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**NONLINEAR VIBRATION OF TRUNCATED CONICAL
SHELLS:
DONNELL, SANDERS AND NEMETH THEORIES**

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École Polytechnique de Montréal

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Abstract

The formulation of nonlinear kinematics of shells in three different shell theories namely Donnell, Sanders and Nemeth including shear deformation for anisotropic materials is presented. A finite element solution for the equilibrium equations of Sander's improved first-approximation theory is developed and has been used to develop the nonlinear finite element amplitude equation of vibration of conical shells of Donnell, Sanders and Nemeth theories using generalized coordinates methods and Lagrange equations of motions. The amplitude equation of nonlinear vibration of conical shell has been solved for multiple cases of isotropic materials with neglecting the shear deformation. Linear vibration frequencies for different conical shells with different materials, geometry and boundary conditions are validated against the existing experimental data in the literature and show excellent agreement. The nonlinear vibration results have been validated against the existing data for cylindrical shells and demonstrate good accordance. The validated model has been used to investigate effect of different parameters including circumferential mode number, cone-half angle, length to radius ratio, thickness to radius ratio and boundary conditions.

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1 Introduction

The primary objective of any structural analysis-design is to predict and avoid catastrophic failures of the structure and subsequently achieving a reliable structure at the lowest possible weight. While linear shell theories have been successfully used for analysis of shell structures when the deformations are small in comparison to the shell thickness, achieving better accuracy requires employing nonlinear shell theories for some practical applications. At the same time the growing widespread application of composite materials with anisotropic properties adds to the complexity of the analysis. Conical shells are important structural element of aerospace vehicles and can be found in different applications ranging from ray-domes up to satellite launch vehicles with large fuel tanks.

In comparison to cylindrical shells and flat plates, nonlinear vibration of conical shells received less attention in scientific literature. One of the earliest studies are the work of Lindholm et al.[1] that was focused on non-symmetric transverse vibrations of isotropic truncated conical shells using Donnell's type of nonlinear kinematics with focus on bending and membrane rigidity. Sun et al. [2] studied the dynamic response of conical and cylindrical shells under ramp and sinusoidal temperature loads using Donnell's nonlinear shallow shell theory. Bendavid et al.[3] developed a nonlinear theory for bending, buckling and vibrations of conical shells based on Sander's nonlinear theory and proposed a variation of generalized coordinates method to predict the shell's response. Kanaka Raju et al. studied the large -amplitude asymmetric vibrations of shells of revolutions employing Sander's nonlinear kinematics to obtain finite element's stiffness matrix. Their proposed iterative method includes iteration over and refinement of stiffness matrix to achieve convergence. Ueda[5] employed an equivalent of Donnell's shallow shell theory to formulate a two degrees of freedom finite element to analyze the free vibrations of conical shells. He employed the same methodology on cylindrical and demonstrated that his results are in good accordance with those of Olson[6].

Liu et al.[7] used Donnell's type of nonlinearities in conjunction with Galerkin's method to study the vibration of shallow conical sandwich shells. They presented effects of variation of several parameters including boundary conditions and the stiffness of the sandwich core on the nonlinear response of the shell. Xu et al.[8] studied the nonlinear vibration of thick conical shells, using Donnell's shallow shell theory and a proposed solution in form of double Fourier series with time dependent coefficients. Their results showed good accordance with the existing data. Fu et al.[9] taking similar approach to Xu et al.[8] in an study presented the relative thickness to radius ratio when the effect of transverse shear can be omitted. Awrejcewicz et al.[10] studied the transition from regular to chaotic vibrations of isotropic spherical and conical shells taking into account Donnell's type of nonlinearities and studied effect of various parameters including cone-half angle, the amplitude and the frequency of the excitation force and the boundary conditions on the transition to chaotic behavior. Chen et al.[11][12] employed Donnell's nonlinear theory to

study nonlinear vibration and stability of rotary truncated conical shells including the coupling of high and low modal responses and stated that right rotation could have hardening effect on the shell's stability. Sofiyev[13] studied the nonlinear vibration of truncated conical shells made of functionally graded materials (FGM) and using Galerkin and Harmonic balance method, the influences of compositional profiles and shell's geometry on the shell dimensionless nonlinear frequency were investigated. In follow up studies Najafov et al.[14][15] expanded that study to FGM conical shells surrounded by an elastic medium on the Paternak-type elastic foundation and truncated cones coated by FGM materials. Sofiyev[16] investigated the large-amplitude vibration of non-homogeneous orthotropic composite truncated conical shells. The Young's modulus and density were assumed to vary exponentially through the thickness of the shell and effect of various parameters including non-homogeneity, orthotropy and shell's geometry on the dynamic response of the shell were presented.

The objective of this study is to develop a hybrid finite element model and solution for the nonlinear vibration of anisotropic conical shells. The solution consisted of two parts:

- The conical shells finite element's displacement functions are developed by using an analytical solution to the Sander's improved first order linear shell theory.
- Substituting the developed finite element formulation in nonlinear strain-displacement relationship (kinematics) of Sanders' nonlinear theory, kinetic and internal strain energies are expressed in terms of nodal displacements. Then employing the generalized coordinates method, the equations of motion of the nonlinear shell are derived through Lagrange method. The numerical solution of these equations of motion has been used for the dynamic analysis of conical shells.

2 Nonlinear Kinematics

For shell structures (where the dimension of the medium in one direction is significantly lower than the other two dimensions), it is common to treat the elastic body as a 2D medium with a reference surface and relating the strain and stress fields of a point at a certain distance from the reference surface to the values of the corresponding point on that reference surface. This approach introduces three important factors on the limitations of any model in shell analysis [17]:

- The relationship that has been used to define the strain-displacement (kinematics) within the reference surface that defines the geometrical nonlinearities;
- The relationship that has been used to relate the displacement, strain and stress at a point with a certain distance from the reference-surface to the values of the corresponding point on the reference surface (shear deformation);

- The relationships that has been used to define the variation and discontinuities of field variables at the interfaces of layers in multilayer shells (layerwise, Zig-Zag and equivalent single layer)

Nemeth[18] in an exposition provided a shear deformation type of shell theory that can provide Donnell's and Sanders' shell theories as an explicit subsets. The kinematics of this theory are developed based on the assumption of small strains and moderate rotations and themselves are a subset of the general derivations of non-orthogonal principle coordinates of [19]. Assuming the reference surface is located at the middle of the shell, the five fundamental unknowns in this formulation are two middle-surface tangential displacements $u_1(\xi_1, \xi_2)$ and $u_2(\xi_1, \xi_2)$, the normal displacement $u_3(\xi_1, \xi_2)$ and two transverse shearing strains $\gamma_{13}^o(\xi_1, \xi_2)$ and $\gamma_{23}^o(\xi_1, \xi_2)$. The tangential and normal displacement fields of a material point $p(\xi_1, \xi_2, \xi_3)$ in a shell expressed in the orthogonal principal-curvature coordinate systems are given as:

$$U_1(\xi_1, \xi_2, \xi_3) = u_1(\xi_1, \xi_2) + \xi_3 [\varphi_1 - \varphi\varphi_2] + F_1(\xi_3) \gamma_{13}^o \quad (1a)$$

$$U_2(\xi_1, \xi_2, \xi_3) = u_2(\xi_1, \xi_2) + \xi_3 [\varphi_2 - \varphi\varphi_1] + F_2(\xi_3) \gamma_{23}^o \quad (1b)$$

$$U_3(\xi_1, \xi_2, \xi_3) = u_3(\xi_1, \xi_2) - \frac{1}{2} \xi_3 (\varphi_1^2 + \varphi_2^2) \quad (1c)$$

where:

- φ_1 , φ_2 and φ are the linear rotation parameters.
- $F_1(\xi_3)$ and $F_2(\xi_3)$ are two analyst-defined functions that specify the through-the thickness distributions of the transverse-shearing strains.

The linear rotation parameters are defined as follows:

$$\varphi_1(\xi_1, \xi_2) = \frac{c_3 u_1(\xi_1, \xi_2)}{R_1} - \frac{1}{A_1} \frac{\partial u_3(\xi_1, \xi_2)}{\partial \xi_1} \quad (2a)$$

$$\varphi_2(\xi_1, \xi_2) = \frac{c_3 u_2(\xi_1, \xi_2)}{R_2} - \frac{1}{A_2} \frac{\partial u_3(\xi_1, \xi_2)}{\partial \xi_2} \quad (2b)$$

$$\varphi(\xi_1, \xi_2) = \frac{1}{2} c_3 \left(\frac{1}{A_1} \frac{\partial u_2(\xi_1, \xi_2)}{\partial \xi_1} - \frac{1}{A_2} \frac{\partial u_1(\xi_1, \xi_2)}{\partial \xi_2} + \frac{u_1(\xi_1, \xi_2)}{\rho_{11}} + \frac{u_2(\xi_1, \xi_2)}{\rho_{22}} \right) \quad (2c)$$

where $A_1, A_2, R_1, R_2, \rho_{11}$ and ρ_{22} respectively are the surface metrics, principle radii of curvature and geodesic radii of curvature in directions of ξ_1 and ξ_2 . In addition the linear deformation parameters are defined as follows:

$$e_{11}^\circ(\xi_1, \xi_2) = \frac{1}{A_1} \frac{\partial u_1(\xi_1, \xi_2)}{\partial \xi_1} - \frac{u_2(\xi_1, \xi_2)}{\rho_{11}} + \frac{u_3(\xi_1, \xi_2)}{R_1} \quad (3a)$$

$$e_{22}^\circ(\xi_1, \xi_2) = \frac{1}{A_2} \frac{\partial u_2(\xi_1, \xi_2)}{\partial \xi_2} + \frac{u_1(\xi_1, \xi_2)}{\rho_{22}} + \frac{u_3(\xi_1, \xi_2)}{R_2} \quad (3b)$$

$$2e_{12}^\circ(\xi_1, \xi_2) = \frac{1}{A_2} \frac{\partial u_1(\xi_1, \xi_2)}{\partial \xi_2} + \frac{1}{A_1} \frac{\partial u_2(\xi_1, \xi_2)}{\partial \xi_1} + \frac{u_1(\xi_1, \xi_2)}{\rho_{11}} - \frac{u_2(\xi_1, \xi_2)}{\rho_{22}} \quad (3c)$$

Using the linear rotation and deformation parameters, the strain-displacement relationship on the reference surface of this theory based on the assumption of small strains and moderate rotations are:

$$\{\boldsymbol{\epsilon}^\circ\} = \begin{Bmatrix} \epsilon_{11}^\circ \\ \epsilon_{22}^\circ \\ \gamma_{12}^\circ \end{Bmatrix} = \begin{Bmatrix} e_{11}^\circ + c_{NL} \left[\frac{1}{2} (\varphi_1^2 + c_2 \varphi^2) + \frac{1}{2} c_1 \left[(e_{11}^\circ)^2 + e_{12}^\circ (e_{12}^\circ + 2\varphi) \right] \right] \\ e_{22}^\circ + c_{NL} \left[\frac{1}{2} (\varphi_2^2 + c_2 \varphi^2) + \frac{1}{2} c_1 \left[(e_{22}^\circ)^2 + e_{12}^\circ (e_{12}^\circ - 2\varphi) \right] \right] \\ 2e_{12}^\circ + c_{NL} \left[\varphi_1 \varphi_2 + c_1 \left[e_{11}^\circ (e_{12}^\circ - \varphi) + e_{22}^\circ (e_{12}^\circ + \varphi) \right] \right] \end{Bmatrix} \quad (4a)$$

$$\{\boldsymbol{\chi}^\circ\} = \begin{Bmatrix} \chi_{11}^\circ \\ \chi_{22}^\circ \\ 2\chi_{12}^\circ \end{Bmatrix} = \begin{Bmatrix} \frac{1}{A_1} \frac{\partial \varphi_1}{\partial \xi_1} - \frac{\varphi_2}{\rho_{11}} \\ \frac{1}{A_2} \frac{\partial \varphi_2}{\partial \xi_2} + \frac{\varphi_1}{\rho_{22}} \\ \frac{1}{A_2} \frac{\partial \varphi_1}{\partial \xi_2} + \frac{1}{A_1} \frac{\partial \varphi_2}{\partial \xi_1} + \left(\frac{\varphi_1}{\rho_{11}} - \frac{\varphi_2}{\rho_{22}} \right) + \left(\frac{1}{R_2} - \frac{1}{R_1} \right) \varphi \end{Bmatrix} \quad (4b)$$

In the above formulation:

- Specifying $c_{NL} = 0$ and $c_3 = 1$ simplifies the kinematics to the improved first-approximation linear shell theory of Sanders [20] and $c_{NL} = 1$ provides different nonlinear theories as described in the following items.
- Specifying $c_{NL} = 1, c_1 = c_2 = c_3 = 1$ defines Nemeth's nonlinear theory [18]
- Specifying $c_{NL} = 1, c_1 = 0$ and $c_2 = c_3 = 1$ retrieves Sanders' kinematics [21] [22].
- Specifying $c_{NL} = 1, c_1 = c_2 = 0$ and $c_3 = 1$ retrieves Sanders' kinematics with the nonlinear rotations about the reference-surface normal neglected [21] [22].

- Specifying $c_{NL} = 1, c_1 = c_2 = c_3 = 0$ defines the Donnell's strain-displacement relationship [23].

The strains at point with ξ_3 distance from the middle surface can be expressed in the following matrix form:

$$\{\boldsymbol{\epsilon}^{\xi_3}\} = \{\epsilon_{11} \quad \epsilon_{22} \quad \gamma_{12} \quad \gamma_{13} \quad \gamma_{23}\}^T = \left(1 + \frac{\xi_3}{R_1}\right)^{-1} \left(1 + \frac{\xi_3}{R_2}\right)^{-1} [\mathbf{S}] \{\boldsymbol{\epsilon}^\circ\} \quad (5)$$

The corresponding strains on the middle surface are defined:

$$\{\boldsymbol{\epsilon}^\circ\} = \left\{ \{\boldsymbol{\epsilon}^\circ\} \quad \{\boldsymbol{\chi}^\circ\} \quad \frac{1}{A_1} \frac{\partial}{\partial \xi_1} \{\boldsymbol{\gamma}^\circ\} \quad \frac{1}{A_2} \frac{\partial}{\partial \xi_2} \{\boldsymbol{\gamma}^\circ\} \quad \{\boldsymbol{\gamma}^\circ\} \right\}^T \quad (6)$$

Moreover $\{\boldsymbol{\gamma}^\circ\}$ is the transverse shear deformation vector or $\{\boldsymbol{\gamma}^\circ\} = \{\gamma_{13}^\circ \quad \gamma_{23}^\circ\}^T$. The definitions of $[\mathbf{S}_i]$ matrices can be found in appendix A.

3 Stress-Strain Relationship and Anisotropy

3.1 Fundamentals and Transformations

The stress-strain relationship for the homogeneous triclinic (fully anisotropic) m^{th} lamina based on the generalized Duhamel-Neumann law can be described as follows [24]:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{Bmatrix} \text{ or } \{\boldsymbol{\sigma}_\circ\} = [\mathbf{C}] \{\boldsymbol{\epsilon}_\circ\} \quad (7)$$

Often the coordinate system that the elasticity tensor is defined is not aligned with the global coordinate system. In such cases to obtain the transformed elasticity matrix we need to employ the transformation between two orthonormal bases (such as Cartesian coordinate system). Such transformation is defined as follows:

$$\begin{Bmatrix} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{i}}_2 \\ \hat{\mathbf{i}}_3 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{i}}_2 \\ \hat{\mathbf{i}}_3 \end{Bmatrix} \text{ or } \{\hat{\mathbf{i}}\} = [\mathbf{a}] \{\hat{\mathbf{i}}\} \quad (8)$$

For individual rotations along each of the three Cartesian axes; matrix $[\mathbf{a}]$ is defined as follows:

$$[\mathbf{a}(\theta_1)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_1) & -\sin(\theta_1) \\ 0 & \sin(\theta_1) & \cos(\theta_1) \end{bmatrix} \quad (\text{for rotation along } x_1) \quad (9a)$$

$$[\mathbf{a}(\theta_2)] = \begin{bmatrix} \cos(\theta_2) & 0 & \sin(\theta_2) \\ 0 & 1 & 0 \\ -\sin(\theta_2) & 0 & \cos(\theta_2) \end{bmatrix} \quad (\text{for rotation along } x_2) \quad (9b)$$

$$[\mathbf{a}(\theta_3)] = \begin{bmatrix} \cos(\theta_3) & -\sin(\theta_3) & 0 \\ \sin(\theta_3) & \cos(\theta_3) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{for rotation along } x_3) \quad (9c)$$

For a combination of rotations along all axes, the combined rotation matrix can be obtained by multiplying single axis rotation matrices of equation (9) in the desired order. For example, for rotation first by an angle of θ_1 along x_1 , then by an angle of θ_2 along x_2 and finally by an angle of θ_3 along x_3 , the total rotation matrix can be calculated as follows:

$$[\mathbf{a}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = [\mathbf{a}(\theta_3)][\mathbf{a}(\theta_2)][\mathbf{a}(\theta_1)] \quad (10)$$

The relationship between the transformed stress-strain tensors and the original stress-strain tensors can be expressed in the matrix form:

$$\{\bar{\boldsymbol{\sigma}}\} = [\mathbf{T}_\sigma]\{\boldsymbol{\sigma}_o\} \quad (11a)$$

$$\{\bar{\boldsymbol{\epsilon}}\} = [\mathbf{T}_\epsilon]\{\boldsymbol{\epsilon}_o\} \quad (11b)$$

where $[\mathbf{T}_\sigma]$ and $[\mathbf{T}_\epsilon]$ are respectively the stress and strain transformation matrices and defined as follows:

$$[\mathbf{T}_\sigma] = \begin{bmatrix} [\mathbf{T}_{1,1}] & 2[\mathbf{T}_{1,2}] \\ [\mathbf{T}_{2,1}] & [\mathbf{T}_{2,2}] \end{bmatrix} \quad (12a)$$

$$[\mathbf{T}_\epsilon] = \begin{bmatrix} [\mathbf{T}_{1,1}] & [\mathbf{T}_{1,2}] \\ 2[\mathbf{T}_{2,1}] & [\mathbf{T}_{2,2}] \end{bmatrix} \quad (12b)$$

where:

$$[\mathbf{T}_{1,1}] = \begin{bmatrix} (a_{\bar{1}1})^2 & (a_{\bar{1}2})^2 & (a_{\bar{1}3})^2 \\ (a_{\bar{2}1})^2 & (a_{\bar{2}2})^2 & (a_{\bar{2}3})^2 \\ (a_{\bar{3}1})^2 & (a_{\bar{3}2})^2 & (a_{\bar{3}3})^2 \end{bmatrix} \quad (13a)$$

$$[\mathbf{T}_{1,2}] = \begin{bmatrix} a_{\bar{1}2}a_{\bar{1}3} & a_{\bar{1}1}a_{\bar{1}3} & a_{\bar{1}1}a_{\bar{1}2} \\ a_{\bar{2}2}a_{\bar{2}3} & a_{\bar{2}1}a_{\bar{2}3} & a_{\bar{2}1}a_{\bar{2}2} \\ a_{\bar{3}2}a_{\bar{3}3} & a_{\bar{3}1}a_{\bar{3}3} & a_{\bar{3}1}a_{\bar{3}2} \end{bmatrix} \quad (13b)$$

$$[\mathbf{T}_{2,1}] = \begin{bmatrix} a_{\bar{2}1}a_{\bar{3}1} & a_{\bar{2}2}a_{\bar{3}2} & a_{\bar{2}3}a_{\bar{3}3} \\ a_{\bar{1}1}a_{\bar{3}1} & a_{\bar{1}2}a_{\bar{3}2} & a_{\bar{1}3}a_{\bar{3}3} \\ a_{\bar{1}1}a_{\bar{2}1} & a_{\bar{1}2}a_{\bar{2}2} & a_{\bar{1}3}a_{\bar{2}3} \end{bmatrix} \quad (13c)$$

$$[\mathbf{T}_{2,2}] = \begin{bmatrix} a_{\bar{2}2}a_{\bar{3}3} + a_{\bar{2}3}a_{\bar{3}2} & a_{\bar{2}1}a_{\bar{3}3} + a_{\bar{2}3}a_{\bar{3}1} & a_{\bar{2}1}a_{\bar{3}2} + a_{\bar{2}2}a_{\bar{3}1} \\ a_{\bar{1}2}a_{\bar{3}3} + a_{\bar{1}3}a_{\bar{3}2} & a_{\bar{1}1}a_{\bar{3}3} + a_{\bar{1}3}a_{\bar{3}1} & a_{\bar{1}1}a_{\bar{3}2} + a_{\bar{1}2}a_{\bar{3}1} \\ a_{\bar{1}2}a_{\bar{2}3} + a_{\bar{1}3}a_{\bar{2}2} & a_{\bar{1}1}a_{\bar{2}3} + a_{\bar{1}3}a_{\bar{2}2} & a_{\bar{1}1}a_{\bar{2}2} + a_{\bar{1}2}a_{\bar{2}1} \end{bmatrix} \quad (13d)$$

The following relationship is defined between the transformed stress and strain fields:

$$\{\bar{\sigma}\} = [\bar{C}]\{\bar{\epsilon}\} \quad (14)$$

The transformed compliance tensor $[\bar{C}]$ of equation (14) for transformed homogeneous fully anisotropic material can be calculated as follows:

$$[\bar{C}] = [\mathbf{T}_\sigma][C][\mathbf{T}_\epsilon]^{-1} \quad (15)$$

3.2 Plane Stress Conditions

For shell structures; usually plane stress condition is applied. Considering the presented coordinate transformations in the previous section, without losing generality we assumed that the thin dimension is aligned along the transformed \bar{x}_3 axis. Two different cases of plane stress condition are formulated. The first one excludes the shear deformation. The second formulation takes into

account the shear deformation.

3.2.1 Plane Stress Reduced Compliance Tensor with No Shear Deformation

In absence of shear deformation, it is assumed that $\bar{\sigma}_{33} = \bar{\sigma}_{23} = \bar{\sigma}_{13} = 0$. Substituting these stresses into (11) and defining the following permutation:

$$\boldsymbol{\pi} = \{1 \ 2 \ 3 \ 4 \ 5 \ 6\} \rightarrow \{1 \ 2 \ 6 \ 3 \ 4 \ 5\} \quad (16)$$

and its associated permutation matrix $[\mathbf{P}_\pi]$, the stress-strain relationship equations can be rearranged as follows:

$$\underbrace{\{\boldsymbol{\sigma}\}}_{[\mathbf{P}_\pi]\{\bar{\boldsymbol{\sigma}}\}} = \underbrace{[\mathbf{C}]}_{[\mathbf{P}_\pi][\bar{\mathbf{C}}][\mathbf{P}_\pi]^{-1}} \underbrace{\{\boldsymbol{\epsilon}\}}_{[\mathbf{P}_\pi]\{\bar{\boldsymbol{\epsilon}}\}} \rightarrow \begin{pmatrix} \bar{\sigma}_{11} \\ \bar{\sigma}_{22} \\ \bar{\sigma}_{12} \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{16} & \bar{C}_{13} & \bar{C}_{14} & \bar{C}_{15} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{26} & \bar{C}_{23} & \bar{C}_{24} & \bar{C}_{25} \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{66} & \bar{C}_{36} & \bar{C}_{46} & \bar{C}_{56} \\ \bar{C}_{13} & \bar{C}_{23} & \bar{C}_{36} & \bar{C}_{33} & \bar{C}_{34} & \bar{C}_{35} \\ \bar{C}_{14} & \bar{C}_{24} & \bar{C}_{46} & \bar{C}_{34} & \bar{C}_{44} & \bar{C}_{45} \\ \bar{C}_{15} & \bar{C}_{25} & \bar{C}_{56} & \bar{C}_{35} & \bar{C}_{45} & \bar{C}_{55} \end{bmatrix} \begin{pmatrix} \bar{\epsilon}_{11} \\ \bar{\epsilon}_{22} \\ 2\bar{\epsilon}_{12} \\ \bar{\epsilon}_{33} \\ 2\bar{\epsilon}_{23} \\ 2\bar{\epsilon}_{13} \end{pmatrix} \quad (17)$$

Using equation (17), it is possible to obtain values of $\bar{\epsilon}_{33}, 2\bar{\epsilon}_{23}$ and $2\bar{\epsilon}_{13}$ in terms of other strains and the elements of the permuted compliance tensor. To do so, first we define the following sub-matrices to solve the linear system that can be defined by equation (17):

$$[\bar{\mathbf{C}}]_{3 \times 3} = [\mathbf{C}]_{1:3,1:3}, \quad [\bar{\mathbf{C}}_{14}]_{3 \times 3} = [\mathbf{C}]_{1:3,4:6}, \quad [\bar{\mathbf{C}}_{41}]_{3 \times 3} = [\mathbf{C}]_{4:6,1:3} \quad \text{and} \quad [\bar{\mathbf{C}}_{44}]_{3 \times 3} = [\mathbf{C}]_{4:6,4:6} \quad (18)$$

Then the plane strain relationship neglecting the shear deformation can be defined as follows:

$$\{\bar{\boldsymbol{\sigma}}\} = \begin{pmatrix} \bar{\sigma}_{11} \\ \bar{\sigma}_{22} \\ \bar{\sigma}_{12} \end{pmatrix} = [\bar{\mathbf{Q}}]\{\bar{\boldsymbol{\epsilon}}\} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{pmatrix} \bar{\epsilon}_{11} \\ \bar{\epsilon}_{22} \\ 2\bar{\epsilon}_{12} \end{pmatrix} \quad (19)$$

Where the $[\bar{\mathbf{Q}}]$ is defined as follows:

$$[\bar{\mathbf{Q}}] = [\bar{\mathbf{C}}] - [\bar{\mathbf{C}}_{14}][\bar{\mathbf{C}}_{44}]^{-1}[\bar{\mathbf{C}}_{41}] \quad (20)$$

Permuted strain-stress relationship can be transformed back into their original form using the

following definitions:

$$[\mathbf{Q}] = \begin{bmatrix} \bar{\mathbf{Q}}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \end{bmatrix} \rightarrow [\bar{\mathbf{Q}}] = [\mathbf{P}_\pi]^{-1} [\mathbf{Q}] [\mathbf{P}_\pi] \quad (21)$$

And in the original form of equation (14) this can be rewritten as follows:

$$\{\bar{\sigma}_{11} \quad \bar{\sigma}_{22} \quad 0 \quad 0 \quad 0 \quad \bar{\sigma}_{12}\}^T = [\bar{\mathbf{Q}}]_{6 \times 6} \{\bar{\epsilon}_{11} \quad \bar{\epsilon}_{22} \quad 0 \quad 0 \quad 0 \quad 2\bar{\epsilon}_{12}\}^T \quad (22)$$

3.2.2 Plane Stress Reduced Compliance Tensor with Shear Deformation

When shear deformation is taken into account, it is assumed that $\bar{\sigma}_{33} = 0$ while $\bar{\sigma}_{23} \neq 0$ and $\bar{\sigma}_{13} \neq 0$. Taking similar approach as the previous section, we start with defining the following permutation:

$$\boldsymbol{\pi} = \{1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6\} \rightarrow \{1 \quad 2 \quad 6 \quad 5 \quad 4 \quad 3\} \quad (23)$$

Using that permutation yields the following stress-strain relationships:

$$\underbrace{\{\boldsymbol{\sigma}\}}_{[\mathbf{P}_\pi] \{\bar{\boldsymbol{\sigma}}\}} = \underbrace{[\bar{\mathbf{C}}]}_{[\mathbf{P}_\pi] [\bar{\mathbf{C}}] [\mathbf{P}_\pi]^{-1}} \underbrace{\{\boldsymbol{\epsilon}\}}_{[\mathbf{P}_\pi] \{\bar{\boldsymbol{\epsilon}}\}} \rightarrow \begin{cases} \bar{\sigma}_{11} \\ \bar{\sigma}_{22} \\ \bar{\sigma}_{12} \\ \bar{\sigma}_{13} \\ \bar{\sigma}_{23} \\ 0 \end{cases} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{16} & \bar{C}_{15} & \bar{C}_{14} & \bar{C}_{13} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{26} & \bar{C}_{25} & \bar{C}_{24} & \bar{C}_{23} \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{66} & \bar{C}_{56} & \bar{C}_{46} & \bar{C}_{36} \\ \bar{C}_{15} & \bar{C}_{25} & \bar{C}_{56} & \bar{C}_{55} & \bar{C}_{45} & \bar{C}_{35} \\ \bar{C}_{14} & \bar{C}_{24} & \bar{C}_{46} & \bar{C}_{45} & \bar{C}_{44} & \bar{C}_{34} \\ \bar{C}_{13} & \bar{C}_{23} & \bar{C}_{36} & \bar{C}_{35} & \bar{C}_{34} & \bar{C}_{33} \end{bmatrix} \begin{cases} \bar{\epsilon}_{11} \\ \bar{\epsilon}_{22} \\ 2\bar{\epsilon}_{12} \\ 2\bar{\epsilon}_{13} \\ 2\bar{\epsilon}_{23} \\ \bar{\epsilon}_{33} \end{cases} \quad (24)$$

