Titre: Title:	Numerical implementation of an elasto-plastic model based on the MSDPu multiaxial criterion
Auteurs: Authors:	Ali Shirazi, Li Li et Michel Aubertin
Date:	2006
Type:	Rapport / Report
Référence: Citation:	Shirazi, A., Li, L. & Aubertin, M. (2006). *Numerical implementation of an elasto-plastic model based on the MSDPu multiaxial criterion* (Rapport technique n° EPM–RT–2006-08).

EPM–RT–2006-08

NUMERICAL IMPLEMENTATION OF AN ELASTO-PLASTIC MODEL BASED ON THE MSDP$_u$ MULTIAXIAL CRITERION

Ali Shirazi, Li LI, Michel Aubertin
Département des Génies civil, géologique et des mines
École Polytechnique de Montréal

Décembre 2006

Poly

ÉCOLE
POLYTECHNIQUE
MONTRÉAL

EPM-RT-2006-08

NUMERICAL IMPLEMENTATION OF AN ELASTO-PLASTIC MODEL BASED ON THE MSDP$_u$ MULTIAXIAL CRITERION

Ali SHIRAZI[1], Li LI[1], Michel AUBERTIN[1,2]

[1]Department of Civil, Geological and Mining Engineering, École Polytechnique de Montréal, C.P. 6079, Succursale Centre-ville, Québec, H3C 3A7, Canada

[2]Industrial NSERC Polytechnique-UQAT Chair on Environment and Mine Wastes Management (http://www.polymtl.ca/enviro-geremi/).

DECEMBER 2006

EPM-RT-2006-08
Numerical implementation of an elasto-plastic model based on the $MSDP_u$ multiaxial criterion
par : Ali SHIRAZI[1], Li Li[1], Michel Aubertin[1,2]

[1]Department of Civil, Geological and Mining Engineering, École Polytechnique de Montréal
[2]Industrial NSERC Polytechnique-UQAT Chair on Environment and Mine Wastes Management

École Polytechnique de Montréal
Bibliothèque – Service de fourniture de documents
Case postale 6079, Succursale «Centre-Ville»
Montréal (Québec)
Canada H3C 3A7

Téléphone :	(514) 340-4846
Télécopie :	(514) 340-4026
Courrier électronique :	biblio.sfd@courriel.polymtl.ca

Ce rapport technique peut-être repéré par auteur et par titre dans le catalogue de la Bibliothèque :
http://www.polymtl.ca/biblio/catalogue/

ABSTRACT

The MSDP$_u$ criterion has been developed to represent transitional states associated with particular types of mechanical response such as yielding, failure, and residual strength. This multiaxial criterion can be applied to a wide variety of geomaterials and loading conditions. To date, its use has been limited to applications relying on relatively simple analytical solutions. In this report, the authors present the method that has been used to introduce the criterion into a well-known, commercially available finite difference code using an elasto-plastic framework. The report starts with a brief review of the MSDP$_u$ formulation, followed by an additional development which includes a hardening component into the original equations. Then, the authors describe the approach that has been used to introduce its main components into the code. Illustrative modelling results obtained with this new elasto-plastic model are shown, and compared to representative laboratory test and analytical results on a circular underground opening. Finally, the main advantages, capabilities, and limitations of the model are briefly discussed, together with ongoing work aimed at analyzing the behavior of underground openings in rock mass.

Key words: Numerical modelling; Elasto-plasticity; Hardening; Thick wall cylinder; Finite difference.

RÉSUMÉ

Le critère MSDP$_u$ a été développé pour représenter les états de transition associés avec certains types de réponse mécanique tels que l'écoulement inélastique, la rupture, et résistance résiduelle. Ce critère multiaxial peut être appliqué à une grande variété de géomatériaux et de conditions de chargements. Jusqu'à présent, son utilisation a été limitée aux applications basées sur des solutions analytiques relativement simples. Dans ce rapport, on montre la méthode utilisée pour introduire ce critère dans un code commercial bien connu, en utilisant un cadre élasto-plastique. Le rapport commence avec une brève revue de la formulation de MSDP$_u$, suivie d'un développement additionnel pour inclure une composante d'écrouissage dans les équations originales. Par la suite, on décrit l'approche utilisée pour implanter les composantes principales dans le code. Des résultats de modélisation obtenus avec ce nouveau modèle élasto-plastique sont montrés et comparés avec des résultats d'essais de laboratoire représentatifs et des résultats analytiques développés pour une ouverture cylindrique. Finalement, les avantages, les capacités, et les limitations du modèle sont brièvement discutés. On résume enfin certains autres travaux en cours destinés à analyser le comportement des ouvertures souterraines dans des massifs rocheux.

Mots clés: Modélisation numérique, Élasto-plasticité, Écrouissage, Cylindre à paroi épaisse, Différence finie.

Shirazi, Li, and Aubertin

TABLE OF CONTENTS

Shirazi, Li, and Aubertin

LIST OF FIGURES

Shirazi, Li, and Aubertin

1. INTRODUCTION

A wide variety of constitutive models exist to describe the mechanical behaviour of geomaterials. Some of these, such as linear elastic models, can be quite simple but they usually cannot capture many of the important responses under realistic loading conditions. The framework of elasto-plastic modelling is more representative in many cases, and it can be used to simulate various facets of the observed behaviour of geological media (such as soils, rocks, tailings, and backfill). Elasto-plastic theory, often used in geotechnique, is generally considered to be the most convenient framework to formulate constitutive models for the practical simulation of geomaterials behaviour (e.g., Potts and Zdravkovic 1999).

The elasto-plastic approach typically involves the concepts of yield function and plastic potential (and flow rule), with or without hardening (softening) rules. Numerous models have been developed in this context, which have been reviewed in monographs and text books including Desai and Siriwardane (1984), Chen and Baladi (1985), Lade (1997, 2005a, 2005b) and Potts and Zdravkovic (1999). Since viscous (rheological) effects related to the porous medium behaviour are not considered, there is no time dependency, so the plastic (stress-strain) response occurs instantaneously. The equations are nevertheless expressed in incremental form so the loading chronology can be taken into account (e.g., Coussy 2000).

Elasto-plastic models have been applied to study a wide range of problems related to stress analysis, ground control, and stability of underground openings. Recent examples include the work of Lee and Rowe (1989, 1991), Anagnostou and Kovari (1993), Eberhardt (2001), Meguid et al. (2003), Callari (2004), Ng and Lee (2005) and Meguid and Rowe (2006). The inelastic criterion used in these and other similar studies are typically based on conventional expressions such as the Mohr-Coloumb and Drucker-Prager yield surfaces, which are available as built-in constitutive models in many commercial codes. However, such commonly used constitutive models are known to have significant limitations as they typically cannot account for many aspects of the actual response of geomaterials subjected to different stress states (e.g., Aubertin et al. 2000). As a result, the numerical calculations may not properly address the multiaxial response of geomaterials, even when the model domain is three-dimensional. Furthermore, these inelastic surfaces, which remain open along the positive mean stress axis cannot account for the inelastic response of geomaterials (i.e. with volumetric straining) under relatively high

Shirazi, Li, and Aubertin

mean pressures. For this purpose, the model should include a "cap" component associated with the initiation and evolution of plastic volumetric strains (e.g., Lade 1997, 2005a, 2005b). Also, many conventional built-in models consider the response of geomaterials as elastic-perfectly plastic (no hardening or softening), which is not realistic in most cases. Some hardening laws may be available in numerical models (e.g., CamClay type of formulation), but these can be difficult to use as the required parameters for the model may not be readily available. Furthermore, hardening models implemented in many codes are essentially based on formulations applicable to clayey soils, which can make them less relevant for other types of geomaterials (such as rock and backfill).

The $MSDP_u$ inelastic criterion (Aubertin et al. 1999a, 2000; Aubertin and Li 2004) can provide a practical and user-friendly framework to account for the inelastic response of different types of geomaterials. A comparative assessment of existing inelastic loci, including the multiaxial $MSDP_u$ criterion, has been presented recently by Li et al. (2005b). Previous publications on $MSDP_u$ have shown that this 3D inelastic criterion is applicable to different materials and stress states. Its application to date, however, has been limited to simple problems, based on idealized analytical solutions that use a perfectly plastic response (e.g., Aubertin et al. 1999a; Aubertin and Li 2004). This seriously limits the practical use of the $MSDP_u$ criterion to model the inelastic response of geomaterials, excluding problems with complex geometries and loading conditions.

