

$$
\hat{u}_i(s) = \quad k_1k_2\sum_{n=0}^{\infty}\frac{s^n}{(2n+1)!} + (k_1+k_2)\sum_{n=0}^{\infty}\frac{s^n}{(2n)!} + \sum_{n=0}^{\infty}\frac{s^{n+1}}{(2n+1)!}\Bigg)\hat{h}_i(s),\ \ i = 1,\ldots,n. \tag{2.100}
$$

Thus, in time domain the control inputs of the system (2.81) *is given by*

$$u_i(t) = k_1 k_2 \sum_{n=0}^{\infty} \frac{h_i^{(n)}(t)}{(2n+1)!} + (k_1 + k_2) \sum_{n=0}^{\infty} \frac{h_i^{(n)}(t)}{(2n)!} + \sum_{n=0}^{\infty} \frac{h_i^{(n+1)}(t)}{(2n+1)!}. \tag{2.101}$$

In order to make sure that the above controller is well-defined, the basic output $h_i(t)$ *should be* C^∞*. We choose then the following function* $\psi(t)$ *as a component of basic outputs:*

$$(t) = \begin{cases} 0, & if\ t \leq 0, \\ \dfrac{\int_0^t exp(-1/(\tau(1-\tau)))^\varepsilon \mathrm{d}\tau}{\int_0^T exp(-1/(\tau(1-\tau)))^\varepsilon \mathrm{d}\tau}, & if\ 0 < t < T, \\ 1, & if\ t \geq T, \end{cases} \tag{2.102}$$

which is known as the Gevrey function of order $\sigma = 1 + \frac{1}{\varepsilon}, \varepsilon > 0$*. Due to the properties of the Gevery function, the convergence of the in-domain controllers* (2.101) *are ensured [112].*

2.4 Adomian Decomposition Method

The Adomian decomposition method (ADM) is a semi-analytic scheme to obtain solutions to ordinary and partial differential equations [1,43,55]. It has been shown that the ADM can achieve a rapid convergence of the series solution compared with other numerical techniques [107]. In general, the ADM deals with a nonlinear abstract equation of the form,

$$w - N(w) = f, \tag{2.103}$$

where N denotes a nonlinear operator, and f is a nonhomogeneous term. To apply the ADM to the nonlinear equation (2.103), we assume that the solution can be written a series solution:

$$w = \sum_{k=0}^{\infty} w_k. \tag{2.104}$$

Then, the nonlinear term $N(w)$ can be expressed as

$$N(w) = \sum_{n=0}^{\infty} A_n(w_0, w_1, \cdots, w_n), \tag{2.105}$$

where [1]

$$A_m = \left[\frac{1}{m!} \frac{\mathrm{d}^m}{\mathrm{d}\lambda^m} N\left(\sum_{i=0}^{\infty} \lambda^i w_i\right) \right] \Bigg|_{\lambda=0}. \tag{2.106}$$

Consequently, we obtain every term w_n in the Adomian series recursively:

$$\begin{cases} w_0 = f, \\ w_1 = A_0(w_0), \\ \vdots \\ w_{n+1} = A_n(w_0, \cdots, w_n), \\ \vdots \end{cases} \tag{2.107}$$

Next, we use two examples to illustrate how to generate numerical solutions by ADM.

Example 2.3. *Consider the initial value problem of the following ordinary differential equation*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x + y(x), \quad y(0) = 0, \tag{2.108}$$

for which the exact solution is given by $y(x) = e^x - x - 1$.

To apply ADM to (2.108), *we introduce the operator* $L_x = \frac{\mathrm{d}}{\mathrm{d}x}$ *and its inverse operator* $L_x^{-1} = \int_0^t (\cdot)\mathrm{d}x$. *The ODE* (2.108) *can then be expressed as*

$$L_x y = x + y(x). \tag{2.109}$$

Apply the operator L_x^{-1} *on both side of the equation* (2.109)

$$y(x) = y(0) + \int_0^x (t + y(t))\mathrm{d}t. \tag{2.110}$$

The recursive relationship is given by

$$y_0 = \frac{x^2}{2}, \quad y_{n+1} = L_x^{-1} y_n, \ n \geq 0. \tag{2.111}$$

Thus, the Adomian series can be expressed as

$$\begin{aligned} y_0 &= \frac{x^2}{2}, \\ y_1 &= L_x^{-1} y_0 = \frac{x^3}{3!}, \\ y_2 &= L_x^{-1} y_1 = \frac{x^4}{4!}, \\ y_3 &= L_x^{-1} y_2 = \frac{x^5}{5!}, \\ &\cdots. \end{aligned} \tag{2.112}$$

The series solution to the ODE (2.108) *is given by*

$$\begin{aligned} y(x) &= y_0 + y_1 + y_2 + y_3 + \cdots \\ &= \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \\ &= e^x - x - 1. \end{aligned} \tag{2.113}$$

Example 2.4. *Consider the initial value problem of Fisher's equation*

$$w_t - w_{xx} = w(1-w), \tag{2.114}$$

subject to the initial condition

$$w(x,0) = x. \tag{2.115}$$

To address the above initial value problem based on the ADM, we rewrite equation (2.114) *in an operator form*

$$L_t w = L_{xx} w + Nw, \tag{2.116}$$

where $L_t = \frac{\partial}{\partial t}, L_{xx} = \frac{\partial^2}{\partial x^2}$, *and the nonlinear term is given by* $Nw = w(1-w)$. *Operating* $L_t^{-1} = \int_0^t (\cdot)\mathrm{dt}$ *on both side of equation* (2.116), *it yields*

$$L_t^{-1} L_t w = L_t^{-1} L_{xx} w + L_t^{-1}(1-w)w, \tag{2.117}$$

$$w - w(0) = L_t^{-1} L_{xx} w + L_t^{-1}(1-w)w. \tag{2.118}$$

Decomposing w *into* $\sum_{n=0}^{\infty} w_n$, *the nonlinear term* $(1-w)w$ *can be expressed in terms of the Adomian polynomials* A_n*:*

$$w = w(0) + L_t^{-1} L_{xx} \sum_{n=0}^{\infty} w_n - L_t^{-1} \sum_{n=0}^{\infty} A_n. \tag{2.119}$$

Then, we can now write

$$\begin{cases} w_0 = x, \\ w_1 = L_t^{-1} L_{xx} w_0 + L_t^{-1} A_0, \\ w_2 = L_t^{-1} L_{xx} w_1 + L_t^{-1} A_1, \\ \vdots \\ w_{n+1} = L_t^{-1} L_{xx} w_n + L_t^{-1} A_n, \\ \vdots \end{cases} \tag{2.120}$$

Thus the components of w *can be computed for* $n \geq 0$*. Specifically, the several terms of Adomian series are given by*

$$\begin{aligned} A_0 &= (1 - w_0) w_0, \\ A_1 &= \frac{\mathrm{d}}{\mathrm{d}\lambda} \left[N(w_0 + w_1 \lambda) \right]_{\lambda=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\lambda} \left[(1 - w_0 - \lambda w_1)(w_0 + \lambda w_1) \right]_{\lambda=0} \\ &= w_1 - 2 w_0 w_1, \\ A_2 &= \frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} \left[N(w_0 + \lambda w_1 + \lambda^2 w_2) \right] \\ &= \frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} \left[(w_0 + \lambda w_1 + \lambda^2 w_2)(1 - w_0 - \lambda w_1 - \lambda^2 w_2) \right] \\ &= 2 w_2 - 4 w_2 w_0 - 2 w_1^2, \\ &\vdots \end{aligned} \tag{2.121}$$

Thus, the solution of (2.114) is $w = x + (x - x^2)t + (-1 + x - \frac{3}{2} + x^3)t^2 + \cdots$.

CHAPTER 3 ARTICLE 1: DEFORMATION CONTROL OF AN EULER-BERNOULLI BEAM BASED ON ZERO-DYNAMICS INVERSE DESIGN AND FLATNESS

This chapter is reproduced from the paper [140].

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Abstract— This paper addresses the problem of deformation control of an Euler-Bernoulli beam with point-wise interior actuation. The method of zero-dynamics inverse design is employed in control synthesis, which allows avoiding early truncations. The method of flat systems is used in the realization of a dynamic control scheme for set-point regulation. The well-posedness and the stability of the considered system are assessed, and the viability of the developed approach is confirmed by numerical simulation.

3.1 Introduction

In this paper, we consider the set-point deformation control of an Euler-Bernoulli beam with point-wise interior (or in-domain) actuation formulated as an output regulation problem, which can be found in such applications as deformable micro-mirrors in adaptive optics systems [127, 132, 133].

Although the control of Euler-Bernoulli beams is a long-time standing topic in the field of partial differential equations (PDEs) control, this problem with different settings and for different applications still draws considerable attention in the recent literature (see, e.g., [6, 66–69, 100]). Due to the fact that the stability of this type of system is a main concern from both theoretical and practical viewpoints, the majority of work on this topic is dedicated to the stabilization of beams via boundary or interior control (see the above cited work and [7, 23, 42, 83, 114]). For the in-domain control of beams, if early truncations are allowed, the model of PDEs can be discretized on space to obtain a system of lumped ordinary differential equations (ODEs) [23, 105]. Then, a variety of techniques developed for the control of finite-dimensional systems can be applied. Nevertheless, it is often of great interest to directly deal with the control of PDE models, in order to avoid the possible instability [17]. A scheme proposed recently in [100, 122] tackles this issue by utilizing the Weierstrass-factorized representation of the spectrum of the input-output dynamics. In this approach, the truncation is still needed in order to obtain a finite-dimensional input-output map. In order to avoid early truncations, the work reported in [16] tried to transform the in-domain actuation to the boundary to enable the application of boundary control techniques. A similar attempt is presented in [137] for the in-domain control of parabolic equations. However, due to the regularity problem, it is almost impossible to bring the point-wise in-domain actuation to the boundary in most cases.

The approach employed in this work is based on zero-dynamic inverse (ZDI) design that is developed in a series of work reported in [27–31, 35] for asymptotic regulation of PDEs. Essentially, this approach amounts to constructing a dynamic control scheme from the zero-dynamics associated with the original system. This allows for the control design to be carried out directly with interior (or eventually boundary) actuation while not requiring any early truncations. Nevertheless, a main issue related to the application of ZDI design is that the implementation of such control schemes requires resolving the corresponding zero-dynamics, which may be very difficult for generic regulation problems, such as set-point control considered in the present work. To overcome this difficulty, we resort to the method of flat systems, which is also a well-established method for PDEs control (see, e.g., [8, 59, 87, 100, 111, 118]). Note that the ZDI design has been applied in a recent work to the in-domain control of a heat equation [145]. Note also that the design of flatness-based feedforward control presented in this paper is greatly inspired by the method developed in [119].

The rest of the paper is organized as follows. Section 3.2 introduces the model of the considered problem and presents the basic properties of this system. Section 3.3 details the control synthesis and the implementation, followed by Section 3.4 that presents motion planning for feedforward control. A simulation study is carried out in Section 3.5. Some concluding remarks are provided in Section 3.6.

3.2 System Modeling and Basic Properties

In this work, we consider a 1-dimensional Euler-Bernoulli beam with constant mass density and flexural rigidity actuated by interior point-wise control located at $\{x_1, x_2, \cdots, x_N\}$. The dynamic transversal displacement of the beam, denoted by $w(x,t)$, in a normalized coordinate, where the variable x is spanned over the domain $(0,1)$, can be described by the following

PDE [42, 53]:

$$w_{tt}(x,t) + w_{xxxx}(x,t) = \sum_{j=1}^{N} \alpha_j(t)\delta(x - x_j), \tag{3.1a}$$

$$x \in (0,1), t > 0,$$

$$w(0,t) = w_{xx}(0,t) = w_x(1,t) = 0, \tag{3.1b}$$

$$w_{xxx}(1,t) = kw_t(1,t),\ k > 0, \tag{3.1c}$$

$$w(x,0) = w^0(x),\ w_t(x,0) = w^1(x), \tag{3.1d}$$

where w_x and w_t denote, respectively, the derivatives of w with respect to its variables x and t, $\delta(x-x_j)$ is the Dirac mass concentrated at the point $x_j \in (0,1)$, and $\alpha_j : t \mapsto \mathbb{R}, j = 1, \cdots, N$, are the control signals. Without loss of generality, we assume that $0 < x_1 < x_2 < ... < x_N < 1$. The initial data defined in (3.1d) are taken as $w^0 \in H^4, w^1 \in H^2$. Note that the feedback control located at $x = 1$ given in (3.1c) is for the purpose of stabilization (see, e.g., [7,42,114]).

For well-posedness and stability analysis, we resort to the abstract linear system of the PDE (3.1). For that, we introduce the state space $Y = \Phi \times L^2(0,1)$, where $\Phi = \{\upsilon \in H^2(0,1) | \upsilon(0) = \upsilon_x(1) = 0\}$. Then, Y is a Hilbert space with the inner product:

$$\left\langle \begin{pmatrix} \psi_1 \\ \varphi_1 \end{pmatrix}, \begin{pmatrix} \psi_2 \\ \varphi_2 \end{pmatrix} \right\rangle_Y = \int_0^1 (\psi_{1xx}\psi_{2xx} + \varphi_1\varphi_2)\, \mathrm{d}x. \tag{3.2}$$

Note that the system (3.1) with boundary feedback control given in (3.1c) can also be expressed as (see, e.g., [42, 114]):

$$\begin{aligned} &w_{tt}(x,t) + w_{xxxx}(x,t) + kw_t(x,t)\delta(x-1) \\ &\quad = \sum_{j=1}^{N} \alpha_j(t)\delta(x - x_j), \quad x \in (0,1), t > 0; \end{aligned} \tag{3.3a}$$

$$w(0,t) = w_{xx}(0,t) = w_x(1,t) = w_{xxx}(1,t) = 0, \tag{3.3b}$$

$$w(x,0) = w^0(x),\ w_t(x,0) = w^1(x), \tag{3.3c}$$

so that the systems (3.1) and (3.3) are equivalent in a weak sense. Hence, based on (3.3) we can define an operator A on Y:

$$A \begin{pmatrix} \psi \\ \varphi \end{pmatrix} = \begin{pmatrix} \varphi \\ -\psi_{xxxx} - k\varphi\delta(x-1) \end{pmatrix} \tag{3.4}$$

with domain

$$
\begin{aligned}
\mathcal{D}(A) =\{&(\psi,\varphi)\in Y; (\psi,\varphi)\in (H^2(0,1)\cap (H^4(0,x_1)\\
&\cup H^4(x_1,x_2)\cup\cdots\cup H^4(x_N,1)))\times H^2(0,1),\\
&\quad(0) = \quad_x(1) = \quad_{xx}(0) = \quad_{xxx}(1) = 0,\\
&\varphi(0)=\varphi_x(1)=0\}.
\end{aligned}
\tag{3.5}
$$

Define another operator $B : \mathbb{R}^N \to \mathcal{D}'(A^*)$, where A^* is the adjoint of A, as

$$
B\alpha = \begin{pmatrix} 0 \\ \sum_{j=1}^{N} \alpha_j \delta(x-x_j) \end{pmatrix}, \tag{3.6}
$$

where $\mathcal{D}'(A^*)$ is the dual space of $\mathcal{D}(A^*)$ and $\alpha = (\alpha_1, ..., \alpha_N)^T$. By a direct computation we obtain that the operator $B^* : \mathcal{D}(A^*) \mapsto \mathbb{R}^N$, the adjoint of B, is given by

$$
B^*\begin{pmatrix}\psi\\ \varphi\end{pmatrix} = (\varphi(x_1),\dots,\varphi(x_N))^T, \quad \forall \begin{pmatrix}\psi\\ \varphi\end{pmatrix} \in \mathcal{D}(A^*). \tag{3.7}
$$

Letting $y = (w, w_t)^T$ and $y_0 = (w^0, w^1)^T$, the abstract control system corresponding to (3.1) (or (3.3)) is of the form:

$$
\dot{y} = Ay + B\alpha, \;\; y(0) = y_0. \tag{3.8}
$$

It is known that A generates an exponentially stable C_0-semigroup $S(t)$ (see, e.g., [7,42,114]), i.e., there exist positive constants m and M such that

$$
\|S(t)\|_{\mathcal{L}(Y,Y)} \le Me^{-mt}, \;\; \forall t \ge 0. \tag{3.9}
$$

Furthermore, System (3.8) (respectively (3.1) or (3.3)) is well-posed (see, e.g., [50]), because B is admissible in the sense that there exist T and $c > 0$ such that for any $\alpha \in L^2(0,T)$ [114],

$$
\left\| \int_0^T S(t-\tau)B\alpha(\tau)\mathrm{d}\tau \right\|_Y \le c\|\alpha\|_{L^2(0,T)}. \tag{3.10}
$$

Letting $U = \mathbb{R}^N$ and considering a test function $z \in Y$, a weak solution to (3.8) (or equivalently to (3.1) or (3.3)) is then given by (see [50], Definition 2.36, p53):

$$
\langle y, z\rangle_Y = \left\langle y_0, S^*(t)z \right\rangle_Y + \int_0^t \langle \alpha(\tau), B^*S^*(t-\tau)z\rangle_U \,\mathrm{d}\tau, \tag{3.11}
$$

where $S^*(t)$ is the adjoint of $S(t)$.

Based on (3.9), we can derive the following property of the weak solution defined in (3.11).

Theorem 3.1. *Assume that there exits a positive constant C such that $\|\alpha\|_{L^\infty} \leq C$. Then for any $t > 0$, the weak solution to* (3.8) *given by* (3.11) *is bounded by*

$$\|y(t)\|_Y \leq Me^{-mt}\|y_0\|_Y + C\frac{M}{m}. \tag{3.12}$$

Proof. Let $y \in Y, t \in [0, +\infty)$. Suppose that there exists a constant $C > 0$, such that $\|\alpha\|_{L^\infty} \leq C$. Then, taking $z \in Y$ we obtain

$$\begin{aligned}
\langle y, z\rangle_Y &= \langle S(t)y_0, z\rangle_Y + \int_0^t \langle S(t-\tau)B\alpha(\tau), z\rangle_U \mathrm{d}\tau \\
&\leq Me^{-mt}\|y_0\|_Y\|z\|_Y + M\|z\|_{L^\infty}\int_0^t Ce^{-m(t-\tau)}\mathrm{d}\tau \\
&\leq Me^{-mt}\|y_0\|_Y\|z\|_Y + C\frac{M}{m}\|z\|_{L^\infty} \\
&\leq Me^{-mt}\|y_0\|_Y\|z\|_Y + C\frac{M}{m}\|z\|_Y,
\end{aligned} \tag{3.13}$$

which leads to (3.12). □

Theorem 3.1 implies that the trajectory of System (3.1) should be bounded if the control inputs are uniformly bounded.

In order to establish the corresponding zero-dynamics, which is an essential step in ZDI design, we present the beam in a serially connected form:

$$w_{tt} + w_{xxxx} = 0,\ x \in \Omega, t > 0, \tag{3.14a}$$

$$w(0,t) = w_{xx}(0,t) = w_x(1,t) = 0, \tag{3.14b}$$

$$w_{xxx}(1,t) = kw_t(1,t), \tag{3.14c}$$

$$\begin{aligned} &w(x_j^+) = w(x_j^-), w_x(x_j^+) = w_x(x_j^-), w_{xx}(x_j^+) = w_{xx}(x_j^-), \\ &\qquad j = 1, \dots, N, \end{aligned} \tag{3.14d}$$

$$[w_{xxx}]_{x_j} = w_{xxx}(x_j^+, t) - w_{xxx}^j(x_j^-, t) = u_j, j = 1, \dots, N, \tag{3.14e}$$

$$w(x,0) = w^0(x), w_t(x,0) = w^1(x), \tag{3.14f}$$

where $\Omega \doteq (0, x_1) \cup (x_1, x_2) \cup \cdots \cup (x_N, 1)$, and x_j^- and x_j^+ denote, respectively, the usual left- and right-hand limits to x_j.

The systems (3.1) and (3.14) are weakly equivalent in the sense that they admit the same weak solution defined by (3.11). Specifically, we have

Lemma 3.1. *Considering weak solutions in* $C([0,T];\Phi)\bigcap C^1([0,T];L^2(0,1))$*,* $T<\infty$*, the systems* (3.1) *and* (3.14) *are equivalent if*

$$\alpha_j(t) = -u_j(t) = -[w_{xxx}]_{x_j},\ j = 1,\dots,N. \tag{3.15}$$

The claim of Lemma 3.1 can be proved by extending the method used in the proof of Theorem 1.1 in [114] to the case of the beam with multiple point-wise interior controls described by the systems (3.1) and (3.14).

3.3 Control Synthesis and Implementation

Let $w^d(x_j,t) \in C^\infty$, for all $t<\infty$, be the reference output at the position x_j, for $j = 1,\dots,N$. Let $e_j(t) = w(x_j,t) - w^d(x_j,t)$, $j = 1,\dots,N$, be the regulation errors. Denote by $e(t) = (e_1(t),\dots,e_N(t))$ the vector of regulation errors and by $u(t) = (u_1(t),\dots,u_N(t))$ the control vector. The considered regulation problem for set-point control is specified as follows.

Problem 3.1. *Find a dynamic control* $u(t)$ *such that the regulation error satisfies* $e(t)\to 0$ *as* $t\to\infty$*.*

To utilize the method of ZDI design, we need first to establish the forced zero-dynamics, or zero dynamics for short, which can be obtained from (3.14) by replacing the input constraints in (3.14e) by the requirement that the regulation errors vanish identically, i.e., $e_j(t) = 0$, for $j = 1,\dots,N$. We obtain then

$$\xi_{tt} + \xi_{xxxx} = 0,\ x \in \Omega, t > 0, \tag{3.16a}$$

$$\xi(0,t) = w_{xx}(0,t) = \xi_x(1,t) = 0, \tag{3.16b}$$

$$\xi_{xxx}(1,t) = k\xi_t(1,t), \tag{3.16c}$$

$$\xi(x_j^+) = \xi(x_j^-),\ \xi_x(x_j^+) = \xi_x(x_j^-),\ \xi_{xx}(x_j^+) = \xi_{xx}(x_j^-),\quad j = 1,\dots,N, \tag{3.16d}$$

$$\xi(x_j,t) = w^d(x_j,t),\ \ j = 1,\dots,N, \tag{3.16e}$$

$$\xi(x,0) = \xi^0(x),\ \xi_t(x,0) = \xi^1(x). \tag{3.16f}$$

Note that, we can always choose appropriate initial data for the reference system so that $\xi^0(x) = \xi^1(x) = 0$. Therefore, in the following, we will consider only the zero-dynamics with null initial condition, which will simplify control design.

To derive the control signal at each control support point, x_j, $j = 1, \dots, N$, we present the zero-dynamics in a *parallel connected* form $\xi(x,t) = \sum_{j=1}^{N} \xi^j(x,t)$, in which for a fixed index j, the sub-system of the zero-dynamics is given by:

$$\xi^j_{tt} + \xi^j_{xxxx} = 0, \; x \in (0, x_j) \cup (x_j, 1), t > 0, \tag{3.17a}$$

$$\xi^j(0,t) = w_{xx}(0,t) = \xi^j_x(1,t) = 0, \tag{3.17b}$$

$$\xi^j_{xxx}(1,t) = k\xi^j_t(1,t), \tag{3.17c}$$

$$\xi^j(x_j^+) = \xi^j(x_j^-), \; \xi^j_x(x_j^+) = \xi^j_x(x_j^-), \; \xi^j_{xx}(x_j^+) = \xi^j_{xx}(x_j^-), \tag{3.17d}$$

$$\xi^j(x_j, t) = w^d_j(t), \tag{3.17e}$$

where $w^d_j(t)$ is the input to the ξ^j-subsystem of the zero-dynamics. As the output of the zero-dynamics satisfies

$$[\xi_{xxx}]_{x_j} = \sum_{i=1}^{N} [\xi^i_{xxx}]_{x_j} = [\xi^j_{xxx}]_{x_j} = u_j, \tag{3.18}$$

for all $j = 1, \dots, N$, $\xi(x,t) = \sum_{j=1}^{N} \xi^j(x,t)$ is a solution to (3.16). Hence, (3.17) and (3.18) form a dynamic control scheme.

The implementation of the above dynamic control scheme requires solving the dynamic system (3.17). For this purpose, we further divide (3.17) into two segments:

$$\xi^{j-}_{tt}(x,t) + \xi^{j-}_{xxxx}(x,t) = 0, \; x \in (0, x_j^-), t > 0, \tag{3.19a}$$

$$\xi^{j-}(0,t) = \xi^{j-}_{xx}(0,t) = 0, \tag{3.19b}$$

and

$$\xi^{j+}_{tt}(x,t) + \xi^{j+}_{xxxx}(x,t) = 0, \; x \in (x_j^+, 1), t > 0, \tag{3.20a}$$

$$\xi^{j+}_x(1,t) = 0, \xi^{j+}_{xxx}(1,t) = k\xi^{j+}_t(1,t), \tag{3.20b}$$

with the joint conditions

$$\xi^j(x_j^-, t) = \xi^j(x_j^+, t) = w^d_j(t), \tag{3.21a}$$

$$\xi^j_x(x_j^-, t) = \xi^j_x(x_j^+, t), \tag{3.21b}$$

$$\xi^j_{xx}(x_j^-, t) = \xi^j_{xx}(x_j^+, t), \tag{3.21c}$$

$$\xi^j_{xxx}(x_j^+, t) - \xi^j_{xxx}(x_j^-, t) = [\xi^j_{xxx}]_{x_j} = u_j. \tag{3.21d}$$

We resort then to the method of flat systems, in particular a standard procedure of Laplace transform-based method, to find the solution to (3.19) and (3.20) subject to (3.21) [8,59,118]. Let $\tilde{x} = x - x_j$. The general solutions to (3.19) and (3.20) in the Laplace domain are given by

$$\begin{aligned}\hat{\xi}^{j-}(x,s) =& \hat{k}_1(s)\hat{C}_1(x,s) + \hat{k}_2(s)\hat{S}_1(x,s) + \hat{k}_3(s)\hat{C}_2(x,s) \\ &+ \hat{k}_4(s)\hat{S}_2(x,s), \qquad (3.22a)\\ \hat{\xi}^{j+}(x,s) =& \hat{k}_5(s)\hat{C}_1(\tilde{x},s) + \hat{k}_6(s)\hat{S}_1(\tilde{x},s) + \hat{k}_7(s)\hat{C}_2(\tilde{x},s) \\ &+ \hat{k}_8(s)\hat{S}_2(\tilde{x},s), \qquad (3.22b)\end{aligned}$$

where

$$\hat{C}_1(x,s) = \frac{\cosh(\sqrt{is}x) + \cos(\sqrt{is}x)}{2}, \qquad (3.23)$$

$$\hat{C}_2(x,s) = \frac{\cosh(\sqrt{is}x) - \cos(\sqrt{is}x)}{2is}, \qquad (3.24)$$

$$\hat{S}_1(x,s) = \frac{\sinh(\sqrt{is}x) + \sin(\sqrt{is}x)}{2\sqrt{is}}, \qquad (3.25)$$

$$\hat{S}_2(x,s) = \frac{\sinh(\sqrt{is}x) - \sin(\sqrt{is}x)}{2(is)^{\frac{3}{2}}}. \qquad (3.26)$$

Note that the derivatives of $\hat{C}_1(x,s)$, $\hat{C}_2(x,s)$, $\hat{S}_1(x,s)$, and $\hat{S}_1(x,s)$ with respect to x are given by:

$$\hat{C}_1' = -s^2\hat{S}_2,\ \hat{C}_2' = \hat{S}_1,\ \hat{S}_1' = \hat{C}_1,\ \hat{S}_2' = \hat{C}_2. \qquad (3.27)$$

The boundary condition (3.19b) results in $\hat{k}_1 = \hat{k}_3 = 0$. From the joint conditions given in (3.21), we get

$$\hat{k}_5 = \hat{k}_2\hat{S}_1(x_j,s) + \hat{k}_4\hat{S}_2(x_j,s), \qquad (3.28a)$$

$$\hat{k}_6 = \hat{k}_2\hat{C}_1(x_j,s) + \hat{k}_4\hat{C}_2(x_j,s), \qquad (3.28b)$$

$$\hat{k}_7 = -s^2\hat{k}_2\hat{S}_2(x_j,s) + \hat{k}_4\hat{S}_1(x_j,s), \qquad (3.28c)$$

$$\hat{k}_8 = -s^2\hat{k}_2\hat{C}_2(x_j,s) + \hat{k}_4\hat{C}_1(x_j,s) - \hat{u}_j(s). \qquad (3.28d)$$

The boundary condition (3.20b) yields:

$$\begin{aligned} 0 =& - s^2\hat{k}_5\hat{S}_2(1-x_j) + \hat{k}_6\hat{C}_1(1-x_j) + \hat{k}_7\hat{S}_1(1-x_j) \\ &+ \hat{k}_8\hat{C}_2(1-x_j), \qquad (3.29a) \\ 0 =& \hat{k}_5(ks\hat{C}_1(1-x_j + s^2\hat{S}_1(1-x_j)) + \hat{k}_6(ks\hat{S}_1(1-x_j) \\ &+ s^2\hat{C}_2(1-x_j)) + \hat{k}_7(ks\hat{C}_2(1-x_j) + s^2\hat{S}_2(1-x_j)) \\ &+ \hat{k}_8(ks\hat{S}_2(1-x_j) - \hat{C}_1(1-x_j)). \qquad (3.29b) \end{aligned}$$

The control can then be derived from (3.28) and (3.29), which reads:

$$\begin{aligned} \hat{u}_j(s) = \Big(&\hat{C}_1(1)(\hat{C}_1(1) - ks\hat{S}_2(1)) \\ &+ s\hat{C}_2(1)(k\hat{S}_1(1) + s\hat{C}_2(1))\Big)\,\hat{y}_j(s), \end{aligned} \qquad (3.30)$$

where $\hat{y}_j(s) \leftrightarrow y_j(t)$ is the so-called basic output, or flat output, which plays a central role in control design.

The control $\hat{u}_j$ can be written in the form

$$\begin{aligned} \hat{u}_j(s) = \sum_{n=0}^{\infty}\Bigg\{&\left(\sum_{k=0}^{n}\frac{(-1)^n}{(4k)!(4(n-k)+3)!}\right)ks^{2n+1} \\ &- \left(\sum_{k=0}^{n}\frac{(-1)^n}{(4k+2)!(4(n-k)+1)!}\right)ks^{2n+1} \\ &- \left(\sum_{k=0}^{n}\frac{(-1)^n}{(4k+2)!(4(n-k)+2)!}\right)s^{2n+2} \\ &- \left(\sum_{k=0}^{n}\frac{(-1)^n}{(4k)!(4(n-k))!}\right)s^{2n}\Bigg\}\,\hat{y}_j(s). \end{aligned} \qquad (3.31)$$

Thus, the time domain control signal is given by

$$\begin{aligned} u_j(t) =& \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^n}{(4k)!(4(n-k)+3)!}\Bigg)ky_j^{(2n+1)}(t) \\ &- \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(-1)^n}{(4k+2)!(4(n-k)+1)!}\right)ky_j^{(2n+1)}(t) \\ &- \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(-1)^n}{(4k+2)!(4(n-k)+2)!}\right)y_j^{(2n+2)}(t) \\ &- \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(-1)^n}{(4k)!(4(n-k))!}\right)y_j^{(2n)}(t). \end{aligned} \qquad (3.32)$$

Note that the full-state trajectory of the zero-dynamics, $\xi_j(x,t)$, can also be expressed in terms of the basic output and its time derivatives. As the control design and analysis do not use the explicit expression of $\xi_j(x,t)$, it is omitted here. Moreover, in flatness-based implementation, the explicit generation of $w^d(x_j,t)$ is not needed.

3.4 Motion Planning and Closed-loop Stability

The closed-loop stability can be derived by virtute of Theorem 3.1.