Again it is possible to solve the value of $\bar{\epsilon}_{33}$ in terms of remaining strains and the elements of elasticity tensor. Defining the following submatrices:

$$[\bar{\mathbf{C}}]_{5 \times 5} = [\bar{\mathbf{C}}]_{1:5,1:5} \quad \text{and} \quad \{\bar{\mathbf{C}}_6\}_{1 \times 5} = [\bar{\mathbf{C}}]_{6,1:5} \quad (25)$$

The plane strain relationship considering the shear deformation can be defined as follows:

$$\{\bar{\boldsymbol{\sigma}}\} = \begin{cases} \bar{\sigma}_{11} \\ \bar{\sigma}_{22} \\ \bar{\sigma}_{12} \\ \bar{\sigma}_{13} \\ \bar{\sigma}_{23} \end{cases} = [\bar{\mathbf{Q}}] \{\bar{\boldsymbol{\epsilon}}\} \triangleq \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} & \bar{Q}_{15} & \bar{Q}_{14} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} & \bar{Q}_{25} & \bar{Q}_{24} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} & \bar{Q}_{56} & \bar{Q}_{46} \\ \bar{Q}_{15} & \bar{Q}_{25} & \bar{Q}_{56} & \bar{Q}_{55} & \bar{Q}_{45} \\ \bar{Q}_{14} & \bar{Q}_{24} & \bar{Q}_{46} & \bar{Q}_{45} & \bar{Q}_{44} \end{bmatrix} \begin{cases} \bar{\epsilon}_{11} \\ \bar{\epsilon}_{22} \\ 2\bar{\epsilon}_{12} \\ 2\bar{\epsilon}_{13} \\ 2\bar{\epsilon}_{23} \end{cases} \quad (26)$$

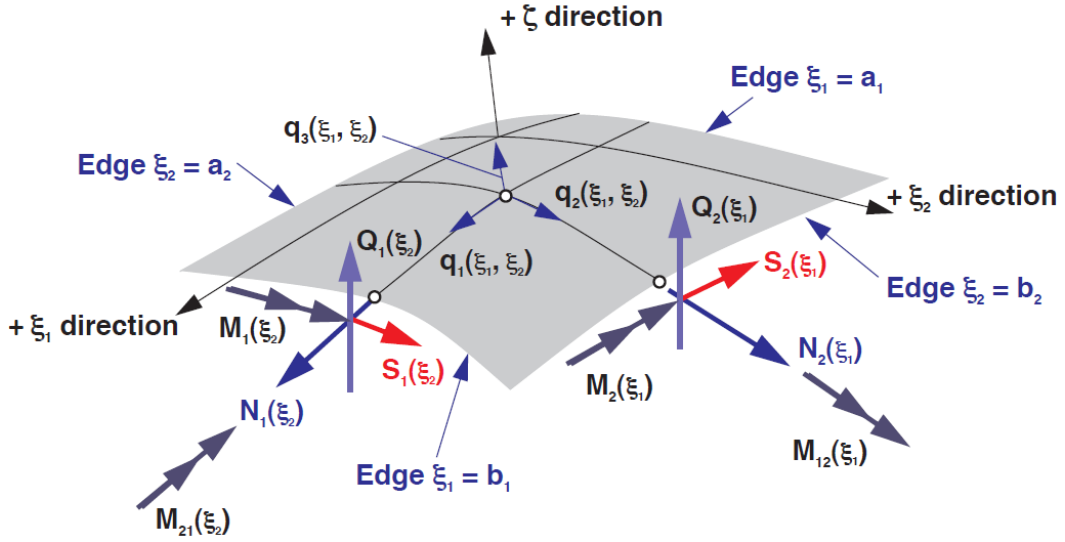


Figure 1. Conventions for applied load, moments and stress resultants [18]

Where the $[\tilde{\mathbf{Q}}]$ can be obtained from:

$$[\tilde{\mathbf{Q}}] = [\tilde{\mathbf{C}}] - \left(\frac{1}{\tilde{C}_{66}} \right) [\text{diag}(\{\tilde{\mathbf{C}}_6\})] [\tilde{\mathbf{C}}] \quad (27)$$

In equation (27), $\text{diag}()$ is the mathematical operator that constructs a diagonal matrix from the elements of a vector. Similar to the previous section, the permuted strain-stress relationship can be transformed back into their original form using the following definitions:

$$[\underline{\mathbf{Q}}] = \begin{bmatrix} \tilde{\mathbf{Q}}_{5 \times 5} & 0 \\ 0 & 0 \end{bmatrix} \rightarrow [\tilde{\mathbf{Q}}] = [\mathbf{P}_\pi]^{-1} [\underline{\mathbf{Q}}] [\mathbf{P}_\pi] \quad (28)$$

And in the original form of equation (14) this can be rewritten as follows:

$$\{\bar{\sigma}_{11} \quad \bar{\sigma}_{22} \quad 0 \quad \bar{\sigma}_{23} \quad \bar{\sigma}_{13} \quad \bar{\sigma}_{12}\}^T = [\tilde{\mathbf{Q}}]_{6 \times 6} \{\bar{\epsilon}_{11} \quad \bar{\epsilon}_{22} \quad 0 \quad 2\bar{\epsilon}_{23} \quad 2\bar{\epsilon}_{13} \quad 2\bar{\epsilon}_{12}\}^T \quad (29)$$

4 Constitutive Equations

The equilibrium equations of the shell are obtained using the principle of virtual work as a function of work-conjugate stress resultants. The conical shells equilibrium equations that are obtained based on Sanders' improved first order linear theory [20] are presented in appendix C.

Work-conjugate stress resultants of equation (C.2) are approximated symmetric stress-resultants. The conventions for loads and stress resultants are shown in figure 1. By substituting stress strain relationship of (22) (or (29) when shear is considered); work-conjugate stress resultants can be expressed in terms of fundamental unknowns ($u_1, u_2, u_3, \gamma_{13}^o, \gamma_{23}^o$). Therefore the the two dimensional constitutive equations of the shell can be defined as follows:

$$\begin{bmatrix} \{\mathbf{n}\} \\ \{\mathbf{m}\} \\ \{\mathbf{f}_1\} \\ \{\mathbf{f}_2\} \\ \{\mathbf{q}\} \end{bmatrix} = [\mathbf{c}\mathbf{c}^0] \{\mathbf{E}^o\} \triangleq \left[\int_{-\frac{h}{2}}^{+\frac{h}{2}} \left(1 + \frac{\xi_3}{R_1}\right)^{-1} \left(1 + \frac{\xi_3}{R_2}\right)^{-1} [\mathbf{s}]^T [\mathbf{Q}] [\mathbf{s}] d\xi_3 \right] \{\mathbf{E}^o\} \quad (30)$$

Details of stress resultants of equation (30) are provided in appendix B. In general, due to presence of principle and geodesic radii of curvature specially in the denominator, the constitutive matrix $[\mathbf{c}\mathbf{c}^0]$ can demonstrate very complex mathematical expressions that can lead to even more complex definitions for work-conjugate stress resultants.

To reduce these terms to some manageable equations; all fractions containing combination of principle and geodesic radii of curvature can be replaced by their second order Taylor series in ξ_3 direction. This approach results in three-parts equations containing zero, first and second orders. First and second order terms can be included or excluded from the equilibrium equations with two flag multipliers such as $\tau_0, \tau \in \{0, 1\}$ accordingly. Using this notation, the elements of 12×12 symmetric $[\mathbf{c}\mathbf{c}^0]$ are presented in appendix C. This formulation provides a flexible approach to compare models with different levels of complexities especially for thick shells. For example setting $\tau_0 = \tau = 0$ results in acquiring a set of equations similar to what was presented in [25]. It should be noted that, after performing through-the-thickness integrations all elements of $[\mathbf{c}\mathbf{c}^0]$ matrix can be expressed in terms of principle and geodesic radii of curvature and elasticity associated coefficients $A_{pq}^k, R_{pq}^{jk}, Q_{pq}^{ijk}, W_{pq}^{jk}, X_{pq}^{ijk}, Y_{pq}^{ijk}$ and W_{pq}^{ijk} that their definitions can be found in appendix E. It is worth mentioning that A_{pq}^0, A_{pq}^1 and A_{pq}^2 are equivalent to [A],[B] and [D] matrices of the classical laminated-plate theory of [26].

Additionally as will be shown in the following sections, to produce homogeneous equilibrium equations for conical shells, the shell is approximated with a linearly variable thickness equivalent truncated cone. For such a case $[\mathbf{c}\mathbf{c}^0]$ shows dependency to ξ_1 . The constitutive matrix with substituted conical principle and geodesic radii of curvature for linearly variable thickness truncated cones is called $[\bar{\mathbf{c}}\bar{\mathbf{c}}^0]$ and presented in appendix H. Related coefficients for this matrix are marked with *bar* (e.g. \bar{A}_{pq}^0) and provided in appendix E.

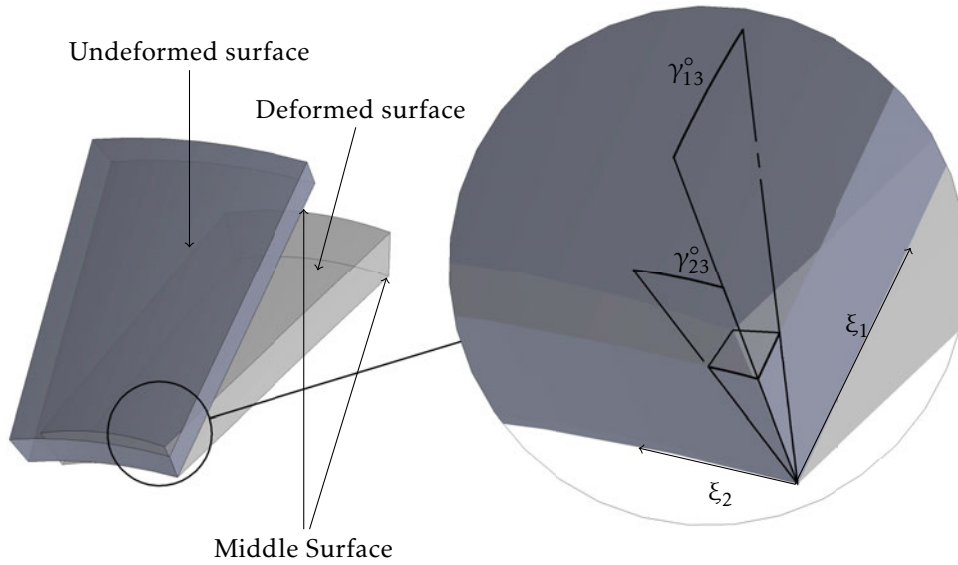


Figure 2. Transverse Shear Deformation

5 Shear Deformation

As has been stated earlier, the through-the-thickness distributions of the transverse shearing strain have been defined by two analyst-defined functions $F_1(\xi_3)$ and $F_2(\xi_3)$. For first order shear deformation theory it can be assumed that:

$$F_1(\xi_3) = F_2(\xi_3) = \xi_3 \quad (31)$$

Therefore the effect of shear results in linear displacement of a point off the middle surface along the angles that have been shown in figure 2.

6 Linear Equilibrium Equations of Conical shells

7 Equilibrium Equations Derivation

Using principle of virtual works five linear equilibrium equations of conical shells based on the already defined kinematics and in terms of stress resultants can be obtained [18]. These five equations are presented in appendix C.

7.1 Solution Form and the Characteristic Equation

Substituting work-conjugate stress resultants of (30) into equilibrium equation (C.2) of appendix C results in obtaining linear equations that are only functions of five fundamental un-

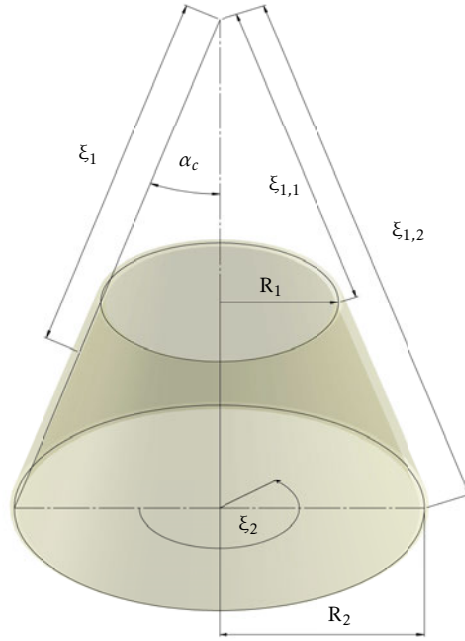


Figure 3. Conical shell element dimensions and coordinates

knowns. Since these equations even for the simplest case of single layer isotropic materials without any shear deformation are found to be non-homogeneous in terms of powers of ξ_1 , additional assumptions are needed to facilitate obtaining an analytical solution. To that end, it is assumed that the thickness of shell varies linearly along the ξ_1 coordinates:

$$t = \delta \xi_1 \quad (32)$$

where δ denotes the thickness variation proportionality. It should be noted that to simulate constant thickness shells, the value of δ is chosen in a way that the constant thickness appears in the middle of any element of the shell. To employ the separation of variable technique the solution is assumed to have the following form:

$$u_1 = u_{1,\bar{x}}(\bar{x}) (\cos(n_c \xi_2)) \quad (33a)$$

$$u_2 = u_{2,\bar{x}}(\bar{x}) (\sin(n_c \xi_2)) \quad (33b)$$

$$u_3 = u_{3,\bar{x}}(\bar{x}) (\cos(n_c \xi_2)) \quad (33c)$$

$$\gamma_{13}^o = u_{4,\bar{x}}(\bar{x}) (\cos(n_c \xi_2)) \quad (33d)$$

$$\gamma_{23}^{\circ} = u_{5,\bar{x}}(\bar{x}) (\sin(n_c \xi_2)) \quad (33e)$$

where

- n_c is the circumferential mode number.
- $\bar{x} = \frac{(\xi_1)}{L}$
- L is a reference length with the suggested value of $L = 0.5(\xi_{1,0} + \xi_{1,1})$ or the ξ_1 coordinates of the middle of the element.

For the longitudinal direction the following solution form that guarantees the homogeneity of the equilibrium equations is chosen for five fundamental unknowns:

$$u_{d,\bar{x}} = C_d(\bar{x})^{\frac{(\lambda-s_d)}{2}} \quad (s_1 = s_2 = s_3 = 1; \quad s_4 = s_5 = 3) \quad (34)$$

where C_d is the arbitrary magnitude of the displacement. Substituting equation (33) in equilibrium equations of (C.2) of appendix C and some lengthy mathematical manipulations yields following five equations:

$$\mathfrak{L}_{i,1} \sin(n_c \xi_2) + \mathfrak{L}_{i,2} \cos(n_c \xi_2) = 0 \quad (i = 1, 2, \dots, 5) \quad (35)$$

$\mathfrak{L}_{i,j}$ (s) are only functions of $u_{d,x}$ and the associated derivatives along ξ_1 direction, circumferential mode number and shell parameters such as elasticity and geometry. In other words they do not contain any dependency to ξ_2 . Since ξ_2 can take any arbitrary value ($0 \leq \xi_2 \leq 2\pi$), for satisfying equations (35), coefficients of both $\sin(n_c \xi_2)$ and $\cos(n_c \xi_2)$ should be zero or $\mathfrak{L}_{i,j} = 0$ ($i = 1 \dots 5, \quad j = 1, 2$) and this process produces a system of 10 equations. By substituting equation (34) in equation (35) and after some mathematical manipulations, the equilibrium equations can be written in the following matrix form:

$$[\mathcal{C}\bar{\mathcal{P}}]_{10 \times 25} \{\mathcal{D}\}_{25 \times 1} = \begin{bmatrix} \mathcal{C}\bar{\mathcal{P}}_{1,1} & \mathcal{C}\bar{\mathcal{P}}_{1,2} & \cdots & \mathcal{C}\bar{\mathcal{P}}_{1,25} \\ \mathcal{C}\bar{\mathcal{P}}_{2,1} & \mathcal{C}\bar{\mathcal{P}}_{2,2} & \cdots & \mathcal{C}\bar{\mathcal{P}}_{2,25} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{C}\bar{\mathcal{P}}_{10,1} & \mathcal{C}\bar{\mathcal{P}}_{10,2} & \cdots & \mathcal{C}\bar{\mathcal{P}}_{10,25} \end{bmatrix} \begin{Bmatrix} \mathcal{D}^0 \\ \mathcal{D}^1 \\ \mathcal{D}^2 \\ \mathcal{D}^3 \\ \mathcal{D}^4 \end{Bmatrix} = 0 \quad (36)$$

The non-zero elements of $[\mathcal{C}\bar{\mathcal{P}}]$ matrix are provided in appendix F. The elements of $\{\mathcal{D}^n\}_{5 \times 1}$ vector are defined as follows:

$$\{\mathcal{D}^n\} = \{D_1^n \quad D_2^n \quad D_3^n \quad D_4^n \quad D_5^n\}^T \quad (37)$$

$$\mathcal{D}_d^n = C_d (\xi_1)^{\left(\frac{s_d-1}{2}+n\right)} \frac{\partial^n}{\partial \xi_1^n} u_{d,\bar{x}} = C_d \left(\frac{\xi_1}{L}\right)^{\left(\frac{\lambda-1}{2}\right)} g(\lambda, s_d, n) \quad (d = 1, 2, \dots, 5) \quad (38)$$

Function $g(\lambda, s_d, n)$ in equation (38) is a polynomial in terms of λ that is defined by the following equation:

$$g(\lambda, s_d, n) = \left(\frac{1}{2}\right)^n L^{\left(\frac{s_d-1}{2}\right)} \prod_{k=1}^n (\lambda - s_d - 2k + 2) \quad (39)$$

Obviously the common term of $\left(\frac{\xi_1}{L}\right)^{\left(\frac{\lambda-1}{2}\right)}$ can be canceled out from $\{\mathcal{D}\}$ vector of system of equations of (36). After removing that term, the system can be rearranged to be defined as a system of equations in terms of arbitrary magnitudes of displacements A_d :

$$[\mathcal{AP}]_{10 \times 5} \{A\}_{5 \times 1} = \begin{bmatrix} \mathcal{AP}_{1,1} & \mathcal{AP}_{1,2} & \cdots & \mathcal{AP}_{1,5} \\ \mathcal{AP}_{2,1} & \mathcal{AP}_{2,2} & \cdots & \mathcal{AP}_{2,5} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{AP}_{10,1} & \mathcal{AP}_{10,2} & \cdots & \mathcal{AP}_{10,5} \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{Bmatrix} = 0 \quad (40)$$

Elements of $[\mathcal{AP}]$ matrix are polynomials of λ and can be obtained from the following equation:

$$\mathcal{AP}_{i,d} = \sum_{n=0}^4 (\mathcal{C}\vec{\mathcal{P}}_{i,s_d+5n}) g(\lambda, s_d, n) \quad (41)$$

Depending on the presence of shear deformation, the order of expansion of the principle and geodesic radii of curvature in calculating the constitutive matrix $[\mathcal{CC}^0]$ (e.g. $\tau_0 \neq 0$ and $\tau \neq 0$), the $[\mathcal{AP}]$ matrix could have sub-column rank or full column-rank. For example for shells consisting of a single layer isotropic material and with the assumptions of no shear-deformation and zero order expansion of radii of curvature, the $[\mathcal{AP}]$ matrix is reduced to a square 3×3 matrix similar to what presented in [25].

To satisfy any arbitrary magnitude of displacements, system of equation (40) should have infinite number of solutions. When after removing zero rows and columns, $[\mathcal{AP}]$ is found to be a square matrix, this requirement can be achieved through equating the determinant of the $[\mathcal{AP}]$ matrix to zero. Since elements of $[\mathcal{AP}]$ matrix are polynomials in terms of λ , that results in obtaining a characteristic equation. In other cases (for non-square forms), the problem can be treated as a linear least square problem that has an exact solution as described in the solution of the following

system:

$$[\mathcal{A}\mathcal{Q}]\{\mathbf{A}\} = [\mathcal{A}\mathcal{P}]^\top [\mathcal{A}\mathcal{P}]\{\mathbf{C}\} = 0 \quad (42)$$

In other words the $[\mathcal{A}\mathcal{Q}]$ matrix can be defined conditionally as follows:

$$[\mathcal{A}\mathcal{Q}] = \begin{cases} \text{rank-submatrix of } [\mathcal{A}\mathcal{P}] & \text{if } [\mathcal{A}\mathcal{P}] \text{ has square rank-submatrix} \\ \text{rank-submatrix of } [\mathcal{A}\mathcal{P}]^\top [\mathcal{A}\mathcal{P}] & \text{otherwise} \end{cases} \quad (43)$$

Using the above definition the characteristic equation can be obtained from the determinant of $[\mathcal{A}\mathcal{Q}]$ matrix:

$$\left| [\mathcal{A}\mathcal{Q}] \right| = P_c(\lambda) = 0 \quad (44)$$

Depending the assumption used, the characteristic polynomial of equation (44) could have between eight up to sixteen distinct roots and the number of distinct roots defines the degrees of the freedom of the system. It is worth mentioning that when the system is treated as a linear least square system, the degrees of the characteristic polynomial could raise up to 32. But the number of distinct roots remains equal to half a that number due to multiplicity of the roots of the characteristic polynomial. Assuming presence of K distinct roots the final solution of the system is obtained by summation of all those solutions.

$$\mathbf{u}_d(\xi_1, \xi_2) = \left(\sum_{k=1}^K C_{d,k} \left(\bar{x}^{\frac{(\lambda_k - s_d)}{2}} \right) \right) (\sin(n_c \xi_2)^{se_d}) (\cos(n_c \xi_2)^{ce_d}) \quad (45a)$$

$$se_d = \begin{cases} 0 & (d = 1, 3, 4) \\ 1 & (d = 2, 5) \end{cases} \quad ce_d = \begin{cases} 1 & (d = 1, 3, 4) \\ 0 & (d = 2, 5) \end{cases} \quad (45b)$$

8 Finite Element Formulation

To preserve the symmetry, the element of current study is bounded by two line nodes such as i and j as shown in figure 4. The nodal degrees of freedom of the elements of the current study are defined as the derivatives (including zero order) of ξ_1 component of unknown displacements at two line nodes. Since depending on the simplification assumptions (e.g. presence or absence of shear deformation and etc.) the number of DOFs can vary, they are chosen from an ordered

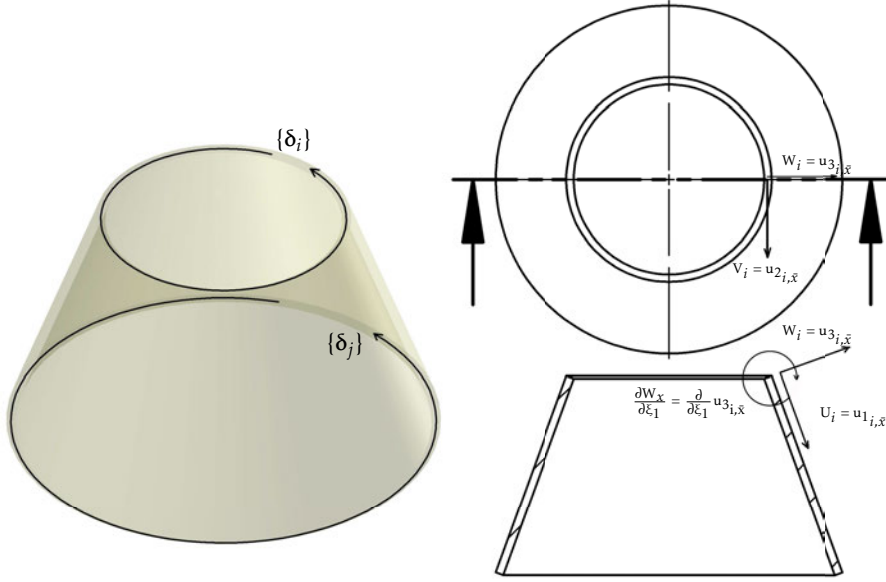


Figure 4. Conical shell element line nodes and translational degrees-of-freedom

list that has been presented in table 1. For example in absence of shear deformation and using isotropic materials, the characteristic equation, has eight distinct roots [25]. In that case each node is assigned with the first four degrees of freedom of table 1 to produce eight total degrees of freedom.

Equation (44) implies linear dependency between elements of $[\mathcal{AP}]$ matrix. Therefore it is expected that $5 - 1 = 4$ elements of $\{C\}$ vector can be expressed in terms of the remaining element. So without losing generality to determine the five unknown arbitrary magnitudes of the displacements ($\{C_{1,k}, C_{2,k}, C_{3,k}, C_{4,k}, C_{5,k}\}^\top$) associated with each λ_k , first we express all of them in terms of $C_{1,k}$. This can be done through solving the following reduced system at $\lambda = \lambda_k$:

$$\begin{bmatrix} \mathcal{A}Q_{2,2} & \mathcal{A}Q_{2,3} & \mathcal{A}Q_{2,4} & \mathcal{A}Q_{2,5} \\ \mathcal{A}Q_{3,2} & \mathcal{A}Q_{3,3} & \mathcal{A}Q_{3,4} & \mathcal{A}Q_{3,5} \\ \mathcal{A}Q_{4,2} & \mathcal{A}Q_{4,3} & \mathcal{A}Q_{4,4} & \mathcal{A}Q_{4,5} \\ \mathcal{A}Q_{5,2} & \mathcal{A}Q_{5,3} & \mathcal{A}Q_{5,4} & \mathcal{A}Q_{5,5} \end{bmatrix}_{|\lambda=\lambda_k} \begin{Bmatrix} \alpha_{k,2} \\ \alpha_{k,3} \\ \alpha_{k,4} \\ \alpha_{k,5} \end{Bmatrix} = \begin{Bmatrix} -\mathcal{A}Q_{2,1} \\ -\mathcal{A}Q_{3,1} \\ -\mathcal{A}Q_{4,1} \\ -\mathcal{A}Q_{5,1} \end{Bmatrix}_{|\lambda=\lambda_k} \quad (46)$$

It should be recalled that to construct the system of equations of (46), only full rank-submatrix of $[\mathcal{A}Q]$ should be considered. For example for isotropic materials with no shear deformation and neglecting principle and geodesic radii of curvature, after removing zero rows and columns the left hand side matrix is reduced to a 2×2 matrix, similar to what presented in [25]. Solving linear

system of equation (46), the relative arbitrary magnitude of displacements associated with each λ_k can be expressed in the following form:

$$\begin{aligned}
 C_{1,k} &= (1)C_{1,k} \\
 C_{2,k} &= \alpha_{2,k}C_{1,k} \\
 C_{3,k} &= \alpha_{3,k}C_{1,k} \\
 C_{4,k} &= \alpha_{4,k}C_{1,k} \\
 C_{5,k} &= \alpha_{5,k}C_{1,k}
 \end{aligned} \tag{47}$$

At this point, the problem is reduced to finding K unknown constants $C_{1,k}$ associated with each λ_k . Assuming δ_m is the nodal degree of freedom of the system placed on the ring located at \bar{x}_m and it is associated with the n^{th} derivative of d_m displacement, its value follows a similar structure as what was presented in equation (37):

$$\delta_m \Big|_{\bar{x}=\bar{x}_m} = \frac{\partial^n}{\partial \xi_1^n} u_{d_m, \bar{x}} \Big|_{\bar{x}=\bar{x}_m} = \sum_{k=1}^K C_{1,k} \left(\alpha_{d_m, k}(\bar{x}_m)^{\left(\frac{\lambda_k-1}{2}-n\right)} g_2(\lambda_k, s_{d_m}, n) \right) = \sum_{k=1}^K C_{1,k} b_{m,k} \tag{48}$$

$$g_2(\lambda, s_d, n) = \left(\frac{1}{2L} \right)^n \prod_{j=1}^n (\lambda - s_d - 2j + 2) \tag{49}$$

Investigating equations (48) and (49) reveals that the only variable parameters are the arbitrary amplitudes of vibrations or $C_{1,k}$ ($k = 1, 2, \dots, k$) because $b_{m,k}$ coefficients are estimated by known parameters. In other words, equation (48) can be used to construct the following system of equations that defines the dependency of unknown amplitudes of vibration of the system to the chosen nodal degrees of freedom:

$$\{\delta\} = \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_K \end{Bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,K} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,K} \\ \vdots & \vdots & \ddots & \vdots \\ a_{K,1} & a_{K,2} & \cdots & a_{K,K} \end{bmatrix} \begin{Bmatrix} C_{1,1} \\ C_{1,2} \\ \vdots \\ C_{1,K} \end{Bmatrix} = [\mathbf{A}]\{\mathbf{C}_1\} \tag{50}$$

Subsequently:

$$\{\mathbf{C}_1\} = [\mathbf{A}]^{-1} \{\delta\} \tag{51}$$

Rearranging to the matrix form and substituting (51) in equation (45) yields the following dis-