This report presents the approach that has been adopted for the numerical implementation of the multiaxial $MSDP_u$ function into a numerical code. The model formulation is modified here to include a hardening component to better reflect the observed behaviour of the materials of interest. The ensuing elasto-plastic model, in which $MSDP_u$ serves as a yield surface and plastic potential, is used for stress analysis of engineering works. The proposed computational scheme is described in detail, and incorporated into the numerical code FLAC 4.0 (Fast Lagrangian Analysis of Continua) developed by Itasca (2002). This code was selected because of its relatively wide use in geomechanics, and because it includes a subroutine (and language) that allows the implementation of new or modified constitutive equations. Through the given explanations, the authors also point out the simplifications that have been adopted to implement the new elasto-plastic model at this stage. To illustrate the model capabilities, and to partly validate its application, the computational scheme is employed to

Shirazi, Li, and Aubertin

model a typical experimental test result obtained on sandstone. The numerical solution is then applied to the case of a circular opening, and the results are compared to a recently proposed analytical solution. The effect of including hardening in the constitutive equations is also illustrated. A brief discussion follows.

2. FORMULATION OF THE MSDP$_u$ CRITERION

The yield function used in the proposed constitutive model follows the multiaxial MSDP$_u$ criterion, which is expressed in terms of commonly used stress invariants (see Aubertin et al. 2000):

[1]
$$F = \sqrt{J_2} - \left\{ \alpha^2 \left(I_1^2 - 2a_1 I_1 \right) + a_2^2 - a_3 \left\langle I_1 - I_c \right\rangle^2 \right\}^{1/2} F_\pi = 0$$

where $I_1 = \mathrm{tr}(\sigma_{ij})$ represents the first invariant of σ_{ij}; $J_2 = (S_{ij} S_{ji})/2$ is the second invariant of the deviatoric stress tensor S_{ij} [$= \sigma_{ij} - (I_1/3)\,\delta_{ij}$; $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$]. Their explicit form can be written as:

[2]
$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

[3]
$$J_2 = \frac{1}{6}\left[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{11} - \sigma_{33})^2\right] + \sigma_{12}^{\,2} + \sigma_{13}^{\,2} + \sigma_{23}^{\,2}$$

The MacCauley brackets are defined as $\langle x \rangle = (x + |x|)/2$ (x is a variable). The criterion parameters α, a_1, a_2, a_3 and I_c are obtained from basic material properties. Parameter α (adapted from the well-known Drucker-Prager criterion) is related to the friction angle ϕ:

[4]
$$\alpha = \frac{2\sin\phi}{\sqrt{3}(3 - \sin\phi)}$$

Parameters a_1, a_2 are defined as follows:

[5]
$$a_1 = \left(\frac{\sigma_c - \sigma_t}{2} \right) - \left(\frac{\sigma_c^2 - (\sigma_t/b)^2}{6\alpha^2(\sigma_c + \sigma_t)} \right)$$

[6]
$$a_2 = \left\{ \left(\frac{\sigma_c + (\sigma_t/b^2)}{3(\sigma_c + \sigma_t)} - \alpha^2 \right) \sigma_c \sigma_t \right\}^{1/2}$$

where σ_t and σ_c are uniaxial strengths in tension (negative) and in compression respectively; b is a shape parameter. On the other hand, parameters a_3 and I_c are related to the material behaviour under relatively high

3 Shirazi, Li, and Aubertin

hydrostatic compression. When the locus closes down toward the mean stress axis, I_c represents the I_1 value where the locus departs from the "low porosity" condition (see Fig. 1). Coefficient a_3 is linked to I_{1n} (also shown in Fig. 1) which corresponds to the intersection of the inelastic locus with the positive I_1 axis. The relationship between a_3, I_{1n} and I_c is expressed as follows:

[7]
$$a_3 = \frac{\alpha^2\left(I_{1n}^2 - 2a_1 I_{1n}\right) + a_2^2}{\left(I_{1n} - I_c\right)^2}$$

The value of I_c and I_{1n} may be obtained experimentally. For dense materials such as hard rocks, the value of I_c or I_{1n} can become very large, so the cap portion of the locus can be neglected (see Fig. 1a).

In eq. [1], function F_π defines the shape of the surface in the π plane (shown in Fig. 1b). It can be expressed as:

[8]
$$F_\pi = \left(\frac{b}{\left[b^2 + (1-b^2)\sin^2(45° - 1.5\theta)\right]^{1/2}}\right)^\nu$$

In this equation, θ is the Lode angle used to define the stress state in the octahedral (π) plane:

[9]
$$\theta = \frac{1}{3}\sin^{-1}\frac{3\sqrt{3}\,J_3}{2\sqrt{J_2^3}}, \quad (-30° \leq \theta \leq 30°)$$

Here, $J_3 = (S_{ij} S_{jk} S_{ki})/3$ is the third invariant of the deviatoric stress tensor S_{ij}. The exponent ν is a mean stress dependent parameter; $\nu=1$ is used here. Parameter b (≤ 1) reflects the ratio of the locus size at $\theta = -30°$ (i.e. reduced triaxial extension) in the π plane when compared to that at $\theta = 30°$. With eq. [8], the value of b, which is equal to $(J_2^{1/2})_{\theta=-30°} \big/ (J_2^{1/2})_{\theta=30°}$ (for $\nu = 1$), can range from 1 (circular shape) to about 0.7 (rounded triangle) (see Fig. 1b). The F_π function can be formulated in a different manner to accommodate cases where $b < 0.7$ (see Li et al. 2005b), but this form is not presented here. The MSDP$_u$ parameters can also be made porosity dependent (Li et al. 2005b); this aspect is not considered in this presentation. More details on the formulation of the MSDP$_u$ criterion and some basic applications can be found in the above mentioned papers from the authors' group.

Shirazi, Li, and Aubertin
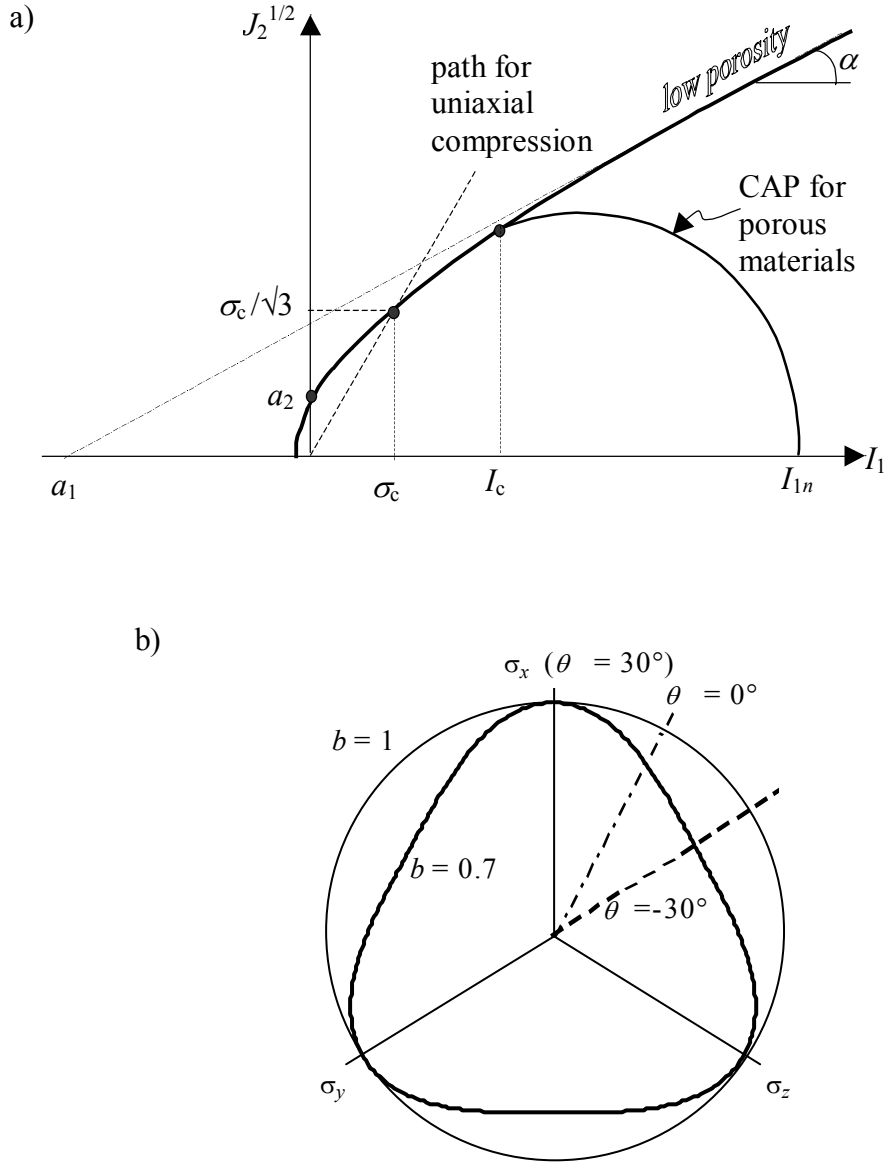
Fig. 1 Schematical representation of the MSDP$_u$ criterion for dense and porous materials under CTC conditions ($\theta = 30°$) (a); view in the π plane (b) (adapted from Aubertin and Li 2004 and Li et al. 2005b).