Corollary 3.1. *If $u_j(t)$ (or $\alpha_j(t)$) $\in L^\infty(0,\infty)$, for all $j=1,\dots,N$, then in closed loop, the system* (3.3) *(or* (3.1)*) with the dynamic control given by* (3.17) *and* (3.18) *is stable in the sense that its trajectory is bounded.*

In order to assure the closed-loop stability of the system, we have to choose appropriate basic outputs $y_j(t)$ so that the control input $u_j(t)$ is uniformly bounded for all $j=1,\dots,N$. For this purpose, we consider a basic output of the form

$$y_j(t) = \bar{y}_j\phi_j(t),\ j=1,\dots,N, \tag{3.33}$$

where $\phi_j(t)$ is a C^∞ function that links 0 and 1. In particular, we choose the following $\phi_j(t)$:

$$\phi_j(t) = \begin{cases} 0, & t \le 0, \\ \dfrac{\int_0^t \exp(-1/(\tau(1-\tau)))^\varepsilon \mathrm{d}\tau}{\int_0^T \exp(-1/(\tau(1-\tau)))^\varepsilon \mathrm{d}\tau}, & 0 < t < T, \\ 1, & t \ge T, \end{cases} \tag{3.34}$$

which is known as the Gevrey function of order $\sigma = 1 + 1/\varepsilon$, $\varepsilon > 0$ (see, e.g., [116]).

The control given in (3.32) becomes then:

$$\begin{aligned} u_j(t) =& \bar{y}_j \left\{ \sum_{n=0}^{\infty} \left. \sum_{k=0}^{n} \frac{(-1)^n}{(4k)!(4(n-k)+3)!} \right) k\phi_j^{(2n+1)}(t) \right. \\ &- \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{(-1)^n}{(4k+2)!(4(n-k)+1)!} \right) k\phi_j^{(2n+1)}(t) \\ &- \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{(-1)^n}{(4k+2)!(4(n-k)+2)!} \right) \phi_j^{(2n+2)}(t) \\ &\left. - \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{(-1)^n}{(4k)!(4(n-k))!} \right) \phi_j^{(2n)}(t) \right\}. \end{aligned} \tag{3.35}$$

Theorem 3.2. *The control* $u_j(t)$ *given in* (3.35) *is uniformly bounded for any Gevrey function* $\phi_j(t)$ *of order* $1 < \sigma < 2$.

The proof of Theorem 3.2 can follow a standard procedure (see, e.g., [8,100,118]) and hence, it is omitted here.

To complete the control design, we need to determine $\bar{y}_j$, $j = 1, \ldots, N$, from the desired reference profile. For a set-point control problem, we can choose the desired steady-state profile of the beam as the solution of the static beam equation corresponding to (3.1), which is of the form

$$\overline{w}^d_{xxxx}(x) = \sum_{j=1}^{N} \overline{\alpha}_j \delta(x - x_j), \; x \in (0, 1), \tag{3.36a}$$

$$\overline{w}^d(0) = \overline{w}^d_{xx}(0) = \overline{w}^d_x(1) = \overline{w}^d_{xxx}(0) = 0, \tag{3.36b}$$

where $\overline{\alpha}_j$, $j = 1, \ldots, N$, are the steady state control inputs.

The solution to the static beam equation can be expressed in terms of its Green's function $G(x, \tilde{x}_j)$:

$$\overline{w}^d = \int_0^1 \sum_{j=1}^{N} G(x, \tilde{x}) \overline{\alpha}_j \delta(x - \tilde{x}) \mathrm{d}\tilde{x} = \sum_{j=1}^{N} G(x, \tilde{x}_j) \overline{\alpha}_j, \tag{3.37}$$

where the Green's function $G(x, \tilde{x}_j)$ is defined as [16]

$$G(x, \tilde{x}_j) = \begin{cases} -\dfrac{x^3}{6} + x\tilde{x}_j \left(1 - \dfrac{\tilde{x}_j}{2}\right), & 0 \le x < \tilde{x}_j; \\ -\dfrac{\tilde{x}_j^3}{6} + \tilde{x}_j x \left(1 - \dfrac{x}{2}\right), & \tilde{x}_j \le x \le 1. \end{cases} \tag{3.38}$$

Now taking N points on $\overline{w}^d(x)$, $\overline{w}^d(x_j), j = 1, ..., N$, we obtain

$$\begin{pmatrix} \overline{w}^d(x_1) \\ \vdots \\ \overline{w}^d(x_N) \end{pmatrix} = \begin{pmatrix} G(x_1, \tilde{x}_1) & \ldots & G(x_N, \tilde{x}_1) \\ \vdots & \ddots & \vdots \\ G(x_1, \tilde{x}_N) & \ldots & G(x_N, \tilde{x}_N) \end{pmatrix} \begin{pmatrix} \overline{\alpha}_1 \\ \vdots \\ \overline{\alpha}_N \end{pmatrix}. \tag{3.39}$$

Due to the invertibility of the matrix in (3.39) formed by $G(x_i, \tilde{x}_j)$, $x_i \neq x_j$, if $i \neq j$ [15], and the fact that $\overline{\alpha}_j = -\lim_{t\to\infty} u_j(t) = \bar{y}_j$, we obtain:

$$\begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_N \end{pmatrix} = \begin{pmatrix} G(x_1, \tilde{x}_1) & \ldots & G(x_N, \tilde{x}_1) \\ \vdots & \ddots & \vdots \\ G(x_1, \tilde{x}_N) & \ldots & G(x_N, \tilde{x}_N) \end{pmatrix}^{-1} \begin{pmatrix} \overline{w}^d(x_1) \\ \vdots \\ \overline{w}^d(x_N) \end{pmatrix}. \tag{3.40}$$

Now, it remains to asses the solution developed for Problem 3.1.

Proposition 3.1. *Assume that the condition given in Theorem 3.1 is fulfilled. Let the basic outputs be given by* (3.33) *with* $\phi_j(t)$ *generated from* (3.34)*, with an order* $1 < \sigma < 2$ *for* $j = 1, \ldots, N$*. Let the reference signals are chosen as* $w^d(x_j, t) = \overline{w}^d(x_j)$*,* $j = 1, \ldots, N$*. Then the dynamic control scheme* (3.17)-(3.18) *with the implementation given by* (3.35) *solves Problem 3.1, i.e.,* $e_j(t) = w_j(x_j, t) - w^d(x_j, t) \to 0$ *as* $t \to \infty$*, for* $j = 1, \ldots, N$*.*

The proof of Proposition 3.1 can follow the development presented in Section III of [31]. Note that a key fact used in the proof is that the homogeneous PDE (with $\alpha_j(t) = 0$, for $j = 1, \ldots, N$) associated with (3.1) (or (3.3)) is exponentially stable.

3.5 Simulation

To illustrate the developed control algorithm, we performed a numerical study on a representative beam. Note that, in the simulation, all the variables are in the normalized coordinates and all the parameters are dimensionless.

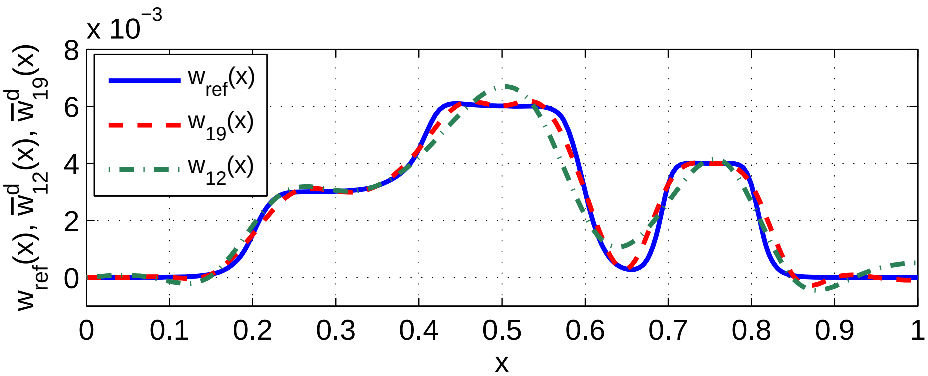
The desired steady state profile considered in the simulation is given by:

$$\begin{aligned} w_{\text{ref}}(x) = & \frac{3 \times 10^{-3}}{1 + \left(\dfrac{10x - c_1}{a_1} \right)^{2b_1}} + \frac{6 \times 10^{-3}}{1 + \left(\dfrac{10x - c_2}{a_2} \right)^{2b_2}} \\ & + \frac{4 \times 10^{-3}}{1 + \left(\dfrac{10x - c_3}{a_3} \right)^{2b_3}} \end{aligned} \tag{3.41}$$

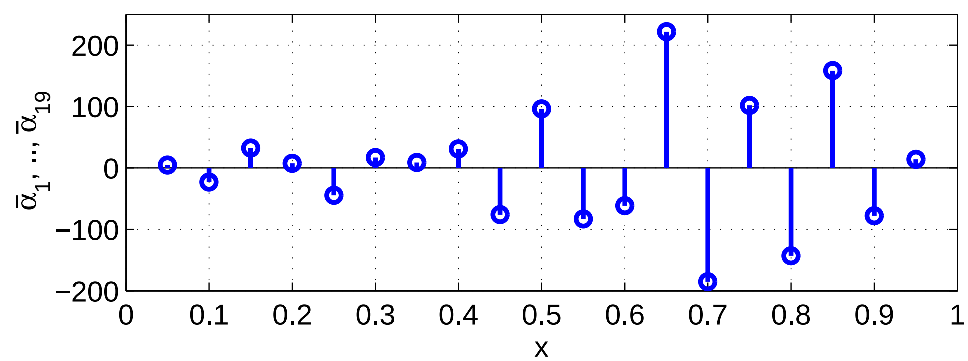
with $a_1 = a_2 = 1$, $a_3 = 0.6$, $b_1 = b_2 = b_3 = 4$, $c_1 = 3$, $c_2 = 5$, and $c_3 = 7.5$, as shown in Fig. 3.1(a).

As in general, the desired steady state profile $w_{\text{ref}}(x)$ may not be a solution to the static beam equation (3.36), we use the Green's functions to interpolate $w_{\text{ref}}(x)$. Indeed, the accuracy of interpolation depends on the number of actuators. As shown in Fig. 3.1(a), the system with 19 actuators offers an adequate performance, which is the setup used in the simulation. The corresponding static controls are shown in Fig. 3.1(b).

A MATLAB Toolbox provided in Chapter 14 of [138] is used in the numerical implementation. The initial data used in the simulation are chosen as $w(x, 0) = -10^{-3} \times \cos(\pi x)$. As the gain-margin of the stabilizing feedback control located at $x = 1$ is $k \in (0, \infty)$, the value of k is determined through numerical experiments, and it is taken as $k = 2$. The interior controls are smooth functions connecting 0 and $\overline{\alpha}_j$, $j = 1, \ldots, N$, as given in (3.34) and (3.35) with



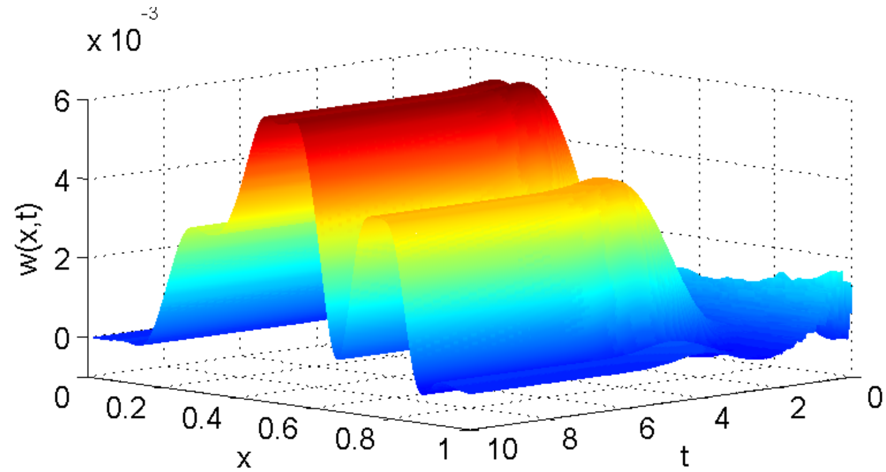
(a)



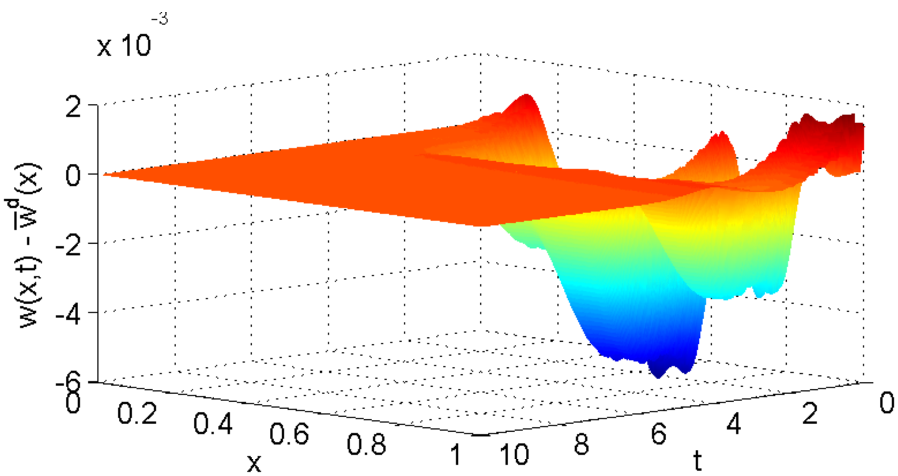
(b)

Figure 3.1 Static profile: (a) interpolation of reference curve; (b) static controls.

$\varepsilon = 1.111$. The evolution of the beam shape and the regulation errors is shown in Fig. 3.2. It can be seen from Fig. 3.2(a) that the initial disturbance has been damped out quickly and the beam has been deformed to the desired form. Furthermore, Fig. 3.2(b) shows that the regulation errors tend to zero identically along the beam. The simulation results confirmed the behavior of the system predicted by theoretical analysis.



(a)



(b)

Figure 3.2 Solution surfaces: (a) beam deformation; (b) regulation error.

3.6 Conclusion

This paper presented a dynamic control scheme for deformation control of an Euler-Bernoulli beam with point-wise interior actuation. The control synthesis is based on the method of ZDI, and the control scheme for a set-point regulation problem is realized by utilizing the method of flat system. It should be noticed that the construction of zero-dynamics for output regulation of many boundary or interiorly controlled PDEs is straightforward and hence, the control design can be carried out in a systematic manner. As the ZDI design does not invoke any early truncations, it can be expected that the performance guaranteed by the design process can be preserved with appropriate implementations. Finally, the method of ZDI provides the possibility to establish a connection with a more generic framework, namely geometric regulation theory (see, e.g., [13]), and is applicable for a wide range of PDE control problems.

CHAPTER 4 ARTICLE 2: ASYMPTOTIC OUTPUT TRACKING FOR A CLASS OF SEMILINEAR PARABOLIC EQUATIONS: A SEMIANALYTICAL APPROACH

This chapter is reproduced from the paper [140].

Authors: Kaijun Yang, Jun Zheng, and Guchuan Zhu.

Abstract— This paper addresses the problem of asymptotic tracking control of a class of semi-linear parabolic equations with pointwise in-domain actuation. First, the assessment of the well-posedness of the considered systems is performed, and then the stability of boundary controlled systems is analysed via Chaffee-Infante equation and Fisher's equation. The application of the zero dynamics inverse design results in a dynamic control scheme that is implemented by using the technique of trajectory planning for flat systems and the Adomian decomposition method. The convergence of the solution of the original systems to that of the corresponding zero dynamics and the convergence of the solution expressed by an Adomian series are also analysed. Numerical simulations are carried out to illustrate the effectiveness of the developed approach.

Index Terms—Semi-linear parabolic equations; Zero dynamics; Adomian decompositon method; Differential flatness.

4.1 Introduction

Asymptotic output tracking is an important method that enables a control system to track reference trajectories with vanishing tracking errors. The theory of asymptotic output regulation was originally developed for the control of finite-dimensional nonlinear systems [39, 40] and then, it has been extended to infinite-dimensional systems described by, e.g., partial differential equations (PDEs) in a series work of Byrnes *et al.* [27–32, 34, 35]. A basic approach for asymptotic output regulation of infinite-dimensional systems is the method of *zero dynamics inverse* that has been applied to the control of a great variety of PDEs, such as heat equations, Burgers' equation, Kuramoto-Sivashinsky equation, etc. [13, 34]. Note that the concept of zero dynamics of infinite-dimensional systems is an extension of the same one in the theory of output regulation of finite-dimensional systems [39, 40]. The method of zero dynamics inverse can be used in the control of PDEs with inputs and outputs located in the domain and/or on the boundary. Essentially, the zero dynamics inverse design results in a dynamic control scheme. A main difficulty in the implementation of such a control scheme is that it requires to solve online the zero dynamics that are also described by a PDE. A solution to get rid of this problem is the use of static control derived from the zero dynamics at the equilibrium, which requires only to solve a steady PDE [13, 34]. Dynamic control schemes derived from the zero dynamics have been developed for the in-domain control of a one-dimensional linear heat equation [145] and the Euler-Bernoulli beam [139]. The dynamic control is implemented by resorting to the method of flat systems, which is also originally developed for the control of finite-dimensional nonlinear systems [58, 89] and has been extended to the control of PDEs [8, 59, 87, 100, 111, 118]. One of the particular feature offered by the method of flat systems is that the trajectory of a PDE can be explicitly computed from the so-called flat outputs and their time-derivatives [100, 118].

The aim of this paper is to extend the approach developed in the previous work [139, 145] to a class of semi-linear parabolic PDEs with in-domain control and a nonlinear term that can be expressed as a polynomial. It should be noted that certain boundary controlled nonlinear parabolic PDEs can be converted to a linear one by using Hopf-Cole transformation [47, 72]. However, it is straightforward to show that this technique is not applicable to systems with in-domain controls. On the other hand, solving online nonlinear PDEs, such as the zero dynamics associated to the problems considered in this work, is of a great computational burden. Therefore, there is a need to develop computationally efficient solutions for the implementation of dynamic control of nonlinear parabolic PDEs. This motivated the work presented in this paper.

The approach developed in this work is to use the technique of flatness-based trajectory planning and Adomian decomposition method (ADM) to solve the zero dynamics involved in the dynamic controller. Adomian decomposition is a semi-analytic method for solving generic functional equations [4, 62]. With ADM, the solution to a functional equation is expressed by a series, which can be seen as a generalization of Taylor series. A particular feature of this method is that the Adomian polynomials in Adomian series can be computed recursively from an initial approximation of the solution to the corresponding equation. It has been shown that Adomian series exhibits a very fast convergence [107], and this technique becomes a powerful tool in numerical analysis of a wide range of both static and evolutional PDEs [1, 43, 55]. In the context of implementation of dynamic in-domain control, we use ADM to solve the zero dynamics by taking the solution to the corresponding linear parabolic equation as the initial approximation, which can be obtained by applying the technique development in the previous work [145]. Note that the technique of flat systems can be straightforwardly applied to solve motion planning problems of nonlinear PDEs under boundary control [95, 101]. However, as with the proposed approaches the ansatz solutions are expressed as a power series, they cannot be applied to systems with multiple in-domain inputs due to the fact that it requires to expend the power series around multiple points. Therefore, there is a need to develop suitable solutions for this type of problems. To assure the validity of the proposed approach, we will assess the well-posedness of the considered semi-linear parabolic PDEs and address the stability, the convergence of the solution to the zero dynamics to that for the original systems, and the convergence of Adomian series solution to the zero dynamics through two typical PDEs, namely Chaffee-Infante equation and Fisher's equation.

The rest of the paper is organized as follows. Section 4.2 presents the setting of the considered semi-linear parabolic PDEs along with their well-poseness and stability analysis. The zero dynamics inverse design process and the method of Adomian decomposition are introduced in Section 4.3. The implementation of the dynamic in-domain control for semi-linear parabolic PDEs is detailed in Section 4.4. Numerical simulation is conducted in Section 4.5, followed by some concluding remarks given in Section 4.6.

4.2 Problem Statement and Stability Analysis

4.2.1 Problem Statement

This work addresses the in-domain control of semi-linear parabolic equations of the following form:

$$\begin{aligned}
&w_t - \alpha w_{xx} = f(w) + \sum_{i=1}^{n} \delta_i(x - x_i)u_i(t),\ x \in (0,1), t > 0,\\
&w_x(0,t) = k_1 w(0,t),\ w_x(1,t) = -k_2 w(1,t),\\
&w(x,0) = \varphi(x),\\
&C_i w = w(x_i,t), i = 1,\dots,n,
\end{aligned} \tag{4.1}$$

where α is a positive constant, $f(w)$ is the nonlinear term on $\mathbb{R}$, the set of all real numbers, which can be expressed as $f(w) = \sum_{i=1}^{P} \beta_i w^i$ with a positive integer $P > 1$. $k_i > 0$, $i = 1, 2$, are the proportional boundary feedback control gains, $\delta_i, i = 1,\dots,n$, are the Dirac functions supported at the point $x_i \in (0,1)$ for $i = 1,\dots,n$, u_i is the in-domain control located at x_i, and $C_i, i = 1,\dots,n$, are the output operators. It is supposed that the initial data $\varphi(x) \in L^2(0,1)$. We denote hereafter by $w(x_i,t)$, $i = 1,\dots,n$, the output signals.

We consider first the well-posedness of the control system (4.1). We introduce first of all some notations. $\mathbb{Z}^+$ denotes the set of all positive integers. Let $L^2(0,1)$ represent the space of 2nd order absolutely integrable real functions defined on $[0,1]$ equipped with the norm $\|v\|_{L^2} = \left(\int_0^1 |v(x)|^2 \mathrm{d}x\right)^{1/2}$, which can be equipped with an inner product defined as $\langle u, v\rangle = \int_0^1 uv\mathrm{d}x$, for any $u, v \in L^2(0,1)$. Let g be a measurable real-valued function. Define the essential supremum of g as $\operatorname{ess\,sup} g = \inf\{a \in \mathbb{R}| \ |\{g > a\}| = 0\}$. Then $\|v\|_\infty = \operatorname{ess\,sup}\{|v(x)|; x \in (0,1)\}$, $\|w\|_{\infty,\infty} = \operatorname{ess\,sup}_{t\in(0,\infty)} \|w(\cdot,t)\|_\infty$. The Sobolev space $H^m(0,1)$ is the space of real functions in $L^2(0,1)$ with derivatives of order less than or equal to m in $L^2(0,1)$. The norm of the Sobolev space $H^1(0,1)$ is defined as

$$\|v\|_{H^1}^2 = \int_0^1 v_x^2 \mathrm{d}x + k_1|v(0)|^2 + k_2|v(1)|^2. \tag{4.2}$$

The Sobolev space $H^{-1}(0,1)$ is defined as the dual space of $H_0^1(0,1)$. Introduce an operator $A = -\alpha\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}$ defined in

$$D(A) = \left\{v \in H^2(0,1) : v_x(0) = k_1 v(0),\ v_x(1) = -k_2 v(1)\right\}. \tag{4.3}$$

Then, A is a positive self-adjoint operator, whose inverse A^{-1} is compact. Due to the Hill-Yosida theorem [142], the linear operator $-A$ is the infinitesimal generator of a strongly continuous semigroup e^{-At}. The eigenvalues of A, denoted by $\lambda_i, i = 1, \ldots, \infty$, can be estimated as $\alpha(i-1)^2\pi^2 < \lambda_i < \alpha i^2\pi^2$ (see, e.g., Reference [41]). Specially, $\alpha\left(\frac{\pi}{2}\right)^2 < \lambda_1 < \alpha\pi^2$ if $\sqrt{k_1 k_2} > \frac{\pi}{2}$. Denote by ψ_i the corresponding eigenvector, which is uniformly bounded [41]. Then, there exists a real-valued constant $L > 0$ such that [28]

$$\left\|e^{-At}\phi\right\|_{L^2} \le e^{-Lt}\|\phi\|_{L^2}, \quad \phi \in L^2(0,1), \quad \forall t \ge 0. \tag{4.4}$$

Note that the parameter L depends on k_1 and k_2. Specifically, $\alpha\left(\frac{\pi}{2}\right)^2 < L < \alpha\pi^2$ if $\sqrt{k_1 k_2} > \frac{\pi}{2}$ [28].

Let A^s, $s \in \mathbb{R}$, be a fractional power of A on the domain

$$D(A^s) = \left\{\varphi \in L^2(0,1) : \sum_{i=1}^{\infty} \lambda_i^{2s}|\langle\varphi, \psi_i\rangle|^2 < \infty\right\}, \tag{4.5}$$

which is defined as for $\varphi \in D(A^s)$

$$A^s\varphi = \sum_{i=1}^{\infty} \lambda_i^s \langle\varphi, \psi_i\rangle\psi_i. \tag{4.6}$$

In particular [41],

$$\sqrt{\alpha}\|\varphi\|_{H^1} = \|A^{1/2}\varphi\|_{L^2}, \tag{4.7}$$

and

$$\|\varphi\|_{L^2}^2 \le \frac{\alpha}{\lambda_1}\|\varphi\|_{H^1}^2. \tag{4.8}$$

Note that the norm $\|\cdot\|_{H^1}$ defined in (4.2) is equivalent to the usual norm of the Sobolev space $H^1(0,1)$ [41], since

$$\frac{1}{\max\left(2(k_0+k_1),(1+k_1+k_2)\right)}\|\varphi\|_{H^1}^2 \le \int_0^1 |\varphi_x|^2\mathrm{d}x + \int_0^1 |\varphi|^2\mathrm{d}x \le (1+\lambda_1^{-1})\|\varphi\|_{H^1}^2. \tag{4.9}$$

We recall the following inequality [70]:

$$\left\|A^s e^{-At}v\right\|_{L^2} \le \kappa t^{-s}e^{-Lt}\|v\|_{L^2}, \forall v \in L^2(0,1),\ \ s,t>0. \tag{4.10}$$

The control system (4.1) can then be expressed as an abstract differential equation:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}w + Aw = Bu + f(w),\\
&w(0) = \varphi,\\
&y = Cw,
\end{aligned} \tag{4.11}$$

where $u = (u_1,\dots,u_n)^T$ and $y = (y_1,\dots,y_n)^T$ represent, respectively, the input and the output, and $B : \mathbb{R}^n \to D'(A)$, the dual space of $D(A)$, $Bv = \sum_{i=1}^n v_i\delta_i(x), v = (v_1,\dots,v_n)^T \in \mathbb{R}^n$, and $C = (C_1,\dots,C_n)^T$ denote, respectively, the input and output operators.

In the subsequent development, the inequalities given in the following lemmas will be extensively used.

Lemma 4.1. *[41] For any $w \in H^1(0,1)$ and $2 \le q \le \infty$, we have*

$$\|w\|_{L^q} \le \eta\|w\|_{H^1}^{\theta}\|w\|_{L^2}^{1-\theta}, \tag{4.12}$$

where $\theta = 1/2 - 1/q$, and $\eta = 2^{3/4}\left(3+2\alpha\lambda_1^{-1}\right)^{1/4}$.

Lemma 4.2. *[143] Let $k>0, g \in C([0,\infty))$ with $g(0)=0$. Assume that a non-negative absolutely continuous function $y : [0,\infty) \to [0,\infty)$ satisfies the following differential inequality*

$$\dot{y} \le -(k-g(y))y, \quad t \ge 0. \tag{4.13}$$

Then, there exists $r^* > 0$ such that for any $y(0) \le r^*$,

$$y(t) \le e^{-\frac{k}{2}t}y(0), \quad t \ge 0. \tag{4.14}$$

We present first the well-posedness of the system (4.1), which can be transformed into an integrated equation by Duhamel's principle [70]. Note that the well-posedness of semi-linear parabolic equations has been intensively addressed in the literature [85, 143]. However, in the considered setting, the non-homogeneous term $\sum_{i=1}^{n} \delta u_i(t)$ belongs to $D'(A)$ rather than $D(A)$. Therefore, the existing results on the well-posedness of semi-linear parabolic equations cannot be directly applied.

Theorem 4.1. *Assume the in-domain controllers $u_i \in C^0(\mathbb{R})$ for $i = 1, \ldots, n$, and the initial data $\varphi \in H^1(0,1)$. Then, there exists $T > 0$ such that the boundary and initial-value problem (BIVP)* (4.1) *admits a unique solution on* $[0,T]$ *and* $w \in C([0,T], H^1(0,1))$.

Proof. By Duhamel's principle [70], the BIVP (4.1) can be reformulated as an integral equation

$$w(t) = e^{-At}\varphi + \int_0^t e^{-A(t-\tau)} \sum_{i=1}^{n} \delta_i u_i(\tau)\mathrm{d}\tau + \int_0^t e^{-A(t-\tau)} f(w(s))\mathrm{d}\tau,\ t > 0. \tag{4.15}$$

The proof is based on the method of successive iterations starting with the solution of the linear part $w_0 = e^{-At}\varphi + \int_0^t e^{-A(t-\tau)} \sum_{i=1}^{n} \delta_i u_i(\tau)\mathrm{d}\tau$. By the definition of the fractional power of A, we obtain

$$\sum_{j=1}^{\infty} \lambda_j^{-\frac{4}{5}} |\langle \delta_i, \psi_j \rangle|^2 = \sum_{j=1}^{\infty} \lambda_j^{-\frac{4}{5}} |\psi_j(x_i)|^2 \leq \Lambda \sum_{j=1}^{\infty} j^{-\frac{8}{5}} < \infty,\ i = 1, \cdots, n, \tag{4.16}$$

where Λ is a positive constant, which implies that $\delta_i \in D(A^{-\frac{2}{5}})$. Thus we have

$$\begin{aligned}
\|w_0 - \varphi\|_{H^1} =& (\sqrt{\alpha})^{-1} \left\| A^{1/2} w_0 - A^{1/2}\varphi \right\|_{L^2} \\
\leq& (\sqrt{\alpha})^{-1} \left\| A^{1/2} e^{-At}\varphi - A^{1/2}\varphi \right\|_{L^2} + (\sqrt{\alpha})^{-1} \int_0^t \left\| A^{1/2} e^{-A(t-\tau)} \sum_{i=1}^{n} \delta_i u_i(\tau) \right\|_{L^2} \mathrm{d}\tau \\
\leq& (\sqrt{\alpha})^{-1} \left\| A^{1/2} e^{-At}\varphi - A^{1/2}\varphi \right\|_{L^2} \\
&+ (\sqrt{\alpha})^{-1} \int_0^t \left\| A^{1/2} A^{\frac{2}{5}} e^{-A(t-\tau)} A^{-\frac{2}{5}} \sum_{i=1}^{n} \delta_i u_i(\tau) \right\|_{L^2} \mathrm{d}\tau \\
\leq& (\sqrt{\alpha})^{-1} \left\| A^{1/2} e^{-At}\varphi - A^{1/2}\varphi \right\|_{L^2} \\
&+ (\sqrt{\alpha})^{-1} \int_0^t \kappa (t-\tau)^{-\frac{9}{10}} e^{-L(t-\tau)} u_i(\tau) \left\| \sum_{i=1}^{n} A^{-\frac{2}{5}} \delta_i \right\|_{L^2} \mathrm{d}\tau.
\end{aligned} \tag{4.17}$$

Note that e^{-At} converges to the identical operator when t tends to 0. As $u_i(t) \in C^0(\mathbb{R}), i = 1, \ldots, n$, we choose a small $t = T_1$ such that $\|w_0 - \varphi\|_{H^1} \leq \frac{\varepsilon}{2}$. The general term w_{m+1} can be defined recursively in terms of w_0 and w_m as following:

$$w_{m+1} = w_0 + \int_0^t e^{-A(t-\tau)} f(w_m(\tau))\mathrm{d}\tau,\ m \geq 0. \tag{4.18}$$

By the induction method, assuming that $\{w_i, i = 1, \ldots, m\} \in B_\varepsilon(\varphi)$, where $B_\varepsilon(\varphi) = \{v \in H^1(0,1) : \|v-\varphi\|_{H^1} < \varepsilon\}$, and due to Lemma 4.1, for all $v \in B_\varepsilon(\varphi)$

$$\begin{aligned}\|v\|_\infty &\leq \eta \|v\|_{L^2}^{1/2}\|v\|_{H^1}^{1/2} \leq \frac{\alpha^{1/4}\eta}{\lambda_1^{1/4}}\|v\|_{H^1} \leq \frac{\alpha^{1/4}\eta}{\lambda_1^{1/4}}\|v-\varphi\|_{H^1} + \frac{\alpha^{1/4}\eta}{\lambda_1^{1/4}}\|\varphi\|_{H^1} \\ &\leq \frac{\alpha^{1/4}\eta}{\lambda_1^{1/4}}\varepsilon + \frac{\alpha^{1/4}\eta}{\lambda_1^{1/4}}\|\varphi\|_{H^1}\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}\|f(v)\|_{H^1} &\leq \sqrt{k_1}|f(v(0))| + \sqrt{k_2}|f(v(1))| + \|f'(v)v_x\|_{L^2} \\ &\leq \sqrt{k_1}M_1 + \sqrt{k_2}M_1 + M_2\|v\|_{H^1} \\ &\leq \sqrt{k_1}M_1 + \sqrt{k_2}M_1 + M_2\left(\varepsilon + \|\varphi\|_{H^1}\right) \\ &< \infty,\end{aligned} \tag{4.20}$$

where

$$\begin{aligned}M_1 &= \max_{|x| \leq \eta\alpha^{1/4}\lambda_1^{-1/4}\varepsilon + \eta\alpha^{1/4}\lambda_1^{-1/4}\|\varphi\|_{H^1}} |f(x)|, \\ M_2 &= \max_{|x| \leq \eta\alpha^{1/4}\lambda_1^{-1/4}\varepsilon + \eta\alpha^{1/4}\lambda_1^{-1/4}\|\varphi\|_{H^1}} |f'(x)|.\end{aligned} \tag{4.21}$$

Therefore, there exists $T < T_1$ such that

$$T(\sqrt{\alpha})^{-1} \sup_{v \in B_\varepsilon(\varphi)} \|f(v)\|_{H^1} \leq \frac{\varepsilon}{2}, \tag{4.22}$$

which yields

$$\begin{aligned}
\sup_{t\in(0,T)} \|w_{m+1}(t)-\varphi\|_{H^1} &\le \sup_{t\in(0,T)} \|w_{m+1}-w_0\|_{H^1} + \|w_0-\varphi\|_{H^1} \\
&\le (\sqrt{\alpha})^{-1} \sup_{t\in(0,T)} \int_0^t \left\| A^{1/2} e^{-A(t-\tau)} f(w_m) \right\|_{L^2} \mathrm{d}\tau + \frac{\varepsilon}{2} \\
&\le (\sqrt{\alpha})^{-1} \sup_{t\in(0,T)} \int_0^t e^{-L(t-\tau)} \|f(w_m)\|_{H^1} \mathrm{d}\tau + \frac{\varepsilon}{2} \\
&\le T(\sqrt{\alpha})^{-1} \sup_{v\in B_\varepsilon(\varphi)} \|f(v)\|_{H^1} + \frac{\varepsilon}{2} \\
&\le \varepsilon.
\end{aligned} \tag{4.23}$$

Therefore, $w_m \in B_\varepsilon(\varphi), i = 0, 1, \dots, \infty$. We define an operator $Q : C([0,T], B_\varepsilon(\varphi)) \to C([0,T], B_\varepsilon(\varphi))$ such that for any $T < T_1$:

$$Qw = w_0 + \int_0^t e^{-A(t-\tau)} f(w(s)) \mathrm{d}\tau. \tag{4.24}$$

As $f \in C^2(\mathbb{R})$ is locally Lipschitz continuous in space $H^1(0,1)$, the operator Q is contracted in a sufficiently small time interval. By the Banach fixed point theorem, there exists a unique solution w to the integral equation in a complete metric space $C([0,T], B_\varepsilon(\varphi))$, which concludes that $w \in C([0,T], H^1(0,1))$. □

Remark 4.1. *From the proof of Theorem 4.1, we know that the theorem can be extended to the case of $f \in C^2$.*

To facilitate the subsequent development, we assume hereafter that k_1 and k_2 satisfy $\sqrt{k_1 k_2} > \frac{\pi}{2}$.