Table 1. List of ordered nodal degrees of freedom

No.	Variable	Description
1	$U_m = u_{1m,\bar{x}}$	Shell longitudinal displacement
2	$W_m = u_{3m,\bar{x}}$	Shell normal displacement
3	$\frac{\partial W_m}{\partial \xi_1} = \frac{\partial}{\partial \xi_1} u_{3m,\bar{x}}$	Shell normal displacement slope
4	$V_m = u_{2m,\bar{x}}$	Shell circumferential displacement
5	$\gamma_{13m}^\circ = u_{4m,\bar{x}}$	Transverse shear deformation along ξ_1
6	$\gamma_{23m}^\circ = u_{5m,\bar{x}}$	Transverse shear deformation along ξ_2
7	$\frac{\partial U_m}{\partial \xi_1} = \frac{\partial}{\partial \xi_1} u_{1m,\bar{x}}$	Longitudinal displacement slope
8	$\frac{\partial V_m}{\partial \xi_1} = \frac{\partial}{\partial \xi_1} u_{2m,\bar{x}}$	Circumferential displacement slope

placement matrix:

$$\{\mathbf{u}\} = \begin{Bmatrix} u_1(\xi_1, \xi_2) \\ u_2(\xi_1, \xi_2) \\ u_3(\xi_1, \xi_2) \\ \gamma_{13}^\circ(\xi_1, \xi_2) \\ \gamma_{23}^\circ(\xi_1, \xi_2) \end{Bmatrix} = \overbrace{[\mathbf{N}]_{5 \times K}}^{\text{displacement matrix}} [\mathbf{A}]^{-1} \{\boldsymbol{\delta}\} = \begin{bmatrix} R_{1,1} & R_{1,2} & \cdots & R_{1,K} \\ R_{2,1} & R_{2,2} & \cdots & R_{2,K} \\ \vdots & \vdots & \ddots & \vdots \\ R_{5,1} & R_{5,2} & \cdots & R_{5,K} \end{bmatrix} \begin{bmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,K} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,K} \\ \vdots & \vdots & \ddots & \vdots \\ c_{K,1} & c_{K,2} & \cdots & c_{K,K} \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_K \end{Bmatrix} \quad (52)$$

where elements of $[\mathbf{R}]$ are defined as follows:

$$R_{d,k}(\xi_1, \xi_2) = \left(\frac{\xi_1}{L}\right)^{\frac{(\lambda_k - s_d)}{2}} (\sin(n_c \xi_2)^{s_d}) (\cos(n_c \xi_2)^{c_d}) \quad (53)$$

Subsequently elements of the displacement matrix $[\mathbf{N}]$ are obtained as follows:

$$N_{d,k}(\xi_1, \xi_2) = \sum_{i=1}^K c_{i,k} R_{d,i} \quad (54)$$

as the n^{th} ($n = 0 \dots$) derivative of d^{th} unknown displacement along the ξ_1 coordinates:

8.1 Notes on the Displacement Matrix Elements

The terms of displacement matrix demonstrate interesting properties that as will be shown, those properties can be exploited to formulate the nonlinear problem in a mathematically eloquent way. First it should be recalled that since the coefficients of the characteristic polynomial are real-valued, the roots should appear either as real numbers or pair of complex conjugate numbers. Therefore if $\lambda_k \in \mathbb{C}$ be found among the roots its complex conjugate such as $\lambda_k^* \in \mathbb{C}$ should be in the roots' set too. It can be proven that the same rule applies to the $c_{i,k}$ elements of $[\mathbf{A}]^{-1}$. In other words if $R_{d,k}$ and R_{d,k^*} are associated with λ_k and its conjugate λ_k^* , the corresponding constant coefficients are also complex conjugates of each other or $c_{i,k} = (c_{i,k^*})^*$. Therefore $N_{d,k}(\xi_1, \xi_2)$ is defined over a function space that its basis function can be defined as follows:

$$H(\beta, c, se, ce) : (\xi_1, \xi_2) \rightarrow \left[c \left((\xi_1/L)^\beta \right) + c^* \left((\xi_1/L)^{\beta^*} \right) \right] (\sin(n_c \xi_2))^{se} (\cos(n_c \xi_2))^{ce} \in \mathbb{R} \quad (55)$$

In special case of real roots when $\lambda_k^* = \lambda_k$ (subsequently $c = c^*$) and there is only a single element associated with that root in each line of $[\mathbf{R}]$ matrix, the corresponding basis function takes the following form:

$$H(\beta, 0.5 * c, se, ce) : (\xi_1, \xi_2) \rightarrow \left[0.5 * c (\xi_1/L)^\beta + 0.5 * c (\xi_1/L)^\beta \right] (\sin(nc \xi_2))^{se} (\cos(nc \xi_2))^{ce} \in \mathbb{R} \quad (56)$$

The evaluation of (55) (or (56)) results in real numbers only because the imaginary parts of the pair of terms cancel each other out. Therefore by expressing the displacement functions in this form, results in obtaining real-valued finite element matrices (stiffness, mass and etc.) that reduces numerical errors. Using this definition, each element of the displacement matrix can be represented in the following form:

$$N_{d,k}(\xi_1, \xi_2) = \sum_{i=1}^M H(\beta_i, c_i, se_i, ce_i) = S_{d,k}(\boldsymbol{\beta}, \mathbf{c}, \mathbf{se}, \mathbf{ce}) \quad (\beta_i, c_i \in \mathbb{C}; se_i, ce_i \in \mathbb{N}) \quad (57)$$

The function space defined by \mathbb{S} demonstrates certain properties that are presented in appendix G. In short these properties mandate that addition, subtraction, multiplication and differentiation with respect to principle surface coordinates generate terms that belong to \mathbb{S} . Moreover surface metrics, principle and geodesic radii of curvature and any real number can be defined over \mathbb{S} . Using these definitions and properties, each displacement can be rewritten in the following form:

$$\mathbf{u}_d = \{\mathbf{S}_d\}_{1 \times K} \{\boldsymbol{\delta}\}_{K \times 1} \quad (\{\mathbf{S}_d\} \in \mathbb{S}^K) = \{S_{d,1} \quad S_{d,2} \quad \cdots \quad S_{d,K}\} \{\delta_1 \quad \delta_2 \quad \cdots \quad \delta_K\}^T \quad (58)$$

Subsequently using equations (G.1e) and (G.1f):

$$\frac{\partial}{\partial \xi_1} u_d = \{\mathbf{S}_d^{\xi_1}\}_{1 \times K} \{\boldsymbol{\delta}\}_{K \times 1} = \left\{ \frac{\partial}{\partial \xi_1} S_{d,1} \quad \frac{\partial}{\partial \xi_1} S_{d,2} \quad \cdots \quad \frac{\partial}{\partial \xi_1} S_{d,K} \right\} \left\{ \delta_1 \quad \delta_2 \quad \cdots \quad \delta_K \right\}^T \quad (59a)$$

$$\frac{\partial}{\partial \xi_2} u_d = \{\mathbf{S}_d^{\xi_2}\}_{1 \times K} \{\boldsymbol{\delta}\}_{K \times 1} = \left\{ \frac{\partial}{\partial \xi_2} S_{d,1} \quad \frac{\partial}{\partial \xi_2} S_{d,2} \quad \cdots \quad \frac{\partial}{\partial \xi_2} S_{d,K} \right\} \left\{ \delta_1 \quad \delta_2 \quad \cdots \quad \delta_K \right\}^T \quad (59b)$$

9 Equations of Motion

In equation (58) the vector defined by $\{\mathbf{S}_d\}$, contains the spatial component of shell motion and if we assume that $\{\boldsymbol{\delta}\} = \{\boldsymbol{\delta}(t)\}$ contains the temporal (time-dependent) component of shell motion, it is possible to obtain equations of motion of the shell using generalized-coordinates method. The Lagrangian equation of motion based on Hamilton's principle can be expressed as follows:

$$\frac{d}{dt} \left[\frac{\partial T}{\partial \dot{\delta}_i} \right] - \frac{\partial T}{\partial \delta_i} + \frac{\partial V}{\partial \delta_i} = q_i, \quad (i = 1, 2, \dots, K) \quad (60)$$

where

- T is the kinetic energy of the system
- V is the elastic strain energy of the system
- q_i is the nodal external force

Equation (60) can be rewritten in the matrix form as follows:

$$\frac{d}{dt} \left[\frac{\partial T}{\partial \dot{\boldsymbol{\delta}}} \right] - \frac{\partial T}{\partial \boldsymbol{\delta}} + \frac{\partial V}{\partial \boldsymbol{\delta}} = \{\mathbf{q}\} \quad (61)$$

9.1 Kinetic Energy

The kinetic energy has three parts; pure translational, cross translational-rotational and pure rotational terms. Translational and rotational velocity components of the shell element can be defined as follows:

$$\{\dot{\mathbf{u}}_{\mathbf{T}}\} = \left\{ \begin{matrix} \dot{\mathbf{u}}_{\mathbf{T}1,2} \\ \dot{\mathbf{u}}_3 \end{matrix} \right\} = \left\{ \begin{matrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{matrix} \right\} = [\mathbf{S}_{\mathbf{T}}] \{\dot{\boldsymbol{\delta}}\} = \begin{bmatrix} [\mathbf{S}_{\mathbf{T}1,2}]_{2 \times K} \\ [\mathbf{S}_{\mathbf{T}3}]_{1 \times K} \end{bmatrix} \{\dot{\boldsymbol{\delta}}\} = \begin{bmatrix} S_{1,1} & S_{1,2} & \cdots & S_{1,K} \\ S_{2,1} & S_{2,2} & \cdots & S_{2,K} \\ S_{3,1} & S_{3,2} & \cdots & S_{3,K} \end{bmatrix} \begin{Bmatrix} \dot{\delta}_1 \\ \vdots \\ \dot{\delta}_K \end{Bmatrix} \quad (62a)$$

$$\{\dot{\mathbf{u}}_{\mathbf{R}}\} = \begin{Bmatrix} \dot{u}_4 \\ \dot{u}_5 \end{Bmatrix} = [\mathbf{S}_{\mathbf{R}}] \{\dot{\boldsymbol{\delta}}\} = \begin{bmatrix} S_{4,1} & S_{4,2} & \cdots & S_{4,K} \\ S_{5,1} & S_{5,2} & \cdots & S_{5,K} \end{bmatrix} \begin{Bmatrix} \dot{\delta}_1 \\ \vdots \\ \dot{\delta}_K \end{Bmatrix} \quad (62b)$$

Using above definitions, the kinetic energy components can be obtained as follows:

$$T_{\mathbf{T}} = \frac{1}{2} \iint_{\Omega} \rho^0 (\{\dot{\mathbf{u}}_{\mathbf{T}}\}^{\top} \cdot \{\dot{\mathbf{u}}_{\mathbf{T}}\}) d\Omega = \frac{1}{2} \{\dot{\boldsymbol{\delta}}\}^{\top} \left(\iint_{\Omega} \rho^0 ([\mathbf{S}_{\mathbf{T}}]^{\top} [\mathbf{S}_{\mathbf{T}}]) A_1 A_2 d\xi_1 d\xi_2 \right) \{\dot{\boldsymbol{\delta}}\} \quad (63a)$$

$$T_{\mathbf{TR}} = \frac{1}{2} \iint_{\Omega} \rho^1 (\{\dot{\mathbf{u}}_{\mathbf{T},2}\}^{\top} \cdot \{\dot{\mathbf{u}}_{\mathbf{R}}\}) d\Omega = \frac{1}{2} \{\dot{\boldsymbol{\delta}}\}^{\top} \left(\iint_{\Omega} \rho^1 ([\mathbf{S}_{\mathbf{T},2}]^{\top} [\mathbf{S}_{\mathbf{R}}]) A_1 A_2 d\xi_1 d\xi_2 \right) \{\dot{\boldsymbol{\delta}}\} \quad (63b)$$

$$T_{\mathbf{R}} = \frac{1}{2} \iint_{\Omega} \rho^2 (\{\dot{\mathbf{u}}_{\mathbf{R}}\}^{\top} \cdot \{\dot{\mathbf{u}}_{\mathbf{R}}\}) d\Omega = \frac{1}{2} \{\dot{\boldsymbol{\delta}}\}^{\top} \left(\iint_{\Omega} \rho^2 ([\mathbf{S}_{\mathbf{R}}]^{\top} [\mathbf{S}_{\mathbf{R}}]) A_1 A_2 d\xi_1 d\xi_2 \right) \{\dot{\boldsymbol{\delta}}\} \quad (63c)$$

The definitions of the areal densities ρ^0 , ρ^1 and ρ^2 are given in appendix E. The total kinetic energy can be obtained from:

$$T = T_{\mathbf{T}} + T_{\mathbf{TR}} + T_{\mathbf{R}} \quad (64)$$

Investigating equation (61) by taking into account (63) yields:

$$\frac{\partial T}{\partial \delta_i} = 0$$

The mass matrices are defined as follows:

$$[\mathbf{M}_{\mathbf{T}}]_{K \times K} = \iint_{\Omega} \rho^0 ([\mathbf{S}_{\mathbf{T}}]^{\top} [\mathbf{S}_{\mathbf{T}}]) A_1 A_2 d\xi_1 d\xi_2 \quad ([\mathbf{M}_{\mathbf{T}}]^{\top} = [\mathbf{M}_{\mathbf{T}}]) \quad (65a)$$

$$[\mathbf{M}_{\mathbf{TR}}]_{K \times K} = \iint_{\Omega} \rho^1 ([\mathbf{S}_{\mathbf{T},2}]^{\top} [\mathbf{S}_{\mathbf{R}}]) A_1 A_2 d\xi_1 d\xi_2 \quad (65b)$$

$$[\mathbf{M}_{\mathbf{R}}]_{K \times K} = \iint_{\Omega} \rho^2 ([\mathbf{S}_{\mathbf{R}}]^{\top} [\mathbf{S}_{\mathbf{R}}]) A_1 A_2 d\xi_1 d\xi_2 \quad ([\mathbf{M}_{\mathbf{R}}]^{\top} = [\mathbf{M}_{\mathbf{R}}]) \quad (65c)$$

It should be noted that over the truncated conical surface integrals of equation (65) can be calculated using equation (G.7) given in appendix G. Subsequently using equation (J.14) yields:

$$\frac{\partial T_T}{\partial \{\dot{\boldsymbol{\delta}}\}} = \frac{1}{2} \frac{\partial}{\partial \{\dot{\boldsymbol{\delta}}\}} (\{\dot{\boldsymbol{\delta}}\}^\top [\mathbf{M}_T] \{\dot{\boldsymbol{\delta}}\}) = \frac{1}{2} ([\mathbf{M}_T] + [\mathbf{M}_T]^\top) \{\dot{\boldsymbol{\delta}}\} = [\mathbf{M}_T] \{\dot{\boldsymbol{\delta}}\} \quad (66a)$$

$$\frac{\partial T_{TR}}{\partial \{\dot{\boldsymbol{\delta}}\}} = \frac{1}{2} \frac{\partial}{\partial \{\dot{\boldsymbol{\delta}}\}} (\{\dot{\boldsymbol{\delta}}\}^\top [\mathbf{M}_{TR}] \{\dot{\boldsymbol{\delta}}\}) = \frac{1}{2} ([\mathbf{M}_{TR}] + [\mathbf{M}_{TR}]^\top) \{\dot{\boldsymbol{\delta}}\} \quad (66b)$$

$$\frac{\partial T_R}{\partial \{\dot{\boldsymbol{\delta}}\}} = \frac{1}{2} \frac{\partial}{\partial \{\dot{\boldsymbol{\delta}}\}} (\{\dot{\boldsymbol{\delta}}\}^\top [\mathbf{M}_R] \{\dot{\boldsymbol{\delta}}\}) = \frac{1}{2} ([\mathbf{M}_R] + [\mathbf{M}_R]^\top) \{\dot{\boldsymbol{\delta}}\} = [\mathbf{M}_R] \{\dot{\boldsymbol{\delta}}\} \quad (66c)$$

Therefore:

$$\frac{d}{dt} \left[\frac{\partial T}{\partial \{\dot{\boldsymbol{\delta}}\}} \right] = [\mathbf{M}_S] \{\ddot{\boldsymbol{\delta}}\} \triangleq \frac{1}{2} (2[\mathbf{M}_T] + [\mathbf{M}_{TR}] + [\mathbf{M}_{TR}]^\top + 2[\mathbf{M}_R]) \{\ddot{\boldsymbol{\delta}}\} \quad (67)$$

9.2 Internal Strain Energy

The internal strain energy over the shell element surface area (Ω) is defined as:

$$V = \frac{1}{2} \iint_{\Omega} \left(\int_{-h/2}^{h/2} \{\boldsymbol{\sigma}^{\xi_3}\}^\top \{\boldsymbol{\epsilon}^{\xi_3}\} \left(1 + \frac{\xi_3}{R_1}\right) \left(1 + \frac{\xi_3}{R_2}\right) d\xi_3 \right) d\Omega \quad (68)$$

Taking into account equation (5) and (7), the through-thickness integral of equation (68) for variable thickness conical shells can be obtained from the following equation:

$$\int_{-h/2}^{h/2} \{\boldsymbol{\sigma}^{\xi_3}\}^\top \{\boldsymbol{\epsilon}^{\xi_3}\} \left(1 + \frac{\xi_3}{R_1}\right) \left(1 + \frac{\xi_3}{R_2}\right) d\xi_3 = \{\mathbf{E}^\circ\}^\top_{1 \times 12} [\mathcal{C}\bar{\mathcal{C}}^\circ]_{12 \times 12} \{\mathbf{E}^\circ\}_{12 \times 1} \quad (69)$$

The elements of $[\mathcal{C}\bar{\mathcal{C}}^\circ]$ matrix for variable thickness conical shells are given in appendix H. It is worthy to mention that for variable thickness conical shells $[\mathcal{C}\bar{\mathcal{C}}^\circ]$ matrix demonstrates dependency to ξ_1 and therefore should be defined in shape of basis function given by equation (56) or in other words they belong to \mathcal{S} function space.

Using the definition for displacements given by equation (58), the differentiation and other mathematical operations defined in appendix G and the original definitions given in equations (2), (3) and (4) it is possible to express the linear rotation and deformation parameters over \mathcal{S} function

space and nodal degrees of freedom in the following general form:

$$\psi = \{\mathbf{S}_\psi\}_{1 \times K} \{\delta\}_{K \times 1} \quad \left(\{\mathbf{S}_\psi\} \in \mathbb{S}^K \text{ and } \psi \in \{\varphi_1, \varphi_2, \varphi, e_{11}^\circ, e_{22}^\circ, e_{12}^\circ, \chi_{11}^\circ, \chi_{22}^\circ, \chi_{12}^\circ\} \right) \quad (70)$$

where for conical shells:

$$\{\mathbf{S}_{\varphi_1}\} = -H_{1/A_1} \{\mathbf{S}_3^{\xi_1}\} \quad (71a)$$

$$\{\mathbf{S}_{\varphi_2}\} = c_3 H_{1/R_2} \{\mathbf{S}_2\} - H_{1/A_2} \{\mathbf{S}_3^{\xi_2}\} \quad (71b)$$

$$\{\mathbf{S}_\varphi\} = 0.5c_3 \left(H_{1/A_1} \{\mathbf{S}_2^{\xi_1}\} - H_{1/A_2} \{\mathbf{S}_1^{\xi_2}\} + H_{1/\rho_{22}} \{\mathbf{S}_2\} \right) \quad (71c)$$

$$\{\mathbf{S}_{e_{11}^\circ}\} = H_{1/A_1} \{\mathbf{S}_2^{\xi_1}\} \quad (71d)$$

$$\{\mathbf{S}_{e_{22}^\circ}\} = H_{1/A_2} \{\mathbf{S}_{32}^{\xi_2}\} + H_{1/\rho_{22}} \{\mathbf{S}_1\} + H_{1/R_2} \{\mathbf{S}_3\} \quad (71e)$$

$$2\{\mathbf{S}_{e_{12}^\circ}\} = H_{1/A_2} \{\mathbf{S}_1^{\xi_2}\} + H_{1/A_1} \{\mathbf{S}_2^{\xi_1}\} - H_{1/\rho_{22}} \{\mathbf{S}_2\} \quad (71f)$$

$$\{\mathbf{S}_{\chi_{11}^\circ}\} = H_{1/A_1} \{\mathbf{S}_{\varphi_1}^{\xi_1}\} \quad (71g)$$

$$\{\mathbf{S}_{\chi_{22}^\circ}\} = H_{1/A_2} \{\mathbf{S}_{\varphi_2}^{\xi_2}\} + H_{1/\rho_{22}} \{\mathbf{S}_{\varphi_1}\} \quad (71h)$$

$$2\{\mathbf{S}_{\chi_{12}^\circ}\} = H_{1/A_2} \{\mathbf{S}_{\varphi_1}^{\xi_2}\} + H_{1/A_1} \{\mathbf{S}_{\varphi_2}^{\xi_1}\} - H_{1/\rho_{22}} \{\mathbf{S}_{\varphi_2}\} + H_{1/R_2} \{\mathbf{S}_\varphi\} \quad (71i)$$

Additionally:

$$\{\boldsymbol{\gamma}^\circ\} = \begin{Bmatrix} \gamma_{13}^\circ \\ \gamma_{23}^\circ \end{Bmatrix} = \{\mathbf{S}_\gamma\} \{\delta\} = \begin{Bmatrix} \mathbf{S}_4 \\ \mathbf{S}_5 \end{Bmatrix} \{\delta\} \quad (72a)$$

$$\frac{1}{A_1} \frac{\partial}{\partial \xi_1} \{\boldsymbol{\gamma}^\circ\} = \{\mathbf{S}_\gamma^{\xi_1}\} \{\delta\} = \frac{1}{A_1} \frac{\partial}{\partial \xi_1} \begin{Bmatrix} \gamma_{13}^\circ \\ \gamma_{23}^\circ \end{Bmatrix} = H_{1/A_1} \begin{Bmatrix} \mathbf{S}_4^{\xi_1} \\ \mathbf{S}_5^{\xi_1} \end{Bmatrix} \{\delta\} \quad (72b)$$

$$\frac{1}{A_2} \frac{\partial}{\partial \xi_2} \{\boldsymbol{\gamma}^\circ\} = \{\mathbf{S}_{\boldsymbol{\gamma}}^{\xi_2}\} \{\boldsymbol{\delta}\} = \frac{1}{A_2} \frac{\partial}{\partial \xi_2} \begin{Bmatrix} \gamma_{13}^\circ \\ \gamma_{23}^\circ \end{Bmatrix} = H_{1/A_2} \left\{ \begin{Bmatrix} \mathbf{S}_4^{\xi_2} \\ \mathbf{S}_5^{\xi_2} \end{Bmatrix} \right\} \{\boldsymbol{\delta}\} \quad (72c)$$

The nonlinear components of strain presented in equation (4a) can be formulated as dot product of two scalars such as ψ_a and ψ_b . Since all those scalars are defined as a vector over \mathbb{S} multiplied by nodal degrees of freedom (similar to equation (70)); their dot products can be formulated using Kronecker product (\otimes) as follows:

$$\begin{aligned} \psi_a \cdot \psi_b &= \{\mathbf{S}_{\psi_a \otimes \psi_b}\}_{1 \times K^2} \{\boldsymbol{\delta}^{\otimes 2}\}_{K^2 \times 1} = (\{\mathbf{S}_{\psi_a}\}_{1 \times K} \otimes \{\mathbf{S}_{\psi_b}\}_{1 \times K}) (\{\boldsymbol{\delta}\}_{1 \times K} \otimes \{\boldsymbol{\delta}\}_{1 \times K}) \\ &= \begin{Bmatrix} S_{\psi_a,1} S_{\psi_b,1} & S_{\psi_a,1} S_{\psi_b,2} & \cdots & S_{\psi_a,1} S_{\psi_b,K} & S_{\psi_a,2} S_{\psi_b,1} & \cdots & S_{\psi_a,K} S_{\psi_b,K} \end{Bmatrix} \\ &\cdot \begin{Bmatrix} \delta_1^2 & \delta_1 \delta_2 & \cdots & \delta_1 \delta_K & \delta_2 \delta_1 & \delta_2^2 & \cdots & \delta_K^2 \end{Bmatrix}^\top \end{aligned} \quad (73)$$

Using the definitions provided in equations (70) to (73) it is possible to decompose the strain vector $\{\boldsymbol{\epsilon}^\circ\}$ into its linear and nonlinear components:

$$\{\boldsymbol{\epsilon}^\circ\} = [\mathbf{S}_{\boldsymbol{\epsilon}_L}^\circ]_{12 \times K} \{\boldsymbol{\delta}\}_{K \times 1} + [\mathbf{S}_{\boldsymbol{\epsilon}_{NL}}^\circ]_{12 \times K^2} \{\boldsymbol{\delta}^{\otimes 2}\}_{K^2 \times 1} \quad (74)$$

The rows of $[\mathbf{S}_{\boldsymbol{\epsilon}_L}^\circ]$ and $[\mathbf{S}_{\boldsymbol{\epsilon}_{NL}}^\circ]$ are provided in appendix I. Defining the following stiffness matrices:

$$[\mathbf{K}_{11}] = [\mathbf{K}_{11}]^\top \triangleq \iint_{\Omega} [\mathbf{S}_{\boldsymbol{\epsilon}_L}^\circ]^\top [\bar{\mathcal{C}}^0] [\mathbf{S}_{\boldsymbol{\epsilon}_L}^\circ] A_1 A_2 d\xi_1 d\xi_2 \quad (75a)$$

$$[\mathbf{K}_{12}] \triangleq \iint_{\Omega} [\mathbf{S}_{\boldsymbol{\epsilon}_L}^\circ]^\top [\bar{\mathcal{C}}^0] [\mathbf{S}_{\boldsymbol{\epsilon}_{NL}}^\circ] A_1 A_2 d\xi_1 d\xi_2 \quad (75b)$$

$$[\mathbf{K}_{21}] \triangleq \iint_{\Omega} [\mathbf{S}_{\boldsymbol{\epsilon}_{NL}}^\circ]^\top [\bar{\mathcal{C}}^0] [\mathbf{S}_{\boldsymbol{\epsilon}_L}^\circ] A_1 A_2 d\xi_1 d\xi_2 = [\mathbf{K}_{12}]^\top \quad (75c)$$

$$[\mathbf{K}_{22}] \triangleq \iint_{\Omega} [\mathbf{S}_{\boldsymbol{\epsilon}_{NL}}^\circ]^\top [\bar{\mathcal{C}}^0] [\mathbf{S}_{\boldsymbol{\epsilon}_{NL}}^\circ] A_1 A_2 d\xi_1 d\xi_2 \quad (75d)$$

the total strain energy can be written as follows:

$$V = \frac{1}{2} (\{\boldsymbol{\delta}\}^\top [\mathbf{K}_{11}] \{\boldsymbol{\delta}\} + \{\boldsymbol{\delta}\}^\top [\mathbf{K}_{12}] \{\boldsymbol{\delta}^{\otimes 2}\} + \{\boldsymbol{\delta}^{\otimes 2}\}^\top [\mathbf{K}_{21}] \{\boldsymbol{\delta}\} + \{\boldsymbol{\delta}^{\otimes 2}\}^\top [\mathbf{K}_{22}] \{\boldsymbol{\delta}^{\otimes 2}\}) \quad (76)$$

Using equations (J.15),(J.43),(J.46) and(J.61) the derivative of the strain energy with respect to the degrees of freedom can be formulated as:

$$\begin{aligned} \frac{\partial V}{\partial \{\delta\}} &= \frac{1}{2} \left([\mathbf{K}_{11}] + [\mathbf{K}_{11}]^T \right) \{\delta\} \\ &+ \frac{1}{2} \left([\tilde{\mathbf{Z}}_{12}] + [\tilde{\mathbf{Z}}_{21}] \right) [\mathbf{V}_{12}] \{\delta^{\otimes 2}\} \\ &+ \frac{1}{2} \left([\tilde{\mathbf{Z}}_{22}] [\mathbf{V}_{22}] \right) \{\delta^{\otimes 3}\} \end{aligned} \quad (77)$$

Taking into account equation (75c) and (75a) yields:

$$\frac{\partial V}{\partial \{\delta\}} = [\mathbf{K}_{11}] \{\delta\} + [\tilde{\mathbf{K}}_{12}] \{\delta^{\otimes 2}\} + \frac{1}{2} [\tilde{\mathbf{K}}_{22}] \{\delta^{\otimes 3}\} \quad (78)$$

9.3 Equations Of Motion In terms of Nodal Displacements

Substituting equation (67) and (78) in (61) results in the following equation of motion:

$$[\mathbf{M}_S] \{\ddot{\delta}\} + [\mathbf{K}_{11}] \{\delta\} + [\tilde{\mathbf{K}}_{12}] \{\delta^{\otimes 2}\} + \frac{1}{2} [\tilde{\mathbf{K}}_{22}] \{\delta^{\otimes 3}\} - \{\mathbf{q}\} = 0 \quad (79)$$