3. ELASTO-PLASTIC CONSTITUTIVE EQUATIONS

3.1 Core formulation

The computational scheme developed here for the numerical implementation of MSDP$_u$ uses an elasto-plastic (EP) constitutive model based on the multiaxial criterion. In order to simplify the equations, following equations will be used:

Shirazi, Li, and Aubertin

[10]
$$I = I_1$$

and

[11]
$$J = \sqrt{J_2} \; .$$

The shear strain increment $\Delta\gamma$ and volumetric strain increment $\Delta\varepsilon$ associated with J and I can be expressed by

[12] $$\Delta\gamma = 2\left\{\frac{1}{6}\left[\left(\Delta\varepsilon_{11} - \Delta\varepsilon_{22}\right)^2 + \left(\Delta\varepsilon_{11} - \Delta\varepsilon_{33}\right)^2 + \left(\Delta\varepsilon_{22} - \Delta\varepsilon_{33}\right)^2\right] + \Delta\varepsilon_{12}^{\;2} + \Delta\varepsilon_{13}^{\;2} + \Delta\varepsilon_{23}^{\;2}\right\}^{1/2}$$

and

[13]
$$\Delta\varepsilon = \Delta\varepsilon_{11} + \Delta\varepsilon_{22} + \Delta\varepsilon_{33}$$

where $\Delta\varepsilon_{ij}$ is the strain increments ($i = 1$ to $3, j = 1$ to 3).

A decomposition of the strain increments assuming small-strain plasticity can be obtained in the following simple incremental form :

[14]
$$\Delta\gamma = \Delta\gamma^{e} + \Delta\gamma^{p}$$

[15]
$$\Delta\varepsilon = \Delta\varepsilon^{e} + \Delta\varepsilon^{p}$$

where the superscripts "e" and "p" refer to the elastic and plastic components of the strain increments. The plastic strain components are non-zero only for the stress states that result in plastic deformations. The assumption of small-strain plasticity neglects higher orders or multiplication terms of the deviatoric components of strain increments. The simple form of decomposition used in eqs. [14] and [15] is convenient and considered acceptable for the initial stage of the numerical implementation of the inelastic locus. The validity of this assumption is partly confirmed later through the numerical results presented below.

The incremental formulation of the Hookean elasticity in terms of the strain increments ($\Delta\gamma^{e}, \Delta\varepsilon^{e}$) takes the form of:

[16]
$$\Delta J = G\Delta\gamma^{e}$$

[17]
$$\Delta I = K\Delta\varepsilon^{e}$$

Shirazi, Li, and Aubertin

The procedure to evaluate the stress state from an elasto-plastic constitutive model includes an elastic predictor, with a plastic corrector to maintain the admissible stress states to be located on the yield surface. A review of the various aspects of this method to obtain the admissible stress states for an elasto-plastic constitutive model was given by Ortiz and Popov (1985). The authors apply here a similar procedure to determine the stress using the $MSDP_u$ criterion. Other more elaborate approaches could also be used for the numerical treatment (e.g., Simo and Hughes 1997; Simo 1998), but these are not deemed necessary at this stage of the model development.

The plastic components of the strain increments (plastic correctors) are determined as follows:

[18]
$$\Delta\gamma^p = \lambda \frac{\partial g}{\partial J}$$

[19]
$$\Delta\varepsilon^p = \lambda \frac{\partial g}{\partial I}$$

where g is the plastic strain potential function, and λ a plastic multiplier. In this first example of implementing the model, the authors are assuming the normality of the strain increment vector to the yield surface; such an associated flow rule (with g equal to the yield function F) admittedly induces some restrictions, particularly when applied to some cohesionless and/or highly frictional materials, but this is considered an acceptable and convenient simplification for this initial application. As a result, the plastic strain increments take the form:

[20]
$$\Delta\gamma^p = \lambda$$

[21]
$$\Delta\varepsilon^p = \lambda(-F_\pi)\left(\frac{\alpha^2(I-a_1) - a_3\langle I - I_c \rangle}{\sqrt{\alpha^2(I^2 - 2a_1 I) + a_2^2}} \right)$$

with $\langle I-I_c \rangle = 0$ for $I \le I_c$.

In eqs. [20] and [21], it is assumed that the Lode angle (θ), included in the function F_π, remains constant during plastic straining. This results in a very useful simplification in the plasticity formulations for the $MSDP_u$ yield criterion. The study conducted by Li et al. (2005b) indicates that this assumption is realistic as the change in θ within the medium is normally less than 10%. Therefore, the use of a constant Lode angle (θ) can be

Shirazi, Li, and Aubertin

considered acceptable with only marginal effects on the determination of the plastic strain increments. It is also appropriate for comparing the numerical results with the analytical solution developed for a circular opening (see below).

From eqs. [14] to [17], the incremental formulation for the elastic response takes the form:

[22]
$$\Delta J = G\Delta\gamma - G\Delta\gamma^{\text{p}}$$

[23]
$$\Delta I = K\Delta\varepsilon - K\Delta\gamma^{\text{p}}$$

Introducing eqs. [20] and [21] into eqs. [22] and [23], one obtains:

[24]
$$\Delta J = G\left(\Delta\gamma - \lambda\right)$$

[25]
$$\Delta I = K\left\{\Delta\varepsilon + \lambda F_\pi\left[\frac{\alpha^2\left(I - a_1\right) - a_3\left\langle I - I_c\right\rangle}{\sqrt{\alpha^2\left(I^2 - 2a_1 I\right) + a_2{}^2}}\right]\right\}$$

Noting $(I^{\text{N}}, J^{\text{N}})$ and $(I^{\text{O}}, J^{\text{O}})$ as the new and old (previous) stress states, before and after strain increment respectively, one can then write:

[26]
$$J^{\text{N}} = J^{\text{O}} + \Delta J$$

[27]
$$I^{\text{N}} = I^{\text{O}} + \Delta I$$

Substituting eqs. [24] and [25] into eqs. [26] and [27], the following relationships are obtained:

[28]
$$J^{\text{N}} = J^{\text{I}} - G\lambda$$

[29]
$$I^{\text{N}} = I^{\text{I}} + H\lambda$$

where H is an intermediate variable expressed as:

[30]
$$H = KF_\pi\left[\frac{\alpha^2\left(I^{\text{I}} - a_1\right) - a_3\left\langle I^{\text{I}} - I_c\right\rangle}{\sqrt{\alpha^2\left(I^{\text{I}2} - 2a_1 I^{\text{I}}\right) + a_2{}^2}}\right]$$

In these equations, I^{I} and J^{I} correspond to the elastic estimates obtained by adding the elastic increments as follows,

[31]
$$J^{\text{I}} = J^{\text{O}} + G\Delta\gamma$$

Shirazi, Li, and Aubertin

[32]
$$I^{\mathrm{I}} = I^{\mathrm{O}} + K\Delta\varepsilon$$

The scalar factor of proportionality λ used as the plastic corrector must be defined so the new stress state remains on the yield surface. This means that:

[33]
$$F\left(I^{\mathrm{N}}, J^{\mathrm{N}}\right) = 0$$

where F is defined by the MSDP$_{\mathrm{u}}$ criterion.

Substituting eqs. [28] and [29] into eq. [33] one obtains the following equation:

[34]
$$A\lambda^2 + B\lambda + C = 0$$

where

[35]
$$A = G^2 - \left(\alpha^2 - a_3 \frac{\langle I^{\mathrm{I}} - I_{\mathrm{c}}\rangle}{I^{\mathrm{I}} - I_{\mathrm{c}}}\right) H^2 F_\pi{}^2$$

[36]
$$B = -2\left\{GJ^{\mathrm{I}} + \left[\alpha^2(I^{\mathrm{I}} - a_1) - a_3\langle I^{\mathrm{I}} - I_{\mathrm{c}}\rangle\right]HF_\pi{}^2\right\}$$

[37]
$$C = J^2 + \left\{-\alpha^2\left[(I^{\mathrm{I}})^2 - 2a_1 I^{\mathrm{I}}\right] - a_2{}^2 + a_3\langle I^{\mathrm{I}} - I_{\mathrm{c}}\rangle^2\right\}F_\pi^2$$

The scalar factor of proportionality λ is the root of eq. [34] (i.e. smallest absolute value); it equals zero if its value is negative.