4.2.2 Stability of Boundary Controlled Semi-linear Parabolic PDEs

It should be noted that one of the essential conditions for the application of the method of zero dynamics inverse is that the homogenous version of the original system (the one without in-domain control in the considered problem) should be (at least locally) exponentially stable [13]. This property can be achieved by suitable boundary feedback control as proposed in the work [28, 35], which is indeed the one used in this paper. Thus, we address first the stability of the boundary controlled semi-linear parabolic PDEs of the following form:

$$\begin{aligned}
&w_t - \alpha w_{xx} = f(w), \ x \in (0,1), t > 0 \\
&w_x(0,t) = k_1 w(0,t), \ w_x(1,t) = -k_2 w(1,t), \\
&w(x,0) = \varphi,
\end{aligned} \tag{4.25}$$

which is the homogenous counterpart of the system (4.1).

Due to the complexity of different PDE systems, it is almost impossible to assess the stability for a general setting of PDEs. For this reason, we consider below two particular PDEs, namely Chaffee-Infante equation and Fisher's equation. Nevertheless, the developed method may be applicable to other types of parabolic PDEs.

4.2.2 Stability of Boundary Controlled Chafee-Infante Equation

Chaffee-Infante equation with boundary feedback control can be expressed as

$$\begin{aligned}
&w_t - \alpha w_{xx} = -\gamma w(w^2 - r), \quad x \in (0,1),\ t \in [0,\infty), \\
&w_x(0,t) = k_1 w(0,t), \quad w_x(1,t) = -k_2 w(1,t), \\
&w(x,0) = \varphi,
\end{aligned} \tag{4.26}$$

where α, γ, and r are positive constants, φ represents the initial data, and k_1, k_2 are proportional feedback control parameters.

Theorem 4.2. *Let the initial data $\varphi \in L^2(0,1)$. Then, the system (4.26) is globally exponentially stable in L^2-norm provided $\lambda_1 > r\gamma$, where λ_1 is the first eigenvalue of the operator A.*

Proof. Multiply both sides of (4.26) by w to obtain

$$\int_0^1 w_t w \mathrm{d}x = \alpha \int_0^1 w_{xx} w \mathrm{d}x - \int_0^1 \gamma w^2 (w^2 - r) \mathrm{d}x. \tag{4.27}$$

Due to the boundary conditions, applying integration by parts yields

$$\int_0^1 w_t w \mathrm{d}x + \alpha \int_0^1 w_x w_x \mathrm{d}x + \alpha k_1 w(0,t)^2 + \alpha k_2 w(1,t)^2 = \int_0^1 -\gamma w^2 (w^2 - r) \mathrm{d}x. \tag{4.28}$$

Then we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 + \alpha \int_0^1 w_x^2 \mathrm{d}x + \alpha k_1 w(0,t)^2 + \alpha k_2 w(1,t)^2 \\
=& r\gamma \int_0^1 w^2 \mathrm{d}x - \gamma \int_0^1 w^4 \mathrm{d}x \\
\le& r\gamma \|w\|_{L^2}^2,
\end{aligned} \tag{4.29}$$

which implies that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 &\le -2\alpha \|w\|_{H^1}^2 + 2r\gamma \|w\|_{L^2}^2 \\
&\le -2(\lambda_1 - r\gamma)\|w\|_{L^2}^2.
\end{aligned} \tag{4.30}$$

Therefore, $\|w\|_{L^2}$ decays to 0 exponentially if $\lambda_1 - r\gamma > 0$. $\square$

4.2.2.2 Stability of Boundary Controlled Fisher's Equation

Fisher's equation is a reaction diffusion equation and can be used to describe some phenomenons in biology [106]. Consider the following boundary controlled Fisher's equation:

$$\begin{aligned}
&w_t - \alpha w_{xx} = rw(1-w), \quad x \in (0,1),\ t \in [0,\infty),\\
&w_x(0,t) = k_1 w(0,t), \quad w_x(1,t) = -k_2 w(1,t),\\
&w(x,0) = \varphi,
\end{aligned} \tag{4.31}$$

where r, k_1 and k_2 are positive constants, and φ is the initial condition.

We provide below a local stability result for Fisher's equation.

Theorem 4.3. *Let $\lambda_1 > r$, where λ_1 is the first eigenvalue of the operator A, and the initial data $\varphi \in L^2(0,1)$. Then, there exists a constant $\rho_1 > 0$ such that for any $\|\varphi\|_{L^2} \le \rho_1$, the system (4.31) is exponentially stable in the sense of L^2-norm.*

Proof. Multiply both sides of (4.31) by w to obtain

$$\int_0^1 w_t w \mathrm{d}x = \alpha \int_0^1 w_{xx} w \mathrm{d}x + \int_0^1 rw^2(1-w)\mathrm{d}x. \tag{4.32}$$

Due to the boundary conditions and making integration by parts, it yields

$$\int_0^1 w_t w \mathrm{d}x + \alpha \int_0^1 w_x w_x \mathrm{d}x + \alpha k_1 w_x(0,t)^2 + \alpha k_2 w_x(1,t)^2 = \int_0^1 rw^2(1-w)\mathrm{d}x. \tag{4.33}$$

Then we have

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 &+ \alpha \int_0^1 w_x^2 \mathrm{d}x + \alpha k_1 w(0,t)^2 + \alpha k_2 w(1,t)^2\\
&= r\int_0^1 w^2 \mathrm{d}x - r\int_0^1 w^3 \mathrm{d}x\\
&\le r\|w\|_{L^2}^2 + r\eta^3 \|w\|_{H^1}^{\frac{1}{2}} \|w\|_{L^2}^{\frac{5}{2}}\\
&\le r\|w\|_{L^2}^2 + \frac{\alpha^{1/4} r\eta^3}{\lambda_1^{1/4}} \|w\|_{H^1} \|w\|_{L^2}^2\\
&\le r\|w\|_{L^2}^2 + \varepsilon\|w\|_{H^1}^2 + \frac{\alpha^{1/2} r^2 \eta^6}{4\varepsilon\lambda_1^{1/2}} \|w\|_{L^2}^4,
\end{aligned} \tag{4.34}$$

which implies that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 &\le -2(\alpha-\varepsilon)\|w\|_{H^1}^2 + 2r\|w\|_{L^2}^2 + 2\frac{\alpha^{1/2}r^2\eta^6}{4\varepsilon\lambda_1^{1/2}}\|w\|_{L^2}^4 \\ &\le -2(\lambda_1 - r - \varepsilon\lambda_1\alpha^{-1})\|w\|_{L^2}^2 + 2\frac{\alpha^{1/2}r^2\eta^6}{4\varepsilon\lambda_1^{1/2}}\|w\|_{L^2}^4.\end{aligned} \tag{4.35}$$

Therefore, we choose a $\varepsilon > 0$ such that $\frac{\alpha(\lambda_1 - r)}{\lambda_1} > \varepsilon$. Applying Lemma 4.2, there exits a constant $\rho_1 > 0$, so that if $\|\varphi\|_{L^2} \le \rho_1$, then $\|w\|_{L^2}$ decays to 0 exponentially. □

4.3 Zero Dynamics Inverse-based In-domain Control Design

For in-domain control, we resort to the method of zero dynamics inverse-based output regulation. In this section, we will establish first the zero dynamics corresponding to the semi-linear PDEs with in-domain control, from which the dynamic control will be deduced. For the implementation of the dynamic control scheme, we need to solve the zero dynamics, which will be achieved by using the Adomian decomposition method.

4.3.1 Asymptotic Output Regulation and Zero Dynamics

To establish the zero dynamics, we first transform (4.1) into an equivalent form:

$$\begin{aligned}&w_t - \alpha w_{xx} = f(w),\ x \in \Omega, t > 0, \\ &w_x(0,t) = k_1 w(0,t),\ w_x(1,t) = -k_2 w(1,t), \\ &[w(x,t)]_{x=x_i} = 0,\ i = 1,\dots,n, \\ &B_i w = [w_x(x,t)]_{x=x_i} = u_i,\ i = 1,\dots,n, \\ &w(x,0) = \varphi, \\ &C_i w = w(x_i,t), i = 1,\dots,n,\end{aligned} \tag{4.36}$$

where $\Omega \triangleq (0,x_1)\cup(x_1,x_2)\cup\cdots\cup(x_{n-1},x_n)\cup(x_n,1)$, and $[v(x)]_{x=x_i} = v(x_i^+) - v(x_i^-)$, with $v(x_i^+)$ and $v(x_i^-)$ denoting, respectively, the right and the left limits of $v(x)$ at the point x_i. Note that the equivalence between the formulations given in (4.1) and (4.36) have been assessed in Reference [145] using the technique presented in Reference [114].

Zero dynamics are defined by a system that constrains its outputs toward the reference signals. Specifically, let y_i^d, $i = 1,\dots,n$, be the desired outputs at the set-point and $y_i^r(t)$, $i = 1,\dots,n$, be the reference signals that the outputs $w(x_i,t), i = 1,\cdots,n$, should track to attain their desired set-point values $y_i^d, i = 1,\cdots,n$, namely, $y_i^r(t) \to y_i^d$ as $t \to \infty$. The objective of output regulation is to find in-domain control laws with which the tracking errors of the system (4.1)

$$e_i(t) = w(x_i, t) - y_i^r(t),\ i = 1, \ldots, n, \tag{4.37}$$

converge to 0 asymptotically:

$$\lim_{t\to\infty} e_i(t) = 0,\ i = 1, \ldots, n. \tag{4.38}$$

The zero dynamics can then be expressed as

$$\begin{aligned}
&\xi_t - \alpha\xi_{xx} = f(\xi),\ x \in \Omega,\ t > 0,\\
&\xi_x(0,t) = k_1\xi(0,t), \quad \xi_x(1,t) = -k_2\xi(1,t),\\
&C_i\xi = \xi(x_i,t) = y_i^r(t),\ i = 1, \ldots, n,\\
&[\xi(x,t)]_{x=x_i} = 0,\ i = 1, \ldots, n,\\
&\xi(x,0) = 0,\\
&B_i\xi = [\xi_x(x,t)]_{x=x_i} = u_i,\ i = 1, \ldots, n.
\end{aligned} \tag{4.39}$$

Note that the initial condition of the zero dynamics can be arbitrary. Obviously, the most convenient one is to set it to 0. Note also that the outputs of the zero dynamics $u_i(t)$, $i = 1, \ldots, n$, are the in-domain control signals. The schematic diagram of the closed-loop control system is shown in Fig. 4.1.

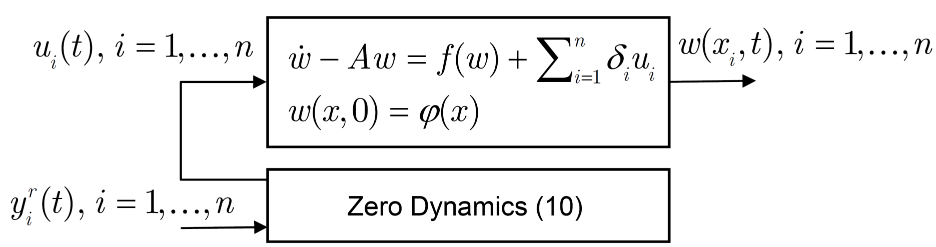


Figure 4.1 Zero-dynamic inverse-based in-domain control.

In order that the in-domain control generated by the zero dynamics can achieve an asymptotic output regulation, we need to guarantee that the original system converge to the corresponding zero dynamics. Again, as it is in general not possible to assess this convergence for the generic setting of semi-linear parabolic PDEs, we will study the cases for Chaffee-Infante equation and Fisher's equation, respectively.

4.3.2 Asymptotic Output Regulation of Chaffee-Infante Equation

For the convergence of the solution to Chaffee-Infante equation to that of its zero dynamics as t tends to ∞, we have the following result.

Theorem 4.4. *Let $\lambda_1 > r\gamma + 3\gamma\|\xi\|^2_{\infty,\infty}$, where λ_1 is the first eigenvalue of the operator A, and the initial data $\varphi \in H_1(0,1)$. Then there exists a constant $\rho_2 > 0$ such that for any $\|\varphi\|_{L^2} \leq \rho_2$, the solution to the in-domain controlled Chaffee-Infante equation exponentially converges to that of its zero dynamics as t tends to ∞ in space $H^1(0,1)$. Furthermore,* $\lim_{t\to\infty} e_i(t) = 0,\ i = 1, \ldots, n$.

Proof. Let w and ξ be the solutions to the in-domain controlled Chaffee-Infante equation and its zero dynamics, respectively. We denote $\widetilde{w} = w - \xi$. By subtracting the in-domain controlled Chaffee-Infante equation to its zero dynamics, we have

$$
\begin{aligned}
&\widetilde{w}_t - \alpha\widetilde{w}_{xx} = r\gamma\widetilde{w} - \gamma\left((\widetilde{w}+\xi)^3 - \xi^3\right),\\
&\widetilde{w}_x(0,t) = k_1\widetilde{w}(0,t), \quad \widetilde{w}_x(1,t) = -k_2\widetilde{w}(1,t),\\
&\widetilde{w}(x,0) = \varphi(x).
\end{aligned}
\tag{4.40}
$$

Multiplying both sides of (4.40) by $\widetilde{w}$ and taking the $L^2(0,1)$ inner product, we have

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{w}\|^2_{L^2} + \alpha\|\widetilde{w}\|^2_{H^1} = r\gamma\|\widetilde{w}\|^2 - \gamma\int_0^1 \left(\widetilde{w}^4 + 3\widetilde{w}^3\xi + 3\widetilde{w}^2\xi^2\right)\mathrm{d}x. \tag{4.41}
$$

By Lemma 4.1 and Young inequality [65], we get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{w}\|^2_{L^2} + 2\alpha\|\widetilde{w}\|^2_{H^1} \leq& 2r\gamma\|\widetilde{w}\|^2_{L^2} + 6\gamma\|\xi\|_\infty\int_0^1 |\widetilde{w}|^3\mathrm{d}x + 6\gamma\|\xi\|^2_\infty\int_0^1 \widetilde{w}^2\mathrm{d}x\\
\leq& \left(2r\gamma + 6\gamma\|\xi\|^2_\infty\right)\|\widetilde{w}\|^2_{L^2} + \varepsilon_1\|\widetilde{w}\|^2_{H^1} + \frac{9\alpha^{1/2}\gamma^2\eta^6\|\xi\|^2_\infty}{\varepsilon_1\lambda_1^{1/2}}\|\widetilde{w}\|^4_{L^2},
\end{aligned}
\tag{4.42}
$$

which implies that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{w}\|^2_{L^2} \leq -\left((2\alpha-\varepsilon_1)\alpha^{-1}\lambda_1 - 2r\gamma - 6\gamma\|\xi\|^2_{\infty,\infty}\right)\|\widetilde{w}\|^2_{L^2} + \frac{9\alpha^{1/2}\gamma^2\eta^6\|\xi\|^2_{\infty,\infty}}{\varepsilon_1\lambda_1^{1/2}}\|\widetilde{w}\|^4_{L^2}. \tag{4.43}
$$

Thus, choosing $\varepsilon_1 > 0$ such that $\frac{2\alpha\lambda_1 - \alpha(2r\gamma + 6\gamma\|\xi\|^2_{\infty,\infty})}{\lambda_1} > \varepsilon_1$, due to Lemma 4.2, there exists a constant $\rho_2, k_3 > 0$ such that for $\|\varphi\|_{L^2} \leq \rho_2$, w converges to 0 exponentially in L^2-norm, namely,

$$\|\widetilde{w}(\cdot,t)\|_{L^2}^2 \le e^{-k_3 t}\|\varphi\|_{L^2}^2, \qquad t\in[0,\infty). \tag{4.44}$$

Multiplying (4.42) by $e^{\frac{1}{2}k_3 t}$ yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{1}{2}k_3 t}\|\widetilde{w}\|_{L^2}^2\right)+\left(2\alpha-\frac{2r\gamma+6\gamma\|\xi\|_{\infty,\infty}^2}{\lambda_1}-\varepsilon_1\right)e^{\frac{1}{2}k_3 t}\|\widetilde{w}\|_{H^1}^2\\
\le&\frac{1}{2}k_3 e^{\frac{1}{2}k_3 t}\|\widetilde{w}\|_{L^2}^2+\frac{9\alpha^{1/2}\gamma^2\eta^6\|\xi\|_{\infty,\infty}^2}{\varepsilon_1\lambda_1^{1/2}}e^{\frac{1}{2}k_3 t}\|\widetilde{w}\|_{L^2}^4 \\
\le&\frac{1}{2}k_3 e^{-\frac{1}{2}k_3 t}\|\varphi\|_{L^2}^2+\frac{9\alpha^{1/2}\gamma^2\eta^6\|\xi\|_{\infty,\infty}^2}{\varepsilon_1\lambda_1^{1/2}}e^{-\frac{3}{2}k_3 t}\|\varphi\|_{L^2}^4.
\end{aligned} \tag{4.45}$$

Integrating from 0 to t, we have

$$\int_0^t e^{\frac{1}{2}k_3 s}\|\widetilde{w}(\cdot,s)\|_{H^1}^2\mathrm{d}s \le M_3(\|\varphi\|_{L^2}),\ t\in[0,\infty), \tag{4.46}$$

where $M_3(\|\varphi\|_{L^2})$ depends on $\|\varphi\|_{L^2}$. Take the L^2-inner product of (4.40) with $-\widetilde{w}_{xx}$ to obtain

$$-\int_0^1 \widetilde{w}_t\widetilde{w}_{xx}\mathrm{d}x+\alpha\int_0^1\widetilde{w}_{xx}^2\mathrm{d}x=-r\gamma\int_0^1\widetilde{w}_{xx}\widetilde{w}\mathrm{d}x+\gamma\int_0^1\left(\widetilde{w}_{xx}\widetilde{w}^3\xi+3\widetilde{w}_{xx}\widetilde{w}^2\xi+3\widetilde{w}_{xx}\widetilde{w}\xi^2\right)\mathrm{d}x. \tag{4.47}$$

Based on Lemma 4.1, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{w}\|_{H^1}^2+2\alpha\|\widetilde{w}_{xx}\|_{L^2}^2 =& -2r\gamma\int_0^1\widetilde{w}_{xx}\widetilde{w}\mathrm{d}x+2\gamma\int_0^1\left(\widetilde{w}_{xx}\widetilde{w}^3\xi+3\widetilde{w}_{xx}\widetilde{w}^2\xi+3\widetilde{w}_{xx}\widetilde{w}\xi^2\right)\mathrm{d}x\\
\le&\varepsilon_2\|\widetilde{w}_{xx}\|_{L^2}^2+\frac{r^2\gamma^2}{\varepsilon_2}\|\widetilde{w}\|_{L^2}^2+\varepsilon_3\|\widetilde{w}_{xx}\|_{L^2}^2+\frac{\gamma^2}{\varepsilon_3}\|\xi\|_\infty^2\int_0^1\widetilde{w}^6\mathrm{d}x\\
&+\varepsilon_4\|\widetilde{w}_{xx}\|_{L^2}^2+\frac{9\gamma^2}{\varepsilon_4}\|\xi\|_\infty^2\int_0^1\widetilde{w}^4\mathrm{d}x+\varepsilon_5\|\widetilde{w}_{xx}\|_{L^2}^2+\frac{9\gamma^2}{\varepsilon_5}\|\xi\|_\infty^4\|\widetilde{w}\|_{L^2}^2\\
\le&\left(\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5\right)\|\widetilde{w}_{xx}\|_{L^2}^2+\left(\frac{r^2\gamma^2}{\varepsilon_2}+\frac{9\gamma^2}{\varepsilon_5}\|\xi\|_\infty^4\right)\|\widetilde{w}\|_{L^2}^2\\
&+\frac{\gamma^2}{\varepsilon_3}\|\xi\|_\infty^2\int_0^1\widetilde{w}^6\mathrm{d}x+\frac{9\gamma^2}{\varepsilon_4}\|\xi\|_\infty^2\int_0^1\widetilde{w}^4\mathrm{d}x\\
\le&\left(\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5\right)\|\widetilde{w}_{xx}\|_{L^2}^2+\left(\frac{r^2\gamma^2}{\varepsilon_2}+\frac{9\gamma^2}{\varepsilon_5}\|\xi\|_\infty^4\right)\|\widetilde{w}\|_{L^2}^2\\
&+\frac{\gamma^2}{\varepsilon_3}\eta^6\|\xi\|_\infty^2\|\widetilde{w}\|_{H^1}^2\|\widetilde{w}\|_{L^2}^4+\frac{9\gamma^2}{\varepsilon_4}\eta^4\|\xi\|_\infty^2\|\widetilde{w}\|_{H^1}\|\widetilde{w}\|_{L^2}^3\\
\le&\left(\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5\right)\|\widetilde{w}_{xx}\|_{L^2}^2+\left(\frac{r^2\gamma^2}{\varepsilon_2}+\frac{9\gamma^2}{\varepsilon_5}\|\xi\|_\infty^4\right)\|\widetilde{w}\|_{H^1}^2\\
&+\left(\frac{\alpha\gamma^2}{\varepsilon_3\lambda_1}\eta^6\|\xi\|_\infty^2\|\varphi\|_{L^2}^2+\frac{9\alpha^{3/2}\gamma^2}{\varepsilon_4\lambda_1^{3/2}}\eta^4\|\xi\|_\infty^2\right)\|\widetilde{w}\|_{H^1}^4,
\end{aligned}
\tag{4.48}
$$

where we have already used the fact that $\int_0^1\widetilde{w}_{tx}\widetilde{w}_x=\int_0^1\widetilde{w}_{xt}\widetilde{w}_x\mathrm{d}x$ in the above inequality, which can be derived as follows: Choose a sequence $\{\widetilde{w}^m\}_{m=1}^\infty\subset C^\infty((0,T)\times(0,1))$ such that $\widetilde{w}^m\to\widetilde{w}$ in $H^1(0,T,H^2(0,1))$. For every $\widetilde{w}^m$, the following equality holds, $\int_0^1\widetilde{w}_{tx}^m\widetilde{w}_x^m=\int_0^1\widetilde{w}_{xt}^m\widetilde{w}_x^m\mathrm{d}x$. Thus, $\int_0^1\widetilde{w}_{tx}\widetilde{w}_x=\int_0^1\widetilde{w}_{xt}\widetilde{w}_x\mathrm{d}x$ as $m\to\infty$.

By choosing the constants $\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5>0$ such that $2\alpha>\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5$, then we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{w}\|_{H^1}^2\le\left(\frac{r^2\gamma^2}{\varepsilon_2}+\frac{9\gamma^2}{\varepsilon_5}\|\xi\|_{\infty,\infty}^4\right)\|\widetilde{w}\|_{H^1}^2+\left(\frac{\alpha\gamma^2}{\varepsilon_3\lambda_1}\eta^6\|\xi\|_{\infty,\infty}^2\|\varphi\|_{L^2}^2+\frac{9\alpha^{3/2}\gamma^2}{\varepsilon_4\lambda_1^{3/2}}\eta^4\|\xi\|_{\infty,\infty}^2\right)\|\widetilde{w}\|_{H^1}^4.
\tag{4.49}
$$

Multiplying by $e^{\frac{1}{2}k_3t}$, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{1}{2}k_3t}\|\widetilde{w}\|_{H^1}^2\right)\le&\left(\frac{r^2\gamma^2}{\varepsilon_2}+\frac{9\gamma^2}{\varepsilon_5}\|\xi\|_{\infty,\infty}^4\right)e^{\frac{1}{2}k_3t}\|\widetilde{w}\|_{H^1}^2+\frac{1}{2}k_3e^{\frac{1}{2}k_3t}\|\widetilde{w}\|_{H^1}^2\\
&+\left(\frac{\alpha\gamma^2}{\varepsilon_3\lambda_1}\eta^6\|\xi\|_{\infty,\infty}^2\|\varphi\|_{L^2}^2+\frac{9\alpha^{3/2}\gamma^2}{\varepsilon_4\lambda^{3/2}}\eta^4\|\xi\|_{\infty,\infty}^2\right)e^{\frac{1}{2}k_3t}\|\widetilde{w}\|_{H^1}^4.
\end{aligned}
\tag{4.50}
$$

By Gronwall's inequality [86] and the inequality (4.46), we get

$$
\begin{aligned}
e^{\frac{1}{2}k_3 t}\|\widetilde{w}\|_{H^1}^2 \leq & \left(\|\varphi\|_{H^1}^2 + \left(\frac{1}{2}k_3 + \frac{r^2\gamma^2}{\varepsilon_2} + \frac{9\gamma^2}{\varepsilon_5}\|\xi\|_{\infty,\infty}^4\right)\int_0^t e^{\frac{1}{2}k_3 s}\|\widetilde{w}(\cdot,s)\|_{H^1}^2 \mathrm{d}s\right) \\
& \times \exp\left(\left(\frac{\alpha\gamma^2}{\varepsilon_3\lambda_1}\eta^6\|\xi\|_{\infty,\infty}^2\|\varphi\|_{L^2}^2 + \frac{9\alpha^{3/2}\gamma^2}{\varepsilon_4\lambda^{3/2}}\eta^4\|\xi\|_{\infty,\infty}^2\right)\int_0^t \|\widetilde{w}(\cdot,s)\|_{H^1}^2 \mathrm{d}s\right) \\
\leq & \left(\|\varphi\|_{H^1}^2 + \left(\frac{1}{2}k_3 + \frac{r^2\gamma^2}{\varepsilon_2} + \frac{9\gamma^2}{\varepsilon_5}\|\xi\|_{\infty,\infty}^4\right) M_3(\|\varphi\|_{L^2})\right) \\
& \times \exp\left(\left(\frac{\alpha\gamma^2}{\varepsilon_3\lambda_1}\eta^6\|\xi\|_{\infty,\infty}^2\|\varphi\|_{L^2}^2 + \frac{9\alpha^{3/2}\gamma^2}{\varepsilon_4\lambda_1^{3/2}}\eta^4\|\xi\|_{\infty,\infty}^2\right) M_3(\|\varphi\|_{L^2})\right),
\end{aligned}
\tag{4.51}
$$

which implies that

$$
\begin{aligned}
\|\widetilde{w}(\cdot,t)\|_{H^1}^2 \leq & \left(\|\varphi\|_{H^1}^2 + \left(\frac{1}{2}k_3 + \frac{r^2\gamma^2}{\varepsilon_2} + \frac{9\gamma^2}{\varepsilon_5}\|\xi\|_{\infty,\infty}^4\right) M_3(\|\varphi\|_{L^2})\right) \\
& \times \exp\left(\left(\frac{\alpha\gamma^2}{\varepsilon_3\lambda_1}\eta^6\|\xi\|_{\infty,\infty}^2\|\varphi\|_{L^2}^2 + \frac{9\alpha^{3/2}\gamma^2}{\varepsilon_4\lambda_1^{3/2}}\eta^4\|\xi\|_{\infty,\infty}^2\right) M_3(\|\varphi\|_{L^2})\right) e^{-\frac{1}{2}k_3 t}, \\
& t \in [0,\infty).
\end{aligned}
\tag{4.52}
$$

Thus, $\|\widetilde{w}\|_{H^1}$ converges to 0 exponentially as t tends to ∞. Due to the fact that $H^1(0,1) \hookrightarrow C(0,1)$, we conclude that $\lim_{t\to\infty} e_i(t) = 0,\ i = 1,\dots,n.$ □

Note that to derive the dynamic in-domain control signals for the system (4.1), we need to solve its zero dynamics (4.39).

4.3.3 Asymptotic Output Regulation of Fisher's Equation

For the convergence of the solution to Fisher's equation to that of its zero dynamics as t tends to ∞, we have the following result.