10 Free Vibration

10.1 Harmonic Motion

Assuming a harmonic solution in form of $\{\delta(\mathbf{t})\} = \{\delta_{\max}\} \sin(\omega t)$ with zero external force ($\{\mathbf{q}\} = 0$), the large amplitude nonlinear free vibration of truncated conical shell can be expressed in the following form:

$$\begin{aligned} &\left([\mathbf{K}_{11}] + [\tilde{\mathbf{K}}_{12}]_{(K \times K)} \text{vecl}(\{\delta_{\max}\}) \sin(\omega t) + \frac{1}{2} [\tilde{\mathbf{K}}_{22}]_{(K^2 \times K)} \text{vecl}(\{\delta_{\max}^{\otimes 2}\}) \sin^2(\omega t) \right) \{\delta_{\max}\} \sin(\omega t) \\ &- \omega^2 [\mathbf{M}_S] \{\delta_{\max}\} \sin(\omega t) = \{\mathbf{R}\} \end{aligned} \quad (80)$$

where the definition of operator $\text{vecl}(\cdot)_{(K \times K)}$ can be found in equation (J.6) of appendix J and $\{\mathbf{R}\}$ is the residual vector. For the linear free vibration case, the nonlinear matrices $[\tilde{\mathbf{K}}_{12}]$ and $[\tilde{\mathbf{K}}_{22}]$ are dropped and equation (80) is reduced to the following equation:

$$[\mathbf{K}_{11}] \{\delta\} - \omega^2 [\mathbf{M}_S] \{\delta\} = 0 \quad (81)$$

Equation (81) can be solved as a classic eigenvalue problem to obtain ω_L^2 . For better presentation, it is convenient to report the dimensionless frequencies that is defined as follows:

$$\Omega = \omega R_2 \sqrt{\frac{\rho(1-\nu)^2}{E}} \quad (82)$$

10.2 Nonlinear Free Vibration

Assuming the period of nonlinear vibration to be T , the maximum displacement $\{\delta_{\max}\}$ occurs at $t = T/4$ where $\omega t = \pi/4$. At such condition (80) is reduced to:

$$\left([\mathbf{K}_{11}] + [\tilde{\mathbf{K}}_{12}]_{(K \times K)} \text{vecI}(\{\delta_{\max}\}) + \frac{1}{2} [\tilde{\mathbf{K}}_{22}]_{(K^2 \times K)} \text{vecI}(\{\delta_{\max}^{\otimes 2}\}) \right) \{\delta_{\max}\} - \omega^2 [\mathbf{M}_S] \{\delta_{\max}\} = \{\mathbf{0}\} \quad (83)$$

It is worthy to mention that some of the earlier studies(e.g. [27] and [28]) have solved equation (83) as an eigenvalue problem and predicted different behavior. Because eigenvalue equation of (83) does not satisfy the nonlinear equilibrium equation of (79) for all the times, the nonlinear frequencies calculated by such approach are not accurate [29][30]. Employing a weighted residual [29] for integration between $t = 0 \rightarrow t = T/4$ that represent the amplitude variation between $\{\mathbf{0}\} \rightarrow \{\delta_{\max}\}$ yields [30]:

$$\int_0^{T/4} \{\mathbf{R}\} \sin(\omega t) dt = 0 \quad (84)$$

Taking into account that $\{\delta_{\max}\}$ is independent of the time and $\int_0^{T/4} \sin^2(\omega t) dt = T/8$, $\int_0^{T/4} \sin^3(\omega t) dt = T/3\pi$ and $\int_0^{T/4} \sin^4(\omega t) dt = 3T/32$, the nonlinear vibration of equation (80) can be transformed to the following eigenvalue problem:

$$\left[[\mathbf{K}_{11}] + \frac{8}{3\pi} [\tilde{\mathbf{K}}_{12}]_{(K \times K)} \text{vecI}(\{\delta_{\max}\}) + \frac{3}{8} [\tilde{\mathbf{K}}_{22}]_{(K^2 \times K)} \text{vecI}(\{\delta_{\max}^{\otimes 2}\}) \right] \{\delta_{\max}\} - \omega^2 [\mathbf{M}_S] \{\delta_{\max}\} = \{\mathbf{0}\} \quad (85)$$

Equation (85) can be solved as an eigenvalue problem using "vector iteration method"[29] in the following steps:

- The mode shapes of the linear frequency of the interest are normalized to have the maximum value of 1 and called $\{\delta_{\max,0}\}$
- The normalized mode shapes is multiplied by a coefficient that makes the largest degree of freedom in the direction of interest (W) equal to the amplitude of the interest (e.g. $\delta_{\max} = 0.2 \times \text{thickness}$).
- The obtained mode shape is used to calculate the amplitude dependent matrices $\text{vecI}_{(K \times K)}(\{\delta_{\max}\})$

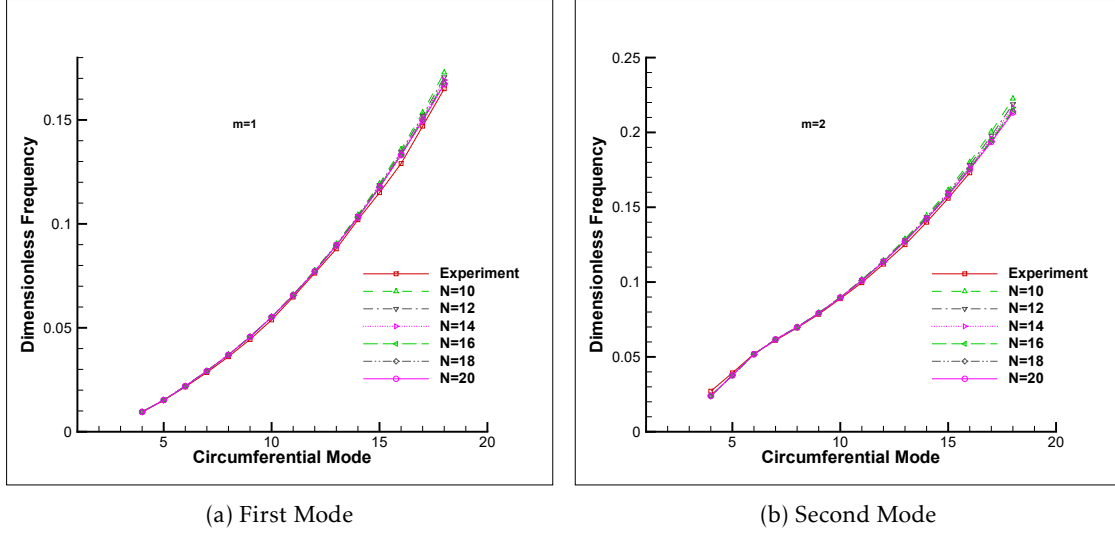


Figure 5. Case 1: FEM linear dimensionless natural frequency (equation (82)) compared to the experimental results of Hu et al.[31] for the first (5a) and second mode (5b) of vibration of a free-free truncated conical shell at different number of elements ($N = 10, 12, 14, 16, 18, 20$)

$$\text{and } \underset{(K^2 \times K)}{\text{vecI}}\left(\left\{\delta_{\max}^{\otimes 2}\right\}\right).$$

- The developed eigenvalue problem is solved and then the calculated mode shape and the nonlinear frequency is used to repeat the process until the residual and the obtained nonlinear frequency converge to a certain threshold that in current study is chosen as relative error less than $1 \times 10^{-6}\%$.

11 Results and Discussion

11.1 Validation: Convergence and Linear Frequencies

To investigate the validity of the model, in the first step the convergence and the accuracy of the linear frequency of vibration in-vacuo has been compared with four available experimental cases in the literature:

- Case 1: free-free vibration of conical shells made of cold-rolled steel reported by Hu et al. [31] where the experimental setup has been reported to be: $\alpha_c = 14.2^\circ$, $R_1 = 0.06919m$, $R_2 = 0.1543m$, $h = 2.54 \times 10^{-4}m$, $E = 203GPa$, $\nu = 0.3$ and $\rho = 7988kg/m^3$ [31–33] and all degrees of freedom were left free (the data has been digitized from Fig. 2. in reference [31]).
- Case 2: vibration of loosely clamped supported cold-rolled steel truncated conical shells with the same material properties as the case 1 reported by Lindholm et al.[1, 33], where the

Table 2. Comparison of linear natural vibration dimensionless frequency Ω for the first mode ($m=1$) with the existing experimental studies

nc	Case 2 (30.2°)		Case 3 (45.1°)		Case 4(60°) Ω (rad/s)			
	Present	Ref. [1]	Present	Ref. [1]	Present		Ref. [34]	
2	0.591447	-	.62666	-	.02928	(255.0)	.03017	(262.8)
3	0.358468	-	.47194	-	.01636	(142.5)	.01689	(147.1)
4	0.228965	-	.32698	-	.01053	(91.8)	.01331	(115.9)
5	0.158844	0.157	.18723	-	.01271	(110.7)	.01524	(132.7)
6	0.118549	0.121	.17681	.165	.01940	(169.0)	.01939	(168.9)
7	0.096123	0.097	.14045	.137	.02423	(211.1)	.02422	(211.0)
8	0.086202	0.089	.11828	.120	.02958	(257.7)	.02957	(257.5)
9	0.085335	0.088	.10680	.112	.03548	(309.0)	.03546	(308.9)
10	0.090378	0.091	.10356	.108	-	-	-	-
11	0.098605	0.099	.10632	.110	-	-	-	-
12	0.108286	0.109	.11300	.117	-	-	-	-
13	0.118805	0.117	.12197	.125	-	-	-	-

other experimental setup parameters have been reported to be: $\alpha_c = 30.2^\circ$, $R_1 = 0.0889m$, $R_2 = 0.2019m$, $h = 2.54 \times 10^{-4}m$, and the boundary condition is set to be $V = W = \partial W/\partial x = 0$ and U free (the data has been digitized from Fig. 4 in reference [1]).

- Case 3: same as the case 2 in terms of material and boundary conditions with the geometrical parameters reported to be: $\alpha_c = 45.1^\circ$, $R_1 = 0.10115m$, $R_2 = 0.2276m$, $h = 2.54 \times 10^{-4}m$ (the data has been digitized from Fig. 5 in reference [1]).
- Case 4: vibration of clamped-free aluminum truncated conical shells reported by Adelman [34] where the experimental setup was: $\alpha_c = 60.0^\circ$, $R_1 = 0.0762m$, $R_2 = 0.6097m$, $h = 6.35 \times 10^{-4}m$, $E = 68.948GPa$, $\nu = 0.315$ and $\rho = 2714kg/m^3$ and the boundary conditions at the clamped small edge where assumed $U = V = W = \partial W/\partial x = 0$ (the data obtained from TABLE IV in reference [34]).

It should be noted that to maintain the accuracy in the current study the linear part of the kinematics equations has been always modeled using the linear part of Sander's kinematics ($c_3 = 1$ in equation (2)). Figure 5 shows the variation of calculated dimensionless frequencies versus the circumferential mode number for different number of elements in case 1 in the first and second mode of vibration. The results show excellent agreement. Using that results for other cases the number of elements has been chosen to 20 elements. The correctness of boundary conditions implementation was investigated in cases 2, 3 and 4. The values of dimensionless linear natural frequencies are presented in table 2 and show good accordance with the experimental results.

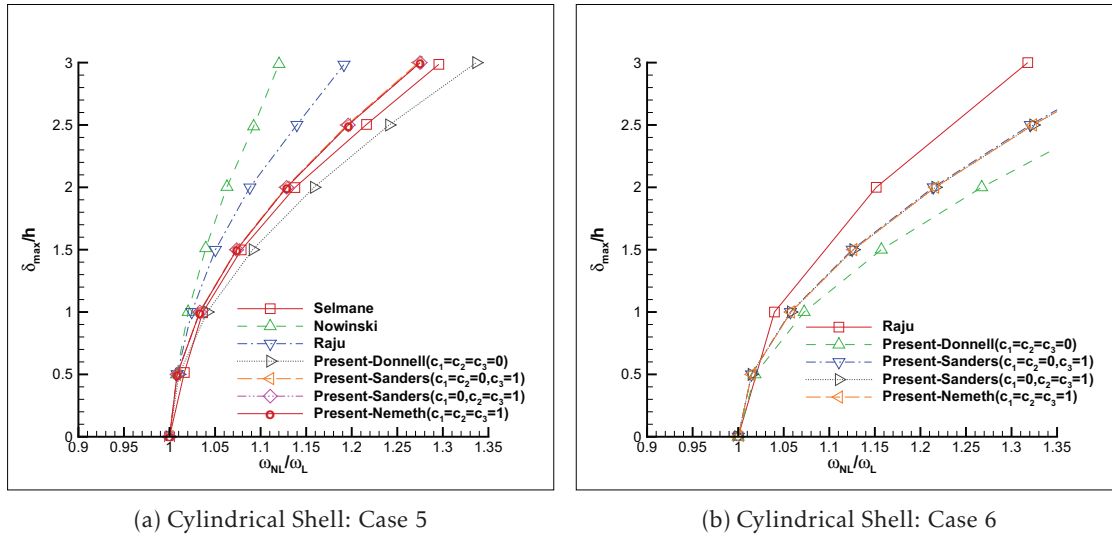


Figure 6. Backbone curve for cylindrical shells: (6a) case 5 reported by Nowinski [35], Raju et al.[36] and Selmane et al.[37]; (6b) case 6 reported by Raju et al.[36]

11.2 Validation: Nonlinear vibration of cylindrical shells

There are limited number of studies on the nonlinear vibration of conical shells and within the existing one some (e.g. [16],[38] and [39]) lack the sufficient data for reproducing the results. Therefore for validation of the nonlinear results of the current study, the cases of cylindrical shell has been simulated using a cone with a small angle. Two cases have been studied namely:

- Case 5: Nonlinear vibration of a cylindrical shell reported by Nowinski [35], Raju et al.[36] and Selmane et al.[37] where the shell parameters where reported to be: $n_c = 4$, $\alpha_c = 0.01^\circ$, $R_1 = 0.0254m$ ($R_2 = 0.025407m$), $L = 0.0399m$, $h = 2.54E - 4 \times 10^{-4}m$, $E = 204.08GPa$, $\nu = 0.3$ and $\rho = 7833.5kg/m^3$. The boundary conditions for this case is $U = V = W = 0$ and $\partial W/\partial x$ left to be free. The natural linear vibration frequency obtained in the current study is $8591.32 \times 2\pi[rad/s]$ vs $8553.74 \times 2\pi[rad/s]$ reported by Raju et al.[36], that shows good accordance.
- Case 6: Nonlinear vibration of a cylindrical shell reported by Raju et al.[36] that is the same as case 5 other than the boundary conditions that was reported to be $V = 0$ and U, W and $\partial W/\partial x$ left to be free. The natural linear vibration frequency obtained in the current study is $6453.46 \times 2\pi[rad/s]$ vs $6428.07 \times 2\pi[rad/s]$ reported by Raju et al.[36] and they only differ by 0.5%.

Figure 6a shows comparison of the backbone curve for case 5 and as can be seen the results of current study for Sanders', Sanders' with nonlinear rotation around the normal to surface neglected

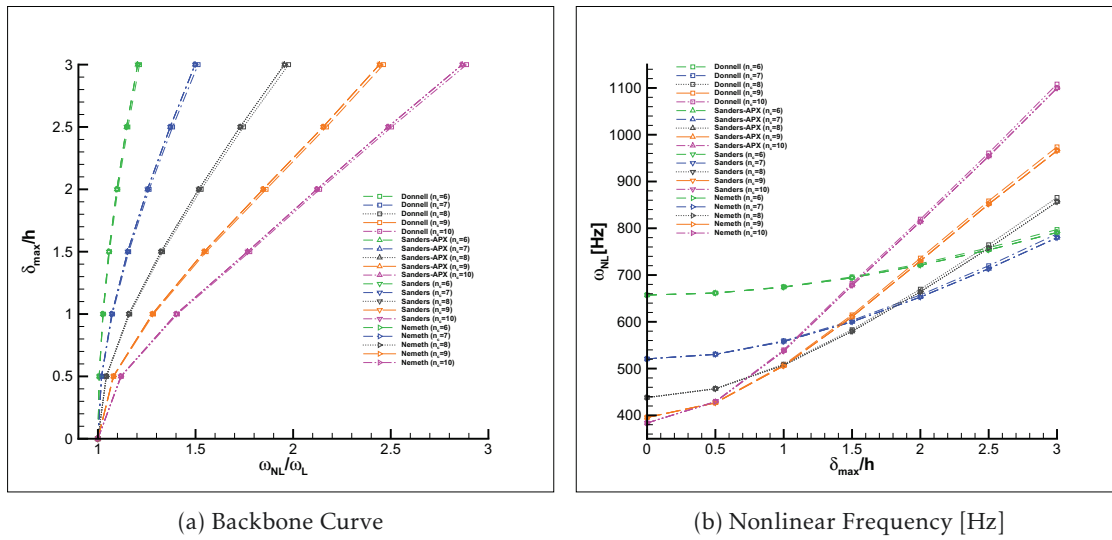


Figure 7. Variation of nonlinear frequency for different circumferential mode numbers ($n_c = 6 - 10$) for truncated cone case 3: (7a) backbone curve; (7b) nonlinear frequency [Hz]

and Nemeth's models lied between those reported by Selmane et al.[37] and Raju et al.[36] and show good accordance. Figure 6b shows comparison of the results of the present study and those reported by Raju et al.[36] for case 6 and it also shows good agreement. It should be noted that while Nowinski [35] used Donnell's type of nonlinearities in his study, Raju et al.[36] and Selmane et al.[37] employed Sander's theory taking into account the nonlinear rotations around the normal to the surface plane ($c_1 = 0$ and $c_2 = c_3 = 1$). Moreover Nowinski [35] assumed the mode shapes to have two components, a harmonic and a time variable component to satisfy the periodicity of the circumferential displacements that behaves roughly similar to a companion mode and might result in lower nonlinear frequencies. Raju et al.[36] formulated the finite element solution in terms of a 12-degrees polynomial that is relatively more loose in comparison to the 8 degrees of freedom of this particular case in the current study. Selmane et al.'s[37] work is more similar to the current study though the in that work the non-diagonal elements of the nonlinear stiffness matrix are neglected. Based on the numerical results (not presented here), omitting the non-diagonal elements of the nonlinear stiffness matrix results in higher nonlinear frequencies but the exact match to those reported by Selmane et al.[37] could not be obtained.

11.3 Nonlinear Vibration of Truncated Conical Shells

11.3.1 Circumferential mode number

Having the developed model sufficiently validated, effect of various parameters can be investigated. Figure 7 shows the variation of nonlinear frequency for different circumferential mode

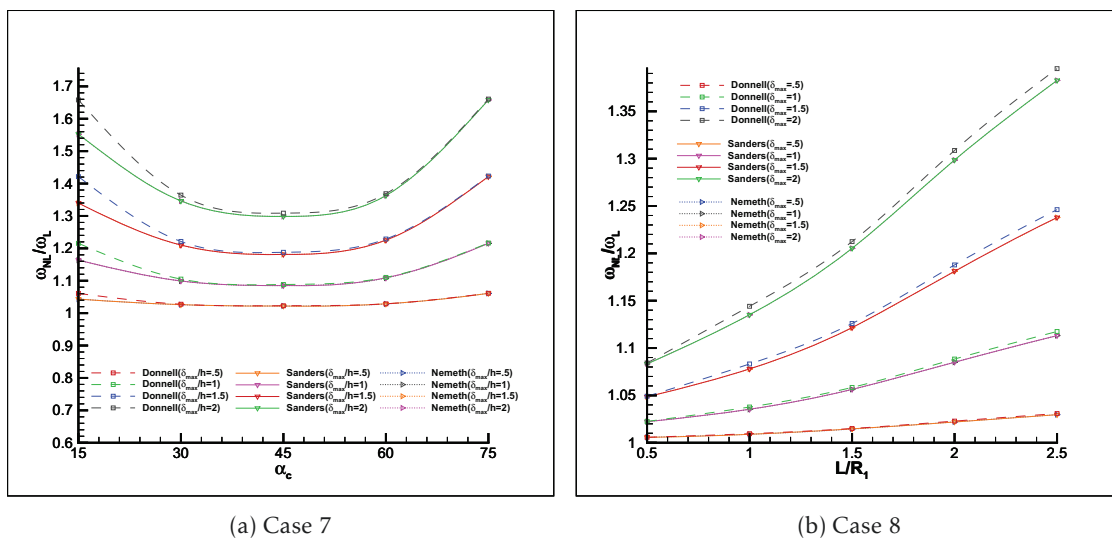


Figure 8. Effects of geometrical parameters: (8a) cone half angle; (8b) L/R_1 ratio

numbers; namely $n_c = 5 - 8$ for the loosely clamped truncated cone case 3 ($\alpha_c = 45.1^\circ$) both in terms of backbone curve and the nonlinear frequency [Hz]. The first observation from figure 7b is the relatively smaller differences between predictions of different theories for conical shells in comparison to the cylindrical shells for this particular case. Investigating the mode shapes revealed that, this can be partially attributed to the fact that the maximum amplitude of nonlinear vibration occurs at nodes closer to the large end of the truncated cone in comparison to the cylindrical shells that it occurs close to the middle of the cylinder. Therefore the effect of the constraint on the amplitude of the vibration is more dominant and it limits the rotational terms in the more complex theories. Moreover it can be seen increasing the circumferential mode number increases the nonlinearity effects on the relative nonlinear frequency. This can be explained by the appearance of n_c as a multiplier in the differentiations with respect to the second principle coordinates ($\partial/\partial\xi_2$) in the linear rotation parameters φ_2 and φ of equation (2). Same phenomenon can be observed more clearly in figure 7b. While the minimum linear frequency for this case happens at $n_c = 9$, the amplifying effect of the circumferential mode number, resulted in demonstrating the highest nonlinear effect and the lowest observable nonlinear frequency occurs at $n_c = 7$. Moreover the same effect resulted in multiple cross-overs between the nonlinear responses of different circumferential mode numbers. Therefore it is important to perform the nonlinear analysis over a wider range of circumferential mode numbers for obtaining the minimum nonlinear frequency for the amplitude of interest instead of relying on the lowest linear frequency.

11.3.2 Geometry

To study the effect of geometrical properties the following cases have been studied:

- Case 7: Loosely clamped truncated cone where geometrical properties are $R_1 = 0.10115m$, $L/R_1 = 2$, $h = 2.54 \times 10^{-4}m$ and the material properties and the boundary conditions same as case 3 other than the density that is chosen to be $\rho = 7833.5kg/m^3$. The circumferential mode number is selected to be $n_c = 7$.
- Case 8: Same as case 3 with $\alpha_c = 45^\circ$ and $n_c = 7$.
- Case 9: Loosely clamped aluminum truncated cone with the same material properties and dimensions (other than the thickness) as case 4 with circumferential mode number $n_c = 7$ and the boundary conditions were chosen to be $U = V = W = 0$ and $\partial W/\partial x$ left to be free.

Figure 8a shows the effect of cone half angle on the relative nonlinear frequency of the truncated cone shell of case 7 for three different theories. The cone half angle values are taken to be $\alpha_c = 15^\circ, 30^\circ, 45^\circ, 60^\circ$ and 75° and the associated first mode natural linear frequencies have been calculated to be 498.5, 491.1, 438.7, 338.8 and 206.3[Hz]. The linear frequency decreases as the cone moves from a cylinder towards a flat plate. The first contributing factor to this behavior is the increase of the lateral (and subsequently the mass of the cone) for this configuration by about 60% when the angle increases from 15° to 75° . Notably the difference between predictions of Donnell's and Sanders' and Nemeth's theories follow the same trend. In case of conical shells in the current formulation $1/R_1 = 1/\rho_{11} = 0$, the appearance of $1/\sin(\alpha_c)$ and $1/\tan(\alpha_c)$ in the denominator of the omitted terms of Donnell's theory (second principle and geodesic radii of curvature) is another contributing factor for this behavior. In other words, the tangential displacement ($u_2 = V$) induces stronger nonlinear behavior for cases closer to cylindrical shells than those of flat plates. On the other hand as can be seen the effect of nonlinearity shows its minimum value at 45° .

Figure 8b shows the effect of variation of the slant length to small radius ratio for cone of case 8. The selected values for are $L/R_1 = 0.5, 1.0, 1.5, 2.0$ and 2.5 and the obtained associated first mode natural linear frequencies are 2636.1, 1428.5, 657.9, 434.5 and 315.1[Hz]. As can be seen the relative nonlinear frequency increases with increasing the length of the cone. Additionally longer cones demonstrate stronger nonlinear response at higher amplitudes.

Figure 9 shows the variation of relative nonlinear frequency with the variation of h/R_1 at $R_1/h=400, 200, 100, 50$ and 25 and the associated linear frequencies were calculated to be 141.5, 147.2, 186.6, 232 and 362.5[Hz]. Skipping the case of very thin shell ($R_1/h=400$), as can be seen the variation of relative frequency ratio with the thickness follows a linear trend. While recalling figure 7a the variation of relative nonlinear frequency with the absolute amplitude of the vibration follows a semi-second order curve; figure 9 can be explained by the fact that the linear frequencies are lin-

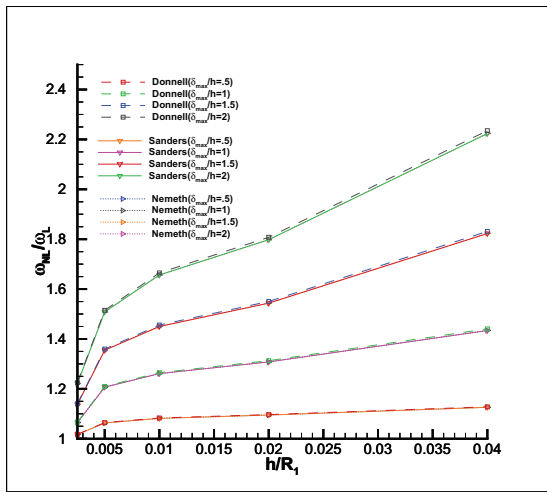


Figure 9. Case 9: Variation of relative nonlinear frequency with thickness to small radius ratio

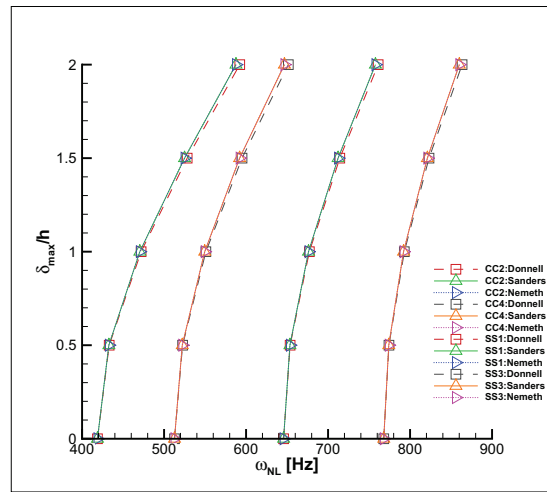
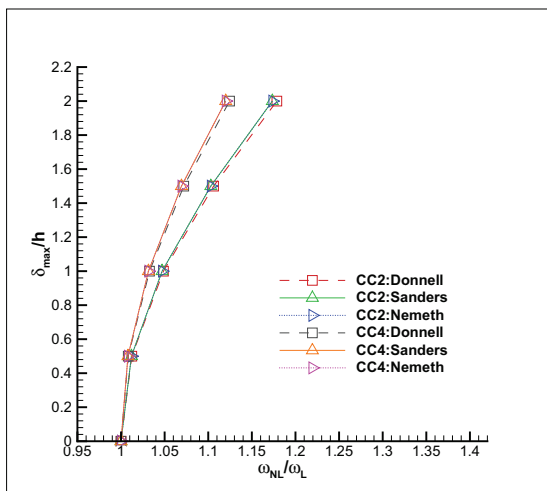
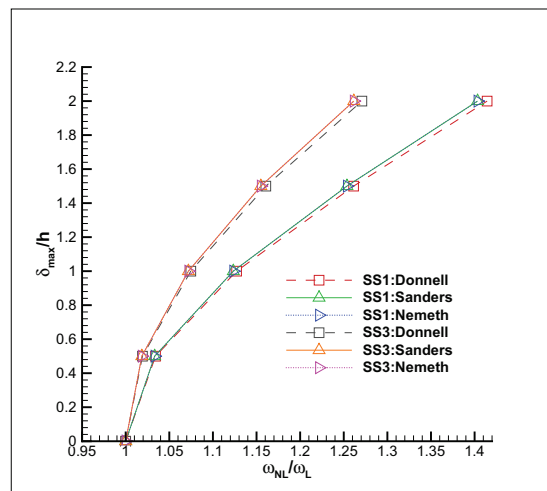


Figure 10. Case 3: Absolute nonlinear frequency[Hz] for different boundary conditions



(a) Clamped



(b) Simply Supported

Figure 11. Effect of boundary conditions on shell of case 3: (11a) clamped; (11b) simply supported

early increased with the thickness and therefore the relative nonlinear frequency becomes linear as a result of the division of a second order function to a first order one.