Once the scalar factor λ is obtained, the new stress state ($I^{\mathrm{N}}, J^{\mathrm{N}}$) can be determined using eqs. [28] to [32]. It should be noted again that the proposed procedure maintains the new stress states ($I^{\mathrm{N}}, J^{\mathrm{N}}$) on the yield surface defined by the MSDP$_{\mathrm{u}}$ criterion.

The new deviatoric stress components can be directly obtained by multiplying the corresponding deviatoric elastic estimate with the ratio $J^{\mathrm{N}}/J^{\mathrm{I}}$. Therefore, the new stress components can be expressed as:

[38]
$$\sigma_{ij}{}^{\mathrm{N}} = \left(\sigma_{ij}{}^{\mathrm{I}} - I^{\mathrm{I}}\delta_{ij}\right)\frac{J^{\mathrm{N}}}{J^{\mathrm{I}}} + I^{\mathrm{N}}\delta_{ij}$$

where δ_{ij} is the Kronecker delta function.

3.2. Hardening behaviour

Shirazi, Li, and Aubertin

The numerical formulation presented above is based on a perfectly plastic response upon yielding. However, porous geomaterials often show a hardening behaviour when loaded beyond their elastic domain (e.g., Prevost 1978; Casey and Naghdi 1981; Grgic et al. 2003). As part of the implementation of the MSDP$_u$ EP-model into a computational scheme, the authors have introduced a simple hardening component. For this purpose, the MSDP$_u$ criterion (eq. [1]) was modified as follows:

[39]
$$F = \sqrt{J_2} - \left\{\alpha^2\left(I_1^2 - 2a_1 I_1\right) + a_2^2 - a_3\left\langle I_1 - I_c\right\rangle^2\right\}^{1/2} F_\pi F_h = 0$$

The added function F_h depends on the plastic strain. It is is expressed as (Desai and Siriwardane 1984; Grgic et al. 2003):

[40]
$$F_h = 1 + \omega\sqrt{|\varepsilon_p|}$$

where $|\varepsilon_p|$ is the norm of the accumulated volumetric plastic strain ($\varepsilon_p = \int d\varepsilon_p$; only the hardening behaviour of contractive materials is considered in this first series of application) and ω is a material parameter that controls the hardening rate. Figure 2 illustrates the effect of the isotropic hardening coefficient ω on the stress-strain curve.
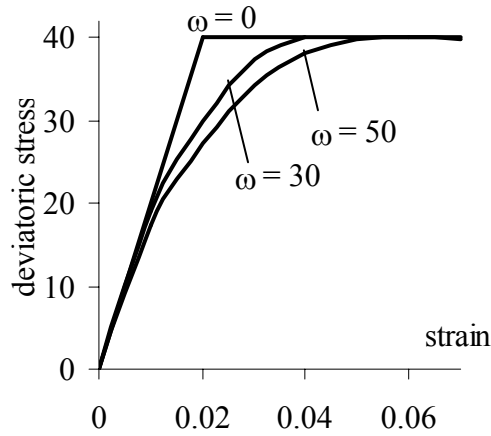

Fig. 2 Effect of ω on the stress-strain curve.

Shirazi, Li, and Aubertin

With this addition, the constitutive formulation and numerical implementation remain exactly the same as for the elasto-perfectly plasticity model described previously, with function F (eq. [1]) replaced by eq. [39]. Thus, the mathematical formulation and numerical implementation procedures are not repeated here.

4. NUMERICAL IMPLEMENTATION

4.1 Selected code

FLAC (Fast Lagrangian Analysis of Continua; Itasca 2002) is a commercial code widely used in geomechanics and geotechnical engineering. Examples of its use are presented in a number of publications, including Detournay and Hart (1999), Billaux et al. (2001) and Brummer et al. (2003). This code is based on an explicit, finite difference method. Its main advantages and disadvantages (compared to the finite element method, for example) are given in the FLAC 4.0 Manual. A review of the application of the numerical technique used in FLAC to plasticity was given by Marti and Cundall (1982) (see also Chen et al. 1999; Chatti et al. 2001; Purwodihardjo and Cambou 2005).

FLAC calculations are based on the dynamic equations of motion. The code can also obtain static solutions by incorporating the appropriate damping variables into the dynamic equations. As a result, time steps must be included as input for the numerical explicit scheme (even though the problem is considered static). As a criterion, the speed of the calculation front should be greater than the maximum speed at which information propagates, to maintain the stability of the explicit scheme existing in FLAC 4.0 (see Cundall 1976 and the user's manual for FLAC 4.0 by Itasca 2002). A time step must therefore be chosen which is smaller than a critical time step related to the stability of the numerical explicit scheme. This can be imposed in FLAC either automatically by the code or by the user.

One of the main advantages of the explicit scheme used in FLAC is that it does not require the convergence of the all equations at each calculation cycle. No iteration process is required, and the numerical scheme converges to the solution explicitly; convergence can be monitored during the calculation cycles through a parameter called the "unbalanced force". A detailed description of the unbalanced force can be found in the user's manual by Itasca (2002).

Shirazi, Li, and Aubertin

On the other hand, the explicit time scheme existing in the code can be less efficient than the implicit schemes often used with the finite element techniques (e.g., Bardet and Choucair 1991; Crisfield 1991; Aubertin 1993; Simo 1998). This may result in a longer time to converge, particularly for problems with different zones of soft and stiff materials or with mixed boundary conditions. With FLAC, smaller time steps may be required and consequently the calculation time may be increased; however, most simulations can be conducted fairly rapidly with modern computers.

4.2 Equations programming

The previously described elasto-plastic constitutive model (herein called the $MSDP_u$ EP-model), has been incorporated using the programming language "FISH" available in FLAC 4.0. The computational scheme developed in this report to incorporate the elasto-plastic model into the numerical code for stress analysis is summarized as follows:

I.   Obtain I^I, J^I (elastic estimates) for the nodal points of each element using I^O, J^O from the previous time step:

$$I^I = I^O + K\Delta\varepsilon$$

$$J^I = J^O + G\Delta\gamma$$

II.   Define the yield equation with $MSDP_u$:

$$F = J - F_0 F_\pi$$

If $F < 0$ (no inelastic response), $J^N = J^I$; $I^N = I^I$, go to (VI); else (inelastic response), go to (III).

III.   Calculate λ (the scalar proportionality factor for plastic strain corrector) from eq. [34].

IV.   Calculate plastic strain:

Shirazi, Li, and Aubertin

$$\Delta\gamma^P = \lambda\frac{\partial F}{\partial J}$$

$$\Delta\varepsilon^p = \lambda\frac{\partial F}{\partial I}$$

V.   Correct the elastic stress ($I^{\mathrm{I}}, J^{\mathrm{I}}$) to obtain the new stress states ($I^{\mathrm{N}}, J^{\mathrm{N}}$) with eqs. [28] to [30] located on the yield surface.

VI. Go to the next time step.

The model implemented into FLAC is currently being used to solve a variety of practical geotechnical problems. Simple examples are shown below.

5. PRELIMINARY VALIDATION OF THE MSDP_u EP-MODEL

Conventional triaxial compression tests are often performed on cylindrical samples to obtain constitutive model parameters. The results of conventional triaxial compression can be used to validate the computational scheme and the hardening component, incorporated in the MSDPu inelastic criterion. For this purpose, a triaxial test reported by Elliott and Brown (1985) on Berea sandstone is simulated.

The first calculation is made with the assumption of an elastic-perfectly plastic response of a geomaterial (without hardening). The numerical simulation is applied to simulate loading and unloading, in order to predict the complete stress-strain characteristics of Berea sandstone.