Theorem 4.5. *Let $\lambda_1 > r + 2r\|\xi\|_{\infty,\infty}$, where λ_1 is the first eigenvalue of the operator A, and the initial data $\varphi \in H^1(0,1)$. Then there exists a constant $\rho_3 > 0$ such that for $\|\varphi\|_{L^2} \leq \rho_3$, the solution to the in-domain controlled Fisher's equation exponentially converges to that of its zero dynamics in H^1-norm, as t tends to ∞. Furthermore, $\lim_{t\to\infty} e_i(t) = 0,\ i = 1,\dots,n.$*

Proof. Let w and ξ be the solution to the in-domain controlled Fisher's equation and its zero dynamics, respectively. We denote $\widetilde{w} = w - \xi$. By subtracting the in-domain controlled

Fisher's equation to its zero dynamics, we have

$$
\begin{aligned}
&\widetilde{w}_t - \alpha \widetilde{w}_{xx} = r\widetilde{w} - r\widetilde{w}(2\xi + \widetilde{w}) \\
&\widetilde{w}_x(0,t) = k_1 \widetilde{w}(0,t), \quad \widetilde{w}_x(1,t) = -k_2 \widetilde{w}(1,t), \\
&\widetilde{w}(x,0) = \varphi(x).
\end{aligned} \tag{4.53}
$$

Multiplying both sides of (4.53) by $\widetilde{w}$ and taking $L^2(0,1)$ the inner product, we have

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{w}\|_{L^2}^2 + \alpha \|\widetilde{w}\|_{H^1}^2 = \int_0^1 \left(r\widetilde{w}^2 - r\widetilde{w}^2(2\xi + \widetilde{w})\right) \mathrm{d}x. \tag{4.54}
$$

By Lemma 4.1, we get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{w}\|_{L^2}^2 + 2\alpha \|\widetilde{w}\|_{H^1}^2 &\le 2\int_0^1 \left(r\widetilde{w}^2 - r\widetilde{w}^2(2\xi + \widetilde{w})\right) \mathrm{d}x \\
&\le \left(2r + 4r\|\xi\|_\infty\right) \|\widetilde{w}\|_{L^2}^2 - 2r \int_0^1 \widetilde{w}^3 \mathrm{d}x \\
&\le \left(2r + 4r\|\xi\|_\infty\right) \|\widetilde{w}\|_{L^2}^2 + 2r\eta^3 \|\widetilde{w}\|_{H^1}^{\frac{1}{2}} \|\widetilde{w}\|_{L^2}^{\frac{5}{2}} \\
&\le \left(2r + 4r\|\xi\|_\infty\right) \|\widetilde{w}\|_{L^2}^2 + \frac{2\alpha^{1/4} r\eta^3}{\lambda_1^{1/4}} \|\widetilde{w}\|_{H^1} \|\widetilde{w}\|_{L^2}^2 \\
&\le \left(2r + 4r\|\xi\|_\infty\right) \|\widetilde{w}\|_{L^2}^2 + \varepsilon_6 \|\widetilde{w}\|_{H^1}^2 + \frac{\alpha^{1/2} r^2 \eta^6}{\varepsilon_6 \lambda_1^{1/2}} \|\widetilde{w}\|_{L^2}^4,
\end{aligned} \tag{4.55}
$$

which implies that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{w}\|_{L^2}^2 &\le -\left(2\alpha - \varepsilon_6\right) \|\widetilde{w}\|_{H^1}^2 + \left(2r + 4r\|\xi\|_{\infty,\infty}\right) \|\widetilde{w}\|_{L^2}^2 + \frac{\alpha^{1/2} r^2 \eta^6}{\varepsilon_6 \lambda_1^{1/2}} \|\widetilde{w}\|_{L^2}^4 \\
&\le -\Big(2\lambda_1 - \left(2r + 4r\|\xi\|_{\infty,\infty} + \alpha^{-1}\lambda_1 \varepsilon_6\right)\Big) \|\widetilde{w}\|_{L^2}^2 + \frac{\alpha^{1/2} r^2 \eta^6}{\varepsilon_6 \lambda_1^{1/2}} \|\widetilde{w}\|_{L^2}^4.
\end{aligned} \tag{4.56}
$$

Therefore, choosing $\varepsilon_6 > 0$ such that $\frac{\alpha(2\lambda_1 - 2r - 4r\|\xi\|_{\infty,\infty})}{\lambda_1} > \varepsilon_6$, by Lemma 4.2, there exists a constant $\rho_3, k_4 > 0$ such that for any $\|\varphi\|_{L^2} \le \rho_3$, $\|\widetilde{w}\|_{L^2} \longrightarrow 0$ exponentially, as $t \longrightarrow \infty$, i.e.

$$
\|\widetilde{w}\|_{L^2}^2 \le e^{-k_4 t} \|\varphi\|_{L^2}^2. \tag{4.57}
$$

By a similar argument to the one in the proof of Theorem 4.4, $\widetilde{w}$ can be estimated by

$$
\int_0^t e^{\frac{1}{2}k_4 s} \|\widetilde{w}(\cdot,s)\|_{H^1}^2 \mathrm{d}s \le M_4(\|\varphi\|_{L^2}), \ t \in [0,\infty), \tag{4.58}
$$

where $M_4(\|\varphi\|_{L^2})$ depends on the norm $\|\varphi\|_{L^2}$.

Taking the L^2-inner product of (4.53) with $-\widetilde{w}_{xx}$ yields

$$-\int_0^1 \widetilde{w}_t\widetilde{w}_{xx}\mathrm{d}x+\alpha\int_0^1 \widetilde{w}_{xx}^2\mathrm{d}x=-r\int_0^1 \widetilde{w}_{xx}\widetilde{w}\mathrm{d}x+r\int_0^1 \widetilde{w}_{xx}\widetilde{w}(2\xi+\widetilde{w})\mathrm{d}x. \tag{4.59}$$

By Lemma 4.1, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{w}\|_{H^1}^2+2\alpha\|\widetilde{w}_{xx}\|_{L^2}^2&=-2r\int_0^1 \widetilde{w}_{xx}\widetilde{w}\mathrm{d}x+2r\int_0^1 \widetilde{w}_{xx}\widetilde{w}(2\xi+\widetilde{w})\mathrm{d}x\\ &\le \varepsilon_7\|\widetilde{w}_{xx}\|_{L^2}^2+\frac{r^2}{\varepsilon_7}\|\widetilde{w}\|_{L^2}^2+\varepsilon_8\|\widetilde{w}_{xx}\|_{L^2}^2+\frac{r^2}{\varepsilon_8}\int_0^1 \widetilde{w}^2(2\xi+\widetilde{w})^2\mathrm{d}x\\ &\le(\varepsilon_7+\varepsilon_8)\|\widetilde{w}_{xx}\|_{L^2}^2+\left(\frac{r^2}{\varepsilon_7}+\frac{4r^2}{\varepsilon_8}\|\xi\|_\infty^2\right)\|\widetilde{w}\|_{L^2}^2+\frac{2r^2}{\varepsilon_8}\int_0^1\widetilde{w}^4\mathrm{d}x\\ &\le(\varepsilon_7+\varepsilon_8)\|\widetilde{w}_{xx}\|_{L^2}^2+\left(\frac{r^2}{\varepsilon_7}+\frac{4r^2}{\varepsilon_8}\|\xi\|_\infty^2\right)\|\widetilde{w}\|_{L^2}^2+\frac{2r^2}{\varepsilon_8}\eta^4\|\widetilde{w}\|_{H^1}\|\widetilde{w}\|_{L^2}^3\\ &\le(\varepsilon_7+\varepsilon_8)\|\widetilde{w}_{xx}\|_{L^2}^2+\left(\frac{\alpha r^2}{\varepsilon_7\lambda_1}+\frac{4\alpha r^2}{\varepsilon_8\lambda_1}\|\xi\|_\infty^2\right)\|\widetilde{w}\|_{H^1}^2+\frac{2\alpha^{3/2}r^2}{\varepsilon_8\lambda_1^{3/2}}\eta^4\|\widetilde{w}\|_{H^1}^4.\end{aligned}\tag{4.60}$$

Choosing appropriate constants $\varepsilon_7,\varepsilon_8>0$ such that $2\alpha>\varepsilon_7+\varepsilon_8$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{w}\|_{H^1}^2\le\left(\frac{\alpha r^2}{\varepsilon_7\lambda_1}+\frac{4\alpha r^2}{\varepsilon_8\lambda_1}\|\xi\|_{\infty,\infty}^2\right)\|\widetilde{w}\|_{H^1}^2+\frac{2\alpha^{3/2}r^2}{\varepsilon_8\lambda_1^{3/2}}\eta^4\|\widetilde{w}\|_{H^1}^4. \tag{4.61}$$

Multiplying by $e^{\frac{1}{2}k_4t}$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}e^{\frac{1}{2}k_4t}\|\widetilde{w}\|_{H^1}^2\le{}&\frac{1}{2}k_4e^{\frac{1}{2}k_4t}\|\widetilde{w}\|_{H^1}^2+\left(\frac{\alpha r^2}{\varepsilon_7\lambda_1}+\frac{4\alpha r^2}{\varepsilon_8\lambda_1}\|\xi\|_{\infty,\infty}^2\right)e^{\frac{1}{2}k_4t}\|\widetilde{w}\|_{H^1}^2\\ &+\frac{2\alpha^{3/2}r^2}{\varepsilon_8\lambda_1^{3/2}}\eta^4e^{\frac{1}{2}k_4t}\|\widetilde{w}\|_{H^1}^4.\end{aligned}\tag{4.62}$$

By Gronwall's inequality [86] and (4.58), the above inequality yields

$$\begin{aligned}e^{\frac{1}{2}k_4t}\|\widetilde{w}\|_{H^1}^2&\le\left(\|\varphi\|_{H^1}^2+\left(\frac{1}{2}k_4+\frac{\alpha r^2}{\varepsilon_7\lambda_1}+\frac{4\alpha r^2}{\varepsilon_8\lambda_1}\|\xi\|_{\infty,\infty}^2\right)\int_0^t e^{\frac{1}{2}k_4s}\|\widetilde{w}(\cdot,s)\|_{H^1}^2\mathrm{d}s\right)\\ &\quad\times\exp\left(\frac{2\alpha^{3/2}r^2}{\varepsilon_8\lambda_1^{3/2}}\eta^4\int_0^t\|\widetilde{w}(\cdot,s)\|_{H^1}^2\mathrm{d}s\right)\\ &\le\left(\|\varphi\|_{H^1}^2+\left(\frac{1}{2}k_4+\frac{\alpha r^2}{\varepsilon_7\lambda_1}+\frac{4\alpha r^2}{\varepsilon_8\lambda_1}\|\xi\|_{\infty,\infty}^2\right)M_4(\|\varphi\|_{L^2})\right)\\ &\quad\times\exp\left(\frac{2\alpha^{3/2}r^2}{\varepsilon_8\lambda_1^{3/2}}\eta^4M_4(\|\varphi\|_{L^2})\right),\end{aligned}\tag{4.63}$$

which implies that

$$
\begin{aligned}
\|\widetilde{w}(\cdot,t)\|_{H^1}^2 \leq & \left(\|\varphi\|_{H^1}^2 + \left(\frac{1}{2}k_4 + \frac{\alpha r^2}{\varepsilon_7\lambda_1} + \frac{4\alpha r^2}{\varepsilon_8\lambda_1}\|\xi\|_{\infty,\infty}^2\right) M_4(\|\varphi\|_{L^2})\right) \\
& \times \exp\left(\frac{2\alpha^{3/2}r^2}{\varepsilon_8\lambda_1^{3/2}}\eta^4 M_4(\|\varphi\|_{L^2})\right) e^{-\frac{1}{2}k_4 t}, t \in [0,\infty).
\end{aligned}
\tag{4.64}
$$

Therefore, the solution to the in-domain controlled Fisher's equation exponentially converges to that of its zero dynamics in H^1-norm, as t tends to ∞. Based on the fact that $H^1(0,1) \hookrightarrow C(0,1)$, we conclude that $\lim_{t\to\infty} e_i(t) = 0,\ i = 1,\ldots,n$, which completes the proof. $\square$

4.3.4 Adomian Decomposition Method for the Solution to Zero Dynamics

Note that as the zero dynamics (4.39) is also a nonlinear PDE, it is almost impossible to obtain an explicit form of its solution. In order to overcome this difficulty, we apply the Adomian decomposition method to obtain a solution to the zero dynamics under a series form.

In next section, we will show the use of the Adomian methodology to obtain the solution to the zero dynamics (4.39) in terms of a infinite series.

4.3.4.1 Introduction to the Adomian Decomposition Method

In this section, we briefly introduce the Adomian decomposition method. The detailed presentation of the ADM and its applications can be found in the literature [4, 62, 107].

Consider an abstract initial-value problem expressed in the following form:

$$
\begin{aligned}
& L\xi + R\xi + N\xi = \phi(x,t), \\
& \xi(0) = \varphi,
\end{aligned}
\tag{4.65}
$$

where $L : X \to Y$ is an invertible linear operator from a Banach space X to a Banach space Y, $R\xi$ and $N\xi$ are linear and nonlinear terms, respectively, and ϕ is the non-homogeneous term. To apply the ADM, we write (4.65) in an operator form

$$
\xi = \psi + L^{-1}\phi - L^{-1}R\xi - L^{-1}N\xi, \tag{4.66}
$$

where the term ψ can be determined by the initial data φ. The nonlinear term can be expressed in terms of a iteration series $N\xi = \sum_{m=0}^{\infty} A_m$, where A_m is given by

$$A_m = \left[\frac{1}{m!} \frac{\mathrm{d}^m}{\mathrm{d}\lambda^m} N \left(\sum_{i=0}^{\infty} \lambda^i \xi_i \right) \right] \Bigg|_{\lambda=0}, \ m = 0, 1, 2, \cdots \tag{4.67}$$

When $\psi + L^{-1}\phi(x,t)$ is defined as ξ_0, we have

$$\begin{aligned} \xi_0 &= \psi + L^{-1}\phi(x,t), \\ \xi_1 &= -L^{-1}R\xi_0 - L^{-1}A_0, \\ &\vdots \\ \xi_{m+1} &= -L^{-1}R\xi_m - L^{-1}A_m. \end{aligned} \tag{4.68}$$

The solution to the nonlinear abstract equation (4.65) can then be expressed as an Adomian series:

$$\xi(x,t) = \sum_{i=0}^{\infty} \xi_i(x,t). \tag{4.69}$$

4.3.4.2 Convergence of the Adomian Series Solution of the Zero Dynamics

Based on the ADM, we express the nonlinear term in (4.39) as

$$f(\xi) = \sum_{i=0}^{\infty} A_i(\xi_0, \xi_1, \ldots, \xi_i), \tag{4.70}$$

and the solution to (4.39) as a series of the form given in (4.69). Let ξ_0 complies with the following linear parabolic equation:

$$\begin{aligned} &\xi_{0t} - \alpha\xi_{0xx} = 0, \ x \in \Omega, t > 0, \\ &\xi_{0x}(0,t) = k_1\xi_0(0,t), \ \xi_{0x}(1,t) = -k_2\xi_0(1,t), \\ &[\xi_0(x,t)]_{x=x_i} = 0, \ i = 1, \ldots, n, \\ &\xi_0(x_i,t) = y^r_{i,0}(t), \ i = 1, \ldots, n, \\ &\xi_0(x,0) = 0. \end{aligned} \tag{4.71}$$

The first component $u_{0,i}$ of in-domain controllers can be obtain as

$$u_{0,i}(t) = [\xi_{0x}(x,t)]_{x=x_i}, \ i = 1, \ldots, n. \tag{4.72}$$

Using the recursive formula (4.67), we obtain the explicit form of ξ_i, for example:

$$\begin{aligned} &A_0(\xi_0) = f(\xi_0), \quad A_1(\xi_0, \xi_1) = \xi_1 f'(\xi_0), \\ &A_2(\xi_0, \xi_1, \xi_2) = \xi_2 f'(\xi_0) + \frac{1}{2} f^{(2)}(\xi_0)(\xi_1)^2. \end{aligned} \tag{4.73}$$

In order to facilitate the analysis of Adomian series, the Adomian's polynomials (4.67) will be rearranged (see, e.g., Reference [55]). Thus, let $\overline{A}$ represent the rearranged Adomian's polynomial, which can be reformulated as follow:

$$\overline{A}_0 = f(\xi_0), \quad \overline{A}_0 + \overline{A}_1 = f(\xi_0 + \xi_1). \tag{4.74}$$

By induction we have for any $m \geq 0$,

$$\sum_{i=0}^{m} \overline{A}_i(\xi_0, \ldots, \xi_i) = f\left(\sum_{i=0}^{m} \xi_i\right), \tag{4.75}$$

which implies that

$$\overline{A}_m = f\left(\sum_{i=0}^{m} \xi_i\right) - f\left(\sum_{i=0}^{m-1} \xi_i\right), \quad m = 1, \ldots, \infty. \tag{4.76}$$

Therefore, the general term $\xi_m, m = 1, \ldots, \infty$, can be written with

$$\xi_m = \int_0^t e^{-A(t-\tau)} \overline{A}_{m-1} \mathrm{d}\tau, \quad m = 1, \ldots, \infty, \tag{4.77}$$

which complies with the following parabolic equation:

$$\begin{aligned} &\xi_{mt} - \alpha \xi_{mxx} = \overline{A}_{m-1}, \quad x \in \Omega,\ t > 0, \\ &\xi_{mx}(0,t) = k_1 \xi_m(0,t),\ \xi_{mx}(1,t) = -k_2 \xi_m(1,t), \\ &[\xi_m(x,t)]_{x=x_i} = 0,\ i = 1, \ldots, n, \\ &\xi_m(x_i,t) = y^r_{i,m}(t),\ i = \ldots, n, \\ &\xi_m(x,0) = 0. \end{aligned} \tag{4.78}$$

The m-th component $u_{m,i}(t)$ of the in-domain controllers $u_i(t)$ can be written as

$$u_{m,i}(t) = [\xi_{mx}(x,t)]_{x=x_i},\ i = 1, \ldots, n. \tag{4.79}$$

The solution to zero dynamics (4.39) and the reference signal $y^r_i(t)$ can be expressed as, respectively,

$$\xi(x,t) = \sum_{m=0}^{\infty} \xi_m(x,t), \quad \xi(x_i,t) = y_i^r(t) = \sum_{m=0}^{\infty} y_{i,m}^r(t), \ i = 1,\dots,n. \tag{4.80}$$

We obtain the in-domain controls u_i from (4.71), (4.78), and (4.80):

$$u_i(t) = \sum_{m=0}^{\infty} u_{m,i}(t) = \sum_{m=0}^{\infty} \left[\xi_{mx}(x,t)\right]_{x=x_i}, \ i = 1,\dots,n. \tag{4.81}$$

Note that the use of the rearranged Adomian polynomials provide a convenient way to prove the convergence of the Aomian series solution to the zero dynamics (4.39).

Theorem 4.6. *Assume that the in-domain control signals $u_i \in C^1(\mathbb{R})$, $i = 1,\dots,n$, are bounded. Let $\xi_m, i = 0,\dots,m$, be defined by (4.68). Define two parameters*

$$M_5 = \max_{|x| \le \eta\alpha^{1/4} 2^{-(k-1)} \lambda_1^{-1/4} + \|\xi_0\|_{\infty,\infty}} |f'(x)|, \tag{4.82}$$

and

$$M = \max\left\{ \frac{4^k \pi \|f(\xi_0)\|_{\infty,\infty}^2}{\alpha}, \ \frac{4\pi M_5^2}{\lambda_1} \right\}. \tag{4.83}$$

There exists $k \in \mathbb{Z}^+$, such that $M < L$. Then the m-th partial sum of the Adomian series $S_m = \sum_{i=0}^{m} \xi_i(x,t)$ converges to ξ in $C([0,\infty), H^1(0,1))$. Furthermore, ξ is the solution to the zero dynamics (4.39).

Remark 4.2. *The nonlinear term in Chaffee-Infante equation ($f(x) = -\gamma x(x^2 - r)$) and that in Fisher's equation ($f(x) = rx(1-x)$) satisfy the assumption of Theorem 4.6 when the quantity of $\|\xi_0\|_\infty$ is small enough. For example, when $0 \le \xi_0(x,t) \le 0.8$, $\alpha = 6$, $r = \gamma = 1$, then choosing $k = 3$, Chaffee-Infante equation meets the conditions of Theorem 4.6.*

Proof. Based on the rearrangement of the Adomain series (4.75), to prove the convergence of the rearranged Adomian series, we should estimate the bound of every term of the series. The procedure to find the first term ξ_0 will be presented in Subsection 4.4.2. According to the proof of Theorem 4.1, $\xi_0 \in C([0,\infty), H^1(0,1))$ is bounded, i.e. $\|\xi_0\|_{\infty,\infty} < \infty$. The estimates of the second term ξ_1 and the third term ξ_2 are given by

$$\begin{aligned}
\|\xi_1\|_{H^1} &= (\sqrt{\alpha})^{-1}\left\|A^{1/2}\xi_1\right\|_{L^2} \le (\sqrt{\alpha})^{-1}\int_0^t \left\|A^{1/2}e^{-A(t-\tau)}\overline{A}_0(\xi_0)\right\|_{L^2}\mathrm{d}\tau \\
&\le (\sqrt{\alpha})^{-1}\int_0^t (t-\tau)^{-1/2}e^{-L(t-\tau)}\left\|\overline{A}_0(\xi_0)\right\|_{L^2}\mathrm{d}\tau \\
&\le (\sqrt{\alpha})^{-1}\int_0^t (t-\tau)^{-1/2}e^{-L(t-\tau)}\left\|f(\xi_0)\right\|_{\infty}\mathrm{d}\tau \\
&\le (\sqrt{\alpha})^{-1}\|f(\xi_0)\|_{\infty,\infty}\int_0^\infty t^{-1/2}e^{-Lt}\mathrm{d}t \\
&\le (\sqrt{\alpha})^{-1}\|f(\xi_0)\|_{\infty,\infty}\sqrt{\frac{\pi}{L}} < \frac{1}{2^k},
\end{aligned} \tag{4.84}$$

and

$$\|\xi_0+\xi_1\|_\infty \le \frac{\eta\alpha^{1/4}}{\lambda_1^{1/4}}\|\xi_1\|_{H^1} + \|\xi_0\|_\infty \le \frac{\eta\alpha^{1/4}}{2^k\lambda_1^{1/4}} + \|\xi_0\|_{\infty,\infty}. \tag{4.85}$$

Applying Lemma 4.1 yields

$$\begin{aligned}
\|\xi_2\|_{H^1} =&(\sqrt{\alpha})^{-1}\left\|A^{1/2}\xi_2\right\|_{L^2} \le (\sqrt{\alpha})^{-1}\int_0^t \left\|A^{1/2}e^{-A(t-\tau)}\overline{A}_1(\xi_0,\xi_1)\right\|_{L^2}\mathrm{d}\tau \\
\le&(\sqrt{\alpha})^{-1}\int_0^t (t-\tau)^{-1/2}e^{-L(t-\tau)}\left\|\overline{A}_1(\xi_0,\xi_1)(\tau)\right\|_{L^2}\mathrm{d}\tau \\
\le&(\sqrt{\alpha})^{-1}\int_0^t (t-\tau)^{-1/2}e^{-L(t-\tau)}\left\|f(\xi_0+\xi_1)-f(\xi_0)\right\|_{L^2}\mathrm{d}\tau \\
\le&(\sqrt{\alpha})^{-1}\int_0^t (t-\tau)^{-1/2}e^{-L(t-\tau)}\left\|f'(\xi_0+\theta_1\xi_1)\xi_1\right\|_{L^2}\mathrm{d}\tau \quad (0<\theta_1<1) \\
\le&(\sqrt{\alpha})^{-1}\int_0^t (t-\tau)^{-1/2}e^{-L(t-\tau)}\|f'(\xi_0+\theta\xi_1)\|_\infty\|\xi_1\|_{L^2}\mathrm{d}\tau \\
\le&\lambda_1^{-1/2}\int_0^t (t-\tau)^{-1/2}e^{-L(t-\tau)}\|f'(\xi_0+\theta_1\xi_1)\|_\infty\|\xi_1\|_{H^1}\mathrm{d}\tau \\
\le&\lambda_1^{-1/2}\sqrt{\frac{\pi}{L}}\frac{M_5}{2^k} \le \frac{1}{2^{k+1}},
\end{aligned} \tag{4.86}$$

and

$$\begin{aligned}
\|\xi_0+\xi_1+\xi_2\|_\infty &\le \eta\|\xi_1+\xi_2\|_{H^1}^{1/2}\|\xi_1+\xi_2\|_{L^2}^{1/2} + \|\xi_0\|_\infty \\
&\le \frac{\eta\alpha^{1/4}}{\lambda_1^{1/4}}\left(\|\xi_1\|_{H^1}+\|\xi_2\|_{H^1}\right) + \|\xi_0\|_\infty \le \frac{3}{2^{k+1}}\frac{\eta\alpha^{1/4}}{\lambda_1^{1/4}} + \|\xi_0\|_{\infty,\infty}.
\end{aligned} \tag{4.87}$$

By induction, we assume that $\|\xi_m\|_{H^1} \le \frac{1}{2^{k+m-1}}$, $\|\sum_{i=0}^m \xi_i\|_\infty \le \frac{\eta\alpha^{1/4}}{2^{k-1}\lambda_1^{1/4}} + \|\xi_0\|_{\infty,\infty}$, $i =$

$1, \ldots, m$, then obtain

$$
\begin{aligned}
\|\xi_{m+1}\|_{H^1} &= (\sqrt{\alpha})^{-1} \left\| A^{1/2}\xi_{m+1} \right\|_{L^2} \leq (\sqrt{\alpha})^{-1} \int_0^t \left\| A^{1/2} e^{-A(t-\tau)} \overline{A}_m(\xi_0, \xi_1, \ldots, \xi_m) \right\|_{L^2} \mathrm{d}\tau \\
&\leq (\sqrt{\alpha})^{-1} \int_0^t (t-\tau)^{-1/2} e^{-L(t-\tau)} \left\| \overline{A}_m(\xi_0, \xi_1, \ldots, \xi_m)(\tau) \right\|_{L^2} \mathrm{d}\tau \\
&\leq (\sqrt{\alpha})^{-1} \int_0^t (t-\tau)^{-1/2} e^{-L(t-\tau)} \left\| f(\xi_0 + \cdots + \xi_m) - f(\xi_0 + \cdots + \xi_{m-1}) \right\|_{L^2} \mathrm{d}\tau \\
&\leq (\sqrt{\alpha})^{-1} \int_0^t (t-\tau)^{-1/2} e^{-L(t-\tau)} \left\| f'(\xi_0 + \cdots + \xi_{m-1} + \theta_m \xi_m)\xi_m \right\|_{L^2} \mathrm{d}\tau \quad (0 < \theta_m < 1) \\
&\leq \lambda_1^{-1/2} \int_0^t (t-\tau)^{-1/2} e^{-L(t-\tau)} \| f'(\xi_0 + \cdots + \theta_m \xi_m) \|_\infty \|\xi_m\|_{H^1} \mathrm{d}\tau \\
&\leq \lambda_1^{-1/2} \sqrt{\frac{\pi}{L}} \frac{M_5}{2^{k+m-1}} \leq \frac{1}{2^{k+m}},
\end{aligned}
\tag{4.88}
$$

and

$$
\left\| \sum_{i=0}^{m+1} \xi_i \right\|_\infty \leq \frac{\eta \alpha^{1/4}}{\lambda_1^{1/4}} \sum_{i=1}^{m+1} \|\xi_i\|_{H^1} + \|\xi_0\|_\infty \leq \frac{\eta \alpha^{1/4}}{\lambda_1^{1/4}} \sum_{i=1}^{m+1} \frac{1}{2^{k+i-1}} + \|\xi_0\|_\infty \leq \frac{\eta \alpha^{1/4}}{2^{k-1}\lambda_1^{1/4}} + \|\xi_0\|_{\infty,\infty}.
\tag{4.89}
$$

The difference between S_m and S_{m+l} can be estimated in space $H^1(0,1)$ as

$$
\|S_{m+l} - S_m\|_{H^1} \leq \sum_{i=1}^{l} \|\xi_{m+i}\|_{H^1} \leq \frac{1}{2^{m+k-1}}, \quad \forall m, l \in \mathbb{Z}^+. \tag{4.90}
$$

Thus, $\xi_m, i = 1, \ldots, \infty$, constitute a Cauchy sequence in $H^1(0,1)$ at almost every time, which implies that there exists a function ξ such that the m-th partial sum of the Adomian series $S_m = \sum_{k=0}^m \xi^k(x,t)$ converges to ξ in $C([0,\infty), H^1(0,1))$. Based on the construction of Aomian series, we have

$$
\xi = w_0 + \int_0^t e^{-A(t-\tau)} f(\xi(\tau)) \mathrm{d}\tau, \ t > 0. \tag{4.91}
$$

Therefore, ξ is the solution to the zero dynamics (4.39). □

Remark 4.3. *From the proof of Theorem 4.6, the theorem can be adapted to the case of $f \in C^\infty$.*

Theorem 4.6 ensures that the solution to the zero dynamics (4.39) can be expressed in terms of an Adomain series. Moreover, the proof of Theorem 4.6 indicates that when the conditions in Theorem 4.6 hold, the solution to the zero dynamics (4.39) generated by ADM will be convergent around the linear part ξ_0 of the Adomian series. The conditions of Theorem 4.6 is indeed weaker than the case where f is globally Lipschitz. Note that although the restriction of Theorem 4.6 is still very conservative, it is enough for the development of the proposed control scheme.

4.4 Differential Flatness-based Set-point Control and Trajectory Planing

Based on Theorems 4.4 and 4.5, the in-domain control problem can be reduced to finding the solution to the corresponding zero dynamics (4.39), which can be expressed by an Adomain series. Due to the recursive procedure, the computation of the Adomain amounts mainly to determining the linear part, ξ_0, which will considerably simplify the implementation of the dynamic control scheme.

4.4.1 Smoothness of the Solution to the Zero Dynamics

We present first the following property on the smoothness of the trajectory of the zero dynamics (4.39) with in-domain actuation, which enables us to obtain the in-domain control of the system (4.96) from the linear part ξ_0 of the Adomian series.

Theorem 4.7. *Assume β_i, $i = 1, \ldots, P$, are some constants, and $u_{0,i} \in C^2(\mathbb{R})$. Then the outputs of the zero dynamics (4.39), $u_i(t), i = 1, \ldots, n$, depend only on the linear part of the Adomian series solution, i.e. $u_i = u_{0,i} = \xi_{0x}(x_i^+, t) - \xi_{0x}(x_i^-, t), i = 1, \ldots, n$.*

Remark 4.4. *It is straightforward to verify that the nonlinear term $f(w)$ specified in Theorem 4.7 can cover the nonlinearity of Chaffee-Infante equation ($f(w) = -\gamma w(w^2 - r)$) and that of Fisher's equation ($f(w) = rw(1 - w)$).*

Proof. In order to obtain the regularity of the solution to (4.39), it suffices to analyse the regularity of every term ξ_i in the corresponding Adomian series. First, we investigate the first term ξ_0, which can be expressed as (4.98). It is easy to see that ξ_0 belongs to $C([0, \infty), H^1(0, 1))$. Consider the second term ξ_1, which is the solution to the following PDE:

$$\begin{aligned}
&\xi_{1t} - \alpha\xi_{1xx} = f(\xi_0), x \in (0, 1),\ t \in [0, \infty), \\
&\xi_{1x}(0, t) = k_1\xi_1(0, t),\ \xi_{1x}(1, t) = -k_2\xi_1(1, t), \\
&\xi_1(x_i, t) = y_{i,1}^r(t),\ i = 1, \ldots, n, \\
&\xi_1(x, 0) = 0.
\end{aligned} \tag{4.92}$$

By Lemma 4.1, the non-homogeneous therm $f(\xi_0)$ of (4.92) can be estimated as

$$\begin{aligned}\|f(\xi_0)\|_{L^2} &= \left\|\sum_{i=1}^{P}\beta_i(\xi_0)^i\right\|_{L^2}\\ &\le \sum_{i=1}^{P}|\beta_i|\|(\xi_0)^i\|_{L^2} \le \sum_{i=1}^{P}|\beta_i|\eta^i\|\xi_0\|_{L^2}^{i\theta_i}\|\xi_0\|_{H^1}^{i(1-\theta_i)},\end{aligned} \tag{4.93}$$

where $\theta_i = \frac{1}{2} - \frac{1}{2i}$, and

$$\begin{aligned}\|f(\xi_0)_x\|_{L^2} &= \left\|\sum_{i=1}^{P}\beta_i(\xi_0)_x^i\right\|_{L^2}\\ &\le \sum_{i=1}^{P}i|\beta_i|\|(\xi_0)^{i-1}\xi_{0x}\|_{L^2} \le \sum_{i=1}^{P}i|\beta_i|\|\xi_0\|_{\infty}^{i-1}\|\xi_{0x}\|_{L^2}\\ &\le \sum_{i=1}^{P}i|\beta_i|\eta^{i-1}\|\xi_0\|_{H^1}^{(i+1)/2}\|\xi_0\|_{L^2}^{(i-1)/2}.\end{aligned} \tag{4.94}$$

Thus, $f(\xi_0)$ belongs to $C([0,\infty),H^1(0,1))$. Based on the classical regularity theory for parabolic equations [85], we conclude that ξ_1 belongs to $C^1([0,\infty),H^2(0,1)) \hookrightarrow C^1([0,\infty),C^1(0,1))$. For the general term ξ_m, by induction and assuming that $\xi_i \in C^1((0,\infty),C^1(0,1))$, $i = 1,\ldots,m-1$, ξ_m is a solution to the following PDE:

$$\begin{aligned}&\xi_{mt} - \alpha\xi_{mxx} = \overline{A}_i(\xi_0,\ldots,\xi_{m-1}), x \in (0,1),\ t \in [0,\infty),\\ &\xi_{mx}(0,t) = k_1\xi_m(0,t),\ \xi_{mx}(1,t) = -k_2\xi_m(1,t),\\ &\xi_m(x,0) = 0.\end{aligned} \tag{4.95}$$

Due to the regularity in $\xi_i, i = 1,\ldots,m-1$, $\overline{A}(\xi_0,\xi_1,\ldots,\xi_{m-1}) = f(\xi_0+\cdots+\xi_{m-1}) - f(\xi_0+\cdots+\xi_{m-2})$ can be dominated by the estimates similar to that of $f(\xi_0)$. Indeed, the non-homogeneous term $\overline{A}(\xi_0,\xi_1,\ldots,\xi_{m-1})$ belongs to $C([0,\infty),H^1(0,1))$. Therefore, ξ_m belongs to $C^1([0,\infty),H^2(0,1)) \hookrightarrow C^1((0,\infty),C^1(0,1))$. By repeating the above process recursively, it can be shown that $\xi_m, m = 1,\ldots,\infty$, will always belong to $C^1((0,\infty),C^1(0,1))$. Therefore, for every internal point $x_i \in (0,1), i = 1,\ldots,n$, $\xi_{mx^+}(x_i,t) - \xi_{mx^-}(x_i,t) = 0, m = 1,\ldots,\infty$, which implies that $\xi_m, m = 1,\ldots,\infty$, have no contribution to the output signals u_i of the zero dynamics (4.39), i.e. $u_{m,i} = 0, i = 1,\ldots,n$. Thus, the in-domain control $u_i, i = 1,\ldots,n$ are generated solely by the linear part ξ_0, i.e., $u_i(t) = u_{0,i} = \xi_{0x}(x_i^+,t) - \xi_{0x}(x_i^-,t), i = 1,\ldots,n$. □

By Theorem 4.7, the implementation of the in-domain controls requires only to solve the linear part ξ_0 of the Adomain series solution to the zero dynamics (4.39), which can be achieved by using the technique of flat systems as shown in the next section.