11.3.3 Boundary conditions

Though various boundary conditions have been applied for the previous cases, the effect of boundary conditions specifically investigated in this subsection. The cone geometry and material

properties were selected to be the same as case 3 and $n_c = 7$. Selected boundary conditions using Tong's[40] convention are:

- CC2: Loosely clamped where $U = W = \partial W/\partial x = 0$ and V left to be free.
- CC4: Fully clamped where $W = V = \partial W/\partial x = 0$.
- SS1: Simply supported where $W = 0$ and V, U and $\partial W/\partial x$ are left to be free.
- SS3: Simply supported where $V = W = 0$ and U and $\partial W/\partial x$ are left to be free.

Figure 11a presents the backbone curves for boundary conditions CC2 and CC4. The nonlinear responses of simply supported shells are presented in figure 11b and demonstrate higher frequencies in comparison to the clamped ones. As can be seen for both cases, constraining more degrees of freedom results in the reduction of the relative nonlinear frequency in both clamped and simply supported cases. Though it should be noted that as it is shown in Figure 10 the more constrained cases demonstrate higher linear frequencies and therefore their reduced nonlinear responses point to higher absolute frequencies.

12 Conclusion and Remarks

In the current study the nonlinear vibration of truncated conical shells for four different shell theories namely Donnell, Sanders with nonlinear rotation along the normal to surface neglected, Sanders and Nemeth have been formulated for anisotropic materials. The formulation employed finite element exact solution of the linear case in conjunction with the generalized coordinates obtained by Lagrange equations to derive the nonlinear amplitude equation of the vibration in the matrix form. The amplitude equation has been solved using a hybrid iterative method to study the effect of various parameters on the nonlinear response of the shell. Results for the linear frequencies have been validated against the existing experimental data in the literature for truncated cones and show good accordance. The nonlinear results have been validated against the existing results of cylindrical shells and found to be in good agreement. Effects of various parameters have been studied as follows:

- In all cases Donnell's theory predicted higher nonlinear response than the other three theories. Moreover in all cases presented in the current study the difference between the predictions of Nemeth's and Sanders' theories found to be very small though Sander's theory predictions found to have the larger value. This can be attributed to the relatively thin shells selected for the current study and the lack of shear deformation since there is a significant emphasis on the shear deformation in the development of Nemeth's theory[18].
- It was found that higher circumferential numbers result in an amplified nonlinear response.

- The relative nonlinear frequency found to demonstrate its minimum when the semi-vertex angle of the cone is equal to 45° degree while the difference between Donnell's and other theories is larger at lower semi-vertex angle. It was found that the nonlinear response increases with increasing the length to small radius ratio and this effect is stronger at higher amplitudes of vibration.
- Other than the case of very thin shells, the variation of relative nonlinear frequency with the thickness found to be linear with higher slopes at higher amplitudes of vibration.
- It was found that the shell demonstrates weaker nonlinear response with adding more constraints, though the absolute value of the nonlinear response is higher at the more constrained cases.

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Appendix A Through-the-thickness Strain Deformation Matrix

Defining the following aliases:

$$z_1 = \left(1 + \frac{\xi_3}{R_1}\right) \quad (\text{A.1a})$$

$$z_2 = \left(1 + \frac{\xi_3}{R_2}\right) \quad (\text{A.1b})$$

$$Z = \frac{1}{2} \left(\frac{\xi_3}{R_2} - \frac{\xi_3}{R_1} \right) \quad (\text{A.1c})$$

Through-the-thickness strain deformation matrix is defined as follows:

$$[\mathbf{S}] = \begin{bmatrix} [\mathbf{S}_0]_{3 \times 3} & [\mathbf{S}_1]_{3 \times 3} & [\mathbf{S}_2]_{3 \times 2} & [\mathbf{S}_3]_{3 \times 2} & [\mathbf{S}_4]_{3 \times 2} \\ [\mathbf{0}]_{2 \times 3} & [\mathbf{0}]_{2 \times 3} & [\mathbf{0}]_{2 \times 2} & [\mathbf{0}]_{2 \times 2} & [\mathbf{S}_5]_{2 \times 2} \end{bmatrix} \quad (\text{A.2})$$

The block matrices of equation (A.2) can be obtained from the following equations [18]:

$$[\mathbf{S}_0] = \begin{bmatrix} z_2 & 0 & 0 \\ 0 & z_1 & 0 \\ 0 & 0 & \frac{1}{2} [z_1 + z_2 + Z] \end{bmatrix} \quad (\text{A.3a})$$

$$[\mathbf{S}_1] = \begin{bmatrix} z_2 & 0 & 0 \\ 0 & z_1 & 0 \\ 0 & 0 & \frac{1}{2} [z_1 + z_2] \end{bmatrix} \quad (\text{A.3b})$$

$$[\mathbf{S}_2] = z_2 \begin{bmatrix} F_1(\xi_3) & 0 \\ 0 & 0 \\ 0 & F_2(\xi_3) \end{bmatrix} \quad (\text{A.3c})$$

$$[\mathbf{S}_3] = z_1 \begin{bmatrix} 0 & 0 \\ 0 & F_2(\xi_3) \\ F_1(\xi_3) & 0 \end{bmatrix} \quad (\text{A.3d})$$

$$[\mathbf{S}_4] = \begin{bmatrix} 0 & -\frac{z_2}{\rho_{11}} F_2(\xi_3) \\ \frac{z_1}{\rho_{22}} F_1(\xi_3) & 0 \\ \frac{z_2}{\rho_{11}} F_1(\xi_3) & -\frac{z_1}{\rho_{22}} F_2(\xi_3) \end{bmatrix} \quad (\text{A.3e})$$

$$[\mathbf{S}_5] = \begin{bmatrix} z_2 \left[z_1 F'_1(\xi_3) - \frac{F_1(\xi_3)}{R_1} \right] & 0 \\ 0 & z_1 \left[z_2 F'_2(\xi_3) - \frac{F_2(\xi_3)}{R_2} \right] \end{bmatrix} \quad (\text{A.3f})$$

Appendix B Work Conjugate Stress-Resultants

The work-conjugate stress resultants are given as [18]:

$$\{\mathbf{n}\} = \begin{bmatrix} n_{11} \\ n_{22} \\ n_{12} \end{bmatrix} \triangleq \int_{-h/2}^{h/2} [\mathbf{S}_0]^\top \begin{Bmatrix} \bar{\sigma}_{11} \\ \bar{\sigma}_{22} \\ \bar{\sigma}_{12} \end{Bmatrix} d\xi_3 \quad (\text{B.1a})$$

$$\{\mathbf{m}\} = \begin{bmatrix} m_{11} \\ m_{22} \\ m_{12} \end{bmatrix} \triangleq \int_{-h/2}^{h/2} [\mathbf{S}_1]^\top \begin{Bmatrix} \bar{\sigma}_{11} \\ \bar{\sigma}_{22} \\ \bar{\sigma}_{12} \end{Bmatrix} d\xi_3 \quad (\text{B.1b})$$

$$\{\mathbf{f}_1\} = \begin{bmatrix} f_{11} \\ f_{12} \end{bmatrix} \triangleq \int_{-h/2}^{h/2} [\mathbf{S}_2]^\top \begin{Bmatrix} \bar{\sigma}_{11} \\ \bar{\sigma}_{22} \\ \bar{\sigma}_{12} \end{Bmatrix} d\xi_3 \quad (\text{B.1c})$$

$$\{\mathbf{f}_2\} = \begin{bmatrix} f_{21} \\ f_{22} \end{bmatrix} \triangleq \int_{-h/2}^{h/2} [\mathbf{S}_3]^\top \begin{Bmatrix} \bar{\sigma}_{11} \\ \bar{\sigma}_{22} \\ \bar{\sigma}_{12} \end{Bmatrix} d\xi_3 \quad (\text{B.1d})$$

$$\{\mathbf{q}\} = \begin{bmatrix} q_{13} \\ q_{23} \end{bmatrix} \triangleq \int_{-h/2}^{h/2} [\mathbf{S}_4]^\top \begin{Bmatrix} \bar{\sigma}_{11} \\ \bar{\sigma}_{22} \\ \bar{\sigma}_{12} \end{Bmatrix} d\xi_3 + \int_{-h/2}^{h/2} [\mathbf{S}_5]^\top \begin{Bmatrix} \bar{\sigma}_{13} \\ \bar{\sigma}_{23} \end{Bmatrix} d\xi_3 \quad (\text{B.1e})$$

Appendix C Conical Shell Linear Equilibrium Equations in Terms of Stress Resultants

The principle parameters of conical shells can be obtained from the following equation:

$$\begin{aligned} A_1(\xi_1, \xi_2) = 1, A_2(\xi_1, \xi_2) = \xi_1 \sin(\alpha_c), \frac{1}{R_1(\xi_1, \xi_2)} = 0, \\ \frac{1}{R_2(\xi_1, \xi_2)} = \frac{1}{\xi_1 \tan(\alpha_c)}, \frac{1}{\rho_{11}} = 0, \frac{1}{\rho_{22}} = \frac{1}{\xi_1} \end{aligned} \quad (C.1)$$

By introducing the geometrical parameters of conical shells into the general equilibrium equations of Sanders' improved linear theory [20]; one obtains the equilibrium equations of a conical shell as follows.:

$$\mathfrak{L}_1: \quad \frac{\partial n_{11}}{\partial \xi_1} + \frac{1}{\xi_1 \sin(\alpha_c)} \frac{\partial n_{12}}{\partial \xi_2} + \frac{1}{\xi_1} (n_{11} - n_{22}) - \frac{1}{2\xi_1^2 \sin(\alpha_c) \tan(\alpha_c)} \frac{\partial m_{12}}{\partial \xi_2} = 0 \quad (C.2a)$$

$$\begin{aligned} \mathfrak{L}_2: \quad \frac{\partial n_{12}}{\partial \xi_1} + \frac{1}{\xi_1 \sin(\alpha_c)} \frac{\partial n_{22}}{\partial \xi_2} + \frac{2n_{12}}{\xi_1} + \frac{3}{2} \frac{1}{\xi_1 \tan(\alpha_c)} \frac{\partial m_{12}}{\partial \xi_1} \\ + \frac{1}{\xi_1^2 \sin(\alpha_c) \tan(\alpha_c)} \frac{\partial m_{22}}{\partial \xi_2} + \frac{3}{2} \frac{1}{\xi_1^2 \tan(\alpha_c)} m_{12} = 0 \end{aligned} \quad (C.2b)$$

$$\begin{aligned} \mathfrak{L}_3: \quad \frac{\partial^2 m_{11}}{\partial \xi_1^2} + \frac{2}{\xi_1^2 \sin(\alpha_c)} \frac{\partial m_{12}}{\partial \xi_2} + \frac{2}{\xi_1 \sin(\alpha_c)} \frac{\partial^2 m_{12}}{\partial \xi_1 \partial \xi_2} + \frac{2}{\xi_1} \frac{\partial m_{11}}{\partial \xi_1} \\ - \frac{1}{\xi_1} \frac{\partial m_{22}}{\partial \xi_1} + \frac{1}{\xi_1^2 \sin(\alpha_c)^2} \frac{\partial^2 m_{22}}{\partial \xi_2^2} - \frac{n_{22}}{\xi_1 \tan(\alpha_c)} = 0 \end{aligned} \quad (C.2c)$$

$$\mathfrak{L}_4: \quad q_{13} - \frac{f_{11}}{\xi_1} - \frac{\partial f_{11}}{\partial \xi_1} - \frac{\partial f_{12}}{\xi_1 \sin(\alpha_c) \partial \xi_2} = 0 \quad (C.2d)$$

$$\mathfrak{L}_5: \quad q_{23} - \frac{f_{12}}{\xi_1} - \frac{\partial f_{12}}{\partial \xi_1} - \frac{\partial f_{22}}{\xi_1 \sin(\alpha_c) \partial \xi_2} = 0 \quad (C.2e)$$

Appendix D Elements of Symmetric Constitutive Matrix

$$\mathcal{C}_{1,1} = +A_{11}^0 + \tau_0 \left(-\frac{1}{R_1} A_{11}^1 + \frac{1}{R_2} A_{11}^1 \right) + \tau \left(+\frac{1}{(R_1^2)} A_{11}^2 - \frac{1}{R_1 R_2} A_{11}^2 \right) \quad (\text{D.1})$$

$$\mathcal{C}_{1,2} = +A_{12}^0 \quad (\text{D.2})$$

$$\begin{aligned} \mathcal{C}_{1,3} = +A_{16}^0 + \tau_0 \left(-0.5 \frac{1}{R_1} A_{16}^1 + 0.5 \frac{1}{R_2} A_{16}^1 \right) + \tau \left(+0.75 \frac{1}{(R_1^2)} A_{16}^2 - \frac{1}{R_1 R_2} A_{16}^2 \right. \\ \left. + 0.25 \frac{1}{(R_2^2)} A_{16}^2 \right) \end{aligned} \quad (\text{D.3})$$

$$\mathcal{C}_{1,4} = +A_{11}^1 + \tau_0 \left(-\frac{1}{R_1} A_{11}^2 + \frac{1}{R_2} A_{11}^2 \right) + \tau \left(+\frac{1}{(R_1^2)} A_{11}^3 - \frac{1}{R_1 R_2} A_{11}^3 \right) \quad (\text{D.4})$$

$$\mathcal{C}_{1,5} = +A_{12}^1 \quad (\text{D.5})$$

$$\mathcal{C}_{1,6} = +A_{16}^1 + \tau_0 \left(-0.5 \frac{1}{R_1} A_{16}^2 + 0.5 \frac{1}{R_2} A_{16}^2 \right) + \tau \left(+0.5 \frac{1}{(R_1^2)} A_{16}^3 - 0.5 \frac{1}{R_1 R_2} A_{16}^3 \right) \quad (\text{D.6})$$

$$\mathcal{C}_{1,7} = +R_{11}^{10} + \tau_0 \left(-\frac{1}{R_1} R_{11}^{11} + \frac{1}{R_2} R_{11}^{11} \right) + \tau \left(+\frac{1}{(R_1^2)} R_{11}^{12} - \frac{1}{R_1 R_2} R_{11}^{12} \right) \quad (\text{D.7})$$

$$\mathcal{C}_{1,8} = +R_{16}^{20} + \tau_0 \left(-\frac{1}{R_1} R_{16}^{21} + \frac{1}{R_2} R_{16}^{21} \right) + \tau \left(+\frac{1}{(R_1^2)} R_{16}^{22} - \frac{1}{R_1 R_2} R_{16}^{22} \right) \quad (\text{D.8})$$

$$\mathcal{C}_{1,9} = +R_{16}^{10} \quad (\text{D.9})$$

$$\mathcal{CC}_{1,10} = +R_{12}^{20} \quad (\text{D.10})$$

$$\begin{aligned} \mathcal{CC}_{1,11} = & +W_{15}^{10} + \tau_0 \left(-\frac{1}{R_1} R_{15}^{10} + \frac{1}{\rho_{11}} R_{16}^{10} + \frac{1}{\rho_{22}} R_{12}^{10} + \frac{1}{(R_1^2)} R_{15}^{11} \right. \\ & - \frac{1}{R_1 R_2} R_{15}^{11} - \frac{1}{R_1 \rho_{11}} R_{16}^{11} + \frac{1}{R_2 \rho_{11}} R_{16}^{11} + \frac{1}{R_2} W_{15}^{11} \left. \right) + \tau \left(-\frac{1}{(R_1^3)} R_{15}^{12} \right. \\ & \left. + \frac{1}{(R_1^2) R_2} R_{15}^{12} + \frac{1}{(R_1^2) \rho_{11}} R_{16}^{12} - \frac{1}{R_1 R_2 \rho_{11}} R_{16}^{12} \right) \end{aligned} \quad (\text{D.11})$$

$$\begin{aligned} \mathcal{CC}_{1,12} = & +W_{14}^{20} + \tau_0 \left(-\frac{1}{R_2} R_{14}^{20} - \frac{1}{\rho_{11}} R_{11}^{20} - \frac{1}{\rho_{22}} R_{16}^{20} + \frac{1}{R_1 \rho_{11}} R_{11}^{21} \right. \\ & \left. - \frac{1}{R_2 \rho_{11}} R_{11}^{21} + \frac{1}{R_2} W_{14}^{21} \right) + \tau \left(-\frac{1}{(R_1^2) \rho_{11}} R_{11}^{22} + \frac{1}{R_1 R_2 \rho_{11}} R_{11}^{22} \right) \end{aligned} \quad (\text{D.12})$$

$$\mathcal{CC}_{2,2} = +A_{22}^0 + \tau_0 \left(+\frac{1}{R_1} A_{22}^1 - \frac{1}{R_2} A_{22}^1 \right) + \tau \left(-\frac{1}{R_1 R_2} A_{22}^2 + \frac{1}{(R_2^2)} A_{22}^2 \right) \quad (\text{D.13})$$

$$\begin{aligned} \mathcal{CC}_{2,3} = & +A_{26}^0 + \tau_0 \left(+0.5 \frac{1}{R_1} A_{26}^1 - 0.5 \frac{1}{R_2} A_{26}^1 \right) + \tau \left(+0.25 \frac{1}{(R_1^2)} A_{26}^2 - \frac{1}{R_1 R_2} A_{26}^2 \right. \\ & \left. + 0.75 \frac{1}{(R_2^2)} A_{26}^2 \right) \end{aligned} \quad (\text{D.14})$$

$$\mathcal{CC}_{2,4} = +A_{12}^1 \quad (\text{D.15})$$

$$\mathcal{CC}_{2,5} = +A_{22}^1 + \tau_0 \left(+\frac{1}{R_1} A_{22}^2 - \frac{1}{R_2} A_{22}^2 \right) + \tau \left(-\frac{1}{R_1 R_2} A_{22}^3 + \frac{1}{(R_2^2)} A_{22}^3 \right) \quad (\text{D.16})$$

$$\mathcal{CC}_{2,6} = +A_{26}^1 + \tau_0 \left(+0.5 \frac{1}{R_1} A_{26}^2 - 0.5 \frac{1}{R_2} A_{26}^2 \right) + \tau \left(-0.5 \frac{1}{R_1 R_2} A_{26}^3 + 0.5 \frac{1}{(R_2^2)} A_{26}^3 \right) \quad (\text{D.17})$$

$$\mathcal{C}\mathcal{C}_{2,7} = +R_{12}^{10} \quad (\text{D.18})$$

$$\mathcal{C}\mathcal{C}_{2,8} = +R_{26}^{20} \quad (\text{D.19})$$

$$\mathcal{C}\mathcal{C}_{2,9} = +R_{26}^{10} + \tau_0 \left(+\frac{1}{R_1} R_{26}^{11} - \frac{1}{R_2} R_{26}^{11} \right) + \tau \left(-\frac{1}{R_1 R_2} R_{26}^{12} + \frac{1}{(R_2^2)} R_{26}^{12} \right) \quad (\text{D.20})$$

$$\mathcal{C}\mathcal{C}_{2,10} = +R_{22}^{20} + \tau_0 \left(+\frac{1}{R_1} R_{22}^{21} - \frac{1}{R_2} R_{22}^{21} \right) + \tau \left(-\frac{1}{R_1 R_2} R_{22}^{22} + \frac{1}{(R_2^2)} R_{22}^{22} \right) \quad (\text{D.21})$$

$$\begin{aligned} \mathcal{C}\mathcal{C}_{2,11} = & +W_{25}^{10} + \tau_0 \left(-\frac{1}{R_1} R_{25}^{10} + \frac{1}{\rho_{11}} R_{26}^{10} + \frac{1}{\rho_{22}} R_{22}^{10} + \frac{1}{R_1 \rho_{22}} R_{22}^{11} \right. \\ & \left. - \frac{1}{R_2 \rho_{22}} R_{22}^{11} + \frac{1}{R_1} W_{25}^{11} \right) + \tau \left(-\frac{1}{R_1 R_2 \rho_{22}} R_{22}^{12} + \frac{1}{(R_2^2) \rho_{22}} R_{22}^{12} \right) \end{aligned} \quad (\text{D.22})$$

$$\begin{aligned} \mathcal{C}\mathcal{C}_{2,12} = & +W_{24}^{20} + \tau_0 \left(-\frac{1}{R_2} R_{24}^{20} - \frac{1}{\rho_{11}} R_{12}^{20} - \frac{1}{\rho_{22}} R_{26}^{20} - \frac{1}{R_1 R_2} R_{24}^{21} \right. \\ & \left. - \frac{1}{R_1 \rho_{22}} R_{26}^{21} + \frac{1}{(R_2^2)} R_{24}^{21} + \frac{1}{R_2 \rho_{22}} R_{26}^{21} + \frac{1}{R_1} W_{24}^{21} \right) + \tau \left(+\frac{1}{R_1 (R_2^2)} R_{24}^{22} \right. \\ & \left. + \frac{1}{R_1 R_2 \rho_{22}} R_{26}^{22} - \frac{1}{(R_2^3)} R_{24}^{22} - \frac{1}{(R_2^2) \rho_{22}} R_{26}^{22} \right) \end{aligned} \quad (\text{D.23})$$

$$\mathcal{C}\mathcal{C}_{3,3} = +A_{66}^0 + 0.75 \frac{1}{(R_1^2)} A_{66}^2 - 1.5 \frac{1}{R_1 R_2} A_{66}^2 + 0.75 \frac{1}{(R_2^2)} A_{66}^2 \quad (\text{D.24})$$

$$\begin{aligned} \mathcal{C}\mathcal{C}_{3,4} = & +A_{16}^1 + \tau_0 \left(-0.5 \frac{1}{R_1} A_{16}^2 + 0.5 \frac{1}{R_2} A_{16}^2 \right) + \tau \left(+0.75 \frac{1}{(R_1^2)} A_{16}^3 - \frac{1}{R_1 R_2} A_{16}^3 \right. \\ & \left. + 0.25 \frac{1}{(R_2^2)} A_{16}^3 \right) \end{aligned} \quad (\text{D.25})$$

$$\begin{aligned} \mathcal{C}\mathcal{C}_{3,5} = & +A_{26}^1 + \tau_0 \left(+0.5 \frac{1}{R_1} A_{26}^2 - 0.5 \frac{1}{R_2} A_{26}^2 \right) + \tau \left(+0.25 \frac{1}{(R_1^2)} A_{26}^3 - \frac{1}{R_1 R_2} A_{26}^3 \right. \\ & \left. + 0.75 \frac{1}{(R_2^2)} A_{26}^3 \right) \end{aligned} \quad (\text{D.26})$$

$$\mathcal{C}\mathcal{C}_{3,6} = +A_{66}^1 + 0.5 \frac{1}{(R_1^2)} A_{66}^3 - \frac{1}{R_1 R_2} A_{66}^3 + 0.5 \frac{1}{(R_2^2)} A_{66}^3 \quad (\text{D.27})$$

$$\begin{aligned} \mathcal{C}\mathcal{C}_{3,7} = & +R_{16}^{10} + \tau_0 \left(-0.5 \frac{1}{R_1} R_{16}^{11} + 0.5 \frac{1}{R_2} R_{16}^{11} \right) + \tau \left(+0.75 \frac{1}{(R_1^2)} R_{16}^{12} - \frac{1}{R_1 R_2} R_{16}^{12} \right. \\ & \left. + 0.25 \frac{1}{(R_2^2)} R_{16}^{12} \right) \end{aligned} \quad (\text{D.28})$$

$$\begin{aligned} \mathcal{C}\mathcal{C}_{3,8} = & +R_{66}^{20} + \tau_0 \left(-0.5 \frac{1}{R_1} R_{66}^{21} + 0.5 \frac{1}{R_2} R_{66}^{21} \right) + \tau \left(+0.75 \frac{1}{(R_1^2)} R_{66}^{22} - \frac{1}{R_1 R_2} R_{66}^{22} \right. \\ & \left. + 0.25 \frac{1}{(R_2^2)} R_{66}^{22} \right) \end{aligned} \quad (\text{D.29})$$

$$\begin{aligned} \mathcal{C}\mathcal{C}_{3,9} = & +R_{66}^{10} + \tau_0 \left(+0.5 \frac{1}{R_1} R_{66}^{11} - 0.5 \frac{1}{R_2} R_{66}^{11} \right) + \tau \left(+0.25 \frac{1}{(R_1^2)} R_{66}^{12} - \frac{1}{R_1 R_2} R_{66}^{12} \right. \\ & \left. + 0.75 \frac{1}{(R_2^2)} R_{66}^{12} \right) \end{aligned} \quad (\text{D.30})$$

$$\begin{aligned} \mathcal{C}\mathcal{C}_{3,10} = & +R_{26}^{20} + \tau_0 \left(+0.5 \frac{1}{R_1} R_{26}^{21} - 0.5 \frac{1}{R_2} R_{26}^{21} \right) + \tau \left(+0.25 \frac{1}{(R_1^2)} R_{26}^{22} - \frac{1}{R_1 R_2} R_{26}^{22} \right. \\ & \left. + 0.75 \frac{1}{(R_2^2)} R_{26}^{22} \right) \end{aligned} \quad (\text{D.31})$$

$$\begin{aligned}
\mathcal{CC}_{3,11} = & +W_{56}^{10} + \tau_0 \left(-\frac{1}{R_1} R_{56}^{10} + \frac{1}{\rho_{11}} R_{66}^{10} + \frac{1}{\rho_{22}} R_{26}^{10} + 0.5 \frac{1}{(R_1^2)} R_{56}^{11} \right. \\
& - 0.5 \frac{1}{R_1 R_2} R_{56}^{11} - 0.5 \frac{1}{R_1 \rho_{11}} R_{66}^{11} + 0.5 \frac{1}{R_1 \rho_{22}} R_{26}^{11} + 0.5 \frac{1}{R_2 \rho_{11}} R_{66}^{11} - 0.5 \frac{1}{R_2 \rho_{22}} R_{26}^{11} \\
& \left. + 0.5 \frac{1}{R_1} W_{56}^{11} + 0.5 \frac{1}{R_2} W_{56}^{11} \right) + \tau \left(-0.75 \frac{1}{(R_1^3)} R_{56}^{12} + \frac{1}{(R_1^2) R_2} R_{56}^{12} + 0.75 \frac{1}{(R_1^2) \rho_{11}} R_{66}^{12} \right. \\
& + 0.25 \frac{1}{(R_1^2) \rho_{22}} R_{26}^{12} - 0.25 \frac{1}{R_1 (R_2^2)} R_{56}^{12} - \frac{1}{R_1 R_2 \rho_{11}} R_{66}^{12} - \frac{1}{R_1 R_2 \rho_{22}} R_{26}^{12} \\
& \left. + 0.25 \frac{1}{(R_2^2) \rho_{11}} R_{66}^{12} + 0.75 \frac{1}{(R_2^2) \rho_{22}} R_{26}^{12} + 0.25 \frac{1}{(R_1^2)} W_{56}^{12} - 0.5 \frac{1}{R_1 R_2} W_{56}^{12} + 0.25 \frac{1}{(R_2^2)} W_{56}^{12} \right)
\end{aligned} \tag{D.32}$$

$$\begin{aligned}
\mathcal{CC}_{3,12} = & +W_{46}^{20} + \tau_0 \left(-\frac{1}{R_2} R_{46}^{20} - \frac{1}{\rho_{11}} R_{16}^{20} - \frac{1}{\rho_{22}} R_{66}^{20} - 0.5 \frac{1}{R_1 R_2} R_{46}^{21} \right. \\
& + 0.5 \frac{1}{R_1 \rho_{11}} R_{16}^{21} - 0.5 \frac{1}{R_1 \rho_{22}} R_{66}^{21} + 0.5 \frac{1}{(R_2^2)} R_{46}^{21} - 0.5 \frac{1}{R_2 \rho_{11}} R_{16}^{21} + 0.5 \frac{1}{R_2 \rho_{22}} R_{66}^{21} \\
& \left. + 0.5 \frac{1}{R_1} W_{46}^{21} + 0.5 \frac{1}{R_2} W_{46}^{21} \right) + \tau \left(-0.25 \frac{1}{(R_1^2) R_2} R_{46}^{22} - 0.75 \frac{1}{(R_1^2) \rho_{11}} R_{16}^{22} - 0.25 \frac{1}{(R_1^2) \rho_{22}} R_{66}^{22} \right. \\
& + \frac{1}{R_1 (R_2^2)} R_{46}^{22} + \frac{1}{R_1 R_2 \rho_{11}} R_{16}^{22} + \frac{1}{R_1 R_2 \rho_{22}} R_{66}^{22} - 0.75 \frac{1}{(R_2^3)} R_{46}^{22} \\
& \left. - 0.25 \frac{1}{(R_2^2) \rho_{11}} R_{16}^{22} - 0.75 \frac{1}{(R_2^2) \rho_{22}} R_{66}^{22} + 0.25 \frac{1}{(R_1^2)} W_{46}^{22} - 0.5 \frac{1}{R_1 R_2} W_{46}^{22} + 0.25 \frac{1}{(R_2^2)} W_{46}^{22} \right)
\end{aligned} \tag{D.33}$$