The discretization of the domain and the prescribed boundary conditions are shown in Fig. 3. The material properties have been estimated based on the experimental results published by Elliott and Brown (1985); these are:

$G = 1200\,\mathrm{MPa}$, $K = 1600\,\mathrm{MPa}$　　　　　　　(elasticity parameters)

$\sigma_{\mathrm{c}} = 17.5\,\mathrm{MPa}$, $\sigma_{\mathrm{t}} = 1\,\mathrm{MPa}$, $I_{\mathrm{c}} = 1\,\mathrm{MPa}$,
$a_3 = 0.75, b = 0.75, \phi = 30°$　　　(MSDP_u parameters)
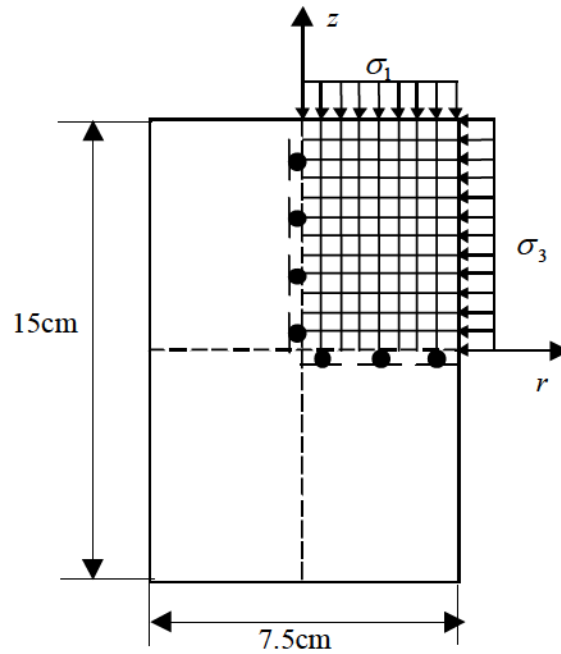
Shirazi, Li, and Aubertin

Fig. 3 The boundary conditions and discretization of the triaxial test (axisymmetric model).

Figure 4 illustrates the comparison between the numerical results and the experimental curve obtained from a triaxial test conducted by Elliott and Brown (1985). The numerical results show a fairly good agreement with the maximum stress and, to a lesser extent, with the strain associated with the inelastic response. As the computational model only accounts for an elastic- perfectly plastic response, the numerical results show some deviations from the experimental results. This is in part due to the fact that such geomaterials can exhibit hardening (and/or softening) resulting from inelastic strain, which influences the modulus, particularly under unloading and re-loading (i.e. in the range of stress states located inside the inelastic criterion,). This illustration nevertheless shows some of the basic capabilities of the model.

The hardening component is then incorporated into the computational scheme to improve the agreement between the numerical simulation and the experimental results. Figure 5 illustrates the numerical simulation of the conventional triaxial compression test conducted on sandstone by Elliott and Brown (1985). The geometry and parameters given with the case of Fig. 3 are also retained. The hardening parameter < is taken as 50 (based

Shirazi, Li, and Aubertin

on the available experimental results). Figure 5 shows that the agreement between the numerical modelling and experimental results is significantly improved, compared to Fig. 4, especially near the onset of inelasticity. The simulated behaviour shown in this case is in good agreement with the physical behaviour of various porous geomaterials, which usually involves a hardening phase when subjected to a CTC test (see Brinkgreve 2005; Carter and Liu 2005; Desai 2005; Lade 2005a, b; Nova, 2005). Nonetheless, the model does not aim at capturing all aspects of the mechanical response (neglecting phenomena such as progressive decreasing of the unloading modulus due to damage and softening, and associated hysteresis effect), as it is intentionally kept relatively simple to limit the number of parameters required for its practical use.
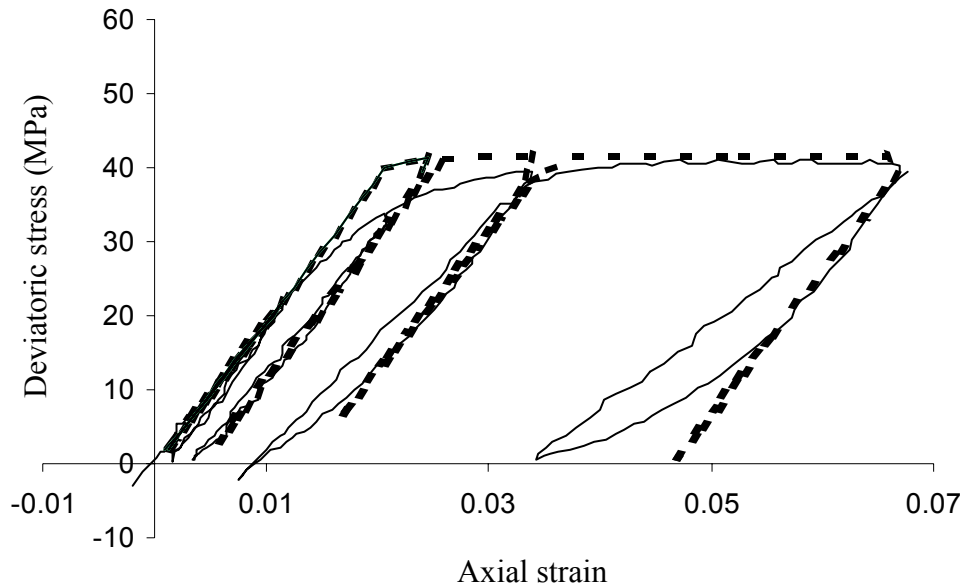

Fig. 4 Numerical results and comparison with the result of a triaxial test conducted by Elliott and Brown (1985) on Berea sandstone.

6. SAMPLE APPLICATIONS

The stress distribution around a circular opening subjected to a far field hydrostatic pressure is examined using the MSDP$_u$ EP-model implemented in the numerical code. This problem is used to further validate the proposed numerical model. The results also provide a basic estimation for the stress distribution around a circular underground opening in a rock mass. The numerical simulation is then used to conduct a short

Shirazi, Li, and Aubertin

parametric study to determine the influence of hardening on the response of the surrounding inelastic rock, which cannot be addressed by the conventional yield criterion (such as Mohr-Coloumb and Drucker-Prager) available in most stress analysis codes.
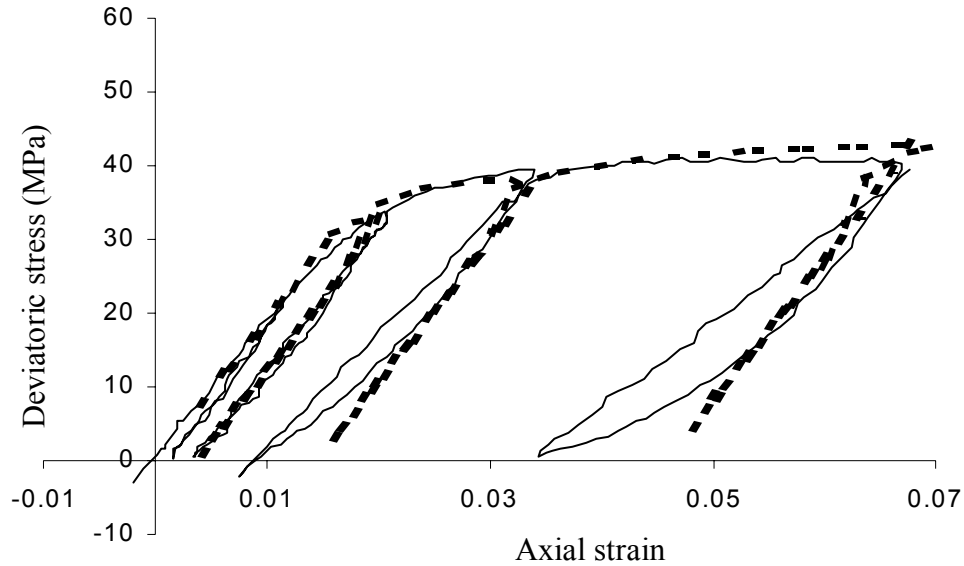

Fig. 5 Comparion between triaxial test results conducted by Elliott and Brown (1985) on Berea sandstone and numerical simulations conducted with the MSDP$_u$ EP-model (hardening parameter $\omega = 50$).