4.4.2 Flatness-Based In-domain Control Design

Differential flatness is a powerful tool for achieving closed-form solutions to a wide class of PDEs. We use this method to obtain the solution to the zero dynamics (4.39) for set-point regulation problem. For this purpose, we change first the time scale from t to αt, while still using t as the time variable. The linear part of (4.71) is converted into the following form:

$$\begin{aligned}
&\widetilde{\xi}_{0t} - \widetilde{\xi}_{0xx} = 0, x \in (0,1),\\
&\widetilde{\xi}_{0x}(0,t) = k_1\widetilde{\xi}_0(0,t),\ \widetilde{\xi}_{0x}(1,t) = -k_2\widetilde{\xi}_0(1,t),\\
&\left[\widetilde{\xi}_0(x,t)\right]_{x=x_i} = 0,\ i = 1,\dots,n,\\
&\widetilde{\xi}_0(x_i,t) = y^r_{0,i}(t/\alpha),\ i = 1,\dots,n,\\
&B_i\widetilde{\xi}_0 = \left[\widetilde{\xi}_{0x}(x,t)\right]_{x=x_i} = v_i,\ i = 1,\dots,n,\\
&\widetilde{\xi}_0(x,0) = 0,
\end{aligned} \tag{4.96}$$

where $v_i(t), i = 1,\dots,n$, are the control inputs in the new time-scale, which can be transformed into the original in-domain control signals $u_i(t), i = 1,\dots,n$, after control synthesis. Due to the linearity of the heat equation, we divide the system $\widetilde{\xi}_0$ into n subsystems $\widetilde{\xi}^i_0(x,t), i = 1,\dots,n$, in a manner that all the subsystem are parallel connected. Hence, $\widetilde{\xi}_0$ can be expressed as $\widetilde{\xi}_0 = \sum_{i=1}^n \widetilde{\xi}^i_0$, where $\widetilde{\xi}^i_0$ is governed by

$$\begin{aligned}
&\widetilde{\xi}^i_{0t} - \widetilde{\xi}^i_{0xx} = 0, \quad x \in (0,x_i) \cup (x_i,1),\\
&\widetilde{\xi}^i_{0x}(0,t) = k_1\widetilde{\xi}^i_0(0,t), \quad \widetilde{\xi}^i_{0x}(1,t) = -k_2\widetilde{\xi}^i_0(1,t),\\
&\widetilde{\xi}^i_0(x_i,t) = y^r_{0,i}(t/\alpha),\ i = 1,\dots,n,\\
&\widetilde{\xi}^i_0(0,t) = 0,\\
&\left[\widetilde{\xi}^i_{0x}\right]_{x=x_i} = v_i(t), i = 1,\dots,n.
\end{aligned} \tag{4.97}$$

In order to generate the trajectory of the system (4.97), we resort to the concept of basic outputs, which are also termed as flat outputs, for flat systems. A particular feature of this method is that the states and the inputs of a flat system can be represented in terms of basic outputs and their time-derivatives without involving integration. Denote by $h_i(t), i =$

$1,\ldots,n$, the basic outputs. The solution to (4.97) is given in a previous work [145]:

$$\begin{aligned}\widetilde{\xi}_0^i = &\Bigg(k_1k_2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{x^{2k+1}(x_i-1)^{2(n-k)+1}}{(2k+1)!(2(n-k)+1)!}h_i^{(n)} - k_1\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{x^{2k+1}(x_i-1)^{2(n-k)}}{(2k-1)!(2(n-k))!}h_i^{n}\\ &+k_2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{x^{2k}(x_i-1)^{2(n-k)+1}}{(2k)!(2(n-k)+1)!}h_i^{(n)} - \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{x^{2k}(x_i-1)^{2(n-k)}}{(2k)!(2(n-k))!}h_i^{(n)}\Bigg)\chi_{(0,x_i)}\\ &+\Bigg(k_1k_2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{x^{2k+1}(x-1)^{2(n-k)+1}}{(2k+1)!(2(n-k)+1)!}h_i^{(n)} - k_1\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{x_i^{2k+1}(x-1)^{2(n-k)}}{(2k+1)!(2(n-k))!}h_i^{(n)}\\ &+k_2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{x_i^{2k}(x-1)^{2(n-k)+1}}{(2k)!(2(n-k)+1)!}h_i^{(n)} - \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{x_i^{2k+1}(x-1)^{2(n-k)}}{(2k)!(2(n-k))!}h_i^{(n)}\Bigg)\chi_{[x_i,1]}\end{aligned} \tag{4.98}$$

where χ is the characteristic function. Thus, the control inputs of the system (4.97) are given by [145]

$$v_i(t) = k_1k_2\sum_{n=0}^{\infty}\frac{h_i^{(n)}(t)}{(2n+1)!} + (k_1+k_2)\sum_{n=0}^{\infty}\frac{h_i^{(n)}(t)}{(2n)!} + \sum_{n=0}^{\infty}\frac{h_i^{(n+1)}(t)}{(2n+1)!},\ i=1,\ldots,n. \tag{4.99}$$

Therefore, the control inputs of the system (4.71) can be expressed as

$$u_i(t) = k_1k_2\sum_{n=0}^{\infty}\frac{h_i^{(n)}(\alpha t)}{(2n+1)!} + (k_1+k_2)\sum_{n=0}^{\infty}\frac{h_i^{(n)}(\alpha t)}{(2n)!} + \sum_{n=0}^{\infty}\frac{h_i^{(n+1)}(\alpha t)}{(2n+1)!},\ i=1,\ldots,n. \tag{4.100}$$

In order to make sure that the above controller is well-defined, the basic output $h_i(t)$ should be C^∞-smooth. We choose then the following function $\psi(t)$ as a component of basic outputs:

$$\psi(t) = \begin{cases} 0, & if\ t\le 0,\\ \dfrac{\int_0^t exp(-1/(\tau(1-\tau)))^\varepsilon \mathrm{d}\tau}{\int_0^T exp(-1/(\tau(1-\tau)))^\varepsilon \mathrm{d}\tau}, & if\ 0<t<T,\\ 1, & if\ t\ge T,\end{cases} \tag{4.101}$$

which is known as Gevrey function of order $\sigma = 1+\frac{1}{\varepsilon}, \varepsilon>0$. The form of basic outputs $h_i(t)$ can be expressed as

$$h_i(t) = \mu_i\psi(t), i=1,\ldots,n, \tag{4.102}$$

where μ_i are constants to be determined later. In order to find appropriate reference trajectories to track the desired set-points y_i^d, the unknown parameters μ_i of flat outputs $h_i(t)$ will be determined by the steady nonlinear differential equation of the system (4.1). Consider the following steady nonlinear differential equation of the system (4.1)

$$
\begin{aligned}
&0 = \alpha \overline{w}_{xx} + f(\overline{w}) + \sum_{i=1}^{n} \delta_i \gamma_i, \\
&\overline{w}_x(0) = k_1 \overline{w}(0), \quad \overline{w}_x(1) = -k_2 \overline{w}(1), \\
&y_i^d = \overline{w}(x_i) = C_i \overline{w}(x), \quad i = 1, \ldots, n,
\end{aligned}
\tag{4.103}
$$

where $\gamma_i, i = 1, \ldots, n$, are the input parameters for the steady-state parabolic equation. The parameter γ_i, $i = 1, \ldots, n$, can be determined by the Green functions G_i, which comply with the following differential equations

$$
\begin{aligned}
&AG_i = \delta_i, \ i = 1, \ldots, n, \\
&G_{ix}(0) = k_1 G_i(0), \quad G_{ix}(1) = -k_2 G_i(1).
\end{aligned}
\tag{4.104}
$$

The Green's functions $G_i(x), i = 1, \ldots, n$, can be explicitly expressed as [145]

$$
G_i(x) = \begin{cases} \dfrac{(1 - k_2 x_i + k_2)(k_1 x + 1)}{\alpha(k_1 + k_2 + k_1 k_2)}, & 0 \le x \le x_i, \\ \dfrac{(1 - k_2 x + k_2)(k_1 x_i + 1)}{\alpha(k_1 + k_2 + k_1 k_2)}, & x_i < x < 1. \end{cases}
\tag{4.105}
$$

Then, using the operator A, the steady system (4.103) can be represented in an abstract form:

$$
A\overline{w} = \sum_{i=1}^{n} \delta_i \gamma_i + f(\overline{w}).
\tag{4.106}
$$

Multiplying the operator A^{-1} on both sides of (4.106) yields

$$
\overline{w} = \sum_{i=1}^{n} A^{-1} \delta_i \gamma_i + A^{-1} f(\overline{w}).
\tag{4.107}
$$

Let $\overline{w}_r$ be a function satisfying

$$
\overline{w}_r = A^{-1} f(\overline{w}),
\tag{4.108}
$$

which gives

$$
-\alpha \frac{\mathrm{d}^2 \overline{w}_r}{\mathrm{d}x^2} = f(\overline{w}).
\tag{4.109}
$$

Furthermore, substituting (4.108) into (4.106) and recalling that $G_i(x) = A^{-1}\delta_i$, we have

$$
\overline{w} = \sum_{i=1}^{n} G_i(x) \gamma_i + \overline{w}_r.
\tag{4.110}
$$

Denoting

$$G = \begin{bmatrix} C_1G_1 & C_1G_2 & C_1G_3 & \dots & C_1G_n \\ C_2G_1 & C_2G_2 & C_2G_3 & \dots & C_2G_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_nG_1 & C_nG_2 & C_nG_3 & \dots & C_nG_n \end{bmatrix} \tag{4.111}$$

and noting that G is an invertible $n \times n$ matrix [145], we obtain

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{bmatrix} = G^{-1} \begin{bmatrix} y_1^d - C_1\overline{w}_r \\ y_2^d - C_2\overline{w}_r \\ \vdots \\ y_n^d - C_n\overline{w}_r \end{bmatrix}. \tag{4.112}$$

According to (4.100), we have

$$\lim_{t\to\infty} u_i(t) = (k_1k_2 + k_1 + k_2)\mu_i = \gamma_i,\ \ i = 1, \dots, n. \tag{4.113}$$

Therefore, we choose the basic inputs as

$$h_i(t) = \frac{\psi(t)}{k_1k_2 + k_1 + k_2}\gamma_i. \tag{4.114}$$

With such basic inputs, the control $u_i(t), i = 1, \dots, n$, given in (4.100) can allow the in-domain controlled Chaffee-Infante equation or Fisher's equation to achieve an asymptotic regulation with respect to the desired outputs $y_i^d, i = 1, \dots, n$. In summary, we have the following result.

Theorem 4.8. *Suppose that the conditions of Theorem 4.4 (resp. Theorem 4.5) and Theorem 4.6 hold and the in-domain controls $u_i(t), i = 1, \dots, n$, are set as (4.100). Then the regulation errors with respect to the desired set-point $y_i^d, i = 1, \dots, n$, for the in-domain controlled Chaffee-Infante equation (resp. Fisher's equation) converge to 0 as $t \to \infty$.*

Remark 4.5. *The developed method can be extended to a wider class of semi-linear parabolic equations provided their stability can be guaranteed.*

4.5 Simulation

In this section, we provide the simulation results for in-domain controlled Chaffee-Infante equation and Fisher's equation to illustrate the effectiveness and the efficiency of the proposed control scheme.

We consider first Fisher's equation with $\alpha = 5$ and $k_1 = k_2 = 7$. The initial condition is set to $w(x,0) = 0.4\sin(\pi x/2)$. The desired steady state profile is a curve expressed by:

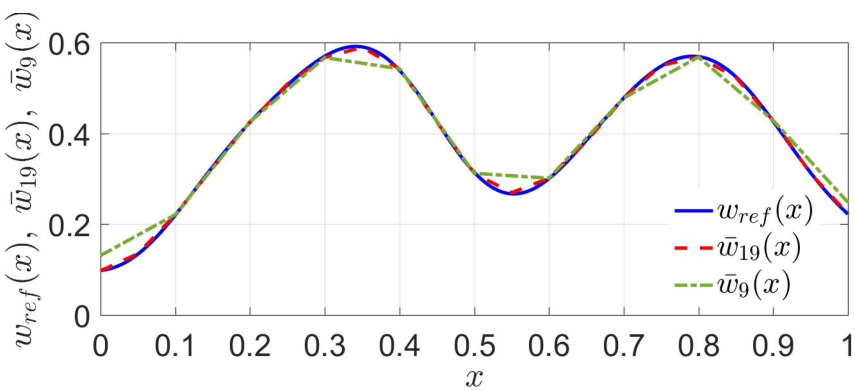
$$w_{\mathrm{ref}}(x) = a_0^1 + \sum_{i=1}^{3} a_i^1 \sin(i\theta x) + b_i^1 \cos(i\theta x) \tag{4.115}$$

where $\theta = 5.4, a_0^1 = 0.36, a_1^1 = 0.13, a_2^1 = -0.17, a_3^1 = 0.04, b_1^1 = 0.03, b_2^1 = 0.04, b_3^1 = 0.01$. The objective is to drive the trajectory of the system to track the prescribed profile given in (4.115).

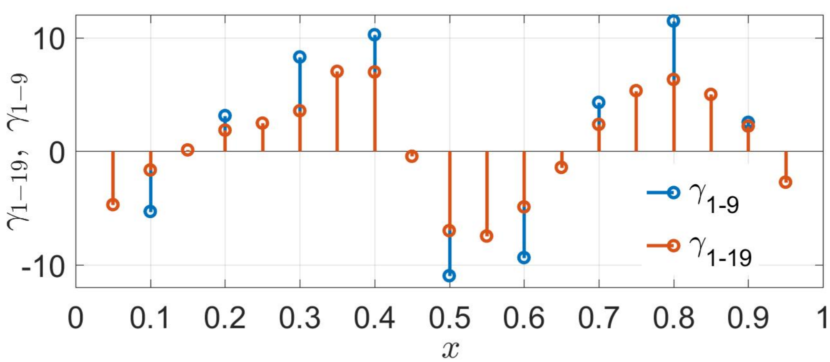
In general, the desired profile $w_{\mathrm{ref}}(x)$ can be any functions that may not be a solution to the static PDE (4.103). Consequently, the regulation accuracy along the domain is affected by the number of in-domain inputs. To illustrate this property, we consider in the simulation 2 different settings with, respectively, 9 and 19 in-domain actuators. The performance of interpellation accuracy of these 2 settings are shown in Fig. 4.2(a). The corresponding static controls are computed by using the numerical scheme presented in Section 4.4.2 as shown in Fig. 4.2(b). It can be seen that the increase of the number of in-domain actuators will improve the regulation accuracy while reduce the amplitude of static control signals. However, it will require more computational effort. The solution surface of Fisher's equation with 19 in-domain actuators is displayed in the Fig. 4.3(a), and the evolution of the deviation between the actual state and its desired static solution given by (4.115) on space and time are shown in Fig. 4.3(b). The simulation results show that that developed in-domain control scheme exhibits a satisfactory performance. In the simulation of in-domain controlled Chaffee-Infante equation, the system parameters are set to $r = \gamma = 1$, $\alpha = 6$, and $k_1 = k_2 = 8$, and the initial condition is chosen to be $w(x,0) = 0.7\cos(\pi x/2)$. The desired static profile $w_{\mathrm{ref}}(x)$ is the same given by (4.115). The interpolation errors for the settings with, respectively, 9 and 19 in-domain controllers and the corresponding static control signals are shown in Fig. 4.4(a) and Fig. 4.4(b), respectively. The setting with 19 actuators is used in the simulation of the in-domain controlled Chaffee-Infante equation. It can be seen from Fig. 4.5(a) and Fig. 4.5(b) that the system asymptotically tracks the desired output and the regulation errors tend to zero identically along the domain.

4.6 Conclusion

To tackle the problem of asymptotic output regulation for some in-domain controlled semi-linear parabolic PDEs, we have developed in this work an approach for control synthesis and implementation, which is a combination of zero dynamics inverse design, Adomian decomposition method, and trajectory planning of flat systems. An advantage of this approach is

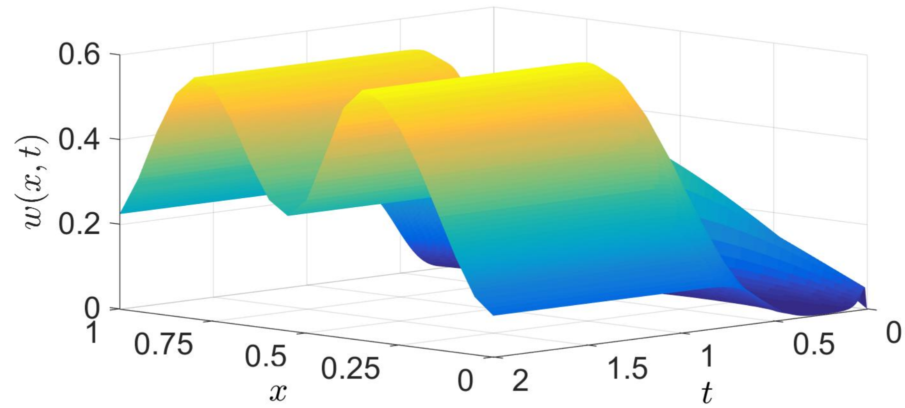


(a)

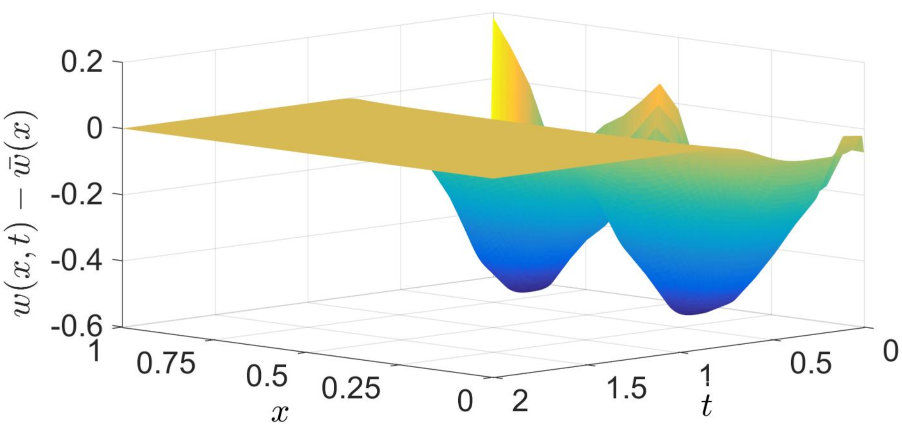


(b)

Figure 4.2 Simulation results: (a) interpolation of the reference signal w_{ref}; (b) steady-state control signals of Fisher's equation.

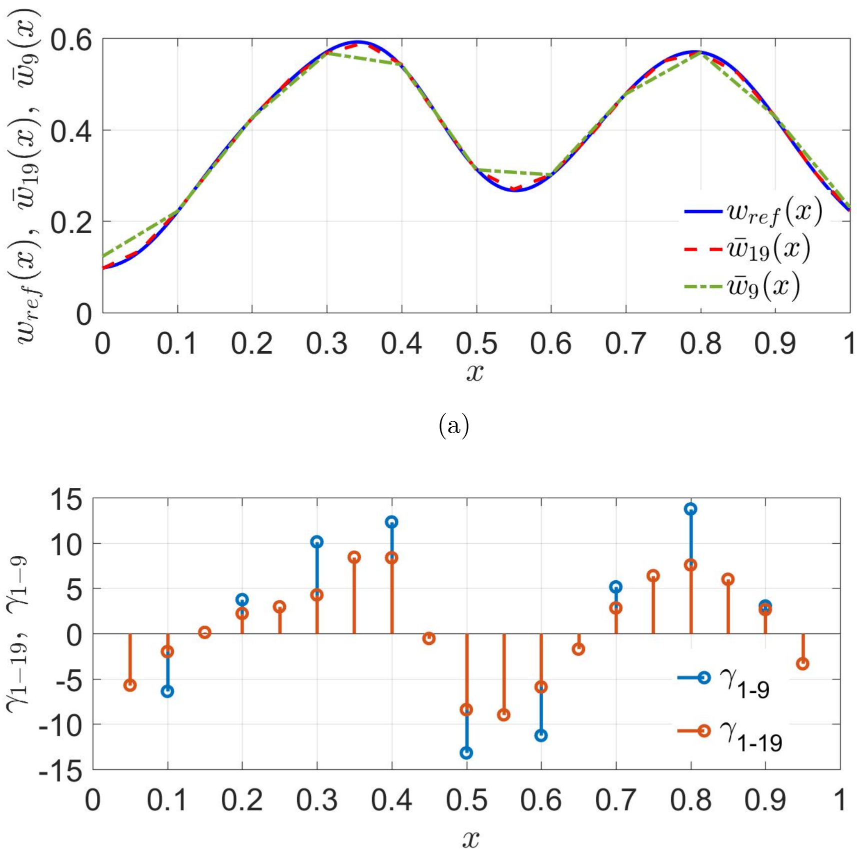


(a)



(b)

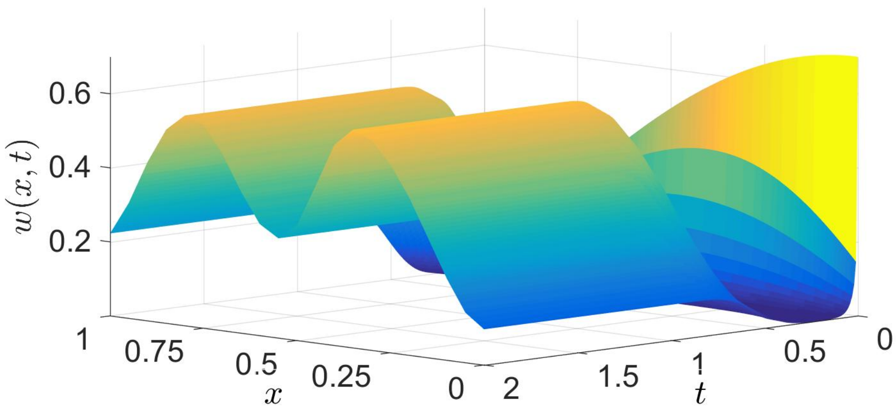
Figure 4.3 Simulation results: (c) solution surface of the in-domain controlled Fisher's equation; (d) surface of regulation errors.



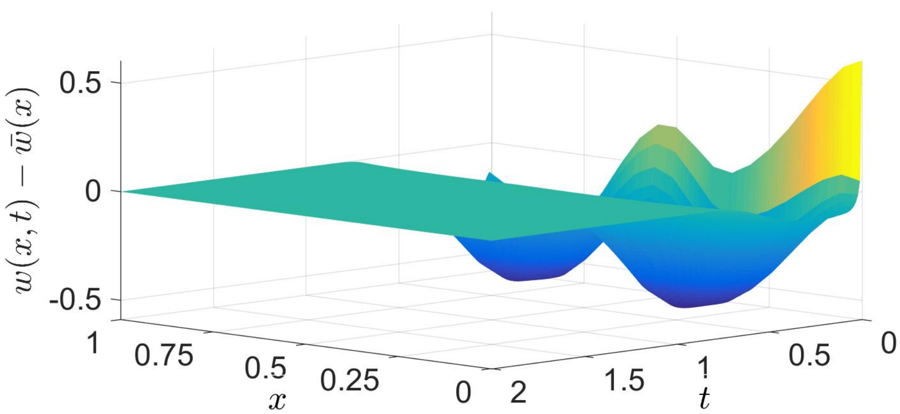
(a)

(b)

Figure 4.4 Simulation results: (a) interpolation of the reference signal; (b) steady-state control signals of Chaffee-Infante equation.



(a)



(b)

Figure 4.5 Simulation results: (c) solution surface of Chaffee-Infante equation; (d) surface of regulation errors.

that for the considered problem, the obtained in-domain controls can be expressed in closed form and hence, early truncations can be avoided in control design. A rigourous analysis on the basic properties of the considered control systems has also been provided to grantee the validity of the developed method. Due to the complexity of PDEs, Chaffee-Infante equation and Fisher's equation are considered in stability and convergence analysis. Nevertheless, the proposed approach is applicable to a wider class of semi-linear parabolic PDEs, which constitutes a subject of our future work. Finally, as the techniques of ADM and flat systems are used in the implementation of the zero dynamics-based control scheme, the developed approach in this paper can also be extended to systems with non-collocated input and output by applying, e.g., the method proposed in the work [33].

CHAPTER 5 ARTICLE 3: A DYNAMIC COMPENSATOR FOR IN-DOMAIN CONTROL OF BURGERS' EQUATION

This chapter is reproduced from the paper [141].

Authors: Kaijun Yang and Guchuan Zhu.

Abstract— This paper addresses the problem of output regulation control of a Burgers' equation under pointwise in-domain actuation. A dynamic compensator that generates the in-domain control is developed for output regulation of the considered system. A nonlinear boundary feedback control is used to ensure the closed-loop stability of the Burgers' equation. The proposed control scheme is implemented by means of Adomian decomposition method and flatness-based trajectory planning. A numerical simulation study is carried out, and the obtained results demonstrate the efficiency and the effectiveness of the proposed approach.

Index Terms—Burgers' equation; Nonlinear feedback control; Dynamic compensator; Adomian decomposition method; Differential flatness.

5.1 Introduction

Burgers' equation is one of the most elaborated parabolic partial differential equations (PDEs), which involves the effects of both nonlinear propagation and diffusion. This PDE was originally developed for modeling a one-dimensional turbulence and has been applied to different problems arising in physics, engineering, mathematical biology, etc. (see, e.g., [98, 135]). The broad range of application of Burgers' equations motivated extensive investigations on the control of this type of PDEs in the literature and many solutions have been developed for different problems, such as linear boundary feedback control [36, 44], backstepping control [80, 92], optimal control [71], and adaptive control [93]. It should be noted that the exponential stability (locally or globally) of PDEs is an essential requirement for asymptotic tracking control [28]. However, as Burgers' equation is a nonlinear PDE, a linear feedback control can usually achieve only a local stability, which may be a performance restriction. In [36, 41], the local exponential stability for Burgers' equation is obtained by using the classical energy method under some assumptions on initial data and the nonhomogeneous terms. A nonlinear boundary feedback control is introduced in [78], which can achieve a global exponential stability of the Burgers' equation.

Most of the existing work on the control of Burgers' equation is concerned with boundary control [28, 29, 32, 36, 46, 78, 84, 102], while where are only few results on in-domain control of this type of PDEs, which is indeed a multiple input-multiple out problem under the setting of multiple pointwise in-domain actuation and sensing, have been reported in the literature. A convenient method, called zero dynamics inversion, that can be used to tackle the problem of asymptotic output regulation of PDEs under in-domain control is the one developed in a series work presented in, e.g., [13, 31, 34, 35]. The method of zero dynamics inversion design will lead to a dynamic compensator that can be applied to PDEs for which the control is located either on the boundary or in the domain. However, the implementation of the dynamic compensator requires to solve inline a PDE of the same type as the original system, which may not be computationally tractable. Thus, static controls derived from the solution of steady-state zero dynamics are often used (see, e.g., [13]). Another solution that allows reducing the computational burden while providing a deterministic implementation scheme is the one proposed in [139, 145], which amounts to combining the zero dynamics inversion design with flatness-based trajectory planning in the implementation of the dynamic compensator for asymptotic output teaching control. This approach is applicable to the in-domain control of a wide range of linear PDEs. Nevertheless, due to the fact that the currently available techniques for flatness-based trajectory planning are limited to linear PDEs [119] or boundary controlled nonlinear PDEs [102], this method is still not applicable to in-domain controlled nonlinear PDEs. To overcome this difficulty, a recent work reported in [140] proposed to employ the Adomian decomposition method (ADM) to achieve a semi-analytical implementation of the dynamic compensator for in-domain control. ADM is a classical numerical approach that provides a means for constructing solutions to nonlinear PDEs expressed as an infinite series with a fast convergence rate [1, 3, 109, 134].

In the present work, we develop a scheme for output regulation control of Burgers' equation based on an approach that can be seen as an extension of the one proposed in [140] in the sense that the dynamic compensator does not correspond exactly to the zero dynamics of the original system. It should be noted that different from the semi-linear parabolic PDEs (Chaffee-Infante equation and Fisher's equation) considered in [140], the nonlinear term in Burgers' equation is not smooth enough and hence, the in-domain control does not depend only on its linear part. To ensure the global exponential stability of the boundary controlled Burgers' equation, a nonlinear boundary feedback control law is used. This dynamic compensator has a simpler structure, and the system performance can be guaranteed by convergence analysis of tracking errors and closed-loop stability.

The rest of the paper is organized as follows. Section 5.2 presents the setting of the considered in-domain controlled Burgers' equation. Section 5.3 details the design of the dynamic compensator along with convergence assessment of tracking errors. Issues related to the implementation of the developed the dynamic compensator are addressed in Section 5.4 and

Section 5.5. Some simulation results are illustrated in Section 5.6, followed by some concluding remarks presented in Section 5.7.

5.2 Problem Formulation

We first introduce the following notations that are frequently used in this paper:

- $L^2(0,1)$ represents the Lebesgue space of square integrable functions on $(0,1)$ with the norm $\|v\|_{L^2} = \left(\int_0^1 |v(x)|^2 \mathrm{d}x\right)^{1/2}$.
- $H^1(0,1)$ denotes the Sobolev space equipped with the norm

$$\|v\|_{H_1} = \sqrt{\|v\|_{L^2}^2 + \|v_x\|_{L^2}^2},\ v \in H^1(0,1). \tag{5.1}$$

- $L^\infty(0,1)$ denotes the space of all bounded measurable functions on $(0,1)$ equipped with the norm $\|v\|_{L^\infty} = \operatorname{ess\,sup}_{x\in(0,1)} |v(x)|$.
- $C^k[0,1]$ is the space of all k-times continuously differentiable functions on the interval $[0,1]$.
- Let X be a Banach space with the norm $\|\cdot\|_X$. Then $L_{((0,\infty),X)}$ denotes the Banach space consisting of all measurable functions on $(0,1)\times(0,\infty)$ with a finite norm

$$\|v\|_{L^\infty((0,\infty);X)} = \operatorname*{ess\,sup}_{t\in(0,\infty)} \|v\|_X,\ v \in L_{((0,\infty),X)}. \tag{5.2}$$

The considered Burgers' equation with in-domain control is of the following form:

$$\begin{aligned}
&w_t - \alpha w_{xx} + w_x w + \sum_{i=1}^{n} \delta_{x_i} u_i(t) = 0,\ x \in (0,1),\ t > 0,\\
&w_x(0,t) = B^l w, \quad w_x(1,t) = B^r w,\\
&w(x,0) = \varphi(x),\\
&y_i(t) = C_i w = w(x_i,t), i = 1,\dots,n.
\end{aligned} \tag{5.3}$$

where $\alpha > 0$, B^l and B^r are, respectively, the left and the right boundary control operators, δ_{x_i} is the Dirac function supported at the points $x_i \in (0,1), i = 1,\dots,n$, and $u_i(t), i = 1,\dots,n$, are the in-domain controls, which are located on the spatial point $x_i \in (0,1), i = 1,\dots,n$, respectively. $C_i, i = 1,\dots,n$, are the output operators and $y_i(t), i = 1,\dots,n$ denotes the outputs.