$$\mathcal{CC}_{4,4} = +A_{11}^2 + \tau_0 \left(-\frac{1}{R_1} A_{11}^3 + \frac{1}{R_2} A_{11}^3 \right) + \tau \left(+\frac{1}{(R_1^2)} A_{11}^4 - \frac{1}{R_1 R_2} A_{11}^4 \right) \tag{D.34}$$

$$\mathcal{CC}_{4,5} = +A_{12}^2 \tag{D.35}$$

$$\mathcal{CC}_{4,6} = +A_{16}^2 + \tau_0 \left(-0.5 \frac{1}{R_1} A_{16}^3 + 0.5 \frac{1}{R_2} A_{16}^3 \right) + \tau \left(+0.5 \frac{1}{(R_1^2)} A_{16}^4 - 0.5 \frac{1}{R_1 R_2} A_{16}^4 \right) \tag{D.36}$$

$$\mathcal{CC}_{4,7} = +R_{11}^{11} + \tau_0 \left(-\frac{1}{R_1} R_{11}^{12} + \frac{1}{R_2} R_{11}^{12} \right) + \tau \left(+\frac{1}{(R_1^2)} R_{11}^{13} - \frac{1}{R_1 R_2} R_{11}^{13} \right) \quad (\text{D.37})$$

$$\mathcal{CC}_{4,8} = +R_{16}^{21} + \tau_0 \left(-\frac{1}{R_1} R_{16}^{22} + \frac{1}{R_2} R_{16}^{22} \right) + \tau \left(+\frac{1}{(R_1^2)} R_{16}^{23} - \frac{1}{R_1 R_2} R_{16}^{23} \right) \quad (\text{D.38})$$

$$\mathcal{CC}_{4,9} = +R_{16}^{11} \quad (\text{D.39})$$

$$\mathcal{CC}_{4,10} = +R_{12}^{21} \quad (\text{D.40})$$

$$\begin{aligned} \mathcal{CC}_{4,11} = & +W_{15}^{11} + \tau_0 \left(-\frac{1}{R_1} R_{15}^{11} + \frac{1}{\rho_{11}} R_{16}^{11} + \frac{1}{\rho_{22}} R_{12}^{11} + \frac{1}{(R_1^2)} R_{15}^{12} \right. \\ & - \frac{1}{R_1 R_2} R_{15}^{12} - \frac{1}{R_1 \rho_{11}} R_{16}^{12} + \frac{1}{R_2 \rho_{11}} R_{16}^{12} + \frac{1}{R_2} W_{15}^{12} \left. \right) + \tau \left(-\frac{1}{(R_1^3)} R_{15}^{13} \right. \\ & \left. + \frac{1}{(R_1^2) R_2} R_{15}^{13} + \frac{1}{(R_1^2) \rho_{11}} R_{16}^{13} - \frac{1}{R_1 R_2 \rho_{11}} R_{16}^{13} \right) \end{aligned} \quad (\text{D.41})$$

$$\begin{aligned} \mathcal{CC}_{4,12} = & +W_{14}^{21} + \tau_0 \left(-\frac{1}{R_2} R_{14}^{21} - \frac{1}{\rho_{11}} R_{11}^{21} - \frac{1}{\rho_{22}} R_{16}^{21} + \frac{1}{R_1 \rho_{11}} R_{11}^{22} \right. \\ & \left. - \frac{1}{R_2 \rho_{11}} R_{11}^{22} + \frac{1}{R_2} W_{14}^{22} \right) + \tau \left(-\frac{1}{(R_1^2) \rho_{11}} R_{11}^{23} + \frac{1}{R_1 R_2 \rho_{11}} R_{11}^{23} \right) \end{aligned} \quad (\text{D.42})$$

$$\mathcal{CC}_{5,5} = +A_{22}^2 + \tau_0 \left(+\frac{1}{R_1} A_{22}^3 - \frac{1}{R_2} A_{22}^3 \right) + \tau \left(-\frac{1}{R_1 R_2} A_{22}^4 + \frac{1}{(R_2^2)} A_{22}^4 \right) \quad (\text{D.43})$$

$$\mathcal{CC}_{5,6} = +A_{26}^2 + \tau_0 \left(+0.5 \frac{1}{R_1} A_{26}^3 - 0.5 \frac{1}{R_2} A_{26}^3 \right) + \tau \left(-0.5 \frac{1}{R_1 R_2} A_{26}^4 + 0.5 \frac{1}{(R_2^2)} A_{26}^4 \right) \quad (\text{D.44})$$

$$\mathcal{CC}_{5,7} = +R_{12}^{11} \quad (\text{D.45})$$

$$\mathcal{C}\mathcal{C}_{5,8} = +R_{26}^{21} \quad (\text{D.46})$$

$$\mathcal{C}\mathcal{C}_{5,9} = +R_{26}^{11} + \tau_0 \left(+\frac{1}{R_1} R_{26}^{12} - \frac{1}{R_2} R_{26}^{12} \right) + \tau \left(-\frac{1}{R_1 R_2} R_{26}^{13} + \frac{1}{(R_2^2)} R_{26}^{13} \right) \quad (\text{D.47})$$

$$\mathcal{C}\mathcal{C}_{5,10} = +R_{22}^{21} + \tau_0 \left(+\frac{1}{R_1} R_{22}^{22} - \frac{1}{R_2} R_{22}^{22} \right) + \tau \left(-\frac{1}{R_1 R_2} R_{22}^{23} + \frac{1}{(R_2^2)} R_{22}^{23} \right) \quad (\text{D.48})$$

$$\begin{aligned} \mathcal{C}\mathcal{C}_{5,11} = & +W_{25}^{11} + \tau_0 \left(-\frac{1}{R_1} R_{25}^{11} + \frac{1}{\rho_{11}} R_{26}^{11} + \frac{1}{\rho_{22}} R_{22}^{11} + \frac{1}{R_1 \rho_{22}} R_{22}^{12} \right. \\ & \left. - \frac{1}{R_2 \rho_{22}} R_{22}^{12} + \frac{1}{R_1} W_{25}^{12} \right) + \tau \left(-\frac{1}{R_1 R_2 \rho_{22}} R_{22}^{13} + \frac{1}{(R_2^2) \rho_{22}} R_{22}^{13} \right) \end{aligned} \quad (\text{D.49})$$

$$\begin{aligned} \mathcal{C}\mathcal{C}_{5,12} = & +W_{24}^{21} + \tau_0 \left(-\frac{1}{R_2} R_{24}^{21} - \frac{1}{\rho_{11}} R_{12}^{21} - \frac{1}{\rho_{22}} R_{26}^{21} - \frac{1}{R_1 R_2} R_{24}^{22} \right. \\ & \left. - \frac{1}{R_1 \rho_{22}} R_{26}^{22} + \frac{1}{(R_2^2)} R_{24}^{22} + \frac{1}{R_2 \rho_{22}} R_{26}^{22} + \frac{1}{R_1} W_{24}^{22} \right) + \tau \left(+\frac{1}{R_1 (R_2^2)} R_{24}^{23} \right. \\ & \left. + \frac{1}{R_1 R_2 \rho_{22}} R_{26}^{23} - \frac{1}{(R_2^3)} R_{24}^{23} - \frac{1}{(R_2^2) \rho_{22}} R_{26}^{23} \right) \end{aligned} \quad (\text{D.50})$$

$$\mathcal{C}\mathcal{C}_{6,6} = +A_{66}^2 + 0.25 \frac{1}{(R_1^2)} A_{66}^4 - 0.5 \frac{1}{R_1 R_2} A_{66}^4 + 0.25 \frac{1}{(R_2^2)} A_{66}^4 \quad (\text{D.51})$$

$$\mathcal{C}\mathcal{C}_{6,7} = +R_{16}^{11} + \tau_0 \left(-0.5 \frac{1}{R_1} R_{16}^{12} + 0.5 \frac{1}{R_2} R_{16}^{12} \right) + \tau \left(+0.5 \frac{1}{(R_1^2)} R_{16}^{13} - 0.5 \frac{1}{R_1 R_2} R_{16}^{13} \right) \quad (\text{D.52})$$

$$\mathcal{C}\mathcal{C}_{6,8} = +R_{66}^{21} + \tau_0 \left(-0.5 \frac{1}{R_1} R_{66}^{22} + 0.5 \frac{1}{R_2} R_{66}^{22} \right) + \tau \left(+0.5 \frac{1}{(R_1^2)} R_{66}^{23} - 0.5 \frac{1}{R_1 R_2} R_{66}^{23} \right) \quad (\text{D.53})$$

$$\mathcal{C}\mathcal{C}_{6,9} = +R_{66}^{11} + \tau_0 \left(+0.5 \frac{1}{R_1} R_{66}^{12} - 0.5 \frac{1}{R_2} R_{66}^{12} \right) + \tau \left(-0.5 \frac{1}{R_1 R_2} R_{66}^{13} + 0.5 \frac{1}{(R_2^2)} R_{66}^{13} \right) \quad (\text{D.54})$$

$$\mathcal{C}\mathcal{C}_{6,10} = +R_{26}^{21} + \tau_0 \left(+0.5 \frac{1}{R_1} R_{26}^{22} - 0.5 \frac{1}{R_2} R_{26}^{22} \right) + \tau \left(-0.5 \frac{1}{R_1 R_2} R_{26}^{23} + 0.5 \frac{1}{(R_2^2)} R_{26}^{23} \right) \quad (\text{D.55})$$

$$\begin{aligned} \mathcal{C}\mathcal{C}_{6,11} = & +W_{56}^{11} + \tau_0 \left(-\frac{1}{R_1} R_{56}^{11} + \frac{1}{\rho_{11}} R_{66}^{11} + \frac{1}{\rho_{22}} R_{26}^{11} + 0.5 \frac{1}{(R_1^2)} R_{56}^{12} \right. \\ & - 0.5 \frac{1}{R_1 R_2} R_{56}^{12} - 0.5 \frac{1}{R_1 \rho_{11}} R_{66}^{12} + 0.5 \frac{1}{R_1 \rho_{22}} R_{26}^{12} + 0.5 \frac{1}{R_2 \rho_{11}} R_{66}^{12} - 0.5 \frac{1}{R_2 \rho_{22}} R_{26}^{12} \\ & \left. + 0.5 \frac{1}{R_1} W_{56}^{12} + 0.5 \frac{1}{R_2} W_{56}^{12} \right) + \tau \left(-0.5 \frac{1}{(R_1^3)} R_{56}^{13} + 0.5 \frac{1}{(R_1^2) R_2} R_{56}^{13} + 0.5 \frac{1}{(R_1^2) \rho_{11}} R_{66}^{13} \right. \\ & \left. - 0.5 \frac{1}{R_1 R_2 \rho_{11}} R_{66}^{13} - 0.5 \frac{1}{R_1 R_2 \rho_{22}} R_{26}^{13} + 0.5 \frac{1}{(R_2^2) \rho_{22}} R_{26}^{13} \right) \end{aligned} \quad (\text{D.56})$$

$$\begin{aligned} \mathcal{C}\mathcal{C}_{6,12} = & +W_{46}^{21} + \tau_0 \left(-\frac{1}{R_2} R_{46}^{21} - \frac{1}{\rho_{11}} R_{16}^{21} - \frac{1}{\rho_{22}} R_{66}^{21} - 0.5 \frac{1}{R_1 R_2} R_{46}^{22} \right. \\ & + 0.5 \frac{1}{R_1 \rho_{11}} R_{16}^{22} - 0.5 \frac{1}{R_1 \rho_{22}} R_{66}^{22} + 0.5 \frac{1}{(R_2^2)} R_{46}^{22} - 0.5 \frac{1}{R_2 \rho_{11}} R_{16}^{22} + 0.5 \frac{1}{R_2 \rho_{22}} R_{66}^{22} \\ & \left. + 0.5 \frac{1}{R_1} W_{46}^{22} + 0.5 \frac{1}{R_2} W_{46}^{22} \right) + \tau \left(-0.5 \frac{1}{(R_1^2) \rho_{11}} R_{16}^{23} + 0.5 \frac{1}{R_1 (R_2^2)} R_{46}^{23} \right. \\ & \left. + 0.5 \frac{1}{R_1 R_2 \rho_{11}} R_{16}^{23} + 0.5 \frac{1}{R_1 R_2 \rho_{22}} R_{66}^{23} - 0.5 \frac{1}{(R_2^3)} R_{46}^{23} - 0.5 \frac{1}{(R_2^2) \rho_{22}} R_{66}^{23} \right) \end{aligned} \quad (\text{D.57})$$

$$\mathcal{C}\mathcal{C}_{7,7} = +Q_{11}^{110} + \tau_0 \left(-\frac{1}{R_1} Q_{11}^{111} + \frac{1}{R_2} Q_{11}^{111} \right) + \tau \left(+\frac{1}{(R_1^2)} Q_{11}^{112} - \frac{1}{R_1 R_2} Q_{11}^{112} \right) \quad (\text{D.58})$$

$$\mathcal{C}\mathcal{C}_{7,8} = +Q_{16}^{120} + \tau_0 \left(-\frac{1}{R_1} Q_{16}^{121} + \frac{1}{R_2} Q_{16}^{121} \right) + \tau \left(+\frac{1}{(R_1^2)} Q_{16}^{122} - \frac{1}{R_1 R_2} Q_{16}^{122} \right) \quad (\text{D.59})$$

$$\mathcal{C}\mathcal{C}_{7,9} = +Q_{16}^{110} \quad (\text{D.60})$$

$$\mathcal{C}\mathcal{C}_{7,10} = +Q_{12}^{120} \quad (\text{D.61})$$

$$\begin{aligned} \mathcal{C}\mathcal{C}_{7,11} = & +Y_{15}^{110} + \tau_0 \left(-\frac{1}{R_1} Q_{15}^{110} + \frac{1}{\rho_{11}} Q_{16}^{110} + \frac{1}{\rho_{22}} Q_{12}^{110} + \frac{1}{(R_1^2)} Q_{15}^{111} \right. \\ & - \frac{1}{R_1 R_2} Q_{15}^{111} - \frac{1}{R_1 \rho_{11}} Q_{16}^{111} + \frac{1}{R_2 \rho_{11}} Q_{16}^{111} + \frac{1}{R_2} Y_{15}^{111} \left. \right) + \tau \left(-\frac{1}{(R_1^3)} Q_{15}^{112} \right. \\ & \left. + \frac{1}{(R_1^2) R_2} Q_{15}^{112} + \frac{1}{(R_1^2) \rho_{11}} Q_{16}^{112} - \frac{1}{R_1 R_2 \rho_{11}} Q_{16}^{112} \right) \end{aligned} \quad (\text{D.62})$$

$$\begin{aligned} \mathcal{C}\mathcal{C}_{7,12} = & +Y_{14}^{210} + \tau_0 \left(-\frac{1}{R_2} Q_{14}^{120} - \frac{1}{\rho_{11}} Q_{11}^{120} - \frac{1}{\rho_{22}} Q_{16}^{120} + \frac{1}{R_1 \rho_{11}} Q_{11}^{121} \right. \\ & \left. - \frac{1}{R_2 \rho_{11}} Q_{11}^{121} + \frac{1}{R_2} Y_{14}^{211} \right) + \tau \left(-\frac{1}{(R_1^2) \rho_{11}} Q_{11}^{122} + \frac{1}{R_1 R_2 \rho_{11}} Q_{11}^{122} \right) \end{aligned} \quad (\text{D.63})$$

$$\mathcal{C}\mathcal{C}_{8,8} = +Q_{66}^{220} + \tau_0 \left(-\frac{1}{R_1} Q_{66}^{221} + \frac{1}{R_2} Q_{66}^{221} \right) + \tau \left(+\frac{1}{(R_1^2)} Q_{66}^{222} - \frac{1}{R_1 R_2} Q_{66}^{222} \right) \quad (\text{D.64})$$

$$\mathcal{C}\mathcal{C}_{8,9} = +Q_{66}^{120} \quad (\text{D.65})$$

$$\mathcal{C}\mathcal{C}_{8,10} = +Q_{26}^{220} \quad (\text{D.66})$$

$$\begin{aligned} \mathcal{C}\mathcal{C}_{8,11} = & +Y_{56}^{120} + \tau_0 \left(-\frac{1}{R_1} Q_{56}^{120} + \frac{1}{\rho_{11}} Q_{66}^{120} + \frac{1}{\rho_{22}} Q_{26}^{120} + \frac{1}{(R_1^2)} Q_{56}^{121} \right. \\ & - \frac{1}{R_1 R_2} Q_{56}^{121} - \frac{1}{R_1 \rho_{11}} Q_{66}^{121} + \frac{1}{R_2 \rho_{11}} Q_{66}^{121} + \frac{1}{R_2} Y_{56}^{121} \left. \right) + \tau \left(-\frac{1}{(R_1^3)} Q_{56}^{122} \right. \\ & \left. + \frac{1}{(R_1^2) R_2} Q_{56}^{122} + \frac{1}{(R_1^2) \rho_{11}} Q_{66}^{122} - \frac{1}{R_1 R_2 \rho_{11}} Q_{66}^{122} \right) \end{aligned} \quad (\text{D.67})$$

$$\begin{aligned} \mathcal{C}\mathcal{C}_{8,12} = & + Y_{46}^{220} + \tau_0 \left(-\frac{1}{R_2} Q_{46}^{220} - \frac{1}{\rho_{11}} Q_{16}^{220} - \frac{1}{\rho_{22}} Q_{66}^{220} + \frac{1}{R_1 \rho_{11}} Q_{16}^{221} \right. \\ & \left. - \frac{1}{R_2 \rho_{11}} Q_{16}^{221} + \frac{1}{R_2} Y_{46}^{221} \right) + \tau \left(-\frac{1}{(R_1^2) \rho_{11}} Q_{16}^{222} + \frac{1}{R_1 R_2 \rho_{11}} Q_{16}^{222} \right) \end{aligned} \quad (\text{D.68})$$

$$\mathcal{C}\mathcal{C}_{9,9} = + Q_{66}^{110} + \tau_0 \left(+\frac{1}{R_1} Q_{66}^{111} - \frac{1}{R_2} Q_{66}^{111} \right) + \tau \left(-\frac{1}{R_1 R_2} Q_{66}^{112} + \frac{1}{(R_2^2)} Q_{66}^{112} \right) \quad (\text{D.69})$$

$$\mathcal{C}\mathcal{C}_{9,10} = + Q_{26}^{120} + \tau_0 \left(+\frac{1}{R_1} Q_{26}^{121} - \frac{1}{R_2} Q_{26}^{121} \right) + \tau \left(-\frac{1}{R_1 R_2} Q_{26}^{122} + \frac{1}{(R_2^2)} Q_{26}^{122} \right) \quad (\text{D.70})$$

$$\begin{aligned} \mathcal{C}\mathcal{C}_{9,11} = & + Y_{56}^{110} + \tau_0 \left(-\frac{1}{R_1} Q_{56}^{110} + \frac{1}{\rho_{11}} Q_{66}^{110} + \frac{1}{\rho_{22}} Q_{26}^{110} + \frac{1}{R_1 \rho_{22}} Q_{26}^{111} \right. \\ & \left. - \frac{1}{R_2 \rho_{22}} Q_{26}^{111} + \frac{1}{R_1} Y_{56}^{111} \right) + \tau \left(-\frac{1}{R_1 R_2 \rho_{22}} Q_{26}^{112} + \frac{1}{(R_2^2) \rho_{22}} Q_{26}^{112} \right) \end{aligned} \quad (\text{D.71})$$

$$\begin{aligned} \mathcal{C}\mathcal{C}_{9,12} = & + Y_{46}^{210} + \tau_0 \left(-\frac{1}{R_2} Q_{46}^{120} - \frac{1}{\rho_{11}} Q_{16}^{120} - \frac{1}{\rho_{22}} Q_{66}^{120} - \frac{1}{R_1 R_2} Q_{46}^{121} \right. \\ & \left. - \frac{1}{R_1 \rho_{22}} Q_{66}^{121} + \frac{1}{(R_2^2)} Q_{46}^{121} + \frac{1}{R_2 \rho_{22}} Q_{66}^{121} + \frac{1}{R_1} Y_{46}^{211} \right) + \tau \left(+\frac{1}{R_1 (R_2^2)} Q_{46}^{122} \right. \\ & \left. + \frac{1}{R_1 R_2 \rho_{22}} Q_{66}^{122} - \frac{1}{(R_2^3)} Q_{46}^{122} - \frac{1}{(R_2^2) \rho_{22}} Q_{66}^{122} \right) \end{aligned} \quad (\text{D.72})$$

$$\mathcal{C}\mathcal{C}_{10,10} = + Q_{22}^{220} + \tau_0 \left(+\frac{1}{R_1} Q_{22}^{221} - \frac{1}{R_2} Q_{22}^{221} \right) + \tau \left(-\frac{1}{R_1 R_2} Q_{22}^{222} + \frac{1}{(R_2^2)} Q_{22}^{222} \right) \quad (\text{D.73})$$

$$\begin{aligned} \mathcal{C}\mathcal{C}_{10,11} = & + Y_{25}^{120} + \tau_0 \left(-\frac{1}{R_1} Q_{25}^{120} + \frac{1}{\rho_{11}} Q_{26}^{120} + \frac{1}{\rho_{22}} Q_{22}^{120} + \frac{1}{R_1 \rho_{22}} Q_{22}^{121} \right. \\ & \left. - \frac{1}{R_2 \rho_{22}} Q_{22}^{121} + \frac{1}{R_1} Y_{25}^{121} \right) + \tau \left(-\frac{1}{R_1 R_2 \rho_{22}} Q_{22}^{122} + \frac{1}{(R_2^2) \rho_{22}} Q_{22}^{122} \right) \end{aligned} \quad (\text{D.74})$$

$$\begin{aligned}
\mathcal{CC}_{10,12} = & + Y_{24}^{220} + \tau_0 \left(-\frac{1}{R_2} Q_{24}^{220} - \frac{1}{\rho_{11}} Q_{12}^{220} - \frac{1}{\rho_{22}} Q_{26}^{220} - \frac{1}{R_1 R_2} Q_{24}^{221} \right. \\
& - \frac{1}{R_1 \rho_{22}} Q_{26}^{221} + \frac{1}{(R_2^2)} Q_{24}^{221} + \frac{1}{R_2 \rho_{22}} Q_{26}^{221} + \frac{1}{R_1} Y_{24}^{221} \left. \right) + \tau \left(+ \frac{1}{R_1 (R_2^2)} Q_{24}^{222} \right. \\
& \left. + \frac{1}{R_1 R_2 \rho_{22}} Q_{26}^{222} - \frac{1}{(R_2^3)} Q_{24}^{222} - \frac{1}{(R_2^2) \rho_{22}} Q_{26}^{222} \right)
\end{aligned} \tag{D.75}$$

$$\begin{aligned}
\mathcal{CC}_{11,11} = & + Z_{55}^{110} + \tau_0 \left(+ \frac{1}{(R_1^2)} Q_{55}^{110} - 2 \frac{1}{R_1 \rho_{11}} Q_{56}^{110} - 2 \frac{1}{R_1 \rho_{22}} Q_{25}^{110} + \frac{1}{(\rho_{11}^2)} Q_{66}^{110} \right. \\
& + 2 \frac{1}{\rho_{11} \rho_{22}} Q_{26}^{110} + \frac{1}{(\rho_{22}^2)} Q_{22}^{110} - 2 \frac{1}{R_1} Y_{55}^{110} + 2 \frac{1}{\rho_{11}} Y_{56}^{110} + 2 \frac{1}{\rho_{22}} Y_{25}^{110} - \frac{1}{(R_1^3)} Q_{55}^{111} \\
& + \frac{1}{(R_1^2) R_2} Q_{55}^{111} + 2 \frac{1}{(R_1^2) \rho_{11}} Q_{56}^{111} - 2 \frac{1}{R_1 R_2 \rho_{11}} Q_{56}^{111} - \frac{1}{R_1 (\rho_{11}^2)} Q_{66}^{111} \\
& + \frac{1}{R_1 (\rho_{22}^2)} Q_{22}^{111} + \frac{1}{R_2 (\rho_{11}^2)} Q_{66}^{111} - \frac{1}{R_2 (\rho_{22}^2)} Q_{22}^{111} - 2 \frac{1}{R_1 R_2} Y_{55}^{111} + 2 \frac{1}{R_1 \rho_{22}} Y_{25}^{111} \\
& + 2 \frac{1}{R_2 \rho_{11}} Y_{56}^{111} + \frac{1}{R_1} Z_{55}^{111} + \frac{1}{R_2} Z_{55}^{111} \left. \right) + \tau \left(+ \frac{1}{(R_1^4)} Q_{55}^{112} - \frac{1}{(R_1^3) R_2} Q_{55}^{112} \right. \\
& - 2 \frac{1}{(R_1^3) \rho_{11}} Q_{56}^{112} + 2 \frac{1}{(R_1^2) R_2 \rho_{11}} Q_{56}^{112} + \frac{1}{(R_1^2) (\rho_{11}^2)} Q_{66}^{112} - \frac{1}{R_1 R_2 (\rho_{11}^2)} Q_{66}^{112} \\
& \left. - \frac{1}{R_1 R_2 (\rho_{22}^2)} Q_{22}^{112} + \frac{1}{(R_2^2) (\rho_{22}^2)} Q_{22}^{112} + \frac{1}{R_1 R_2} Z_{55}^{112} \right)
\end{aligned} \tag{D.76}$$

$$\begin{aligned}
\mathcal{C}\mathcal{C}_{11,12} = & + Z_{45}^{120} + \tau_0 \left(+ \frac{1}{R_1 R_2} Q_{45}^{120} + \frac{1}{R_1 \rho_{11}} Q_{15}^{120} + \frac{1}{R_1 \rho_{22}} Q_{56}^{120} \right. \\
& - \frac{1}{R_2 \rho_{11}} Q_{46}^{120} - \frac{1}{R_2 \rho_{22}} Q_{24}^{120} - \frac{1}{(\rho_{11}^2)} Q_{16}^{120} - \frac{1}{\rho_{11} \rho_{22}} Q_{12}^{120} - \frac{1}{\rho_{11} \rho_{22}} Q_{66}^{120} \\
& - \frac{1}{(\rho_{22}^2)} Q_{26}^{120} - \frac{1}{R_1} Y_{45}^{210} + \frac{1}{\rho_{11}} Y_{46}^{210} + \frac{1}{\rho_{22}} Y_{24}^{210} - \frac{1}{R_2} Y_{45}^{120} - \frac{1}{\rho_{11}} Y_{15}^{120} \\
& - \frac{1}{\rho_{22}} Y_{56}^{120} - \frac{1}{(R_1^2) \rho_{11}} Q_{15}^{121} + \frac{1}{R_1 R_2 \rho_{11}} Q_{15}^{121} - \frac{1}{R_1 R_2 \rho_{22}} Q_{24}^{121} \\
& + \frac{1}{R_1 (\rho_{11}^2)} Q_{16}^{121} - \frac{1}{R_1 (\rho_{22}^2)} Q_{26}^{121} + \frac{1}{(R_2^2) \rho_{22}} Q_{24}^{121} - \frac{1}{R_2 (\rho_{11}^2)} Q_{16}^{121} + \frac{1}{R_2 (\rho_{22}^2)} Q_{26}^{121} \quad (D.77) \\
& - \frac{1}{R_1 R_2} Y_{45}^{211} + \frac{1}{R_1 \rho_{22}} Y_{24}^{211} + \frac{1}{R_2 \rho_{11}} Y_{46}^{211} - \frac{1}{R_1 R_2} Y_{45}^{121} - \frac{1}{R_1 \rho_{22}} Y_{56}^{121} \\
& - \frac{1}{R_2 \rho_{11}} Y_{15}^{121} + \frac{1}{R_1} Z_{45}^{121} + \frac{1}{R_2} Z_{45}^{121} \left. \right) + \tau \left(+ \frac{1}{(R_1^3) \rho_{11}} Q_{15}^{122} - \frac{1}{(R_1^2) R_2 \rho_{11}} Q_{15}^{122} \right. \\
& - \frac{1}{(R_1^2) (\rho_{11}^2)} Q_{16}^{122} + \frac{1}{R_1 (R_2^2) \rho_{22}} Q_{24}^{122} + \frac{1}{R_1 R_2 (\rho_{11}^2)} Q_{16}^{122} + \frac{1}{R_1 R_2 (\rho_{22}^2)} Q_{26}^{122} \\
& \left. - \frac{1}{(R_2^3) \rho_{22}} Q_{24}^{122} - \frac{1}{(R_2^2) (\rho_{22}^2)} Q_{26}^{122} + \frac{1}{R_1 R_2} Z_{45}^{122} \right)
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}\mathcal{C}_{12,12} = & + Z_{44}^{220} + \tau_0 \left(+ \frac{1}{(R_2^2)} Q_{44}^{220} + 2 \frac{1}{R_2 \rho_{11}} Q_{14}^{220} + 2 \frac{1}{R_2 \rho_{22}} Q_{46}^{220} + \frac{1}{(\rho_{11}^2)} Q_{11}^{220} \right. \\
& + 2 \frac{1}{\rho_{11} \rho_{22}} Q_{16}^{220} + \frac{1}{(\rho_{22}^2)} Q_{66}^{220} - 2 \frac{1}{R_2} Y_{44}^{220} - 2 \frac{1}{\rho_{11}} Y_{14}^{220} - 2 \frac{1}{\rho_{22}} Y_{46}^{220} + \frac{1}{R_1 (R_2^2)} Q_{44}^{221} \\
& + 2 \frac{1}{R_1 R_2 \rho_{22}} Q_{46}^{221} - \frac{1}{R_1 (\rho_{11}^2)} Q_{11}^{221} + \frac{1}{R_1 (\rho_{22}^2)} Q_{66}^{221} - \frac{1}{(R_2^3)} Q_{44}^{221} - 2 \frac{1}{(R_2^2) \rho_{22}} Q_{46}^{221} \\
& + \frac{1}{R_2 (\rho_{11}^2)} Q_{11}^{221} - \frac{1}{R_2 (\rho_{22}^2)} Q_{66}^{221} - 2 \frac{1}{R_1 R_2} Y_{44}^{221} - 2 \frac{1}{R_1 \rho_{22}} Y_{46}^{221} - 2 \frac{1}{R_2 \rho_{11}} Y_{14}^{221} \quad (D.78) \\
& + \frac{1}{R_1} Z_{44}^{221} + \frac{1}{R_2} Z_{44}^{221} \left. \right) + \tau \left(+ \frac{1}{(R_1^2) (\rho_{11}^2)} Q_{11}^{222} - \frac{1}{R_1 (R_2^3)} Q_{44}^{222} \right. \\
& - 2 \frac{1}{R_1 (R_2^2) \rho_{22}} Q_{46}^{222} - \frac{1}{R_1 R_2 (\rho_{11}^2)} Q_{11}^{222} - \frac{1}{R_1 R_2 (\rho_{22}^2)} Q_{66}^{222} + \frac{1}{(R_2^4)} Q_{44}^{222} \\
& \left. + 2 \frac{1}{(R_2^3) \rho_{22}} Q_{46}^{222} + \frac{1}{(R_2^2) (\rho_{22}^2)} Q_{66}^{222} + \frac{1}{R_1 R_2} Z_{44}^{222} \right)
\end{aligned}$$