The numerical results are first compared to an analytical solution given by Li et al. (2005b) for the stress distribution around a circular opening located in an elasto-plastic medium, subjected to a far field hydrostatic pressure σ_h (of 40 MPa in this case; see Fig. 6). The analytical solution developed with the MSDP$_u$ inelastic criterion is not presented here (see detailed equations in Li et al. 2005b). In Fig. 6, r_0 is the initial radius of the opening, p_0 is an internal pressure in the circular opening (p_0=0 in this example), R is the radius of the interface between yielding (plastic) and unyielding (elastic) regions, r and ψ are the cylindrical co-ordinates of the calculation point, and σ_r and σ_ψ are the radial and tangential stresses, respectively.
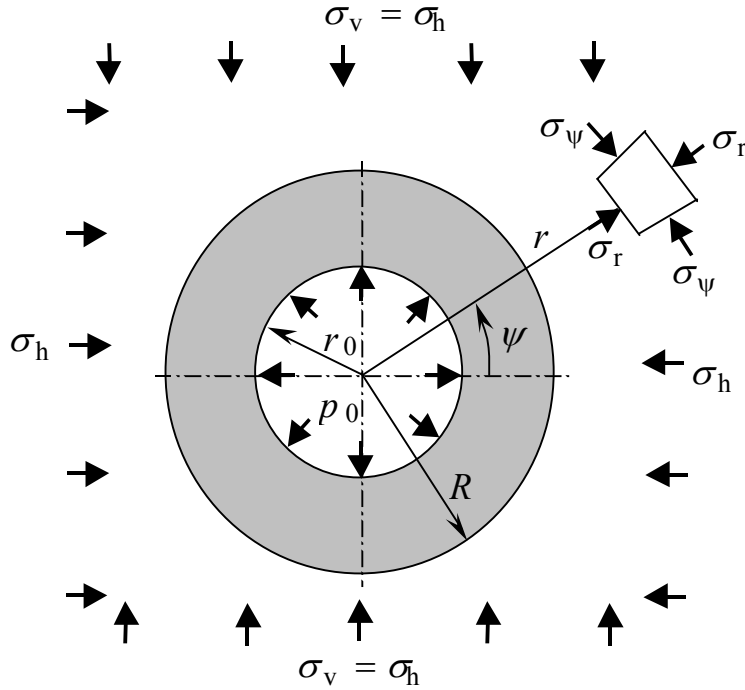
Shirazi, Li, and Aubertin

Fig. 6 Circular opening located in an elasto-plastic medium subjected to a far field hydrostatic pressure ($\sigma_v = \sigma_h$).

Figure 7 illustrates the discretization over one quarter of the domain (using symmetry). In order to have a good representation of the infinite region, the discretized domain is extended to 40 times the initial radius ($r_0 = 3$ m) of the circular opening.

The medium is elasto-plastic, and governed by the $MSDP_u$ inelastic criterion. The elastic parameters, corresponding to those of a rock mass, are: $G = 137.5$MPa, $K = 187.5$MPa. The material parameters for the $MSDP_u$ criterion are: $\sigma_c = 45$MPa, $\sigma_t = 1$MPa, $I_c = 44$MPa, $a_3 = 0.75$, $b = 0.75$, $\phi = 30°$.

Figure 8 shows a comparison between the FLAC simulation and the analytical results given by Li et al. (2005*b*) for the radial and tangential stress distribution around the circular opening. It can be seen that the agreement between the numerical and the analytical results is excellent. Other comparisons between the computational results and existing solutions (not shown here) have also confirmed the validity of the numerical implementation of the $MSDP_u$ EP-model.
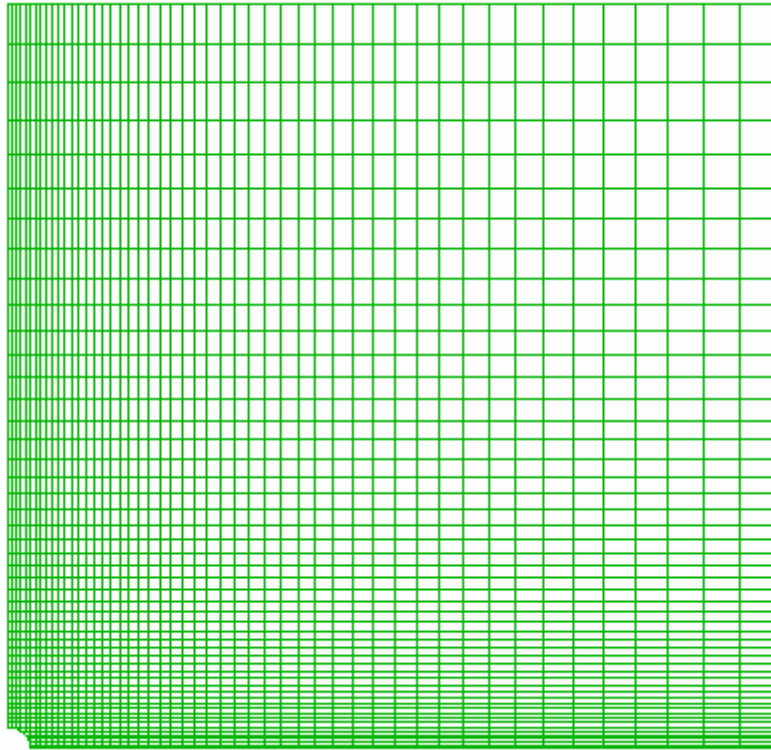
Shirazi, Li, and Aubertin

Fig. 7   Discretization of the problem domain using the symmetry axes (plane strain conditions).
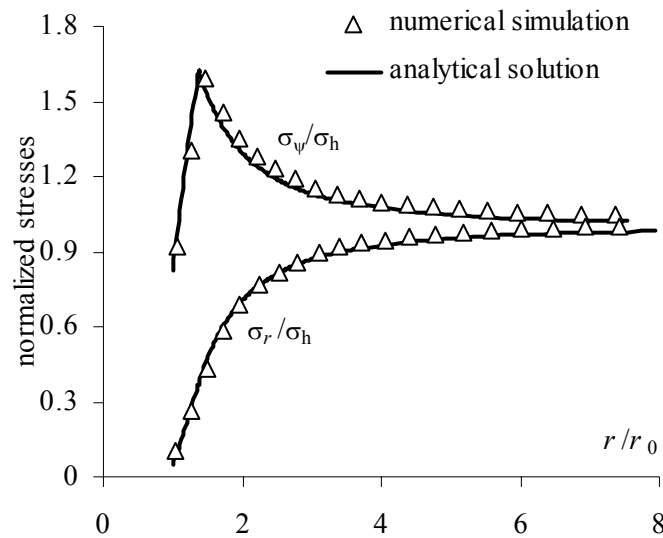

Fig. 8   Comparison bewteen numerical results and the analytical solution given by Li et al. (2005*b*) for the
normalized radial and tangential stresses around the circular opening.

18   Shirazi, Li, and Aubertin

Figure 9 illustrates some of the results of a parametric study using the MSDP$_u$ EP-model with hardening. This figure shows the tangential stress distribution around the cylindrical opening, using the same geometry and initial conditions as shown in Fig. 6. It can be seen that the stress distribution tends toward the elastic solution given by the Hiramatsu and Oka (1962, 1968) analytical formulation when the hardening parameter ω increases from 0 (perfectly plastic) to a relatively large value (of 140 in this case).

The authors have also performed a number of other calculations aimed at evaluating the response of underground openings, with an emphasis on the case of backfilled stopes (to extend the work of Aubertin et al. 2003a and Li et al. 2003, which was based on the classical models described above). These results will appear elsewhere.
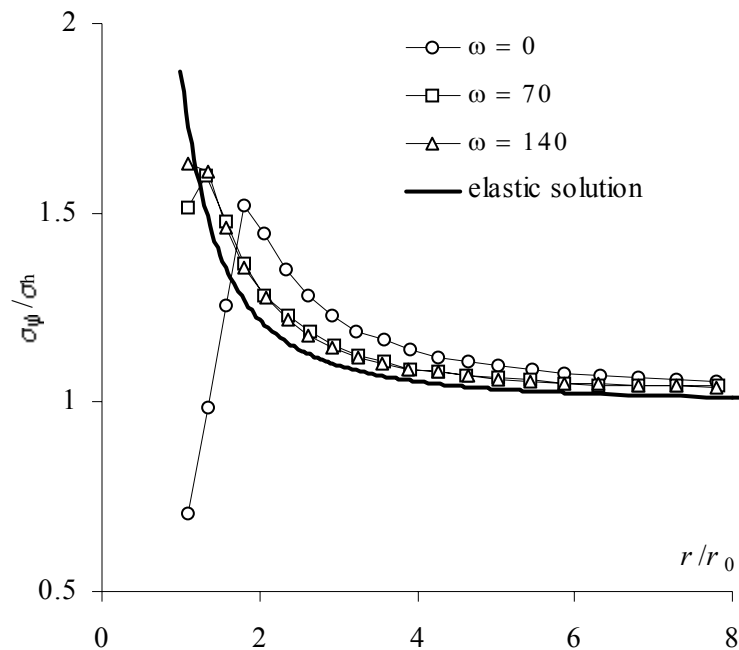

Fig. 9   Calculated values of the tangential stress distribution around a circular opening evaluated with the MSDP$_u$ EP-model and the elastic solution.