The objective is to establish in-domain control laws that can drive the system (5.3) to track the desired set-points $\overline{y}_i^d, i = 1, \ldots, n$. Specifically, let

$$e_i(t) = y_i(t) - \overline{y}_i^d, i = 1, \ldots, n, \tag{5.4}$$

be the output regulation errors. Then in closed-loop, we should have

$$\lim_{t\to\infty} e_i(t) = 0,\ i = 1, \ldots, n. \tag{5.5}$$

To facilitate the dynamic compensator design, the in-domain controlled Burgers' equation (5.3) can be reformulated under a serially connected form:

$$\begin{aligned}
&w_t - \alpha w_{xx} + w_x w = 0,\ i = 1, \ldots, n,\ x \in \Omega,\ t > 0,\\
&[w(x,t)]_{x=x_i} = 0,\ i = 1, \ldots, n,\\
&[w_x(x,t)]_{x=x_i} = u_i(t),\\
&w_x(0,t) = B^l w,\ w_x(1,t) = B^r w,\\
&w(x,0) = \varphi(x),\\
&y_i(t) = C_i w = w(x_i,t), i = 1, \ldots, n.
\end{aligned} \tag{5.6}$$

where $\Omega \triangleq (0,x_1)\cup(x_1,x_2)\cup\cdots\cup(x_{n-1},x_n)\cup(x_n,1)$. $[w(x,t)]_{x=x^i} = w(x_i^+,t) - w(x_i^-,t)$, where $w(x_i^+,t)$ and $w(x_i^-,t)$ represent the right and left limits of w at the point (x_i,t), respectively. Note that the equivalence between (5.3) and (5.6) can be verified by the technique in the references [114, 145].

5.3 Dynamic Compensator for In-domain Control

Asymptotic set-point control of in-domain controlled PDEs can be achieved by using a scheme of input-output inversion. Specifically, let $y_i^r(t), 1, \ldots, n$, be reference trajectories satisfying $y_i^r(t) \to \overline{y}_i^d$, as $t \to \infty$. Then, the in-domain control $u_i(t), i = 1, \ldots, n$, may be generated by a dynamic system, usually of infinite dimension, with $y_i^r(t), 1, \ldots, n$, as inputs. It has been shown that zero-dynamics inversion is a convenient tool for this purpose (see, e.g., [13, 28, 31, 139, 140, 145]). It is shown in [32] that the solution of the zero dynamics of a boundary controlled Burgers' equation with linear boundary feedback converges to the solution to the original system. As the considered problem in this work is an in-domain controlled Burgers' equation with nonlinear boundary feedback, the results given in [32] cannot be directly applied. For this raison, we consider a dynamic compensator of the following form:

$$\begin{aligned}
&\xi_t - \alpha\xi_{xx} + \xi\xi_x = 0,\ x \in \Omega,\ t > 0, \\
&[\xi(x,t)]_{x=x_i} = 0, i = 1, \ldots, n, \\
&\xi(0,t) = 0, \quad \xi(1,t) = 0, \\
&\xi(x_i,t) = y_i^r(t), i = 1, \ldots, n, \\
&\xi(x,0) = 0, \\
&[\xi_x(x,t)]_{x=x_i} = u_i(t), i = 1, \ldots, n,
\end{aligned} \tag{5.7}$$

which is indeed the zero dynamics of the system (5.3) (or (5.6)) with (forced) homogeneous boundary conditions. It should be noted that the inputs to the dynamic compensator are the reference trajectories $y_i^r, i = 1, \ldots, n$, and its outputs are the in-domain controls $u_i, i = 1, \ldots, n$.

In the following, we will first investigate the stability of the dynamic compensator (5.7) in L^2-norm and H^1-norm, which is crucial for stability analysis of the controlled Burgers' equation (5.3).

Proposition 5.1. *Assume that $y_i^r(t)$ and $y_{it}^r(t), i = 1, \cdots, n$, are uniformly bounded in $[0,\infty)$. Then the system (5.7) is stable in L^2-norm, i.e.,*

$$\begin{aligned}
\|\xi\|_{L^2}^2 \le & C_1 \sum_{i=1}^{n} |y_i^r(0)|^2 e^{-\frac{\alpha}{2}t} + C_2 \sum_{i=1}^{n} \|y_{it}^r(\cdot)\|_{L^\infty(0,\infty)}^2 \\
& + C_3 \sum_{i=1}^{n} \|y_i^r(\cdot)\|_{L^\infty(0,\infty)}^2 + C_4 \sum_{i=1}^{n} \|y_i^r(\cdot)\|_{L^\infty(0,\infty)}^4
\end{aligned} \tag{5.8}$$

where C_1, C_2, C_3, and C_4 are constants independent of $y_i^r(t), i = 1, \cdots, n$.

Proof. The dynamic compensator (5.7) consists of n serially connected PDEs, which can be formulated as:

$$\begin{cases} \xi^0_t - \alpha\xi^0_{xx} + \xi^0\xi^0_x = 0, \quad x \in (0, x_1), \ t > 0, \\ \xi^0(0,t) = 0, \quad \xi^0(x_1,t) = y^r_1(t), \\ \xi^0(x,0) = 0; \end{cases} \tag{5.9}$$

$$\begin{cases} \xi^i_t - \alpha\xi^i_{xx} + \xi^i\xi^i_x = 0, \quad x \in (x_i, x_{i+1}), \ t > 0, \\ \xi^i(x_i,t) = y^r_i(t), \quad \xi^i(x_{i+1},t) = y^r_{i+1}(t), \\ \xi^i(x,0) = 0, \\ i = 1, \dots, n-1; \end{cases} \tag{5.10}$$

$$\begin{cases} \xi^n_t - \alpha\xi^n_{xx} + \xi^n\xi^n_x = 0, \quad x \in (x_n, 1), \ t > 0, \\ \xi^n(1,t) = 0, \quad \xi^n(x_n,t) = y^r_n(t). \\ \xi^n(x,0) = 0; \end{cases} \tag{5.11}$$

Therefore, the solution to the dynamic compensator (5.7) can be expressed as:

$$\xi = \chi_{(0,x_1]}\xi^0 + \sum_{i=1}^{n-1} \chi_{(x_i,x_{i+1}]}\xi^i + \chi_{(x_n,1)}\xi^n, \tag{5.12}$$

where χ_I denotes the characteristic function supported on the interval I.

We will then estimate the upper bound of $\|\xi^i\|_{L^2}$, $i = 0, 1, \dots, n$, by using the standard procedure for establishing the *a prior* estimates of the solutions to parabolic PDEs (see [85]).

First, we deal with ξ^0-system by transforming the inhomogeneous boundary at $x = x_1$ of (5.9) into homogeneous one. Using a new variable $\zeta_0 = \xi^0 - \frac{x}{x_1}y^r_1(t)$, we obtain

$$\zeta_{0t} - \zeta_{0xx} = -\zeta_{0x}\zeta_0 - \frac{x}{x_1}y^r_{1t}(t) - \frac{x}{x_1^2}(y^r_1(t))^2 - \frac{1}{x_1}y^r_1(t)\zeta_0 - \frac{x}{x_1}y^r_1(t)\zeta_{0x}, \tag{5.13}$$

with the boundary condition and the initial value

$$\zeta_0(0,t) = \zeta_0(x_1,t) = 0, \tag{5.14}$$

$$\zeta_0(x,0) = \frac{x}{x_1}y^r_1(0). \tag{5.15}$$

Multiplying (5.13) by ζ_0 and integrating from 0 to x_1 yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{x_1}\zeta_0^2\mathrm{d}x-\alpha\int_0^{x_1}\zeta_{0xx}\zeta_0\mathrm{d}x\\
=&-\int_0^{x_1}\zeta_{0x}\zeta_0^2\mathrm{d}x-\int_0^{x_1}\frac{x}{x_1}y_{1t}^r(t)\zeta_0\mathrm{d}x-\int_0^{x_1}\frac{x}{x_1^2}(y_1^r(t))^2\zeta_0\mathrm{d}x\\
&-\int_0^{x_1}\frac{1}{x_1}y_1^r(t)\zeta_0^2\mathrm{d}x-\int_0^{x_1}\frac{x}{x_1}y_1^r(t)\zeta_{0x}\zeta_0\mathrm{d}x
\end{aligned}\tag{5.16}$$

Making integration by parts and applying the boundary conditions, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{x_1}\zeta_0^2\mathrm{d}x+\alpha\int_0^{x_1}(\zeta_{0x})^2\mathrm{d}x\\
=&-\int_0^{x_1}\frac{x}{x_1}y_{1t}^r(t)\zeta_0\mathrm{d}x-\int_0^{x_1}\frac{x}{x_1^2}(y_1^r(t))^2\zeta_0\mathrm{d}x\\
&-\int_0^{x_1}\frac{1}{x_1}y_1^r(t)\zeta_0^2\mathrm{d}x-\int_0^{x_1}\frac{x}{x_1}y_1^r(t)(\zeta_0^2)_x\mathrm{d}x\\
\le&\frac{\alpha}{4}\int_0^{x_1}\zeta_0^2\mathrm{d}x+\frac{y_{1t}^r(t)^2x_1}{3\alpha}+\frac{\alpha}{4}\int_0^{x_1}\zeta_0^2\mathrm{d}x+\frac{y_1^r(t)^4}{3\alpha x_1}
\end{aligned}\tag{5.17}$$

Due to the fact that $\int_0^{x_1}\zeta_0^2\mathrm{d}x\le\int_0^{x_1}\zeta_{0x}^2\mathrm{d}x$, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{x_1}\zeta_0^2\mathrm{d}x+\frac{\alpha}{2}\int_0^{x_1}\zeta_0^2\mathrm{d}x\le\frac{y_{1t}^r(t)^2x_1}{3\alpha}+\frac{y_1^r(t)^4}{3\alpha x_1}.\tag{5.18}$$

Multiplying $e^{\frac{\alpha}{2}t}$, it yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{\alpha}{2}t}\int_0^{x_1}\zeta_0^2\mathrm{d}x\right)\le\left(\frac{y_{1t}^r(t)^2x_1}{3\alpha}+\frac{y_1^r(t)^4}{3\alpha x_1}\right)e^{\frac{\alpha}{2}t}.\tag{5.19}$$

Integrating from 0 to t, we obtain

$$\begin{aligned}
\int_0^{x_1}\zeta_0^2\mathrm{d}x&\le\frac{x_1|y_1^r(0)|^2}{3}e^{-\frac{\alpha}{2}t}+\int_0^t\left(\frac{y_{1t}^r(\tau)^2x_1}{3\alpha}+\frac{y_1^r(\tau)^4}{3\alpha x_1}\right)e^{-\frac{\alpha}{2}(t-\tau)}\mathrm{d}\tau\\
&\le\frac{x_1|y_1^r(0)|^2}{3}e^{-\frac{\alpha}{2}t}+\frac{2}{\alpha}\left(\frac{x_1\|y_{1t}^r(\cdot)\|^2_{L^\infty(0,\infty)}}{3\alpha}+\frac{\|y_1^r(\cdot)\|^4_{L^\infty(0,\infty)}}{3\alpha x_1}\right),
\end{aligned}\tag{5.20}$$

which implies that

$$\begin{aligned}
\int_0^{x_1}(\xi^0)^2\mathrm{d}x\le&\frac{2x_1|y_1^r(0)|^2}{3}e^{-\frac{\alpha}{2}t}+\frac{4x_1}{3\alpha^2}\|y_{1t}^r(\cdot)\|^2_{L^\infty(0,\infty)}+\frac{2x_1}{3}\|y_1^r(\cdot)\|^2_{L^\infty(0,\infty)}\\
&+\frac{4}{3\alpha^2x_1}\|y_1^r(\cdot)\|^4_{L^\infty(0,\infty)}.
\end{aligned}\tag{5.21}$$

Using the transformation

$$\zeta_i = \xi^i - \left(\frac{x_{i+1} - x}{x_{i+1} - x_i} y_i^r(t) + \frac{x - x_i}{x_{i+1} - x_i} y_{i+1}^r(t) \right) \tag{5.22}$$

and

$$\zeta_n = \xi^n - \frac{1 - x}{1 - x_n} y_n^r(t) \tag{5.23}$$

for the subsystems defined in $x \in (x_i, x_{i+1})$, $i = 1, \ldots, n-1$, and $x \in (x_n, 1)$, respectively, (5.10) and (5.11) can be transformed into

$$\begin{aligned} \zeta_{it} - \zeta_{ixx} = & - \zeta_{ix}\zeta_i - \zeta_{ix} \left(\frac{x_{i+1} - x}{x_{i+1} - x_i} y_i^r(t) + \frac{x - x_i}{x_{i+1} - x_i} y_{i+1}^r(t) \right) \\ & - \frac{\zeta_i}{x_{i+1} - x_i} \left(y_{i+1}^r(t) - y_i^r(t) \right) \\ & - \frac{1}{x_{i+1} - x_i} \left(y_{i+1}^r(t) - y_i^r(t) \right) \left(\frac{x_{i+1} - x}{x_{i+1} - x_i} y_i^r(t) + \frac{x - x_i}{x_{i+1} - x_i} y_{i+1}^r(t) \right) \\ & - \frac{x_{i+1} - x}{x_{i+1} - x_i} y_{it}^r(t) - \frac{x - x_i}{x_{i+1} - x_i} (y_{i+1}^r)_t(x_{i+1}, t), \end{aligned} \tag{5.24}$$

with the homogeneous conditions and the initial value

$$\zeta_i(x_i, t) = \zeta_i(x_{i+1}, t) = 0, \tag{5.25}$$

$$\zeta_i(x, 0) = -\frac{x_{i+1} - x}{x_{i+1} - x_i} y_i^r(0) - \frac{x - x_i}{x_{i+1} - x_i} y_{i+1}^r(0), \tag{5.26}$$

and

$$\begin{aligned} \zeta_{nt} - \zeta_{nxx} = & - \zeta_{nx}\zeta_n - \zeta_{nx} \frac{1 - x}{1 - x_n} y_n^r(t) + \frac{\zeta_n}{1 - x_n} y_n^r(t) \\ & + \frac{1 - x}{(1 - x_n)^2} (y_n^r(0))^2 - \frac{1 - x}{1 - x_n} y_{nt}^r(0), \end{aligned} \tag{5.27}$$

with the homogeneous conditions and the initial value

$$\zeta_n(x_n, t) = \zeta_n(1, t) = 0, \tag{5.28}$$

$$\zeta_n(x, 0) = -\frac{1 - x}{1 - x_n} y_n^r(0). \tag{5.29}$$

Applying the similar technique, we obtain the similar estimates for $\xi^i, i = 1, \cdots, n$. Hence,

$$
\begin{aligned}
\|\xi\|_{L^2}^2 \leq & \int_0^{x_1} (\xi^0)^2 \mathrm{d}x + \sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1}} (\xi^i)^2 \mathrm{d}x + \int_{x_n}^1 (\xi^n)^2 \mathrm{d}x \\
\leq & C_1 \sum_{i=1}^n |y_i^r(0)|^2 e^{-\frac{\alpha}{2}t} + C_2 \sum_{i=1}^n \|y_{it}^r(\cdot)\|_{L^\infty(0,\infty)}^2 \\
& + C_3 \sum_{i=1}^n \|y_i^r(\cdot)\|_{L^\infty(0,\infty)}^2 + C_4 \sum_{i=1}^n \|y_i^r(\cdot)\|_{L^\infty(0,\infty)}^4,
\end{aligned}
\tag{5.30}
$$

where C_1, C_2, C_3, and C_4 are constants independent of $y_i^r(t)$. This completes the proof. $\square$

The following result is to confirm that the dynamic compensator is also stable in the sense ISS (input-to-state stable). Specifically, a function $\gamma(x)$ is said to belong to class $\mathcal{K}$ if $\gamma : [0, s) \to [0, \infty)$ is strictly increasing and $\gamma(0) = 0$, and a continuous function $\beta(x, t)$ is said to belong to class $\mathcal{KL}$ if $\beta(\cdot, t)$ belongs to $\mathcal{K}$ and $\beta(x, \cdot)$ is monotonically decreasing in t with $\lim_{t\to\infty} \beta(x, t) = 0$. We have then

Proposition 5.2. *Under the assumptions of Proposition 5.1, the dynamic compensator given in* (5.7) *is stable in H^1-norm. Furthermore, there exist a function of class $\mathcal{K}$, γ, and a function of class $\mathcal{KL}$, β, such that*

$$
\|\xi\|_{H^1} \leq \beta\left(\sum_{i=1}^n |y_1^r(0)|, t\right) + \gamma\left(\sum_{i=1}^n \|y_{it}^r(\cdot)\|_{L^\infty(0,\infty)} + \|y_i^r(\cdot)\|_{L^\infty(0,\infty)}\right). \tag{5.31}
$$

Proof. Due to Proposition 5.1 and applying the similar technique used in [28], the conclusion on the claim follows directly. $\square$

Remark 5.1. *Due to the Sobolev embedding theorem $H^1(0,1) \hookrightarrow C(0,1)$ and Proposition 5.2, we can also conclude that if $y_i^r(t)$ and $y_{it}^r(t), i = 1, \cdots, n$, are bounded in $[0, \infty)$, then*

$$
\|\xi\|_{L^\infty(0,\infty;L^\infty(0,1))} < \infty. \tag{5.32}
$$

In order to stabilise the controlled Burgers' equation (5.6) around the dynamic compensator (5.7), we introduce the following nonlinear control:

$$
\begin{aligned}
B^l w &= (3\alpha)^{-1} w(0,t)^2 + k_1 w(0,t) + \xi_x(0,t), \\
B^r w &= (3\alpha)^{-1} w(1,t)^2 - k_2 w(1,t) + \xi_x(1,t)
\end{aligned}
\tag{5.33}
$$

By using the nonlinear boundary feedback control given in (5.33), the closed-loop system can be expressed as

$$
\begin{aligned}
& w_t - \alpha w_{xx} + w_x w = 0,\ x \in \Omega,\ t > 0, \\
& [w(x,t)]_{x=x_i} = 0,\ i = 1, \cdots, n, \\
& [w_x(x,t)]_{x=x_i} = u_i(t),\ i = 1, \cdots, n, \\
& w_x(0,t) = (3\alpha)^{-1} w(0,t)^2 + k_1 w(0,t) + \xi_x(0,t), \\
& w_x(1,t) = (3\alpha)^{-1} w(1,t)^2 - k_2 w(1,t) + \xi_x(1,t), \\
& w(x,0) = \varphi(x), \\
& y_i(t) = C_i w = w(x_i,t), i = 1, \ldots, n.
\end{aligned} \tag{5.34}
$$

To facilitate stability analysis of (5.34), we introduce the following norm of the Sobolev space $H^1(0,1)$:

$$
\|v\|_{H^1} = \sqrt{\int_0^1 v_x^2 \mathrm{d}x + k_1|v(0)|^2 + k_2|v(1)|^2},\ \forall v \in H^1(0,1). \tag{5.35}
$$

The following Sobolev inequality will be used in stability analysis of Burgers' equation.

Lemma 5.1. *[41] For any $v \in H^1(0,1)$ and $2 \leq q \leq \infty$, we have*

$$
\|v\|_{L^q} \leq \eta \|v\|_{H^1}^{\theta} \|v\|_{L^2}^{1-\theta}, \tag{5.36}
$$

where $\theta = 1/2 - 1/q$, and η is independent of v.

Note that the norm defined in (5.35) is equivalent to the usual norm of the Sobolev space $H^1(0,1)$ given early [41].

To validate the proposed in-domain control scheme, we need to ensure that the solution of the dynamic compensator (5.7) converges to that to the in-domain controlled Burgers' equation (5.34). Before proceeding to study the stability of the controlled Burgers' equation, we introduce the operator $A_k = -\alpha \frac{\mathrm{d}^2}{\mathrm{d}t^2}$ with the domain $D(A_k) = \{v \in H^2(0,1) : v_x(0) = k_1 v(0),\ v_x(1) = -k_2 v(1)\}$. Let λ_1 be the first eigenvalue associated to A_k. Then, for $k_1 k_2 > \sqrt{\pi}$, $\alpha \frac{\pi}{2} < \lambda_1 < \alpha \pi^2$ [41]. Moreover, $\|v\|_{L^2}^2 \leq \frac{\alpha}{\lambda_1} \|v\|_{H^1}^2$ [41].

Theorem 5.1. *Let $\alpha^{1/2} \lambda_1^{1/2} > \|\xi\|_{L^\infty((0,\infty), L^\infty(0,1))}$, $k_1 k_2 > \sqrt{\pi}$, and the initial data $\varphi \in H^1(0,1)$. Then the solution to the in-domain controlled Burgers' equation* (5.34) *converges to that of the dynamic compensator(5.7) as t tends to ∞ in space $H^1(0,1)$. Furthermore, $\lim\limits_{t \to \infty} e_i(t) = 0$.*

Proof. Let w and ξ be the solution to the in-domain controlled Burgers' equation (5.34) and the dynamic compensator (5.7), respectively. We denote $\eta = w - \xi$. Subtracting the in-domain controlled Burgers' equation (5.34) to the dynamic compensator (5.7), we have

$$\begin{aligned}
&\eta_t - \alpha\eta_{xx} = -\eta\eta_x - \eta\xi_x - \eta_x\xi \\
&\eta_x(0,t) = (3\alpha)^{-1}\eta^2(0,t) + k_1\eta(0,t), \\
&\eta_x(1,t) = (3\alpha)^{-1}\eta^2(1,t) - k_2\eta(1,t), \\
&\eta(x,0) = \varphi.
\end{aligned} \tag{5.37}$$

Consider the Lyapunov function candidate:

$$V = \frac{1}{2}\int_0^1 \eta(x,t)^2 \mathrm{d}x. \tag{5.38}$$

Taking the time derivative and integrating by parts, by Lemma 5.1 and Young's inequality (see, e.g., [65]), we have

$$\begin{aligned}
\dot{V} =& \int_0^1 \eta\left(\alpha\eta_{xx} - \eta\eta_x - \eta\xi_x - \eta_x\xi\right) \\
\leq& -\alpha\int_0^1 \eta_x^2 \mathrm{d}x + \alpha\eta\eta_x\Big|_0^1 - \frac{1}{3}\int_0^1 \left(\eta^3\right)_x \mathrm{d}x - \int_0^1 (\eta^2\xi)_x \mathrm{d}x + \int_0^1 \eta\eta_x\xi \mathrm{d}x \\
\leq& -\alpha\int_0^1 \eta_x^2 \mathrm{d}x - \eta(0,t)\left(\alpha\eta_x(0,t) - \frac{1}{3}\eta(0,t)^2\right) + \eta(1,t)\left(\alpha\eta_x(1,t) - \frac{1}{3}\eta(1,t)^2\right) \\
&+ \|\xi\|_{L^\infty}\|\eta\|_{L^2}\|\eta_x\|_{L^2}.
\end{aligned} \tag{5.39}$$

Applying the nonlinear boundary control (5.33) and the boundary conditions given in (5.37), we get

$$\begin{aligned}
\dot{V} &\leq -\alpha\int_0^1 \eta_x^2 \mathrm{d}x - \alpha k_1\eta(0,t)^2 - \alpha k_2\eta(1,t)^2 + \alpha^{1/2}\lambda_1^{-1/2}\|\xi\|_{L^\infty(0,\infty;L^\infty(0,1))}\|\eta\|_{H^1}^2 \\
&\leq -\alpha\|\eta\|_{H^1} + \alpha^{1/2}\lambda_1^{-1/2}\|\xi\|_{L^\infty(0,\infty;L^\infty(0,1))}\|\eta\|_{H^1}^2 \\
&\leq -\left(\alpha - \alpha^{1/2}\lambda_1^{-1/2}\|\xi\|_{L^\infty(0,\infty;L^\infty(0,1))}\right)\|\eta\|_{H^1}^2 \\
&\leq -2\left(\alpha - \alpha^{1/2}\lambda_1^{-1/2}\|\xi\|_{L^\infty(0,\infty;L^\infty(0,1))}\right)\lambda_1\alpha^{-1}V.
\end{aligned} \tag{5.40}$$

Therefore, $\|\eta\|_{L^2}$ is exponentially stable, i.e.,

$$\|\eta\|_{L^2}^2 \leq e^{-k_3 t}\|\varphi\|_{L^2}^2, \tag{5.41}$$

where $k_3 = 2\left(\alpha - \alpha^{1/2}\lambda_1^{-1/2}\|\xi\|_{L^\infty(0,\infty;L^\infty(0,1))}\right)\lambda_1\alpha^{-1}$.

Multiplying (5.40) by $e^{\frac{1}{2}k_3 t}$, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{1}{2}k_3 t}\|\eta\|_{L^2}^2\right)+\left(2\alpha-2\alpha^{1/2}\lambda_1^{-1/2}\|\xi\|_{L^\infty(0,\infty;L^\infty(0,1))}\right)e^{\frac{1}{2}k_3 t}\|\eta\|_{H^1}\\&\quad\leq\frac{1}{2}k_3 e^{\frac{1}{2}k_3 t}\|\eta\|_{L^2}^2.\end{aligned}\tag{5.42}$$

Integrating from 0 to t, we obtain

$$\begin{aligned}&e^{\frac{1}{2}k_3 t}\|\eta\|_{L^2}^2+\left(2\alpha-2\lambda^{1/2}\lambda_1^{-1/2}\|\xi\|_{L^\infty(0,\infty;L^\infty(0,1))}\right)\int_0^t e^{\frac{1}{2}k_3\tau}\|\eta(\cdot,\tau)\|_{H^1}\mathrm{d}\tau\\&\leq\frac{1}{2}k_3\int_0^t e^{\frac{1}{2}k_3\tau}\|\eta(\cdot,\tau)\|_{L^2}^2\mathrm{d}\tau\\&\leq\frac{k_3}{2}\|\varphi\|_{L^2}^2\int_0^t e^{-\frac{1}{2}k_3\tau}\mathrm{d}\tau,\end{aligned}\tag{5.43}$$

which implies that

$$\int_0^t e^{\frac{1}{2}k_3\tau}\|\eta(\cdot,\tau)\|_{H^1}\mathrm{d}\tau\leq M_1,\quad t\in[0,\infty),\tag{5.44}$$

where M_1 is a positive constant depending only on $\|\varphi\|_{L^2}$. In the following, we estimate the bound of $\|\eta\|_{H^1}$. Multiplying $-\eta_{xx}$ on both sides of (5.37) yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\eta\|_{H^1}^2+\alpha\|\eta_{xx}\|_{L^2}^2=\int_0^1\left(\eta\eta_x\eta_{xx}+\eta_{xx}\eta\xi_x+\eta_{xx}\eta_x\xi\right)\mathrm{d}x\\&\quad\leq\|\eta_{xx}\|_{L^2}\|\eta\eta_x\|_{L^2}+\|\eta_{xx}\|_{L^2}\|\eta\xi_x\|_{L^2}+\|\eta_{xx}\|_{L^2}\|\eta_x\xi\|_{L^2}\\&\quad\leq\varepsilon_1\|\eta_{xx}\|_{L^2}^2+(4\varepsilon_1)^{-1}\|\eta\eta_x\|_{L^2}^2+\varepsilon_2\|\eta_{xx}\|_{L^2}^2+(4\varepsilon_2)^{-1}\|\eta\xi_x\|_{L^2}^2\\&\qquad+\varepsilon_3\|\eta_{xx}\|_{L^2}+(4\varepsilon_3)^{-1}\|\eta_x\xi\|_{L^2}^2\\&\quad\leq(\varepsilon_1+\varepsilon_2+\varepsilon_3)\|\eta_{xx}\|_{L^2}^2+(4\varepsilon_1)^{-1}\|\eta\|_{L^\infty}^2\|\eta_x\|_{L^2}^2+(4\varepsilon_2)^{-1}\|\eta\|_{L^\infty}^2\|\xi_x\|_{L^2}^2\\&\qquad+(4\varepsilon_3)^{-1}\|\xi\|_{L^\infty}^2\|\eta_x\|_{L^2}^2\\&\quad\leq(\varepsilon_1+\varepsilon_2+\varepsilon_3)\|\eta_{xx}\|_{L^2}^2+\frac{\eta}{4\varepsilon_1}\|\eta\|_{L^2}\|\eta\|_{H^1}\|\eta_x\|_{L^2}^2\\&\qquad+(4\varepsilon_2)^{-1}\|\eta\|_{L^2}\|\eta\|_{H^1}\|\xi\|_{H^1}^2+(4\varepsilon_3)^{-1}\|\xi\|_{L^\infty}^2\|\eta\|_{H^1}^2\\&\quad\leq(\varepsilon_1+\varepsilon_2+\varepsilon_3)\|\eta_{xx}\|_{L^2}^2+\frac{\eta\alpha^{1/2}}{4\lambda_1^{1/2}\varepsilon_1}\|\eta\|_{H^1}^4+\frac{\alpha^{1/2}}{4\lambda_1^{1/2}\varepsilon_2}\|\xi\|_{L^\infty(0,\infty;H^1(0,1))}^2\|\eta\|_{H^1}^2\\&\qquad+(4\varepsilon_3)^{-1}\|\xi\|_{L^\infty(0,\infty;L^\infty(0,1))}^2\|\eta\|_{H^1}^2.\end{aligned}\tag{5.45}$$

Next, we will estimate the H^1-norm of η to show that $\|\eta\|_{H^1}$ exponentially converges to 0 as

t tends to ∞. Choosing constants $\varepsilon_1, \varepsilon_2, \varepsilon_3$ such that $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 < \alpha$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\eta\|_{H^1}^2 \leq & \frac{\eta\alpha^{1/2}}{2\lambda_1^{1/2}\varepsilon_1}\|\eta\|_{H^1}^4 + \frac{\alpha^{1/2}}{2\lambda_1^{1/2}\varepsilon_2}\|\xi\|^2_{L^\infty(0,\infty;H^{(}0,1))}\|\eta\|^2_{H^1} \\ & + (2\varepsilon_3)^{-1}\|\xi\|^2_{L^\infty(0,\infty;L^\infty(0,1))}\|\eta\|^2_{H^1}.\end{aligned} \tag{5.46}$$

Multiplying (5.46) by $e^{\frac{1}{2}k_3 t}$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{1}{2}k_3 t}\|\eta\|^2_{H^1}\right) \leq & \left(\frac{1}{2}k_3 + \frac{\lambda^{1/2}}{2\lambda_1^{1/2}\varepsilon_2}\|\xi\|^2_{L^\infty(0,\infty;H^1(0,1))} + (2\varepsilon_3)^{-1}\|\xi\|^2_{L^\infty(0,\infty;L^\infty(0,1)}\right) \\ & \times e^{\frac{1}{2}k_3 t}\|\eta\|^2_{H^1} + \frac{\eta\alpha^{1/2}}{2\lambda_1^{1/2}\varepsilon_1}e^{\frac{1}{2}k_3 t}\|\eta\|^4_{H^1}.\end{aligned} \tag{5.47}$$

By Gronwall's inequality it yields

$$\begin{aligned}e^{\frac{1}{2}k_3 t}\|\eta\|_{H^1} \leq & \left(\|\varphi\|_{H^1} + \left(\frac{1}{2}k_3 + \frac{\alpha^{1/2}}{2\lambda_1^{1/2}\varepsilon_2}\|\xi\|^2_{L^\infty(0,\infty;H^1(0,1))} + (2\varepsilon_3)^{-1}\|\xi\|^2_{L^\infty(0,\infty;L^\infty(0,1)}\right)\right. \\ & \left. \times \int_0^t e^{\frac{1}{2}k_3\tau}\|\eta(\cdot,\tau)\|^2_{H^1}\mathrm{d}\tau\right)\exp\left(\frac{\eta\alpha^{1/2}}{2\lambda_1^{1/2}\varepsilon_1}\int_0^t\|\eta(\cdot,\tau)\|^2_{H^2}\mathrm{d}\tau\right),\end{aligned} \tag{5.48}$$

which implies that

$$\begin{aligned}\|\eta\|^2_{H^1} \leq & \left(\|\varphi\|_{H^1} + M_1\left(\frac{1}{\text{ffl}2}k_3 + \frac{\alpha^{1/2}}{2\lambda_1^{1/2}\varepsilon_2}\|\xi\|^2_{L^\infty(0,\infty;H^1(0,1))} + (2\varepsilon_3)^{-1}\|\xi\|^2_{L^\infty(0,\infty;L^\infty(0,1)}\right)\right) \\ & \times \exp\left(\frac{\eta\alpha^{1/2}}{2\lambda_1^{1/2}\varepsilon_1}M_1\right)e^{-\frac{1}{2}k_3 t}, \quad t \in [0,\infty).\end{aligned} \tag{5.49}$$

Therefore, $\|\eta\|_{H^1}$ exponentially converges to 0 as t tends to ∞. Furthermore, based on the fact that $H^1(0,1) \hookrightarrow C(0,1)$, we can conclude that $\lim_{t\to\infty} e_i(t) = 0, i = 1, \ldots, n$. □

Remark 5.2. *As the dynamic compensator and the difference between the controlled Burgers' equation and the dynamics of the compensator are stable in L^2-norm and H^1-norm, we can deduce from the fact $\|w\|_{H_1} \leq \|\xi\|_{H^1} + \|\eta\|_{H^1}$ that the system* (5.34) *is also stable in H^1-norm.*

5.4 ADM-based Control Implementation

The dynamic compensator (5.7) is a nonlinear PDE for which it is very difficult or almost impossible to obtain the solution in closed form. In order to overcome this difficulty, we apply the Adomian decomposition method, which allows expressing the solution of the dynamic compensator in a series form.