Appendix E Through-the-thickness Elasticity Coefficient Integrals

E.1 Compliance Tensor

The through-the-thickness integrals of the compliance tensor are defined as follows ($F' = dF/d\xi_3$):

$$A_{pq}^k = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \bar{Q}_{pq}(\xi_3)^k d\xi_3 \quad (\text{E.1a})$$

$$R_{pq}^{jk} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \bar{Q}_{pq}(\xi_3)^k F_j d\xi_3 \quad (\text{E.1b})$$

$$Q_{pq}^{ijk} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \bar{Q}_{pq}(\xi_3)^k F_i F_j d\xi_3 \quad (\text{E.1c})$$

$$W_{pq}^{jk} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \bar{Q}_{pq}(\xi_3)^k F'_j d\xi_3 \quad (\text{E.1d})$$

$$Y_{pq}^{ijk} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \bar{Q}_{pq}(\xi_3)^k F'_i F_j d\xi_3 \quad (\text{E.1e})$$

$$Z_{pq}^{ijk} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \bar{Q}_{pq}(\xi_3)^k F'_i F'_j d\xi_3 \quad (\text{E.1f})$$

If shell consists of N_{LYRS} layers, each with a constant thickness such as t_l , that each one presents constant properties through the thickness of the shell element ($\bar{Q}_{pq,l} = cte$), integrals of equation (E.1) can be replaced with summations. The A_{pq}^k for zero and first order shear deformation model is the same. For zero order shear deformation (when $F_i = F_j = 0$), $R_{pq}^{jk} = Q_{pq}^{ijk} = W_{pq}^{jk} = Y_{pq}^{ijk} = Z_{pq}^{ijk} = 0$. For first order shear deformation model (when $F_i = F_j = \xi_3$), the through-the-thickness integrals

can be obtained from the following summations:

$$A_{pq}^k = \sum_{l=1}^{N_{LYRS}} \left(I^{A,k}(\xi_{3,l}) - I^{A,k}(\xi_{3,l-1}) \right) \bar{Q}_{pq,l}; \quad I^{A,k}(\xi_3) = \frac{\xi_3^{k+1}}{k+1} \quad (\text{E.2a})$$

$$R_{pq}^{jk} = \sum_{l=1}^{N_{LYRS}} \left(I^{R,k}(\xi_{3,l}) - I^{R,k}(\xi_{3,l-1}) \right) \bar{Q}_{pq,l}; \quad I^{R,k}(\xi_3) = \frac{\xi_3^{k+2}}{k+2} \quad (\text{E.2b})$$

$$Q_{pq}^{ijk} = \sum_{l=1}^{N_{LYRS}} \left(I^{Q,k}(\xi_{3,l}) - I^{Q,k}(\xi_{3,l-1}) \right) \bar{Q}_{pq,l}; \quad I^{Q,k}(\xi_3) = \frac{\xi_3^{k+3}}{k+3} \quad (\text{E.2c})$$

$$W_{pq}^k = \sum_{l=1}^{N_{LYRS}} \left(I^{W,k}(\xi_{3,l}) - I^{W,k}(\xi_{3,l-1}) \right) \bar{Q}_{pq,l}; \quad I^{Q,k}(\xi_3) = \frac{\xi_3^{k+1}}{k+1} \quad (\text{E.2d})$$

$$Y_{pq}^k = \sum_{l=1}^{N_{LYRS}} \left(I^{Y,k}(\xi_{3,l}) - I^{Y,k}(\xi_{3,l-1}) \right) \bar{Q}_{pq,l}; \quad I^{Y,k}(\xi_3) = \frac{\xi_3^{k+2}}{k+2} \quad (\text{E.2e})$$

$$Z_{pq}^k = \sum_{l=1}^{N_{LYRS}} \left(I^{Z,k}(\xi_{3,l}) - I^{Z,k}(\xi_{3,l-1}) \right) \bar{Q}_{pq,l}; \quad I^{Z,k}(\xi_3) = \frac{\xi_3^{k+1}}{k+1} \quad (\text{E.2f})$$

where:

$$\xi_{3,l} = \xi_{3,0} + \sum_{m=1}^l (t_m) \quad (\text{E.3a})$$

$$\xi_{3,0} = -\frac{1}{2} \sum_{m=1}^{N_{LYRS}} (t_m) \quad (\text{E.3b})$$

When a constant thickness lamina is replaced with its equivalent linearly-variable thickness layer that presents the same thickness at a location such as $\xi_1 = \xi_{1,m}$, the coordinates-free through-

the-thickness coefficients can be obtained from the following equation:

$$\begin{aligned}\bar{A}_{pq}^k &= \xi_{1,m}^{-(k+1)} A_{pq}^k; & \bar{R}_{pq}^k &= \xi_{1,m}^{-(k+2)} R_{pq}^k; & \bar{Q}_{pq}^k &= \xi_{1,m}^{-(k+3)} Q_{pq}^k; \\ \bar{W}_{pq}^k &= \xi_{1,m}^{-(k+1)} W_{pq}^k; & \bar{Y}_{pq}^k &= \xi_{1,m}^{-(k+2)} Y_{pq}^k; & \bar{Z}_{pq}^k &= \xi_{1,m}^{-(k+1)} Z_{pq}^k;\end{aligned}\tag{E.4}$$

E.2 Density

Assuming the volumetric density of the shell through the thickness is defined as $\rho(\xi_3)$ function, the areal densities introduced in equation (63) are defined as follows:

$$\rho^k = \int_{-\frac{h}{2}}^{+\frac{h}{2}} [\rho(\xi_3)] (\xi_3)^k \left(1 + \frac{\xi_3}{R_1}\right) \left(1 + \frac{\xi_3}{R_2}\right) d\xi_3 \quad (k = 0, 1, 2)\tag{E.5}$$

Similar to the previous section, if the shell consists of N_{LYRS} layers, each with a constant thickness such as t_1 , that each one presents a constant density through the thickness of the shell element ($\rho_1 = cte$), above integrals converts to the following summations:

$$\rho^k = \rho_0^k + \tau_0 \rho_{\tau_0}^k + \tau_0 \tau \rho_{\tau}^k\tag{E.6}$$

where:

$$\rho_0^k = \sum_{l=1}^{N_{LYRS}} \left(I^{A,k}(\xi_{3,l}) - I^{A,k}(\xi_{3,l-1}) \right) \rho_1\tag{E.7a}$$

$$\rho_{\tau_0}^k = \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \sum_{l=1}^{N_{LYRS}} \left(I^{A,k+1}(\xi_{3,l}) - I^{A,k+1}(\xi_{3,l-1}) \right) \rho_1\tag{E.7b}$$

$$\rho_{\tau}^k = \left(\frac{1}{R_1 R_2} \right) \sum_{l=1}^{N_{LYRS}} \left(I^{A,k+2}(\xi_{3,l}) - I^{A,k+2}(\xi_{3,l-1}) \right) \rho_1\tag{E.7c}$$

The definition of $\xi_{3,l}$ is as same as what has been given in equation (E.3), the definition of $I^{A,k}(\xi_3)$ is same as (E.2a) and $\tau_0, \tau \in \{0, 1\}$ are flag parameters to neglect or to consider the effects of principle radii of curvature at zero, one and third order.

When a constant thickness lamina is replaced with its equivalent linearly-variable thickness layer that presents the same thickness at a location such as $\xi_1 = \xi_{1,m}$, the areal densities demonstrate

dependency to ξ_1 coordinates and can be obtained from the following equation:

$$\rho^k = \left(\xi_{1,m}^{-(k+1)} \right) \left[\rho_{c,0}^k \right] \left(\xi_1^{k+1} \right) + \tau_0 \left(\xi_{1,m}^{-(k+2)} \right) \left[\rho_{c,\tau_0}^k \right] \left(\xi_1^{k+2} \right) + \tau_0 \tau \left(\xi_{1,m}^{-(k+3)} \right) \left[\rho_{c,\tau}^k \right] \left(\xi_1^{k+3} \right) \quad (\text{E.8})$$

where ρ_c^k is the calculated areal density for the original constant thickness shell using equation (E.6).

Appendix F Matrix of Equilibrium Equations for Variable Thickness Conical Shells

Introducing the following aliases:

$$sn = \sin(\alpha_c) \quad (\text{F.1a})$$

$$cs = \cos(\alpha_c) \quad (\text{F.1b})$$

$$tn = \tan(\alpha_c) \quad (\text{F.1c})$$

The non-zero elements of $[\mathcal{CP}]$ matrix for linearly variable thickness truncated cones can be formulated as follows:

$$\begin{aligned} \mathcal{CP}_{1,1} = & -\bar{A}_{16}^0 \frac{n_c}{sn} + 0.5\bar{A}_{16}^1 \frac{n_c cs}{(sn^2)} - 0.5\bar{A}_{26}^1 \frac{n_c cs}{(sn^2)} + 0.5\bar{A}_{26}^1 \frac{n_c}{sntn} + \tau_0 \left[\right. \\ & -0.5\bar{A}_{16}^1 \frac{n_c}{sntn} + 0.25\bar{A}_{16}^2 \frac{n_c cs}{(sn^2)tn} - 0.25\bar{A}_{26}^2 \frac{n_c}{sn(tn^2)} + 0.25\bar{A}_{26}^2 \frac{n_c cs}{(sn^2)tn} \left. \right] + \tau \left[\right. \\ & \left. -0.25\bar{A}_{16}^2 \frac{n_c}{sn(tn^2)} + 0.25\bar{A}_{26}^3 \frac{n_c}{sn(tn^3)} - 0.25\bar{A}_{26}^3 \frac{n_c cs}{(sn^2)(tn^2)} \right] \end{aligned} \quad (\text{F.2})$$

$$\begin{aligned} \mathcal{CP}_{1,2} = & -\bar{A}_{16}^0 + \bar{A}_{26}^0 - \bar{A}_{26}^0 \frac{(n_c^2)}{(sn^2)} - 1.5\bar{A}_{16}^1 \frac{cs}{sn} - 0.5\bar{A}_{26}^1 \frac{(n_c^2)}{(sn^2)tn} \\ & + 1.5\bar{A}_{26}^1 \frac{cs}{sn} + 0.5\bar{A}_{26}^2 \frac{(n_c^2)}{(sn^2)(tn^2)} + \tau_0 \left[-0.5\bar{A}_{16}^1 \frac{1}{tn} + 0.5\bar{A}_{26}^1 \frac{(n_c^2)}{(sn^2)tn} - 0.5\bar{A}_{26}^1 \frac{1}{tn} \right. \\ & \left. - 0.75\bar{A}_{16}^2 \frac{cs}{sntn} + 0.25\bar{A}_{26}^2 \frac{(n_c^2)}{(sn^2)(tn^2)} - 0.75\bar{A}_{26}^2 \frac{cs}{sntn} - 0.25\bar{A}_{26}^3 \frac{(n_c^2)}{(sn^2)(tn^3)} \right] + \tau \left[\right. \\ & \left. - 0.25\bar{A}_{16}^2 \frac{1}{(tn^2)} - 0.75\bar{A}_{26}^2 \frac{(n_c^2)}{(sn^2)(tn^2)} + 0.75\bar{A}_{26}^2 \frac{1}{(tn^2)} + 0.75\bar{A}_{26}^3 \frac{cs}{sn(tn^2)} \right. \\ & \left. - 0.5\bar{A}_{26}^3 \frac{(n_c^2)}{(sn^2)(tn^3)} + 0.25\bar{A}_{26}^4 \frac{(n_c^2)}{(sn^2)(tn^4)} \right] \end{aligned} \quad (\text{F.3})$$

$$\begin{aligned}
\mathcal{CP}_{1,3} = & -\bar{A}_{26}^0 \frac{n_c}{sntn} - 2\bar{A}_{16}^1 \frac{n_c}{sn} + 2\bar{A}_{26}^1 \frac{n_c}{sn} - \bar{A}_{26}^1 \frac{(n_c^3)}{(sn^3)} \\
& + 0.5\bar{A}_{26}^1 \frac{n_c}{sn(tn^2)} + 0.5\bar{A}_{26}^2 \frac{(n_c^3)}{(sn^3)tn} + \tau_0 \left[+0.5\bar{A}_{26}^1 \frac{n_c}{sn(tn^2)} - \bar{A}_{16}^2 \frac{n_c}{sntn} \right. \\
& - 0.25\bar{A}_{26}^2 \frac{n_c}{sn(tn^3)} - \bar{A}_{26}^2 \frac{n_c}{sntn} + 0.5\bar{A}_{26}^2 \frac{(n_c^3)}{(sn^3)tn} - 0.25\bar{A}_{26}^3 \frac{(n_c^3)}{(sn^3)(tn^2)} \left. \right] + \tau \left[\right. \\
& - 0.75\bar{A}_{26}^2 \frac{n_c}{sn(tn^3)} + 0.25\bar{A}_{26}^3 \frac{n_c}{sn(tn^4)} - 0.75\bar{A}_{26}^3 \frac{(n_c^3)}{(sn^3)(tn^2)} + \bar{A}_{26}^3 \frac{n_c}{sn(tn^2)} \\
& \left. + 0.25\bar{A}_{26}^4 \frac{(n_c^3)}{(sn^3)(tn^3)} \right]
\end{aligned} \tag{F.4}$$

$$\begin{aligned}
\mathcal{CP}_{1,4} = & -2\bar{R}_{16}^{10} \frac{n_c}{sn} + \bar{R}_{26}^{10} \frac{n_c}{sn} - \bar{W}_{56}^{10} \frac{n_c}{sn} + 0.5\bar{W}_{56}^{11} \frac{n_c}{sntn} + \tau_0 \left[\right. \\
& - \bar{R}_{26}^{10} \frac{n_c}{sn} - 0.25\bar{R}_{26}^{12} \frac{n_c}{sn(tn^2)} - 0.5\bar{W}_{56}^{11} \frac{n_c}{sntn} + 0.25\bar{W}_{56}^{12} \frac{n_c}{sn(tn^2)} \left. \right] + \tau \left[\right. \\
& \left. + 0.25\bar{R}_{26}^{12} \frac{n_c}{sn(tn^2)} + 0.25\bar{R}_{26}^{13} \frac{n_c}{sn(tn^3)} - 0.25\bar{W}_{56}^{12} \frac{n_c}{sn(tn^2)} \right]
\end{aligned} \tag{F.5}$$

$$\begin{aligned}
\mathcal{CP}_{1,5} = & -\bar{R}_{26}^{20} \frac{(n_c^2)}{(sn^2)} + 0.5\bar{R}_{26}^{21} \frac{(n_c^2)}{(sn^2)tn} + 2\bar{W}_{14}^{20} - \bar{W}_{24}^{20} + \tau_0 \left[\right. \\
& - 2\bar{R}_{14}^{20} \frac{1}{tn} - 2\bar{R}_{16}^{20} + \bar{R}_{24}^{20} \frac{1}{tn} + \bar{R}_{26}^{20} - \bar{R}_{24}^{21} \frac{1}{(tn^2)} + 0.5\bar{R}_{26}^{21} \frac{(n_c^2)}{(sn^2)tn} \\
& - \bar{R}_{26}^{21} \frac{1}{tn} - 0.25\bar{R}_{26}^{22} \frac{(n_c^2)}{(sn^2)(tn^2)} + 2\bar{W}_{14}^{21} \frac{1}{tn} \left. \right] + \tau \left[+\bar{R}_{24}^{22} \frac{1}{(tn^3)} \right. \\
& \left. - 0.75\bar{R}_{26}^{22} \frac{(n_c^2)}{(sn^2)(tn^2)} + \bar{R}_{26}^{22} \frac{1}{(tn^2)} + 0.25\bar{R}_{26}^{23} \frac{(n_c^2)}{(sn^2)(tn^3)} \right]
\end{aligned} \tag{F.6}$$

$$\begin{aligned}
\mathcal{CP}_{1,6} = & -2\bar{A}_{16}^0 \frac{n_c}{sn} + 0.5\bar{A}_{16}^1 \frac{n_c}{sntn} + 0.5\bar{A}_{16}^1 \frac{n_c CS}{(sn^2)} + \tau_0 \left[-\bar{A}_{16}^1 \frac{n_c}{sntn} \right. \\
& \left. + 0.25\bar{A}_{16}^2 \frac{n_c}{sn(tn^2)} + 0.25\bar{A}_{16}^2 \frac{n_c CS}{(sn^2)tn} \right] + \tau \left[-0.5\bar{A}_{16}^2 \frac{n_c}{sn(tn^2)} \right]
\end{aligned} \tag{F.7}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{1,7} = & +\bar{A}_{16}^0 - \bar{A}_{26}^0 + 1.5\bar{A}_{16}^1 \frac{cs}{sn} - 1.5\bar{A}_{26}^1 \frac{cs}{sn} + \tau_0 \left[+0.5\bar{A}_{16}^1 \frac{1}{tn} \right. \\
& + 0.5\bar{A}_{26}^1 \frac{1}{tn} + 0.75\bar{A}_{16}^2 \frac{cs}{sntn} + 0.75\bar{A}_{26}^2 \frac{cs}{sntn} \left. \right] + \tau \left[+0.25\bar{A}_{16}^2 \frac{1}{(tn^2)} - 0.75\bar{A}_{26}^2 \frac{1}{(tn^2)} \right. \\
& \left. - 0.75\bar{A}_{26}^3 \frac{cs}{sn(tn^2)} \right] \quad (F.8)
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{1,8} = & + 2\bar{A}_{16}^1 \frac{n_c}{sn} - \bar{A}_{26}^1 \frac{n_c}{sn} - 0.5\bar{A}_{26}^2 \frac{n_c}{sntn} + \tau_0 \left[+\bar{A}_{16}^2 \frac{n_c}{sntn} \right. \\
& \left. + 0.5\bar{A}_{26}^2 \frac{n_c}{sntn} + 0.25\bar{A}_{26}^3 \frac{n_c}{sn(tn^2)} \right] + \tau \left[-0.25\bar{A}_{26}^3 \frac{n_c}{sn(tn^2)} - 0.25\bar{A}_{26}^4 \frac{n_c}{sn(tn^3)} \right] \quad (F.9)
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{1,9} = & - 2\bar{R}_{16}^{10} \frac{n_c}{sn} + 0.5\bar{R}_{16}^{11} \frac{n_c}{sntn} + \tau_0 \left[-0.5\bar{R}_{16}^{11} \frac{n_c}{sntn} + 0.25\bar{R}_{16}^{12} \frac{n_c}{sn(tn^2)} \right] + \tau \left[\right. \\
& \left. - 0.25\bar{R}_{16}^{12} \frac{n_c}{sn(tn^2)} \right] \quad (F.10)
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{1,10} = & + 3\bar{R}_{16}^{20} - \bar{R}_{26}^{20} + \bar{W}_{14}^{20} + \tau_0 \left[-\bar{R}_{14}^{20} \frac{1}{tn} - \bar{R}_{16}^{20} + 3\bar{R}_{16}^{21} \frac{1}{tn} \right. \\
& \left. + \bar{W}_{14}^{21} \frac{1}{tn} \right] \quad (F.11)
\end{aligned}$$

$$\mathcal{C}\bar{\mathcal{P}}_{1,12} = +\bar{A}_{16}^0 + 1.5\bar{A}_{16}^1 \frac{cs}{sn} + \tau_0 \left[+0.5\bar{A}_{16}^1 \frac{1}{tn} + 0.75\bar{A}_{16}^2 \frac{cs}{sntn} \right] + \tau \left[+0.25\bar{A}_{16}^2 \frac{1}{(tn^2)} \right] \quad (F.12)$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{1,13} = & + 3\bar{A}_{16}^1 \frac{n_c}{sn} - 0.5\bar{A}_{16}^2 \frac{n_c}{sntn} + \tau_0 \left[+1.5\bar{A}_{16}^2 \frac{n_c}{sntn} - 0.25\bar{A}_{16}^3 \frac{n_c}{sn(tn^2)} \right] + \tau \left[\right. \\
& \left. + 0.25\bar{A}_{16}^3 \frac{n_c}{sn(tn^2)} \right] \quad (F.13)
\end{aligned}$$

$$\mathcal{C}\bar{\mathcal{P}}_{1,15} = +\bar{R}_{16}^{20} + \tau_0 \left[+\bar{R}_{16}^{21} \frac{1}{tn} \right] \quad (F.14)$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{2,1} = & +\bar{A}_{12}^0 - \bar{A}_{22}^0 - \bar{A}_{66}^0 \frac{(n_c^2)}{(sn^2)} + 0.5\bar{A}_{66}^1 \frac{(n_c^2)}{(sn^2)tn} \\
& + 0.5\bar{A}_{66}^1 \frac{(n_c^2)cs}{(sn^3)} - 0.25\bar{A}_{66}^2 \frac{(n_c^2)cs}{(sn^3)tn} + \tau_0 \left[+\bar{A}_{22}^1 \frac{1}{tn} \right] + \tau \left[-\bar{A}_{22}^2 \frac{1}{(tn^2)} \right. \\
& \left. - 0.75\bar{A}_{66}^2 \frac{(n_c^2)}{(sn^2)(tn^2)} + 0.25\bar{A}_{66}^3 \frac{(n_c^2)}{(sn^2)(tn^3)} + 0.25\bar{A}_{66}^3 \frac{(n_c^2)cs}{(sn^3)(tn^2)} - 0.0625\bar{A}_{66}^4 \frac{(n_c^2)cs}{(sn^3)(tn^3)} \right]
\end{aligned} \tag{F.15}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{2,2} = & +\bar{A}_{12}^0 \frac{n_c}{sn} - \bar{A}_{22}^0 \frac{n_c}{sn} - \bar{A}_{66}^0 \frac{n_c}{sn} + \bar{A}_{12}^1 \frac{n_c}{sntn} \\
& - \bar{A}_{22}^1 \frac{n_c}{sntn} - 1.5\bar{A}_{66}^1 \frac{n_c cs}{(sn^2)} + 0.5\bar{A}_{66}^1 \frac{n_c}{sntn} + 0.75\bar{A}_{66}^2 \frac{n_c cs}{(sn^2)tn} + \tau_0 \left[+\bar{A}_{22}^1 \frac{n_c}{sntn} \right. \\
& \left. + \bar{A}_{22}^2 \frac{n_c}{sn(tn^2)} \right] + \tau \left[-\bar{A}_{22}^2 \frac{n_c}{sn(tn^2)} - 0.75\bar{A}_{66}^2 \frac{n_c}{sn(tn^2)} - \bar{A}_{22}^3 \frac{n_c}{sn(tn^3)} \right. \\
& \left. - 0.75\bar{A}_{66}^3 \frac{n_c cs}{(sn^2)(tn^2)} + 0.25\bar{A}_{66}^3 \frac{n_c}{sn(tn^3)} + 0.1875\bar{A}_{66}^4 \frac{n_c cs}{(sn^2)(tn^3)} \right]
\end{aligned} \tag{F.16}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{2,3} = & +\bar{A}_{12}^0 \frac{1}{tn} - \bar{A}_{22}^0 \frac{1}{tn} + \bar{A}_{12}^1 \frac{(n_c^2)}{(sn^2)} - \bar{A}_{22}^1 \frac{(n_c^2)}{(sn^2)} \\
& - 2\bar{A}_{66}^1 \frac{(n_c^2)}{(sn^2)} + \bar{A}_{66}^2 \frac{(n_c^2)}{(sn^2)tn} + \tau_0 \left[+\bar{A}_{22}^1 \frac{1}{(tn^2)} + \bar{A}_{22}^2 \frac{(n_c^2)}{(sn^2)tn} \right] + \tau \left[-\bar{A}_{22}^2 \frac{1}{(tn^3)} \right. \\
& \left. - \bar{A}_{22}^3 \frac{(n_c^2)}{(sn^2)(tn^2)} - \bar{A}_{66}^3 \frac{(n_c^2)}{(sn^2)(tn^2)} + 0.25\bar{A}_{66}^4 \frac{(n_c^2)}{(sn^2)(tn^3)} \right]
\end{aligned} \tag{F.17}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{2,4} = & -\bar{R}_{66}^{10} \frac{(n_c^2)}{(sn^2)} + 0.5\bar{R}_{66}^{11} \frac{(n_c^2)}{(sn^2)tn} + 2\bar{W}_{15}^{10} - \bar{W}_{25}^{10} + \tau_0 \left[+2\bar{R}_{12}^{10} \right. \\
& \left. - \bar{R}_{22}^{10} + \bar{R}_{22}^{11} \frac{1}{tn} + 0.5\bar{R}_{66}^{11} \frac{(n_c^2)}{(sn^2)tn} - 0.25\bar{R}_{66}^{12} \frac{(n_c^2)}{(sn^2)(tn^2)} + 2\bar{W}_{15}^{11} \frac{1}{tn} \right] + \tau \left[\right. \\
& \left. - \bar{R}_{22}^{12} \frac{1}{(tn^2)} - 0.75\bar{R}_{66}^{12} \frac{(n_c^2)}{(sn^2)(tn^2)} + 0.25\bar{R}_{66}^{13} \frac{(n_c^2)}{(sn^2)(tn^3)} \right]
\end{aligned} \tag{F.18}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{2,5} = & + 2\bar{R}_{12}^{20} \frac{n_c}{sn} - \bar{R}_{22}^{20} \frac{n_c}{sn} + \bar{W}_{46}^{20} \frac{n_c}{sn} - 0.5\bar{W}_{46}^{21} \frac{n_c}{sntn} + \tau_0 \left[\right. \\
& - \bar{R}_{46}^{20} \frac{n_c}{sntn} - \bar{R}_{66}^{20} \frac{n_c}{sn} + \bar{R}_{22}^{21} \frac{n_c}{sntn} + \bar{R}_{46}^{21} \frac{n_c}{sn(tn^2)} + \bar{R}_{66}^{21} \frac{n_c}{sntn} \\
& \left. - 0.25\bar{R}_{46}^{22} \frac{n_c}{sn(tn^3)} - 0.25\bar{R}_{66}^{22} \frac{n_c}{sn(tn^2)} + 0.5\bar{W}_{46}^{21} \frac{n_c}{sntn} - 0.25\bar{W}_{46}^{22} \frac{n_c}{sn(tn^2)} \right] + \tau \left[\right. \\
& - \bar{R}_{22}^{22} \frac{n_c}{sn(tn^2)} - 0.75\bar{R}_{46}^{22} \frac{n_c}{sn(tn^3)} - 0.75\bar{R}_{66}^{22} \frac{n_c}{sn(tn^2)} + 0.25\bar{R}_{46}^{23} \frac{n_c}{sn(tn^4)} \\
& \left. + 0.25\bar{R}_{66}^{23} \frac{n_c}{sn(tn^3)} + 0.25\bar{W}_{46}^{22} \frac{n_c}{sn(tn^2)} \right] \tag{F.19}
\end{aligned}$$