7. DISCUSSION

The results presented above are part of an ongoing investigation into the geomechanical response of porous materials (soils and fills) when placed in a confined volume, such as the case with backfill placed in mine stopes

Shirazi, Li, and Aubertin

or in narrow trenches (see Li et al. 2003, 2005*a*, 2006). For such situations, analytical solutions have been developed for simplified conditions, but as mentioned previously these bear serious limitations in terms of the type of constitutive equations that can be used, the geometry that can be considered, and the initial and boundary conditions that can be applied.

In view of these very restrictive attributes of analytical solutions, it is clear that numerical tools offer much more flexibility. Nonetheless, commonly used codes also have limitations, in part because the nature of the models already implemented, which are not readily representative of the variety of geomaterials that need to be analyzed, i.e. from hard rock, to fractured rock mass, to cemented fill, to very soft porous media. The $MSDP_u$ criterion used in this report has been developed with this broad range of material behaviour in mind. Since its initial formulation and application to brittle materials (Aubertin and Simon 1996, 1998; Aubertin et al. 1999*a*), it has been extended to deal with a variety of physical processes, including time and scale effects (Aubertin et al. 2000, 2001), strength anisotropy (Li and Aubertin 2000), the effect of low to high cohesion with different pressure (mean stress) dependencies (Aubertin and Li 2004), and influence of initial porosity for different materials (rock, concrete, soils, etc; see Li et al. 2005*b*). The above mentioned publications from the authors summarize the main features, with the advantages (and limitations) of this inelastic criterion. Its modular nature allows the $MSDP_u$ formulation to be adapted to suit different needs. Work is still in progress to include other phenomena such as the influence of saturated and unsaturated conditions (with pore pressure; see Fredlund and Rahardjo 1993, Wheeler and Sivakumar 1995, Aubertin et al. 2003*b,* and Alonso and Olivella 2006), progressive damage (with an evolution law and softening) on the elastic moduli and on inelastic behaviour (see Aubertin et al. 1998, 1999*b*), and the possible use of mixed (isotropic and kinematic) hardening for semi-brittle materials based on internal state variables (Yahya et al. 2000; Aubertin et al. 1999*b*). Other issues that are considered include the use of a non-associated flow law (i.e. $F \neq g$), calculations with a varying Lode angle θ (when needed), and also the evaluation of more elaborate numerical integration schemes to benefit from recent developments in this field (e.g., Cristfield 1991; Aubertin 1993; Simo 1998). These aspects have not however been included in this presentation, which rather aims at proposing a simple but practical means to include the key features of the $MSDP_u$ locus into numerical analyses.

Shirazi, Li, and Aubertin

For the types of problems most often encountered in geotechnical engineering, it is very useful to have a flexible but still relatively simple model that can represent a range of material behaviour, and which can be used in different numerical modelling situations.

The results shown in this report illustrate some of the steps that have been taken to help resolve these types of problems. The $MSDP_u$ EP-model, as implemented here into a commercial code, provides a practical approach to tackle many of the most critical issues in geomechanics. This work will continue to further advance with the development of constitutive equations, numerical code, and solution strategies for practical engineering problems.

8. CONCLUSION

In this report, an elasto-plastic constitutive model has been developed using the general multiaxial $MSDP_u$ criterion. The proposed model includes an isotropic hardening component that was added to better represent the behaviour of porous media. The ensuing $MSDP_u$ EP-model has been implemented into a commercial code commonly used in geomechanics and geotechnical engineering. The proposed model and numerical implementation are described with the adopted simplifications, and sample calculation results are shown and validated through a comparison of the stress distribution around a circular opening in an inelastic mass submitted to hydrostatic pressure, as obtained from existing analytical solutions previously developed by the authors. It has also been shown that the proposed model can describe fairly well the experimental results of triaxial compression tests. The proposed model is presented as a new practical tool among the available models to solve geomechanical problems.

ACKNOWLEDGEMENT

Shirazi, Li, and Aubertin

REFERENCES

Alonso, E.E., and Olivella, S. 2006. Unsaturated soil mechanics applied to geotechnical problems. *In* Proceedings of the 4th International Conference on Unsaturated Soils, April 2006, Carefree, Arizona. *Edited by* G.A. Miller, C.E. Zapata, S.L. Houston, and D.G. Fredlund. ASCE Geotechnical Special Publication. Vol. 147, pp. 1-35.

Anagnostou, G., and Kovari, K. 1993. Significant parameters in elastoplastic analysis of underground openings. Journal of Geotechnical Engineering, **119**(3): 401-419.

Aubertin, M. 1993. Numerical integration schemes for models describing the viscoplastic behaviour of soft rocks with state variables. *In* Proceedings of the 1st Canadian Symposium on Numerical Modelling Application in Mining and Geomechanics, March 1993, Montreal, QC, Canada. *Edited by* H. Mitri. McGill University Press, Montreal, pp. 50-65.

Aubertin, M., and Li, L. 2004. A porosity-dependent inelastic criterion for engineering materials. International Journal of Plasticity, **20**(12): 2179-2208.

Aubertin, M., and Simon, R. 1996. A multiaxial failure criterion that combines two quadric surfaces. *In* Rock Mechanics-Tools and Techniques. *Edited by* M. Aubertin, F. Hassani, H. Mitri. A.A. Balkema, Rotterdam, The Netherlands. Vol. 2, pp. 1729-1736.

Aubertin, M., and Simon, R. 1998. Un critère de rupture multiaxial pour matériaux fragiles. Canadian Journal of Civil Engineering, **25**(2): 277-290.

Aubertin, M., Julien, M.R., and Li, L. 1998. Keynote Lecture: The semi-brittle behavior of low porosity rocks. *In* Proceedings of the 3rd North America Rock Mechanics Symposium, Cancun, 3–5 June 1998. Vol. 2, pp. 65–90.

Aubertin, M., Li, L., Simon, R., and Khalfi, S. 1999*a*. Formulation and application of a short term strength criterion for isotropic rocks. Canadian Geotechnical Journal, **36**(5): 947-960.

Aubertin, M., Yahya, O.M.L., and Julien, M. 1999*b*. Modeling mixed hardening of alkali halides with a modified version of an internal state variables model. International Journal of Plasticity, **15**: 1067-1088.

Aubertin, M., Li, L., and Simon, R. 2000. A multiaxial stress criterion for short term and long term strength of isotropic rock media. International Journal of Rock Mechanics and Mining Sciences, **37**: 1169-1193.

Aubertin, M., Li, L., and Simon, R. 2001. Evaluating the large scale strength of rock mass with the $MSDP_u$ criterion. *In* Rock Mechanics in The National Interest: 38th US Rock Mechanics Symposium (DC Rocks 2001). *Edited by* D. Elsworth, J.P. Tinucci, K.A. Heasley. A.A. Balkema, Rotterdam, The Netherlands, pp. 1209-1216.

Aubertin, M., Li, L., Arnoldi, S., Belem, T., Bussière, B., Benzaazoua, M., and Simon, R. 2003*a*. Interaction between backfill and rock mass in narrow stopes. *In* Soil and Rock America 2003. *Edited by* P.J. Culligan, H.H. Einstein, A.J. Whittle. Verlag Glückauf Essen (VGE), Essen, Germany. Vol. 1, pp. 1157-1164.

Shirazi, Li, and Aubertin

Aubertin, M., Mbonimpa, M., Bussière, B., and Chapuis, R.P. 2003*b*. A model to predict the water retention curve from basic geotechnical properties. Canadian Geotechnical Journal, **40**(6): 1104-1122.

Bardet, J.P., and Choucair, W. 1991. A linearized integration technique for incremental constitutive equations. International Journal for Numerical and Analytical Methods in Geomechanics, **15**: 1-19.

Billaux, D., Rachez, X., Detournay, C., and Hart, R. (eds) 2001. FLAC and numerical modeling in geomechanics - 2001. Proceedings of the 2nd International FLAC Conference, Lyon, France, October 2001. Rotterdam: Balkema.

Brinkgreve, R.B.J. 2005. Selection of soil models and parameters for geotechnical engineering application. *In* Soil Constitutive Models: Evaluation, Selection, and Calibration. *Edited by* J.A. Yamamuro, V.N. Kaliakin. ASCE Geotechnical Special Publication. Vol. 128, pp. 69-98.

Brummer, R., Andrieux, P., Detournay, C., and Hart, R. (eds) 2003. FLAC and numerical modeling in geomechanics - 2003. Proceedings of the 3rd International FLAC Symposium, Sudbury, Ontario, Canada, October 2003. Lisse: Balkema.