5.4.1 Introduction to the Adomian Decomposition Method

Consider the following abstract operator equation:

$$\begin{aligned} &L\xi + R\xi + N\xi = f, \\ &\xi(0) = \varphi, \end{aligned} \tag{5.50}$$

where $L : X \to X$ is a linear operator from a Banach space X to a Banach space Y and is reversible. Ru and Nu are linear and nonlinear terms in space X, respectively. f is the homogeneous term. A variety of ordinary differential equations and partial differential equations can be expressed by this abstract equation, such as the nonlinear Sturm-Liouville equation, Fisher's equation, and Burgers' equation. Applying the inverse operator L^{-1} to (5.50) yields

$$\xi = \psi + L^{-1}f - L^{-1}R\xi - L^{-1}N\xi, \tag{5.51}$$

where ψ is determined by the initial data φ. The scheme of Adomian decomposition can generate the solution of the abstract equation (5.50) as an infinite series:

$$\xi = \sum_{m=0}^{\infty} \xi_m, \tag{5.52}$$

where

$$\begin{cases} \xi_0 = \psi + L^{-1}\xi, \\ \xi_1 = -L^{-1}R\xi_0 - L^{-1}A_0, \\ \quad \vdots \\ \xi_{m+1} = -L^{-1}R\xi_m - L^{-1}A_m. \end{cases} \tag{5.53}$$

The nonlinear term can be expressed in terms of a iteration series, i.e., $N\xi = \sum_{m=0}^{\infty} A_m(\xi_0, \ldots, \xi_m)$, where the polynomial A_m can be computed from $\xi_i, i = 1, \ldots, m$, as follows [3, 134]:

$$A_m(\xi_0, \ldots, \xi_m) = \frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}\lambda^m}\left[N\left(\sum_{i=0}^{\infty} \lambda^i \xi_i\right)\right]_{\lambda=0}. \tag{5.54}$$

5.4.2 Implementation of Dynamic Compensator

Based on the ADM, the nonlinear term of the dynamic compensator (5.7) can be expressed as

$$\xi_x \xi = \sum_{i=0}^{\infty} A_i(\xi_0, \xi_1, \ldots, \xi_i), \tag{5.55}$$

where

$$A_m(\xi_0, \cdots, \xi_m) = \xi_m \xi_{0x} + \xi_{m-1}\xi_{1x} + \cdots + \xi_1 \xi_{(m-1)x} + \xi_0 \xi_{mx}. \tag{5.56}$$

We chose a ξ_0 that complies with the following linear parabolic equation:

$$\begin{aligned}
&\xi_{0t} - \alpha \xi_{0xx} = 0,\ x \in \Omega, t > 0,\\
&\xi_0(0,t) = 0,\ \xi_0(1,t) = 0,\\
&[\xi_0(x,t)]_{x=x_i} = 0,\ i = 1, \ldots, n,\\
&\xi_0(x_i,t) = y^r_{i,0}(t),\ i = 1, \ldots, n,\\
&\xi_0(x,0) = 0.
\end{aligned} \tag{5.57}$$

The corresponding component $u_{0,i}(t)$ of the in-domain control is then given by

$$u_{0,i}(t) = [\xi_{0x}(x,t)]_{x=x_i},\ i = 1, \ldots, n. \tag{5.58}$$

Using the recursive formula (5.54), ξ_1 complies with

$$\begin{aligned}
&\xi_{1t} - \alpha \xi_{1xx} + \xi_{0x}\xi_0 = 0,\ x \in \Omega, t > 0,\\
&\xi_1(0,t) = 0,\ \xi_1(1,t) = 0,\\
&[\xi_1(x,t)]_{x=x_i} = 0,\ i = 1, \ldots, n,\\
&\xi_1(x,0) = 0.
\end{aligned} \tag{5.59}$$

The corresponding component $u_{1,i}(t)$ of the in-domain control is:

$$u_{1,i}(t) = [\xi_{1x}(x,t)]_{x=x_i},\ i = 1, \ldots, n. \tag{5.60}$$

Therefore, the general term $\xi_m, m = 1, \ldots, \infty$, complies with the following parabolic equation:

$$\begin{aligned}
&\xi_{mt} - \alpha \xi_{mxx} + A_{m-1}(\xi_0, \ldots, \xi_{m-1}) = 0, \quad x \in \Omega,\ t > 0,\\
&\xi_m(0,t) = 0,\ \xi_m(1,t) = 0,\\
&[\xi_m(x,t)]_{x=x_i} = 0,\ i = 1, \ldots, n,\\
&\xi_m(x,0) = 0,
\end{aligned} \tag{5.61}$$

and the m-th component $u_{m,i}(t)$ of the in-domain control $u_i(t)$ is given by

$$u_{m,i}(t) = [\xi_{mx}(x,t)]_{x=x_i}, \ i = 1, \dots, n. \tag{5.62}$$

Thus, the solution to the dynamic compensator (5.7) and the reference signal $y_i^r(t)$ can be expressed, respectively, as

$$\xi(x,t) = \sum_{m=0}^{\infty} \xi_m(x,t), \ y_i^r(t) = y_{0,i}^r + \sum_{i=1}^{\infty} \xi_m(x_i,t), i = 1, \dots, n. \tag{5.63}$$

Finally, the in-domain control u_i can be derived from (5.58), (5.60), and (5.62), which takes the form:

$$u_i(t) = \sum_{m=0}^{\infty} u_{m,i}(t) = \sum_{m=0}^{\infty} [\xi_{mx}(x,t)]_{x=x_i}, \ i = 1, \dots, n. \tag{5.64}$$

5.5 Motion Planning

The recursive procedure of the ADM starts by solving the linear parabolic equation (5.57). As the considered problem is a set-point control, we resort to flatness-based trajectory planning to compute ξ_0. For this purpose, we first change the time scale from t to αt, while still using t as the time variable, which leads to the following linear parabolic equation:

$$\begin{aligned}
&\xi_{0t} - \xi_{0xx} = 0, x \in (0,1), \ t > 0 \\
&\xi_0(0,t) = 0, \ \xi_0(1,t) = 0, \\
&[\xi_0(x,t)]_{x=x_i} = 0, \ i = 1, \dots, n, \\
&\xi_0(x_i,t) = y_{0,i}^r(t/\alpha), \ i = 1, \dots, n, \\
&B_i\xi_0 = [\xi_{0x}(x,t)]_{x=x_i} = v_i(t), \ i = 1, \dots, n, \\
&\xi_0(x,0) = 0,
\end{aligned} \tag{5.65}$$

where $v_i(t), i = 1, \dots, n$, are the control inputs in the new time-scale. Due to the linearity of (5.65), we can divide ξ_0 into $\xi_0^i(x,t), i = 1, \dots, n$, in a manner that all the subsystems are parallel connected. Hence, ξ_0 can be expressed as $\xi_0 = \sum_{i=1}^{n} \xi_0^i$, where ξ_0^i is governed by

$$
\begin{aligned}
&\xi_{0t}^i - \xi_{0xx}^i = 0, \quad x \in (0, x_i) \cup (x_i, 1), \\
&\xi_0^i(0,t) = 0, \quad \xi_0^i(1,t) = 0, \\
&\xi_0^i(x_i, t) = y_{0,i}^r(t/\alpha), \\
&\xi_0^i(0,t) = 0, \\
&\left[\xi_{0x}^i\right]_{x=x_i} = v_i(t), \\
&i = 1, \dots, n.
\end{aligned}
\tag{5.66}
$$

We split every system ξ_0^i into two sub-systems, i.e., for fixed $x_i \in (0,1)$,

$$\xi_{it}^-(x,t) - \xi_{ixx}^-(x,t) = 0, \quad x \in (0, x_i), \tag{5.67}$$

$$\xi_i^-(0,t) = 0, \quad \xi_{ix}^-(0,t) = 0, \tag{5.68}$$

and

$$\xi_{it}^+(x,t) - \xi_{ixx}^+(x,t) = 0, \quad x \in (x_i, 1), \tag{5.69}$$

$$\xi_i^+(0,t) = 0, \quad \xi_i^+(1,t) = 0, \tag{5.70}$$

with the joint conditions

$$\xi_i^-(x_i,t) = \xi_i^+(x_i,t), \tag{5.71}$$

$$\xi_{ix}^+(x_i,t) - \xi_{ix}^-(x_i,t) = v_i(t). \tag{5.72}$$

Applying Laplace transform to both sides of (5.67) with boundary conditions (5.68) and to (5.69) with boundary conditions (5.70), it yields

$$
\begin{aligned}
&s\widehat{\xi_i^-}(x,s) = \widehat{\xi_{ixx}^-}(x,s), x \in (0, x_i), \\
&\widehat{\xi_i^-}(0,s) = 0,
\end{aligned}
\tag{5.73}
$$

and

$$
\begin{aligned}
&s\widehat{\xi_i^+}(x,s) = \widehat{\xi_{ixx}^+}(x,s), x \in (x_i, 1), \\
&\widehat{\xi_i^+}(1,s) = 0.
\end{aligned}
\tag{5.74}
$$

Then, (5.73) and (5.74) can be expressed as

$$\begin{aligned} \widehat{\xi_i^-}(x,s) &= \hat{C}_1(s)\phi_1(x,s) + \hat{C}_2(s)\phi_2(x,s), \\ \widehat{\xi_i^+}(x,s) &= \hat{C}_3(s)\phi_1(\zeta,s) + \hat{C}_4(s)\phi_2(\zeta,s), \end{aligned} \tag{5.75}$$

where $\zeta = x - x_i$, and

$$\phi_1(x,s) = \frac{\sinh(\sqrt{s}x)}{\sqrt{s}}, \quad \phi_2(x,s) = \cosh(\sqrt{s}x). \tag{5.76}$$

Applying the technique used in [119], we derive

$$\begin{aligned} \hat{v}_i(s) &= \left(\phi_1(1-x_i,s)\phi_2(x_i,s) + \phi_1(x_i,s)\phi_2(1-x_i,s)\right)\hat{h}_i(s) \\ &= \phi_1(1,s)\hat{h}_i(s), \end{aligned} \tag{5.77}$$

where $\hat{h}_i(s) \leftrightarrow h_i(t)$ is the so-called flat output. Recalling that

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}, \quad \cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, \tag{5.78}$$

we obtain

$$\hat{v}_i(s) = \sum_{m=0}^{\infty} \frac{s^m}{(2m+1)!}\hat{h}_i(s). \tag{5.79}$$

Thus, applying the inverse Laplace transform, the in-domain control in time domain is given by

$$v_i(t) = \sum_{m=0}^{\infty} \frac{h_i^{(m)}(t)}{(2m+1)!}, \; i = 1, \ldots, n. \tag{5.80}$$

Therefore, the in-domain control in the original time-scale can be expressed as

$$u_{0,i}(t) = \sum_{m=0}^{\infty} \frac{h_i^{(m)}(\alpha t)}{(2m+1)!}, \; i = 1, \ldots, n. \tag{5.81}$$

In order to assure the convergence of the above series, the flat outputs $h_i(t), i = 1, \ldots, n$, should be of C^∞-smooth. We choose then the following function $\psi(t)$ as a component of basic outputs:

$$\psi(t) = \begin{cases} 0, & \text{if} \quad t \leq 0, \\ \dfrac{\int_0^t exp(-1/(\tau(1-\tau)))^{\varepsilon} \mathrm{d}\tau}{\int_0^T exp(-1/(\tau(1-\tau)))^{\varepsilon} \mathrm{d}\tau}, & \text{if} \quad 0 < t < T, \\ 1, & \text{if} \quad t \geq T, \end{cases} \tag{5.82}$$

which is known as the Gevrey function of order $\sigma = 1 + \frac{1}{\varepsilon}, \varepsilon > 0$. The form of basic outputs $h_i(t)$ can be expressed as

$$h_i(t) = \mu_i \psi(t), i = 1, \ldots, n, \tag{5.83}$$

where μ_i are constants.

In order to find appropriate reference trajectories to track the desired set-points y_i^d, the unknown parameters μ_i of flat outputs $h_i(t)$ can be determined by the steady-state equation corresponding to the system (5.7), which is of the following form:

$$\begin{aligned} & -\alpha\bar{\xi}_{xx} + \bar{\xi}\bar{\xi}_x = 0, \ x \in \Omega, \\ & \left[\bar{\xi}(x)\right]_{x=x_i} = 0, i = 1, \ldots, n, \\ & \bar{\xi}(0) = 0, \quad \bar{\xi}(1) = 0, \\ & \bar{\xi}(x_i) = y_i^d, i = 1, \ldots, n, \\ & \left[\bar{\xi}_x(x)\right]_{x=x_i} = -\gamma_i, i = 1, \ldots, n, \end{aligned} \tag{5.84}$$

To find the solution to the static PDE (5.84), we still use the ADM with which the solution is expressed as

$$\bar{\xi} = \sum_{m=0}^{\infty} \bar{\xi}_m, \tag{5.85}$$

and the nonlinear term can be written as

$$\bar{\xi}_m \bar{\xi}'_m = \overline{A}_m(\bar{\xi}_0, \cdots, \bar{\xi}_m) = \bar{\xi}_m \bar{\xi}'_0 + \bar{\xi}_{m-1}\bar{\xi}'_1 + \ldots + \bar{\xi}_1 \bar{\xi}'_{m-1} + \bar{\xi}_0 \bar{\xi}'_m. \tag{5.86}$$

Therefore, the first term $\bar{\xi}_0$ complies with the following linear parabolic equation:

$$\begin{aligned}
&-\alpha\bar{\xi}_{0xx} = 0,\ x \in \Omega,\\
&\bar{\xi}_0(0) = 0,\ \bar{\xi}_0(1) = 0,\\
&\left[\bar{\xi}_0(x)\right]_{x=x_i} = 0,\ i = 1,\dots,n,\\
&\bar{\xi}(x_i) = y_i^d,\ i = 1,\dots,n,\\
&[\bar{\xi}_{0x}]_{x=x_i} = \bar{u}_{0,i},\ i = 1,\dots,n.
\end{aligned} \tag{5.87}$$

Using the recursive formula (5.54), $\bar{\xi}_1$ is governed by

$$\begin{aligned}
&-\alpha\bar{\xi}_{1xx} + \bar{\xi}_{0x}\bar{\xi}_0 = 0,\ x \in \Omega,\\
&\bar{\xi}_1(0) = 0,\ \bar{\xi}_1(1) = 0,\\
&\left[\bar{\xi}_1(x)\right]_{x=x_i} = 0,\ i = 1,\dots,n.
\end{aligned} \tag{5.88}$$

The second component $\bar{u}_{1,i}$ in γ_i is given by

$$\bar{u}_{1,i} = [\bar{\xi}_{1x}(x)]_{x=x_i},\ i = 1,\dots,n. \tag{5.89}$$

Therefore, the general term $\bar{\xi}_m, m = 1,\dots,\infty$, complies with the following parabolic equation:

$$\begin{aligned}
&-\alpha\bar{\xi}_{mxx} + A_{m-1}(\bar{\xi}_0,\dots,\bar{\xi}_{m-1}) = 0, \quad x \in \Omega,\\
&\bar{\xi}_m(0) = 0,\ \bar{\xi}_m(1) = 0,\\
&\left[\bar{\xi}_m(x)\right]_{x=x_i} = 0,\ i = 1,\dots,n.
\end{aligned} \tag{5.90}$$

hus, The m-th component $\bar{u}_{m,i}$ in γ_i is given by

$$\bar{u}_{m,i} = \left[\bar{\xi}_{mx}(x)\right]_{x=x_i},\ i = 1,\dots,n. \tag{5.91}$$

The parameter $\gamma_i, i = 1,\dots,n$, can be decomposed as

$$\gamma_i = \sum_{m=0}^{\infty}\bar{u}_{m,i} = \sum_{m=0}^{\infty}\left[\bar{\xi}_{mx}(x)\right]_{x=x_i},\ i = 1,\dots,n. \tag{5.92}$$

Therefore, according to (5.81), we choose the parameters $\mu_i, i = 1,\dots,n$, such that

$$\lim_{t\to\infty} u_{0,i}(t) = \mu_i = \bar{u}_{0,i},\ i = 1,\dots,n. \tag{5.93}$$

Finally, the basic inputs are set as

$$h_i(t) = \bar{u}_{0,i}\psi(t),\ \ i = 1, \ldots, n. \tag{5.94}$$

With such basic outputs, the control $u_i(t), i = 1, \ldots, n$, given in (5.61), (5.64), and (5.81), can allow the in-domain controlled Burgers' equation to achieve an asymptotic regulation with respect to the desired outputs $y_i^d, i = 1, \ldots, n$.

5.6 Simulation Study

To illustrate the proposed control scheme, we conduct a numerical simulation study of the in-domain controlled Burgers' equation with Matlab. In the simulation, the initial condition is chosen as $\varphi = 0.4\sin(\pi x)$, and the parameters are set as $k_1 = k_2 = 7$, and $\alpha = 1$.

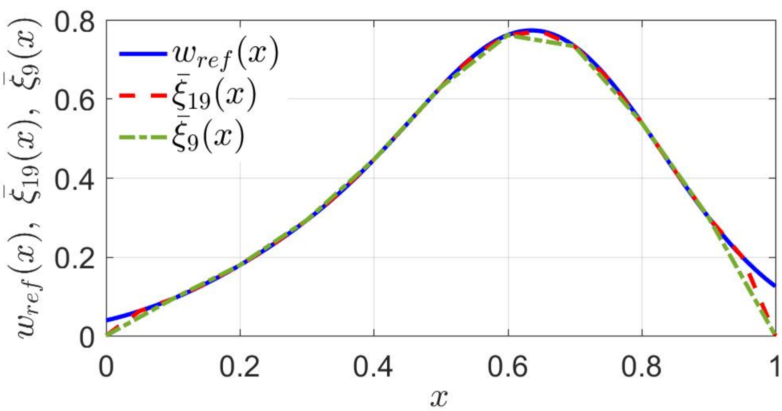
The desired static profile is described as

$$w_{ref}(x) = a_1 \exp\left(-\left(\frac{x - b_1}{c_1}\right)^2\right) + a_2 \exp\left(-\left(\frac{x - b_2}{c_2}\right)^2\right) \tag{5.95}$$

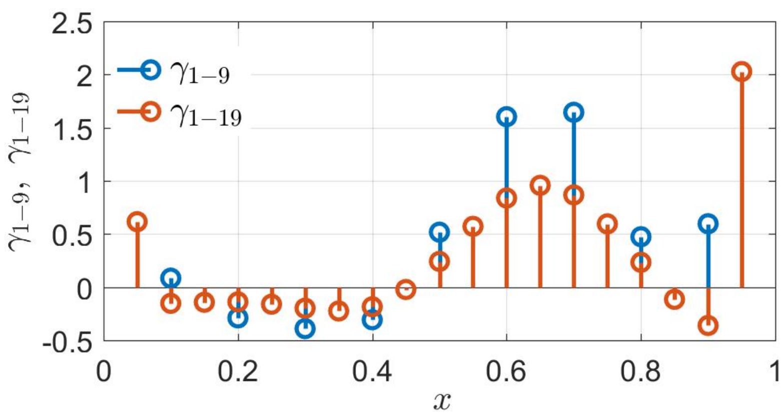
where the parameters are set to $a_1 = 0.75$, $b_1 = 0.65$, $c_1 = 0.26$, $a_2 = 0.17$, $b_2 = 0.31$, and $c_2 = 0.25$. Our aim is to drive the trajectory of the Burgers' equation to track the desired profile $w_{\mathrm{ref}}(x)$ via in-domain control. Due to the fact that $w_{\mathrm{ref}}(x)$ may not be the solution of the Burgers' equation (5.3) in steady-state, we illustrate in Fig. 5.1(a) the accuracy of interpolation of a system with, respectively 9 and 19 in-domain actuators corresponding to the desired curve (5.95). Clearly, the enhancement of the tracking performance of Burgers' equation necessitates an adequate number of in-domain actuators. The setting with 19 in-domain actuators at the point $x_i = \frac{i}{20}, i = 1, \ldots, 19$, is used in performance evaluation. In the simulation, we choose the 4-th partial sum of the Adomian series $\xi^4 = \sum_{i=0}^{4} \xi_i$ to approximate solution of the dynamic compensator. Accordingly, the parameters $\gamma_i, i = 1, \ldots, 19$, is also computed by a 4-th order approximation of (5.92). The obtained control in steady state is given in Fig. 5.1(b). The solution profile $w(x,t)$ is shown in Fig. 5.2(a), and the evolution of the regulation errors is depicted in Fig. 5.2(b). The simulation results show that the developed control scheme can achieve the objective of asymptotic set-point control.

5.7 Conclusion

The present work dealt with the in-domain control of Burgers' equation for output regulation. A nonlinear boundary feedback control is used and, inspired by the zero dynamic inverse



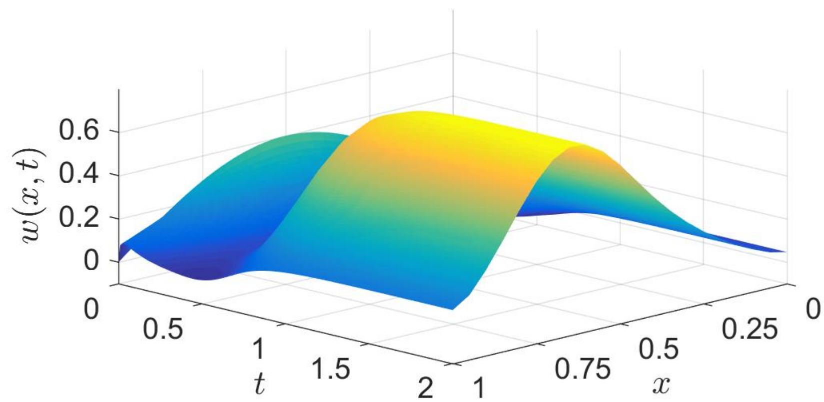
(a)



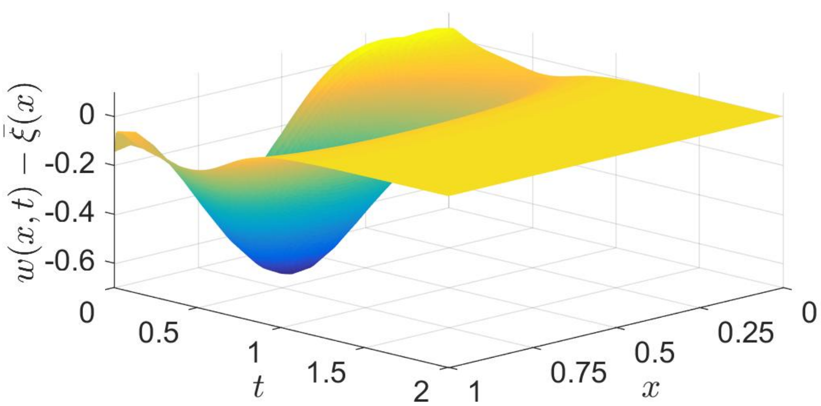
(b)

Figure 5.1 Simulation results: (a) interpolation of the reference signal w_{ref}; (b) static control signals.

design, a dynamic compensator with homogeneous is applied to generate the in-domain control signals. The closed-loop stability of the proposed control scheme is assessed through a rigorous analysis. The method of Adomian composition combined with flatness-based trajectory planning is used to implement the dynamic compensator. The results obtained from numerical simulations illustrate the performance of the developed control scheme, which confirms the validity of the proposed method.



(a)



(b)

Figure 5.2 Simulation results: (c) solution surface; (d) surface of regulation errors.

CHAPTER 6 GENERAL DISCUSSION

In-domain control of PDE systems has been developed over the last few decades. However, several problems remain to be investigated. One difficulty arose in in-domain control of PDE systems is that the in-domain input operators are unbounded in the Sobolev space, such as $H^1(\Omega)$, to which the solutions to PDE systems belong. This means that the control techniques developed for boundary control of PDE systems cannot be applied directly to the in-domain control problems. To overcome this difficulty, dual spaces of the Sobolev space have been introduced, on which the interior input operators are bounded, which makes it possible to analyze the well-posedness and the regularity and enables control synthesis to achieve the stability of linear and nonlinear PDE systems with multiple in-domain inputs. Another difficulty is that the equivalence between the boundary control and the in-domain control may not be established. Thus, the existing boundary control techniques cannot be directly applied to in-domain control. We need to develop new methods to address in-domain control of PDEs.

This thesis addresses the in-domain control of linear and nonlinear PDE systems, such as Euler-Bernoulli equation and semi-linear parabolic equation, including Fisher's equation, Chaffee-Infante equation, and Burger' equation. The classification of the considered PDEs and the methods developed in this thesis for tackling the corresponding problems are summarized in Fig.6.1. For the linear cases, zero dynamics are linear PDEs, whose solutions can be easily derived using classical mathematical methods, such as Laplace transform, Fourier transform, and separation of variables. For example, the solutions to the zero dynamics of heat equations and Euler-Bernoulli equations with multiple interior actuations can be explicitly expressed as analytical series in terms of flat outputs. To improve in-domain control design, our earlier work utilized differential flatness to generate the desired trajectory and then employed the Green's function method to solve the motion planning problem. For nonlinear cases, the zero dynamics of nonlinear PDEs with multiple interior actuations are governed by nonlinear PDEs. We resort then the Adomian decomposition method (ADM) to investigate the regularity of the zero dynamics of nonlinear semi-linear parabolic equations, which in turn allows determining the effect that interior actuations exert on the nonlinear term of semi-linear parabolic equations. Indeed, if the nonlinear term of a semi-linear parabolic equation is sufficiently smooth, such as in Fisher's equation and Chafee-Infante equation, then the iterated terms of an Adomian series will preserve the smoothness at the level of the approximation obtained in the previous steps. While the nonlinear term of semi-linear parabolic equations is not smooth and is subject to in-domain control, the regularity of iterated terms

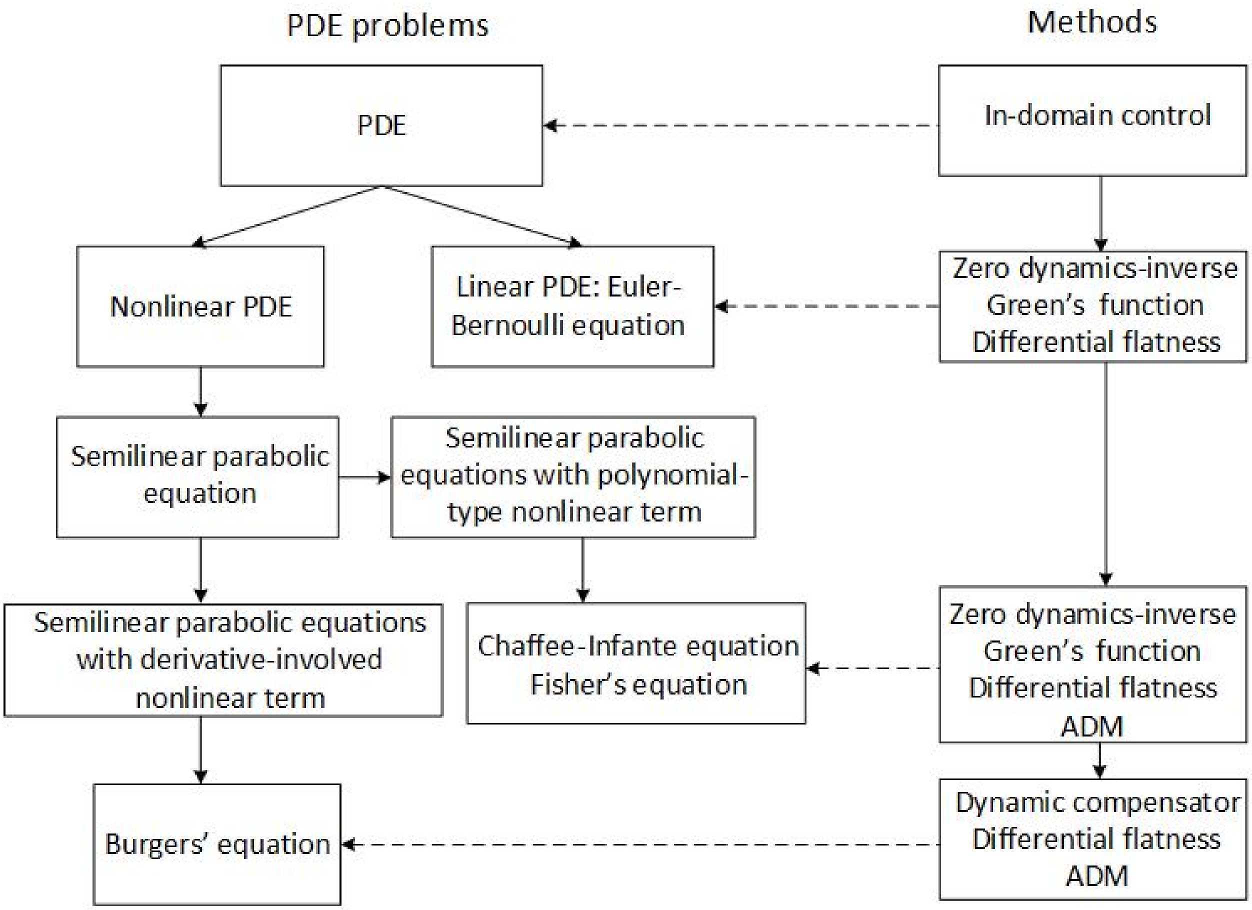


Figure 6.1 In-domain control problems and corresponding methods.

will not exceed that of the previous items. For example, the nonlinear term of a Burgers' equation involves the derivative on space, which indicates that the second or the consecutive terms of the Adomian series of the Burgers' equation degrades the smoothness, making the solution much more oscillating compared with the Fisher's equation or the Chafee-Infante equation. This is why a different approach has to be developed to deal with the in-domain control of Burgers' equation. Due to the regularity of Fisher's equation and Chafee-Infante equation, higher-order terms of Adomian series have no impact on the regularity of their systems. Therefore, in-domain control of Fisher's equations and Chafee-Infanter equations is reduced to the in-domain control design of their linear counterpart, which facilitates the in-domain control design and implementation.