Callari, C. 2004. Coupled numerical analysis of strain localization induced by shallow tunnels in saturated soils. Computers and Geotechnics, **31**(3): 193-207

Carter, J.P., and Liu, M.D. 2005. Review of the structured Cam Clay model. *In* Soil Constitutive Models: Evaluation, Selection, and Calibration. *Edited by* J.A. Yamamuro, V.N. Kaliakin. ASCE Geotechnical Special Publication. Vol. 128, pp. 99-132.

Casey, J., and Naghdi, P.M. 1981. On the characterization of strain-hardening in plasticity. Journal of Applied Mechanics, Transactions ASME, **4**8(2): 285-296.

Chatti, S., Dogui, A., Dubujet, P., and Sidoroff, F. 2001. An objective incremental formulation for the solution of anisotropic elastoplastic problems at finite strain. Communications in Numerical Methods in Engineering, **17**(12): 845-862.

Chen, W.F., and Baladi, G.Y. 1985. Soil plasticity: Theory and implementation. Elsevier Science.

Chen, X, Tan, C.P., and Haberfield, C.M. 1999. Solutions for the deformations and stability of elastoplastic hollow cylinders subjected to boundary pressures. International Journal for Numerical and Analytical Methods in Geomechanics, **23**(8): 779-800.

Coussy, O. 2000. Poromechanics. John Wiley & Sons.

Crisfield, M.A. 1991. Non-linear finite element analysis of solids and structures. John Wiley and Sons, Inc. Vols. 1 and 2.

Cundall, P.A. 1976. Explicit finite difference methods in geomechanics. *In* Proceedings of the EF Conference on Numerical Methods in Geomechanics, Blacksburg, Virginia. American Society of Mechanical Engineers, Applied Mechanics Division. Vol. 1, pp. 132-150.

Shirazi, Li, and Aubertin

Desai, C.S. 2005. Constitutive modeling for geologic materials: significance and directions. International Journal of Geomechanics, **5**(2): 81-84.

Desai, C.S., and Siriwardane, H.J. 1984. Constitutive laws for engineering materials with emphasis on geologic materials. Prentice-Hall Inc.

Detournay, C., and Hart, R. (eds) 1999. FLAC and numerical modeling in geomechanics. *In* Proceedings of the Conference, Minneapolis, September 1999. Rotterdam: Balkema.

Eberhardt, E. 2001. Numerical modelling of three-dimension stress rotation ahead of an advancing tunnel face. International Journal of Rock Mechanics and Mining Sciences, 38(4): 499-518.

Elliott, G.M., and Brown, E.T. 1985. Yield of a soft, high porosity rock. Géotechique, **35**(4): 413-423.

Fredlund, D.G., and Rahardjo, H. 1993. Soil Mechanics for Unsaturated Soils. John Wiley and Sons, New York.

Grgic, D., Homand, F., and Hoxha, D. 2003. A short- and long-term rheological model to understand the collapses of iron mines in Lorraine, France. Computers and Geotechnics, **30**(7): 557-570.

Hiramatsu, Y., and Oka, Y. 1962. Stress around a shaft or level excavated in ground with a three-dimensional stress state. Memoirs of the Faculty of Engineering, Kyoto University, **24**: 56-76.

Hiramatsu, Y., and Oka, Y. 1968. Determination of the stress in rock unaffected by boreholes or drifts, from measured strains or deformations. International Journal of Rock Mechanics and Mining Sciences, **5**(1): 337-353.

Itasca, 2002. FLAC - Fast Lagrangian Analysis of Continua, User's Guide. Itasca Consulting Group, Inc, Minneapolis, MN.

Lade, P.V. 1977. Elasto-plastic stress-strain theory for cohesionless soil with curved yield surfaces. International Journal of Solids and Structures, **13**: 1019-1035.

Lade, P.V. 2005*a*. Overview of constitutive model for soils. *In* Soil Constitutive Models: Evaluation, Selection, and Calibration. *Edited by* J.A. Yamamuro, V.N. Kaliakin. ASCE Geotechnical Special Publication. Vol. 128, pp. 1-34.

Lade, P.V. 2005*b*. Single hardening model for soils: parameter determination and typical values. *In* Soil Constitutive Models: Evaluation, Selection, and Calibration. *Edited by* J.A. Yamamuro, V.N. Kaliakin. ASCE Geotechnical Special Publication. Vol. 128, pp. 290-309.

Lee, K.M., and Rowe, R.K. 1989. Deformations caused by surface loading and tunnelling: the role of elastic anisotropy. Geotechnique, **39**(1): 125-140.

Lee, K.M., and Rowe, R.K. 1991. Analysis of three-dimensional ground movements. The Thunder Bay tunnel. Canadian Geotechnical Journal, **28**(1): 25-41.

Li, L., and Aubertin, M. 2000. Un critère de rupture multiaxial pour les roches avec une anisotropie planaire. *In* Proceedings of the 53rd Canadian Geotechnical Conference, Montreal. Canadian Geotechnical Society. pp. 357-364.

Li, L., Aubertin, M., Simon, R., Bussière, B., and Belem, T. 2003. Modeling arching effects in narrow backfilled stopes with FLAC. *In* FLAC and Numerical Modeling in Geomechanics – 2003. *Edited by* R. Brummer, P. Andrieux, C. Detournay, R. Hart. A.A. Balkema, Rotterdam, The Netherlands. pp. 211-219.

Li, L., Aubertin, M., and Belem, T. 2005*a*. Formulation of a three dimensional analytical solution to evaluate stress in backfilled vertical narrow openings. Canadian Geotechnical Journal, **42**(6): 1705–1717.

Li, L., Aubertin, M., Simon, R., and Bussière, B. 2005*b*. Formulation and application of a general inelastic locus for geomaterials with variable porosity. Canadian Geotechnical Journal, **42**(2): 601–623.

Li, L., Aubertin, M., and Belem, T. 2006. Erratum: Formulation of a three dimensional analytical solution to evaluate stress in backfilled vertical narrow openings. Canadian Geotechnical Journal, **43**(3): 338–339.

Marti, J., and Cundall, P. 1982. Mixed discretization procedure for accurate modeling of plastic collapse. International Journal for Numerical and Analytical Methods in Geomechanics, **6**(1): 129-139.

Meguid, M.A., and Rowe, R.K. 2006. Stability of D-shaped tunnels in a Mohr–Coulomb material under anisotropic stress conditions. Canadian Geotechnical Journal, **43**(2): 273–281 (2006)

Meguid, M.A., Rowe, R.K., and Lo, K.Y. 2003. Three-dimensional analysis of unlined tunnels in rock subjected to high horizontal stress. Canadian Geotechnical Journal, **40**(6): 1208-1224.

Ng, C.W.W., and Lee, G.T.K. 2005. Three-dimensional ground settlements and stress-transfer mechanisms due to open-face tunneling. Canadian Geotechnical Journal, **42**(4): 1015-1029.

Nova, R. 2005. A simple elastoplastic model for soils and soft rocks. *In* Soil Constitutive Models: Evaluation, Selection, and Calibration. *Edited by* J.A. Yamamuro, V.N. Kaliakin. ASCE Geotechnical Special Publication. Vol. 128, pp. 380-399.

Ortiz, M., and Popov, E.P. 1985. Accuracy and stability integration algorithms for elastoplastic constitutive relations. International Journal for Numerical Methods in Engineering, **21**(9): 1561-1576.

Potts, D.M., and Zdravkovic, L. 1999. Finite element analysis in geotechnical engineering: Theory. Thomas Telford.

Prevost, J.H. 1978. Plasticity theory for soil stress-strain behavior. ASCE Journal of Engineering Mechanics Division, **104**(5): 1177-1194.

Purwodihardjo, A., and Cambou, B. 2005. Time-dependent modelling for soils and its application in tunnelling. International Journal for Numerical and Analytical Methods in Geomechanics, **29**(1): 49-71.

Simo, J.C. 1998. Numerical analysis and simulation of plasticity. Handbook of Numerical Analysis. Elsevier Science, Amsterdam. Vol. 4, pp. 183-499.

Simo, J.C., and Hughes, T.J.R. 1997. Computational inelasticity. Springer, New York.

Wheeler, S.J., and Sivakumar, V. 1995. An elasto-plastic critical state framework for unsaturated soil, *Géotechnique*, **45**(1): 35-53.

Shirazi, Li, and Aubertin

Yahya, O.M.L., Aubertin, M., and Julien, M.R. 2000. A unified representation of the plasticity, creep and relaxation behavior of rocksalt. International Journal of Rock Mechanics and Mining Sciences, **37**(5): 787-800.

L'École Polytechnique se spécialise dans la formation d'ingénieurs et la recherche en ingénierie depuis 1873