Systematic in-domain control design procedures for linear and semi-linear PDE systems are proposed in this thesis to solve PDE systems with interior actuations. In-domain control of Euler-Bernoulli equation, Fisher's equation, Chafee-Infante equations, and Burgers' equation are solved by using the proposed control algorithms. Furthermore, the proposed approaches can be applied to other classes of PDE systems. Specifically, the proposed algorithms for in-domain control of PDE systems can be generalized as follows. For linear cases, based on the zero dynamics inverse, in-domain control design of linear PDEs can be reduced to that of its zero dynamics. Then, using Laplace transform, the zero dynamics can be decomposed into several subsystems governed by linear PDEs. The methods of differential flatness and Green's function are employed to produce the desired trajectories to solve the motion planning. For nonlinear cases, the ADM is used to analyze the regularity of the zero dynamics of nonlinear PDE systems. When the nonlinear terms of the PDE systems are smooth, the in-domain control of nonlinear PDEs can be reduced to that of their linear counterpart. When the nonlinear terms of PDE systems involve high order derivative terms, the in-domain control will depend on both their linear and nonlinear terms, which requires the calculation of more terms of Adomian series of the zero dynamics of the nonlinear PDEs to derive the in-domain control. Furthermore, if nonlinear boundary feedback control is used to enhance the performance of the system, such as the stability, dynamic compensators of particular form should be considered to enable well-posedness and stability analysis. The design and implementation of the in-domain control have to be adjusted accordingly. Finally, the approaches developed in this thesis belong to the category of late-lumping, which means that we deal with the original PDE models in control design without using any approximation. Thus, the truncations may happen only at the stage of implementation, which makes the control design and implementation procedures computationally tractable with guaranteed performance.

CHAPTER 7 CONCLUSION

7.1 Conclusion

In-domain control design is a challenging problem while representing a promising trend in PDE control theory. A key aspect related to in-domain control is that it eliminates the restriction on the location of actuations inherent to boundary control and simultaneously provides profound insights into the manipulation of PDE systems. This thesis addresses in-domain control of an Euler-Bernoulli equation and a class of semi-linear parabolic equations, including Fisher's equation, Chaffee-Infante equation, and Burgers' equation.

Chapter 1 introduces the in-domain control of PDE systems and some tools that enable systematic procedures for PDE control design, including backstepping control, boundary feedback control, differential flatness, and zero dynamics inverse. This chapter includes also a summary of the advances in in-domain control of PDE systems.

Chapter 2 presents an overview of the basis on PDE theory and introduces some notions and tools used throughout this thesis, including the Sobolev space, the Hölder inequality and the Sobolev inequalities, which play essential roles in establishing the *a priori* estimates of the solutions to the PDEs. Some well-known results on the existence, the uniqueness, and the regularity of linear elliptic equations and linear parabolic equations are recalled in this chapter. The basic concepts of semigroup analysis are presented, and their applications are illustrated with an example on how the semigroup theory can be leveraged for the control of infinite-dimensional systems. Differential flatness is introduced, and in-domain control of a heat equation is used to demonstrate differential flatness-based trajectory planning. This chapter also discusses the Adomian decomposition method (ADM), and illustrates the application of the ADM to a Fisher's equation.

Chapter 3 addresses in-domain control of an Euler-Bernoulli equation using zero dynamics inverse and differential flatness. Euler-Bernoulli equation features the flatness property, which allows expressing the state variables and the control inputs in terms of flat outputs and their time derivatives, thereby constructing the solution to the Euler-Bernoulli equation and generating the in-domain control. To complete motion planning, the Green's function of the static equation of the Euler-Bernoulli equation is exploited to produce static control. The desired control signals are constructed by the static control along with the desired trajectory developed using differential flatness. The simulation results illustrate the validity and the effectiveness of the proposed control strategy.

Chapter 4 is devoted to in-domain control of a class of semi-linear parabolic equations, including Fisher's equation and Chaffee-Infante equation. The method of zero dynamics inverse is employed to establish the in-domain control. The Adomian decomposition method is leveraged to obtain a semi-analytic solution to the zero dynamics. Convergence analysis of the Adomian series is conducted, which can guarantee that the solution derived by the ADM can approach the exact solution to the zero dynamics. Due to the regularity analysis of the considered semi-linear parabolic equations, we conclude that the interior actuations depends only on the linear term of the Adomian series. This indicates that in-domain control can be achieved by only the linear term of the Adomian series, greatly facilitating the control design procedure.

Chapter 5 presents a novel in-domain control scheme based on a dynamic compensator for a Burgers' equation under nonlinear boundary feedback control. A dynamic compensator is introduced to generate in-domain control, which can track the desired outputs. A Lyapunov stability analysis confirms that the error between the outputs of the controlled Burgers' equation and the desired output signals converge to zero over the time. Finally, the ADM and the flatness are utilized in the implementation of the resulting in-domain control.

Chapter 6 provides a general discussion on in-domain control of PDE systems, the methodology employed along with the research of this thesis, and the achievements obtained in this work.

7.2 Perspectives on Future Work

7.2.1 In-domain control of higher-dimensional PDEs

This thesis addressed in-domain control of one-dimensional PDE systems with multiple interior actuations. A future work may be to extend the approaches developed for in-domain control of one-dimensional PDEs to higher-dimensional PDEs. In-domain control of 2-dimensional linear PDEs can be examined first, and in-domain control of other types of PDEs will also be considered.

7.2.2 Tracking/Rejection of time-dependent signals

The objective of this thesis is to treat the multiple interior set-point problems of PDE systems by dynamic control laws. However, large numbers of problems are formulated as tracking and rejection of time-dependent desired signals. There is a significant difference between those two types of problems from both theoretical and practical perspectives. Indeed, set-point control problems require only the PDE systems to retain the desired prescribed outputs by manipulating the control inputs. Whereas, tracking the desired time-dependent signals and disturbances rejection need to allow the controlled PDE systems to adjust the control signals while varying the time. Hence, the control techniques developed in this thesis cannot yet be directly extended to the cases where the prescribed outputs are time-dependent. More efforts should be devoted to formulating in-domain control of PDE systems with time-dependent references and to obtaining control laws based on available control techniques in order to tackle these types of problems.

REFERENCES

[1] A. Abdelrazec and D. Pelinovsky. Convergence of the adomian decomposition method for initial-value problems. *Numerical Methods for Partial Differential Equations*, 27(4):749–766, 2011.

[2] R.A. Adams and J.J.F. Fournier. *Sobolev Spaces*, volume 140. Elsevier, 2003.

[3] G. Adomian. A review of the decomposition method in applied mathematics. *Journal of Mathematical Analysis and Applications*, 135(2):501–544, 1988.

[4] G. Adomian. A review of the decomposition method and some recent results for nonlinear equation. *Math. Comput. Modeling*, 13(7):17–43, 1990.

[5] G. Adomian. *Solving Frontier Problems of Physics: the Decomposition Method*, volume 60. Springer Science & Business Media, 2013.

[6] K. Ammari, D. Mercier, V. Regnier, and J. Valein. Spectral analysis and stabilization of a chain of serially connected Euler-Bernoulli beam and strings. *Communications on Pure and Applied Analysis*, 11(2):785–807, 2012.

[7] K. Ammari and M. Tucsnak. Stabilization of Bernoulli-Euler beams by means of a pointwise feedback force. *SIAM J. Control Optimal*, 39(4):1160–1181, 2000.

[8] Y. Aoustin, M. Fliess, H. Mounier, P. Rouchon, and J. Rudolph. Theory and practice in the motion planning and control of a flexible robot arm using Mikusinski operators. In *Proc. of the Fifth IFAC Symposium on Robot Control*, pages 287–293, France, Nantes, 1997.

[9] A. Armaou and P.D. Christofides. Nonlinear feedback control of parabolic partial differential equation systems with time-dependent spatial domains. *Journal of Mathematical Analysis and Applications*, 239(1):124–157, 1999.

[10] P. Ascencio, A. Astolfi, and T. Parisini. Backstepping pde design: A convex optimization approach. *IEEE Transactions on Automatic Control*, 63(7):1943–1958, 2018.

[11] E. Aulisa and D.S. Gilliam. Regulation of a controlled burgers' equation: Tracking and disturbance rejection for general time dependent signals. In *IEEE American Control Conference (ACC)*, pages 1290–1295, 2013.

[12] E. Aulisa and D.S. Gilliam. A numerical algorithm for set-point regulation of non-linear parabolic control systems. *International Journal of Numerical Analysis and Modeling*, 11(1):54–85, 2014.

[13] E. Aulisa and D.S. Gilliam. *A Practical Guide to Geometric Regulation for Distributed Parameter Systems*. CRC Press, Boca Raton, FL, 2016.

[14] A. Badkoubeh, J. Zheng, and G. Zhu. Flatness-based deformation control of an euler–bernoulli beam with in-domain actuation. *IET Control Theory & Applications*, 10(16):2110–2118, 2016.

[15] A. Badkoubeh, J. Zheng, and G. Zhu. Flatness-based deformation control of an Euler-Bernoulli beam with in-domain actuation. *IET Control Theory & Applications*, 2016.

[16] A. Badkoubeh and G. Zhu. A Green's function-based design for deformation control of a microbeam with in-domain actuation. *Journal of Dynamic Systems, Measurement, and Control*, 136(1):011018, 2014.

[17] M. J. Balas. Active control of flexible dynamic systems. *Journal of Optimization Theory and Applications*, 25(3):415–436, 1978.

[18] M.J. Balas. Feedback control of linear diffusion processes. In *Information Linkage Between Applied Mathematics and Industry*, pages 513–525. Elsevier, 1979.

[19] M.J. Balas. The galerkin method and feedback control of linear distributed parameter systems. *Journal of Mathematical Analysis and Applications*, 91(2):527–546, 1983.

[20] A. Balogh and M. Krstic. Infinite dimensional backstepping-style feedback transformations for a heat equation with an arbitrary level of instability. *European Journal of Control*, 8(2):165–175, 2002.

[21] G. Bastin and J.M. Coron. On boundary feedback stabilization of non-uniform linear 2× 2 hyperbolic systems over a bounded interval. *Systems & Control Letters*, 60(11):900–906, 2011.

[22] O.A. Bauchau and J.I. Craig. Euler-Bernoulli beam theory. In *Structural Analysis*, pages 173–221. Springer, 2009.

[23] A. Bensoussan, G. Da Prato, M. Delfor, and S. K. Mitter. *Representation and Control of Infinite-Dimensional Systems*. Birkhauser, 2006.

[24] T. Berger, A. Ilchmann, and F. Wirth. Zero dynamics and stabilization for analytic linear systems. *Acta Applicandae Mathematicae*, 138(1):17–57, 2015.

[25] T.G. Bifano, J.A. Perreault, P.A. Bierden, and C.E. Dimas. Micromachined deformable mirrors for adaptive optics. In *High-resolution wavefront control: methods, devices, and applications IV*, volume 4825, pages 10–14. International Society for Optics and Photonics, 2002.

[26] M.P. Bonkile, A. Awasthi, C. Lakshmi, V. Mukundan, and V.S. Aswin. A systematic literature review of burgers' equation with recent advances. *Pramana*, 90(6):69, 2018.

[27] C. I. Byrnes and D. S. Gilliam. On the transfer function of the zero dynamics for a boundary controlled heat problem in two dimensions. In *Proc. of the 48th IEEE Conference on Decision and Control*, pages 2369–2374, Shanghai, China, Dec. 2009.

[28] C. I. Byrnes, D. S. Gilliam, C. Hu, and V. I. Shubov. Asymptotic regulation for distributed parameter systems via zero dynamics inverse design. *Int. J. of Robust and Nolinear Control*, 23(3):305–333, 2013.

[29] C. I. Byrnes, D. S. Gilliam, A. Isidori, and V. I. Shubov. Set-point boundary control for a nonlinear distributed parameter system. In *Proc. of the 42nd IEEE Conference on Decision and Control*, pages 312–317, Maui, Hawaii, Dec. 2003.

[30] C. I. Byrnes, D. S. Gilliam, A. Isidori, and V. I. Shubov. Static and dynamic controllers for boundary controlled distributed parameter systems. In *Proc. of the 43rd IEEE Conference on Decision and Control*, pages 3324–3325, Nassau, Bahamas, Dec. 2004.

[31] C. I. Byrnes, D. S. Gilliam, A. Isidori, and V. I. Shubov. Interior point control of a heat equation using zero dynamics design. In *Proc. of the 2006 American Control Conference*, pages 1138–1143, Minneapolis, Minnesota, June 14-16 2006.

[32] C.I. Byrnes, D. Gilliam, A. Isidori, and V.I. Shubov. Set-point boundary control for a viscous Burgers equation. In W. P. Dayawansa, A. Lindquist, and Y. Zhou, editors, *New Directions in Control Theory and Applications*, chapter 4, pages 43–60. Springer Verlag, Berlin Heidelberg, New York, 2005.

[33] C.I. Byrnes and D.S. Gilliam. Zero dynamics inverse design for asymptotic regulation of the heat equation: the non-colocated case. Proc. of the 2009 American Control Conference, pages 832–836, St. Louis, MO, USA, June 10-2 2009.

[34] C.I. Byrnes, D.S. Gilliam, C. Hu, and V.I. Shubov. Zero dynamics boundary control for regulation of the kuramoto–sivashinsky equation. *Mathematical and Computer Modelling*, 52(5-6):875–891, 2010.

[35] C.I. Byrnes, D.S. Gilliam, A. Isidori, and V.I. Shubov. Zero dynamics modeling and boundary feedback design for parabolic systems. *Mathematical and Computer Modelling*, 44(9-10):857–869, 2006.

[36] C.I. Byrnes, D.S. Gilliam, and V.I. Shubov. On the global dynamics of a controlled viscous burgers' equation. *Journal of Dynamical and Control Systems*, 4(4):457–519, 1998.

[37] C.I. Byrnes, D.S. Gilliam, and V.I. Shubov. Boundary control, stabilization and zero-pole dynamics for a non-linear distributed parameter system. *International Journal of Robust and Nonlinear Control: IFAC-Affiliated Journal*, 9(11):737–768, 1999.

[38] C.I. Byrnes and A. Isidori. A frequency domain philosophy for nonlinear systems, with applications to stabilization and to adaptive control. In *IEEE 23rd Conference on Decision and Control*, pages 1569–1573, 1984.

[39] C.I. Byrnes and A. Isidori. Limit sets, zero dynamics, and internal models in the problem of nonlinear output regulation. *IEEE Transactions on Automatic Control*, 48(10):1712–1723, 2003.

[40] C.I. Byrnes and A. Isidori. Nonlinear internal models for output regulation. *IEEE Trans. on Automatic Control*, 49:2244–2247, 2004.

[41] C.I. ByrnesI, D.S. Gilliam, and V.I. Shubov. Boundary control for a viscous burgers' equation. *Identification and Control for Systems Governed by Partial Differential Equations, SIAM*, pages 171–185, 1993.

[42] G. Chen, M.C. Delfour, A.M. Krall, and G. Payre. Modeling, stabilization and control of serially connected beams. *SIAM Journal on Control and Optimization*, 25(3):526–546, 1987.

[43] Y. Cherruault and G. Adomian. Decomposition methods: a new proof of convergence. *Mathematical and Computer Modelling*, 18(12):103–106, 1993.

[44] H. Choi, R. Temam, P. Moin, and J. Kim. Feedback control for unsteady flow and its application to the stochastic Burgers equation. *Journal of Fluid Mechanics*, 253:509–543, 1993.

[45] P.D. Christofides and A. Armaou. Global stabilization of the kuramoto–sivashinsky equation via distributed output feedback control. *Systems & Control Letters*, 39(4):283–294, 2000.

[46] P.D. Christofides and J. Baker. Robust output feedback control of quasi-linear parabolic pde systemsfn1. *Systems & Control Letters*, 36(5):307–316, 1999.

[47] J.D. Cole. On a quasi-linear parabolic equation occurring in aerodynamics. *Quarterly of Applied Mathematics*, 9(3):225–236, 1951.

[48] J. Coron, B. d'Andrea Novel, and G. Bastia. A lyapunov approach to control irrigation canals modeled by saint-venant equations. In *Control Conference (ECC), 1999 European*, pages 3178–3183. IEEE, 1999.

[49] J. Coron, B. d'Andrea Novel, and G. Bastin. A strict lyapunov function for boundary control of hyperbolic systems of conservation laws. *IEEE Transactions on Automatic control*, 52(1):2–11, 2007.

[50] J. M. Coron. *Control and Nonlinearity*. American Mathematical Society, 2007.

[51] R.F. Curtain and H. Zwart. *An Introduction to Infinite-dimensional Linear Systems Theory*, volume 21. Springer Science & Business Media, 2012.

[52] M.C. Delfour and S.K. Mitter. Controllability and observability for infinite-dimensional systems. *SIAM Journal on Control*, 10(2):329–333, 1972.

[53] M.A. Demetriou. Activation policy of smart controllers for flexible structures with multiple actuator/sensor pairs. In *Proceedings of the 14th International Symposium on Mathematical Theory of Networks and Systems, Perpignan, France*, 2000.

[54] R.C. Dorf and R.H. Bishop. *Modern Control Systems*. Pearson Addison-Wesley, 1998.

[55] I.L. El-Kalla. Error analysis of adomian series solution to a class of nonlinear differential equations. *Applied Mathematics E-Notes*, 7:214–221, 2007.

[56] L.C. Evans. *Partial Differential Equations*. American Mathematical Society, 2010.

[57] R.A. Fisher. The wave of advance of advantageous genes. *Annals of eugenics*, 7(4):355–369, 1937.

[58] M. Fliess, J. Lévine, P. Martin, and P. Rouchon. Flatness and defect of nonlinear system introductory theory and examples. *Int. J. of Control*, 61:1327–1361, 1995.

[59] M. Fliess, P. Martin, N. Petit, and P. Rouchon. Active restoration of a signal by precompensation technique. In *the 38th IEEE Conf. on Decision and Control*, pages 1007–1011, Phoenix, AZ, Dec. 1999.

[60] G.F. Franklin, J.D. Powell, and A. Emami-Naeini. *Feedback Control of Dynamic Systems*, volume 3. Addison-Wesley Reading, MA, 1994.

[61] A.V. Fursikov. Stabilizability of two-dimensional navier—stokes equations with help of a boundary feedback control. *Journal of Mathematical Fluid Mechanics*, 3(3):259–301, 2001.

[62] G.Adomian. On the solution of partial differential equations with specified boundary conditions. *J. of Mathemaical Analysis and Applications*, 140(2):569–581, 1989.

[63] S.S. Ge and J. Zhang. Neural-network control of nonaffine nonlinear system with zero dynamics by state and output feedback. *IEEE Transactions on Neural Networks*, 14(4):900–918, 2003.

[64] G.G.Rigatos. *Nonlinear Control and Filtering Using Differential Flatness Approaches: Applications to Electromechanical Systems*, volume 25. Springer, 2015.

[65] H.Brezis. *Functional analysis, Sobolev Spaces and Partial Differential Equations*. Springer Science & Business Media, 2010.

[66] W. He, S. Ge, B.V.E. How, Y.S. Choo, and K.S. Hong. Robust adaptive boundary control of a flexible marine riser with vessel dynamics. *Automatica*, 47(4):722–732, 2011.

[67] W. He and C. Liu. Vibration control of atimoshenko beam system with input backlash. *IET Control Theory & Applications*, 9(12):1802–1809, 2015.

[68] W. He, C. Sun, and S.S. Ge. Top tension control of a flexible marine riser by using integral-barrier Lyapunov function. *Mechatronics, IEEE/ASME Transactions on*, 20(2):497–505, 2015.

[69] J. Henikl, W. Kemmetmüller, T. Meurer, and A. Kugi. Infinite-dimensional decentralized damping control of large-scale manipulators with hydraulic actuation. *Automatica*, 63:101–115, 2016.

[70] D. Henry. *Geometric Theory of Semilinear Parabolic Equations*, volume 840. Springer, 2006.

[71] M. Hinze and K. Kunisch. Second order methods for optimal control of time-dependent fluid flow. *SIAM Journal on Control and Optimization*, 40(3):925–946, 2001.

[72] E. Hopf. The partial differential equation $u_t + uu_x = \mu u_{xx}$. *Communications on Pure and Applied Mathematics*, 3(3):201–230, 1950.

[73] I.A.Raptis and K.P.Valavanis. *Linear and Nonlinear Control of Small-scale Unmanned Helicopters*, volume 45. Springer Science & Business Media, 2010.

[74] A. Isidori, A. Krener, C. Gori-Giorgi, and S. Monaco. Nonlinear decoupling via feedback: a differential geometric approach. *IEEE Transactions on Automatic Control*, 26(2):331–345, 1981.

[75] V. Jurdjevic, J. Velimir, and V. Đurđević. *Geometric Control Theory*, volume 52. Cambridge university press, 1997.

[76] H.K. Khalil. *Nonlinear Control.* Pearson New York, 2015.

[77] C. Kravaris. Input/output linearization: a nonlinear analog of placing poles at process zeros. *AIChE Journal*, 34(11):1803–1812, 1988.

[78] M. Krstic. On global stabilization of burgers' equation by boundary control. *Systems & Control Letters*, 37(3):123–141, 1999.

[79] M. Krstic, B.Z. Guo, A. Balogh, and A. Smyshlyaev. Output-feedback stabilization of an unstable wave equation. *Automatica*, 44(1):63–74, 2008.

[80] M. Krstic, L. Magnis, and R. Vazquez. Nonlinear control of the viscous burgers equation: Trajectory generation, tracking, and observer design. *Journal of Dynamic Systems, Measurement, and Control*, 131(2):021012, 2009.

[81] M. Krstic, A.A. Siranosian, and A. Smyshlyaev. Backstepping boundary controllers and observers for the slender timoshenko beam: Part i-design. In *IEEE American Control Conference*, pages 2412–2417, 2006.

[82] M. Krstic and A. Smyshlyaev. Backstepping boundary control for first-order hyperbolic pdes and application to systems with actuator and sensor delays. *Systems & Control Letters*, 57(9):750–758, 2008.

[83] M. Krstić and A. Smyshyaev. *Boundry Control of PDEs: A Course on Backstepping Designs.* SIAM, New York, 2008.

[84] S. Kundu and A.K. Pani. Finite element approximation to global stabilization of the burgers' equation by neumann boundary feedback control law. *Adv. Comput. Math.*, 44:541–570, 2018.

[85] O.A. Ladyzhenskaia, V.A. Solonnikov, and N.N. Ural'tseva. *Linear and Quasi-linear Equations of Parabolic Type*, volume 23. American Mathematical Soc., 1988.

[86] V. Lakshmikantham, S. Leela, and A.A. Martynyuk. *Stability analysis of nonlinear systems*. Springer, 1989.

[87] B. Laroche, P. Martin, and P. Rouchon. Motion planning for the heat equation. *Int. J. of Robust Nonlinear Control*, 10:629–643, 2000.

[88] C.T. Leondes. *Advances in Control Systems: Theory and Applications*, volume 5. Elsevier, 2014.

[89] J. Lévine. *Analysis and Control of Nonlinear Systems: A Flatness-based Approach*. Springer-Verlag, Berlin, 2009.

[90] F.B. Li, Q. Li, and Z.K. Li. Research on the transverse displacement sensors using linear array ccd. In *Advanced Materials Research*, volume 317, pages 988–991. Trans Tech Publ, 2011.

[91] W.J. Liu. Boundary feedback stabilization of an unstable heat equation. *SIAM Journal on Control and Optimization*, 42(3):1033–1043, 2003.

[92] W.J. Liu and M. Krstic. Backstepping boundary control of burgers' equation with actuator dynamics. *Systems & Control Letters*, 41(4):291–303, 2000.

[93] W.J. Liu and M. Krstić. Adaptive control of Burgers' equation with unknown viscosity. *International Journal of Adaptive Control and Signal Processing*, 15(7):745–766, 2001.

[94] A.F. Lynch and J. Rudolph. Flatness-based boundary control of a nonlinear parabolic equation modelling a tubular reactor. In *Nonlinear control in the year 2000 volume 2*, pages 45–54. Springer, 2001.

[95] A.F. Lynch and J. Rudolph. Flatness-based boundary control of a class of quasi-linear parabolic distributed parameter systems. *International Journal of Control*, 75(15):1219–1230, 2002.

[96] M.A.Högele. Metastability of the chafee-infante equation with small heavy-tailed lévy noise. 2011.

[97] P. Martin, L. Rosier, and P. Rouchon. Null controllability of the heat equation using flatness. *Automatica*, 50(12):3067–3076, 2014.

[98] J. Martinus, F.T. Nieuwstadt, and J. Steketee. *Selected Papers of J.M. Burgers*. Kluwer Academic Publishers, 1995.

[99] T. Meurer. Flatness-based trajectory planning for diffusion–reaction systems in a parallelepipedon—a spectral approach. *Automatica*, 47(5):935–949, 2011.

[100] T. Meurer. *Control of Higher-Dimensional PDEs: Flatness and Backstepping Designs*. Springer, Berlin, 2013.

[101] T. Meurer and M. Krstic. Finite-time multi-agent deployment: A nonlinear pde motion planning approach. *Automatica*, 47(11):2534–2542, 2011.

[102] T. Meurer and A. Kugi. Tracking control for boundary controlled parabolic pdes with varying parameters: Combining backstepping and differential flatness. *Automatica*, 45(5):1182–1194, 2009.

[103] T. Meurer and M. Zeitz. A novel design approach to flatness-based feedback boundary control of nonlinear reaction-diffusion systems with distributed parameters. In *New Trends in Nonlinear Dynamics and Control and their Applications*, pages 221–235. Springer, 2003.

[104] S. Monaco and D. Normand-Cyrot. Zero dynamics of sampled nonlinear systems. *Systems & Control Letters*, 11(3):229–234, 1988.

[105] K. Morris. Control of systems governed by partial differential equations. *The Control Theory Handbook*, 2010.

[106] J.D. Murray. *Mathematical Biology: I. An Introduction*. Springer, 2003.

[107] N. Ngarhasta, B. Some, K. Abbaoui, and Y. Cherruault. New numerical study of adomian method applied to a diffusion model. *Kybernetes*, 31(1):61–75, 2002.

[108] P. Orlandi. *Fluid Flow Phenomena: A Numerical Toolkit*, volume 55. Springer Science & Business Media, 2012.

[109] L. ougoffa and R.C. Rach. Solving nonlocal initial-boundary value problems for linear and nonlinear parabolic and hyperbolic partial differential equations by the adomian decomposition method. *Applied Mathematics and Computation*, 225:50–61, 2013.

[110] A.A. Paranjape, J.Guan, S.J. Chung, and M. Krstic. Pde boundary control for flexible articulated wings on a robotic aircraft. *IEEE Transactions on Robotics*, 29(3):625–640, 2013.

[111] N. Petit and P. Rouchon. Flatness of heavy chain systems. *SIAM J. Control and Optimization*, 40(2):475–495, 2001.

[112] P.Rouchon. Motion planning, equivalence, infinite dimensional systems. *Applied Mathematics and Computer Science*, 11(1):165–188, 2001.

[113] P.V. Ramana and B.K.R. Prasad. Modified adomian decomposition method for van der pol equations. *International Journal of Non-Linear Mechanics*, 65:121–132, 2014.

[114] R. Rebarber. Exponential stability of coupled beams with dissipative joints: a frequency domain approach. *SIAM Journal on Control and Optimization*, 33(1):1–28, 1995.

[115] G. Rigatos, P. Siano, P. Wira, and M. Sayed-Mouchaweh. Control of dc–dc converter and dc motor dynamics using differential flatness theory. *Intelligent Industrial Systems*, 2(4):371–380, 2016.

[116] L. Rodino. *Linear Partial Differential Operators in Gevrey Spaces*. World Scientific, River Edge, NJ, 1993.

[117] R. Rothfuss, J. Rudolph, and M. Zeitz. Flatness based control of a nonlinear chemical reactor model. *Automatica*, 32(10):1433–1439, 1996.

[118] J. Rudolph. *Flatness Based Control of Distributed Parameter Systems*. Shaker-Verlag, Aachen, 2003.

[119] J. Rudolph and F. Woittennek. Flachheitsbasierte randsteuerung von elastischen balken mit piezoaktuatoren (flatness based boundary control of piezoelectric benders). *at-Automatisierungstechnik Methoden und Anwendungen der Steuerungs-, Regelungs-und Informationstechnik*, 50(9/2002):412, 2002.

[120] J. Rudolph and F. Woittennek. Motion planning and open loop control design for linear distributed parameter systems with lumped controls. *International Journal of Control*, 81(3):457–474, 2008.

[121] B. Schörkhuber, T. Meurer, and A. Jüngel. Flatness of semilinear parabolic pdes—a generalized cauchy–kowalevski approach. *IEEE Transactions on Automatic Control*, 58(9):2277–2291, 2013.

[122] J. Schröck, T. Meurer, and A. Kugi. Motion planning for piezo-actuated flexible structures: Modeling, design, and experiment. *IEEE Trans. Contr. Syst. Technol.*, 21(3):807–819, 2012.

[123] S. Skogestad and I. Postlethwaite. *Multivariable Feedback Control: Analysis and Design*, volume 2. Wiley New York, 2007.

[124] A. Smyshlyaev and M. Krstic. Backstepping observers for a class of parabolic pdes. *Systems & Control Letters*, 54(7):613–625, 2005.

[125] A. Smyshlyaev and M. Krstic. Boundary control of an anti-stable wave equation with anti-damping on the uncontrolled boundary. *Systems & Control Letters*, 58(8):617–623, 2009.

[126] E.D. Sontag. *Mathematical Control Theory: Deterministic Finite Dimensional Systems*, volume 6. Springer Science & Business Media, 2013.

[127] J.B. Stewart, A. Diouf, Y. Zhou, and T.G. Bifano. Open-loop control of a MEMS deformable mirror for large-amplitude wavefront control. *JOSA A*, 24(12):3827–3833, 2007.

[128] S.E. Street. Analysis and control of nonlinear infinite dimensional systems. *IEEE Transactions on Automatic Control*, 40(4):787, 1995.

[129] S. Tang and R.O. Weber. Numerical study of fisher's equation by a petrov-galerkin finite element method. *The ANZIAM Journal*, 33(1):27–38, 1991.

[130] R. Triggiani. Boundary feedback stabilizability of parabolic equations. *Applied Mathematics and Optimization*, 6(1):201–220, 1980.

[131] M. Tucsnak and G. Weiss. *Observation and Control for Operator Semigroups*. Springer Science & Business Media, 2009.

[132] R. K. Tyson. *Principles of Adaptive Optics*. CRC Press, Boca Raton, FL, 3rd edition, 2011.

[133] C.R. Vogel and Q. Yang. Modeling, simulation, and open-loop control of a continuous facesheet MEMS deformable mirror. *JOSA A*, 23(5):1074–1081, 2006.

[134] A.M. Wazwaz and S.M. El-Sayed. A new modification of the adomian decomposition method for linear and nonlinear operators. *Applied Mathematics and Computation*, 122(3):393–405, 2001.

[135] G.B. Whitham. *Linear and Nonlinear Waves.* Modern Book Incorporated, 1975.

[136] F. Woittennek. On flatness and controllability of simple hyperbolic distributed parameter systems. In *Proc. 18th IFAC World Congress*, pages 14452–14457, 2011.

[137] F. Woittennek, S. Wang, and T. Knüppel. Backstepping design for parabolic systems with in-domain actuation and Robin boundary conditions. In *Proc. of the 19th IFAC World Congress*, pages 5175–5180, Cape Town, South Africa, 2014.

[138] B. Yang. *Stress, Strain, and Structural Dynamics: An Interactive Handbook of Formulas, Solutions, and MATLAB Toolboxes.* Elsevier Academic Press, Burlington, 2005.

[139] K. Yang, J. Zheng, and G. Zhu. Deformation control of an Euler-Bernoulli beam based on zero-dynamics inverse design and flatness. 2016 IEEE 55th Conference on Decision and Control (CDC), pages 4791–4796, Las Vegas, NV, USA, 12-14 Dec. 2016.

[140] K. Yang, J. Zheng, and G. Zhu. Asymptotic output tracking for a class of semi-linear parabolic equations: A semi-analytical approach. *Int. J. of Robust and Nonlinear Control*, 2019. In press.

[141] K. Yang and G. Zhu. A dynamic compensator for in-domain control of burgers' equation. *International Journal of Control*, 2019.

[142] K. Yosida. *Functional Analysis.* Springer-Verlag, Berlin Heidelberg New York, 1980.

[143] X. Yu, C. Xu, and J. Chu. Local exponential stabilization of fisher's equation using the backstepping technique. *Systems & Control Letters*, 74:1–7, 2014.

[144] Z. Zhang, Y. Wu, and J. Huang. Differential-flatness-based finite-time anti-swing control of underactuated crane systems. *Nonlinear Dynamics*, 87(3):1749–1761, 2017.

[145] J. Zheng and G. Zhu. In-domain control of a heat equation: an approach combining zero-dynamics inverse and differential flatness. *Mathematical Problems in Engineering*, 2015, 2015.