$$\mathcal{C}\bar{\mathcal{P}}_{2,6} = + 2\bar{A}_{11}^0 + \tau_0 \left[+ 2\bar{A}_{11}^1 \frac{1}{tn} \right] \tag{F.20}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{2,7} = & + \bar{A}_{12}^0 \frac{n_c}{sn} + \bar{A}_{66}^0 \frac{n_c}{sn} + \bar{A}_{12}^1 \frac{n_c}{sntn} + 1.5\bar{A}_{66}^1 \frac{n_c cs}{(sn^2)} \\
& - 0.5\bar{A}_{66}^1 \frac{n_c}{sntn} - 0.75\bar{A}_{66}^2 \frac{n_c cs}{(sn^2)tn} + 0.75\bar{A}_{66}^2 \frac{n_c}{sn(tn^2)} - 0.25\bar{A}_{66}^3 \frac{n_c}{sn(tn^3)} \\
& + 0.75\bar{A}_{66}^3 \frac{n_c cs}{(sn^2)(tn^2)} - 0.1875\bar{A}_{66}^4 \frac{n_c cs}{(sn^2)(tn^3)} \tag{F.21}
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{2,8} = & + \bar{A}_{12}^0 \frac{1}{tn} + \bar{A}_{12}^1 \frac{(n_c^2)}{(sn^2)} - 2\bar{A}_{12}^1 + \bar{A}_{22}^1 + 2\bar{A}_{66}^1 \frac{(n_c^2)}{(sn^2)} \\
& - \bar{A}_{66}^2 \frac{(n_c^2)}{(sn^2)tn} + \tau_0 \left[-\bar{A}_{22}^2 \frac{1}{tn} \right] + \tau \left[+\bar{A}_{22}^3 \frac{1}{(tn^2)} + \bar{A}_{66}^3 \frac{(n_c^2)}{(sn^2)(tn^2)} - 0.25\bar{A}_{66}^4 \frac{(n_c^2)}{(sn^2)(tn^3)} \right] \tag{F.22}
\end{aligned}$$

$$\mathcal{C}\bar{\mathcal{P}}_{2,9} = + 3\bar{R}_{11}^{10} - \bar{R}_{12}^{10} + \bar{W}_{15}^{10} + \tau_0 \left[+\bar{R}_{12}^{10} + 3\bar{R}_{11}^{11} \frac{1}{tn} + \bar{W}_{15}^{11} \frac{1}{tn} \right] \tag{F.23}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{2,10} = & + \bar{R}_{12}^{20} \frac{n_c}{sn} + \bar{R}_{66}^{20} \frac{n_c}{sn} - 0.5\bar{R}_{66}^{21} \frac{n_c}{sntn} + \tau_0 \left[+0.5\bar{R}_{66}^{21} \frac{n_c}{sntn} \right. \\
& \left. - 0.25\bar{R}_{66}^{22} \frac{n_c}{sn(tn^2)} \right] + \tau \left[+0.25\bar{R}_{66}^{22} \frac{n_c}{sn(tn^2)} \right] \tag{F.24}
\end{aligned}$$

$$\mathcal{C}\bar{\mathcal{P}}_{2,11} = +\bar{A}_{11}^0 + \tau_0 \left[+\bar{A}_{11}^1 \frac{1}{tn} \right] \quad (\text{F.25})$$

$$\mathcal{C}\bar{\mathcal{P}}_{2,13} = -3\bar{A}_{11}^1 + \tau_0 \left[-3\bar{A}_{11}^2 \frac{1}{tn} \right] \quad (\text{F.26})$$

$$\mathcal{C}\bar{\mathcal{P}}_{2,14} = +\bar{R}_{11}^{10} + \tau_0 \left[+\bar{R}_{11}^{11} \frac{1}{tn} \right] \quad (\text{F.27})$$

$$\mathcal{C}\bar{\mathcal{P}}_{2,18} = -\bar{A}_{11}^1 + \tau_0 \left[-\bar{A}_{11}^2 \frac{1}{tn} \right] \quad (\text{F.28})$$

$$\begin{aligned} \mathcal{C}\bar{\mathcal{P}}_{3,1} = & -\bar{A}_{22}^0 \frac{n_c}{sn} - 2\bar{A}_{66}^0 \frac{n_c}{sn} - \bar{A}_{22}^1 \frac{n_c}{sntn} + \bar{A}_{66}^1 \frac{n_c cs}{(sn^2)} \\ & - 3\bar{A}_{66}^1 \frac{n_c}{sntn} + 1.5\bar{A}_{66}^2 \frac{n_c cs}{(sn^2)tn} + \tau_0 \left[+\bar{A}_{22}^1 \frac{n_c}{sntn} + \bar{A}_{22}^2 \frac{n_c}{sn(tn^2)} \right] + \tau \left[-\bar{A}_{22}^2 \frac{n_c}{sn(tn^2)} \right. \\ & - 1.5\bar{A}_{66}^2 \frac{n_c}{sn(tn^2)} - \bar{A}_{22}^3 \frac{n_c}{sn(tn^3)} - 1.5\bar{A}_{66}^3 \frac{n_c}{sn(tn^3)} + 0.5\bar{A}_{66}^3 \frac{n_c cs}{(sn^2)(tn^2)} \\ & \left. + 0.375\bar{A}_{66}^4 \frac{n_c cs}{(sn^2)(tn^3)} \right] \end{aligned} \quad (\text{F.29})$$

$$\begin{aligned} \mathcal{C}\bar{\mathcal{P}}_{3,2} = & -\bar{A}_{22}^0 \frac{(n_c^2)}{(sn^2)} - 2\bar{A}_{66}^0 - 2\bar{A}_{22}^1 \frac{(n_c^2)}{(sn^2)tn} - 3\bar{A}_{66}^1 \frac{cs}{sn} \\ & - 3\bar{A}_{66}^1 \frac{1}{tn} - \bar{A}_{22}^2 \frac{(n_c^2)}{(sn^2)(tn^2)} - 4.5\bar{A}_{66}^2 \frac{cs}{sntn} + \tau_0 \left[+\bar{A}_{22}^1 \frac{(n_c^2)}{(sn^2)tn} \right. \\ & \left. + 2\bar{A}_{22}^2 \frac{(n_c^2)}{(sn^2)(tn^2)} + \bar{A}_{22}^3 \frac{(n_c^2)}{(sn^2)(tn^3)} \right] + \tau \left[-\bar{A}_{22}^2 \frac{(n_c^2)}{(sn^2)(tn^2)} - 1.5\bar{A}_{66}^2 \frac{1}{(tn^2)} \right. \\ & - 2\bar{A}_{22}^3 \frac{(n_c^2)}{(sn^2)(tn^3)} - 1.5\bar{A}_{66}^3 \frac{cs}{sn(tn^2)} - 1.5\bar{A}_{66}^3 \frac{1}{(tn^3)} - \bar{A}_{22}^4 \frac{(n_c^2)}{(sn^2)(tn^4)} \\ & \left. - 1.125\bar{A}_{66}^4 \frac{cs}{sn(tn^3)} \right] \end{aligned} \quad (\text{F.30})$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{3,3} = & -\bar{A}_{22}^0 \frac{n_c}{sntn} - \bar{A}_{22}^1 \frac{n_c}{sn(tn^2)} - \bar{A}_{22}^1 \frac{(n_c^3)}{(sn^3)} - 4\bar{A}_{66}^1 \frac{n_c}{sn} \\
& - \bar{A}_{22}^2 \frac{(n_c^3)}{(sn^3)tn} - 6\bar{A}_{66}^2 \frac{n_c}{sntn} + \tau_0 \left[+\bar{A}_{22}^1 \frac{n_c}{sn(tn^2)} + \bar{A}_{22}^2 \frac{n_c}{sn(tn^3)} + \bar{A}_{22}^2 \frac{(n_c^3)}{(sn^3)tn} \right. \\
& \left. + \bar{A}_{22}^3 \frac{(n_c^3)}{(sn^3)(tn^2)} \right] + \tau \left[-\bar{A}_{22}^2 \frac{n_c}{sn(tn^3)} - \bar{A}_{22}^3 \frac{(n_c^3)}{(sn^3)(tn^2)} - \bar{A}_{22}^3 \frac{n_c}{sn(tn^4)} \right. \\
& \left. - 2\bar{A}_{66}^3 \frac{n_c}{sn(tn^2)} - \bar{A}_{22}^4 \frac{(n_c^3)}{(sn^3)(tn^3)} - 1.5\bar{A}_{66}^4 \frac{n_c}{sn(tn^3)} \right]
\end{aligned} \tag{F.31}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{3,4} = & -3\bar{R}_{66}^{10} \frac{n_c}{sn} - 4.5\bar{R}_{66}^{11} \frac{n_c}{sntn} - \bar{W}_{25}^{10} \frac{n_c}{sn} - \bar{W}_{25}^{11} \frac{n_c}{sntn} + \tau_0 \left[\right. \\
& \left. - \bar{R}_{22}^{10} \frac{n_c}{sn} + 1.5\bar{R}_{66}^{11} \frac{n_c}{sntn} + \bar{R}_{22}^{12} \frac{n_c}{sn(tn^2)} + 2.25\bar{R}_{66}^{12} \frac{n_c}{sn(tn^2)} \right] + \tau \left[\right. \\
& \left. - \bar{R}_{22}^{12} \frac{n_c}{sn(tn^2)} - 2.25\bar{R}_{66}^{12} \frac{n_c}{sn(tn^2)} - \bar{R}_{22}^{13} \frac{n_c}{sn(tn^3)} - 2.25\bar{R}_{66}^{13} \frac{n_c}{sn(tn^3)} \right]
\end{aligned} \tag{F.32}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{3,5} = & -\bar{R}_{22}^{20} \frac{(n_c^2)}{(sn^2)} - \bar{R}_{22}^{21} \frac{(n_c^2)}{(sn^2)tn} + 3\bar{W}_{46}^{20} + 4.5\bar{W}_{46}^{21} \frac{1}{tn} + \tau_0 \left[\right. \\
& - 3\bar{R}_{46}^{20} \frac{1}{tn} - 3\bar{R}_{66}^{20} + \bar{R}_{22}^{21} \frac{(n_c^2)}{(sn^2)tn} - 3\bar{R}_{46}^{21} \frac{1}{(tn^2)} - 3\bar{R}_{66}^{21} \frac{1}{tn} \\
& \left. + \bar{R}_{22}^{22} \frac{(n_c^2)}{(sn^2)(tn^2)} + 2.25\bar{R}_{46}^{22} \frac{1}{(tn^3)} + 2.25\bar{R}_{66}^{22} \frac{1}{(tn^2)} + 1.5\bar{W}_{46}^{21} \frac{1}{tn} + 2.25\bar{W}_{46}^{22} \frac{1}{(tn^2)} \right] + \tau \left[\right. \\
& - \bar{R}_{22}^{22} \frac{(n_c^2)}{(sn^2)(tn^2)} - 2.25\bar{R}_{46}^{22} \frac{1}{(tn^3)} - 2.25\bar{R}_{66}^{22} \frac{1}{(tn^2)} - \bar{R}_{22}^{23} \frac{(n_c^2)}{(sn^2)(tn^3)} \\
& \left. - 2.25\bar{R}_{46}^{23} \frac{1}{(tn^4)} - 2.25\bar{R}_{66}^{23} \frac{1}{(tn^3)} + 0.75\bar{W}_{46}^{22} \frac{1}{(tn^2)} \right]
\end{aligned} \tag{F.33}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{3,6} = & -\bar{A}_{12}^0 \frac{n_c}{sn} - \bar{A}_{66}^0 \frac{n_c}{sn} - \bar{A}_{12}^1 \frac{n_c}{sntn} + 0.5\bar{A}_{66}^1 \frac{n_c c s}{(sn^2)} \\
& - 1.5\bar{A}_{66}^1 \frac{n_c}{sntn} + 0.75\bar{A}_{66}^2 \frac{n_c c s}{(sn^2)tn} - 0.75\bar{A}_{66}^2 \frac{n_c}{sn(tn^2)} + 0.25\bar{A}_{66}^3 \frac{n_c c s}{(sn^2)(tn^2)} \\
& - 0.75\bar{A}_{66}^3 \frac{n_c}{sn(tn^3)} + 0.1875\bar{A}_{66}^4 \frac{n_c c s}{(sn^2)(tn^3)}
\end{aligned} \tag{F.34}$$

$$\begin{aligned} \mathcal{CP}_{3,7} = & + 2\bar{A}_{66}^0 + 3\bar{A}_{66}^1 \frac{cs}{sn} + 3\bar{A}_{66}^1 \frac{1}{tn} + 4.5\bar{A}_{66}^2 \frac{cs}{sntn} + 1.5\bar{A}_{66}^2 \frac{1}{(tn^2)} \\ & + 1.5\bar{A}_{66}^3 \frac{1}{(tn^3)} + 1.5\bar{A}_{66}^3 \frac{cs}{sn(tn^2)} + 1.125\bar{A}_{66}^4 \frac{cs}{sn(tn^3)} \end{aligned} \quad (\text{F.35})$$

$$\begin{aligned} \mathcal{CP}_{3,8} = & + \bar{A}_{22}^1 \frac{n_c}{sn} + 4\bar{A}_{66}^1 \frac{n_c}{sn} + \bar{A}_{22}^2 \frac{n_c}{sntn} + 6\bar{A}_{66}^2 \frac{n_c}{sntn} + \tau_0 \left[\right. \\ & - \bar{A}_{22}^2 \frac{n_c}{sntn} - \bar{A}_{22}^3 \frac{n_c}{sn(tn^2)} \left. \right] + \tau \left[+ \bar{A}_{22}^3 \frac{n_c}{sn(tn^2)} + 2\bar{A}_{66}^3 \frac{n_c}{sn(tn^2)} + \bar{A}_{22}^4 \frac{n_c}{sn(tn^3)} \right. \\ & \left. + 1.5\bar{A}_{66}^4 \frac{n_c}{sn(tn^3)} \right] \end{aligned} \quad (\text{F.36})$$

$$\begin{aligned} \mathcal{CP}_{3,9} = & - \bar{R}_{12}^{10} \frac{n_c}{sn} - \bar{R}_{66}^{10} \frac{n_c}{sn} - \bar{R}_{12}^{11} \frac{n_c}{sntn} - 1.5\bar{R}_{66}^{11} \frac{n_c}{sntn} + \tau_0 \left[\right. \\ & \left. + 0.5\bar{R}_{66}^{11} \frac{n_c}{sntn} + 0.75\bar{R}_{66}^{12} \frac{n_c}{sn(tn^2)} \right] + \tau \left[-0.75\bar{R}_{66}^{12} \frac{n_c}{sn(tn^2)} - 0.75\bar{R}_{66}^{13} \frac{n_c}{sn(tn^3)} \right] \end{aligned} \quad (\text{F.37})$$

$$\begin{aligned} \mathcal{CP}_{3,10} = & + 4\bar{R}_{66}^{20} + 6\bar{R}_{66}^{21} \frac{1}{tn} + \bar{W}_{46}^{20} + 1.5\bar{W}_{46}^{21} \frac{1}{tn} + \tau_0 \left[-\bar{R}_{46}^{20} \frac{1}{tn} - \bar{R}_{66}^{20} \right. \\ & - \bar{R}_{46}^{21} \frac{1}{(tn^2)} + \bar{R}_{66}^{21} \frac{1}{tn} + 0.75\bar{R}_{46}^{22} \frac{1}{(tn^3)} + 3.75\bar{R}_{66}^{22} \frac{1}{(tn^2)} + 0.5\bar{W}_{46}^{21} \frac{1}{tn} \\ & \left. + 0.75\bar{W}_{46}^{22} \frac{1}{(tn^2)} \right] + \tau \left[-0.75\bar{R}_{46}^{22} \frac{1}{(tn^3)} + 0.25\bar{R}_{66}^{22} \frac{1}{(tn^2)} - 0.75\bar{R}_{46}^{23} \frac{1}{(tn^4)} - 0.75\bar{R}_{66}^{23} \frac{1}{(tn^3)} \right. \\ & \left. + 0.25\bar{W}_{46}^{22} \frac{1}{(tn^2)} \right] \end{aligned} \quad (\text{F.38})$$

$$\begin{aligned} \mathcal{CP}_{3,12} = & + \bar{A}_{66}^0 + 1.5\bar{A}_{66}^1 \frac{cs}{sn} + 1.5\bar{A}_{66}^1 \frac{1}{tn} + 2.25\bar{A}_{66}^2 \frac{cs}{sntn} + 0.75\bar{A}_{66}^2 \frac{1}{(tn^2)} \\ & + 0.75\bar{A}_{66}^3 \frac{cs}{sn(tn^2)} + 0.75\bar{A}_{66}^3 \frac{1}{(tn^3)} + 0.5625\bar{A}_{66}^4 \frac{cs}{sn(tn^3)} \end{aligned} \quad (\text{F.39})$$

$$\begin{aligned} \mathcal{CP}_{3,13} = & + \bar{A}_{12}^1 \frac{n_c}{sn} + 2\bar{A}_{66}^1 \frac{n_c}{sn} + \bar{A}_{12}^2 \frac{n_c}{sntn} + 3\bar{A}_{66}^2 \frac{n_c}{sntn} \\ & + \bar{A}_{66}^3 \frac{n_c}{sn(tn^2)} + 0.75\bar{A}_{66}^4 \frac{n_c}{sn(tn^3)} \end{aligned} \quad (\text{F.40})$$

$$\begin{aligned} \mathcal{C}\bar{\mathcal{P}}_{3,15} = & + \bar{R}_{66}^{20} + 1.5\bar{R}_{66}^{21} \frac{1}{tn} + \tau_0 \left[+0.5\bar{R}_{66}^{21} \frac{1}{tn} + 0.75\bar{R}_{66}^{22} \frac{1}{(tn^2)} \right] + \tau \left[\right. \\ & \left. + 0.25\bar{R}_{66}^{22} \frac{1}{(tn^2)} \right] \end{aligned} \quad (\text{F.41})$$

$$\begin{aligned} \mathcal{C}\bar{\mathcal{P}}_{4,1} = & + 2\bar{A}_{26}^0 - \bar{A}_{26}^0 \frac{(n_c^2)}{(sn^2)} + 3\bar{A}_{26}^1 \frac{1}{tn} + 0.5\bar{A}_{26}^1 \frac{(n_c^2)cs}{(sn^3)} \\ & - \bar{A}_{26}^1 \frac{(n_c^2)}{(sn^2)tn} + 0.5\bar{A}_{26}^2 \frac{(n_c^2)cs}{(sn^3)tn} + \tau_0 \left[+0.5\bar{A}_{26}^1 \frac{(n_c^2)}{(sn^2)tn} - \bar{A}_{26}^1 \frac{1}{tn} \right. \\ & \left. - 0.25\bar{A}_{26}^2 \frac{(n_c^2)cs}{(sn^3)tn} + 0.5\bar{A}_{26}^2 \frac{(n_c^2)}{(sn^2)(tn^2)} - 1.5\bar{A}_{26}^2 \frac{1}{(tn^2)} - 0.25\bar{A}_{26}^3 \frac{(n_c^2)cs}{(sn^3)(tn^2)} \right] + \tau \left[\right. \\ & \left. + 1.5\bar{A}_{26}^2 \frac{1}{(tn^2)} - 0.75\bar{A}_{26}^2 \frac{(n_c^2)}{(sn^2)(tn^2)} - 0.75\bar{A}_{26}^3 \frac{(n_c^2)}{(sn^2)(tn^3)} + 0.25\bar{A}_{26}^3 \frac{(n_c^2)cs}{(sn^3)(tn^2)} \right. \\ & \left. + 1.5\bar{A}_{26}^3 \frac{1}{(tn^3)} + 0.25\bar{A}_{26}^4 \frac{(n_c^2)cs}{(sn^3)(tn^3)} \right] \end{aligned} \quad (\text{F.42})$$

$$\begin{aligned} \mathcal{C}\bar{\mathcal{P}}_{4,2} = & + \bar{A}_{26}^0 \frac{n_c}{sn} + 4\bar{A}_{26}^1 \frac{n_c}{sntn} - 1.5\bar{A}_{26}^1 \frac{n_c cs}{(sn^2)} - 1.5\bar{A}_{26}^2 \frac{n_c cs}{(sn^2)tn} \\ & + 3\bar{A}_{26}^2 \frac{n_c}{sn(tn^2)} + \tau_0 \left[-0.5\bar{A}_{26}^1 \frac{n_c}{sntn} - 2\bar{A}_{26}^2 \frac{n_c}{sn(tn^2)} + 0.75\bar{A}_{26}^2 \frac{n_c cs}{(sn^2)tn} \right. \\ & \left. + 0.75\bar{A}_{26}^3 \frac{n_c cs}{(sn^2)(tn^2)} - 1.5\bar{A}_{26}^3 \frac{n_c}{sn(tn^3)} \right] + \tau \left[+0.75\bar{A}_{26}^2 \frac{n_c}{sn(tn^2)} + 2.25\bar{A}_{26}^3 \frac{n_c}{sn(tn^3)} \right. \\ & \left. - 0.75\bar{A}_{26}^3 \frac{n_c cs}{(sn^2)(tn^2)} + 1.5\bar{A}_{26}^4 \frac{n_c}{sn(tn^4)} - 0.75\bar{A}_{26}^4 \frac{n_c cs}{(sn^2)(tn^3)} \right] \end{aligned} \quad (\text{F.43})$$

$$\begin{aligned} \mathcal{C}\bar{\mathcal{P}}_{4,3} = & + 2\bar{A}_{26}^0 \frac{1}{tn} + 3\bar{A}_{26}^1 \frac{1}{(tn^2)} + \bar{A}_{26}^2 \frac{(n_c^2)}{(sn^2)tn} + \tau_0 \left[-\bar{A}_{26}^1 \frac{1}{(tn^2)} \right. \\ & \left. - 1.5\bar{A}_{26}^2 \frac{1}{(tn^3)} - 0.5\bar{A}_{26}^3 \frac{(n_c^2)}{(sn^2)(tn^2)} \right] + \tau \left[+1.5\bar{A}_{26}^2 \frac{1}{(tn^3)} + 0.5\bar{A}_{26}^3 \frac{(n_c^2)}{(sn^2)(tn^2)} \right. \\ & \left. + 1.5\bar{A}_{26}^3 \frac{1}{(tn^4)} + 0.5\bar{A}_{26}^4 \frac{(n_c^2)}{(sn^2)(tn^3)} \right] \end{aligned} \quad (\text{F.44})$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{4,4} = & -\bar{R}_{26}^{10} \frac{(n_c^2)}{(sn^2)} - \bar{R}_{26}^{11} \frac{(n_c^2)}{(sn^2)tn} + 3\bar{W}_{56}^{10} + 4.5\bar{W}_{56}^{11} \frac{1}{tn} + \tau_0 \left[\right. \\
& + 3\bar{R}_{26}^{10} + \bar{R}_{26}^{11} \frac{(n_c^2)}{(sn^2)tn} + 3\bar{R}_{26}^{11} \frac{1}{tn} + \bar{R}_{26}^{12} \frac{(n_c^2)}{(sn^2)(tn^2)} \\
& \left. - 2.25\bar{R}_{26}^{12} \frac{1}{(tn^2)} + 1.5\bar{W}_{56}^{11} \frac{1}{tn} + 2.25\bar{W}_{56}^{12} \frac{1}{(tn^2)} \right] + \tau \left[+2.25\bar{R}_{26}^{12} \frac{1}{(tn^2)} - \bar{R}_{26}^{12} \frac{(n_c^2)}{(sn^2)(tn^2)} \right. \\
& \left. + 2.25\bar{R}_{26}^{13} \frac{1}{(tn^3)} - \bar{R}_{26}^{13} \frac{(n_c^2)}{(sn^2)(tn^3)} + 0.75\bar{W}_{56}^{12} \frac{1}{(tn^2)} \right]
\end{aligned} \tag{F.45}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{4,5} = & + 3\bar{R}_{26}^{20} \frac{n_c}{sn} + 4.5\bar{R}_{26}^{21} \frac{n_c}{sntn} + \bar{W}_{24}^{20} \frac{n_c}{sn} + \bar{W}_{24}^{21} \frac{n_c}{sntn} + \tau_0 \left[\right. \\
& \left. - \bar{R}_{24}^{20} \frac{n_c}{sntn} - \bar{R}_{26}^{20} \frac{n_c}{sn} - 1.5\bar{R}_{26}^{21} \frac{n_c}{sntn} + \bar{R}_{24}^{22} \frac{n_c}{sn(tn^3)} - 1.25\bar{R}_{26}^{22} \frac{n_c}{sn(tn^2)} \right] + \tau \left[\right. \\
& \left. - \bar{R}_{24}^{22} \frac{n_c}{sn(tn^3)} + 1.25\bar{R}_{26}^{22} \frac{n_c}{sn(tn^2)} - \bar{R}_{24}^{23} \frac{n_c}{sn(tn^4)} + 1.25\bar{R}_{26}^{23} \frac{n_c}{sn(tn^3)} \right]
\end{aligned} \tag{F.46}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{4,6} = & + 3\bar{A}_{16}^0 + \bar{A}_{26}^0 + 4.5\bar{A}_{16}^1 \frac{1}{tn} + 1.5\bar{A}_{26}^1 \frac{1}{tn} + \tau_0 \left[+1.5\bar{A}_{16}^1 \frac{1}{tn} \right. \\
& \left. - 0.5\bar{A}_{26}^1 \frac{1}{tn} + 2.25\bar{A}_{16}^2 \frac{1}{(tn^2)} - 0.75\bar{A}_{26}^2 \frac{1}{(tn^2)} \right] + \tau \left[+0.75\bar{A}_{16}^2 \frac{1}{(tn^2)} + 0.75\bar{A}_{26}^2 \frac{1}{(tn^2)} \right. \\
& \left. + 0.75\bar{A}_{26}^3 \frac{1}{(tn^3)} \right]
\end{aligned} \tag{F.47}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{4,7} = & + 2\bar{A}_{26}^0 \frac{n_c}{sn} + 3.5\bar{A}_{26}^1 \frac{n_c}{sntn} + 1.5\bar{A}_{26}^1 \frac{n_c cs}{(sn^2)} + 1.5\bar{A}_{26}^2 \frac{n_c cs}{(sn^2)tn} \\
& + 1.5\bar{A}_{26}^2 \frac{n_c}{sn(tn^2)} + \tau_0 \left[-\bar{A}_{26}^1 \frac{n_c}{sntn} - 0.75\bar{A}_{26}^2 \frac{n_c cs}{(sn^2)tn} - 1.75\bar{A}_{26}^2 \frac{n_c}{sn(tn^2)} \right. \\
& \left. - 0.75\bar{A}_{26}^3 \frac{n_c cs}{(sn^2)(tn^2)} - 0.75\bar{A}_{26}^3 \frac{n_c}{sn(tn^3)} \right] + \tau \left[+1.5\bar{A}_{26}^2 \frac{n_c}{sn(tn^2)} + 0.75\bar{A}_{26}^3 \frac{n_c cs}{(sn^2)(tn^2)} \right. \\
& \left. + 2.25\bar{A}_{26}^3 \frac{n_c}{sn(tn^3)} + 0.75\bar{A}_{26}^4 \frac{n_c cs}{(sn^2)(tn^3)} + 0.75\bar{A}_{26}^4 \frac{n_c}{sn(tn^4)} \right]
\end{aligned} \tag{F.48}$$

$$\begin{aligned}
\mathcal{CP}_{4,8} = & + \bar{A}_{26}^0 \frac{1}{tn} + 1.5\bar{A}_{26}^1 \frac{1}{(tn^2)} - 3\bar{A}_{26}^1 + 3\bar{A}_{26}^1 \frac{(n_c^2)}{(sn^2)} \\
& - 4.5\bar{A}_{26}^2 \frac{1}{tn} + 3.5\bar{A}_{26}^2 \frac{(n_c^2)}{(sn^2)tn} + \tau_0 \left[-0.5\bar{A}_{26}^1 \frac{1}{(tn^2)} + 1.5\bar{A}_{26}^2 \frac{1}{tn} - 1.5\bar{A}_{26}^2 \frac{(n_c^2)}{(sn^2)tn} \right. \\
& - 0.75\bar{A}_{26}^2 \frac{1}{(tn^3)} - 1.75\bar{A}_{26}^3 \frac{(n_c^2)}{(sn^2)(tn^2)} + 2.25\bar{A}_{26}^3 \frac{1}{(tn^2)} \left. \right] + \tau \left[+0.75\bar{A}_{26}^2 \frac{1}{(tn^3)} \right. \\
& - 2.25\bar{A}_{26}^3 \frac{1}{(tn^2)} + 1.75\bar{A}_{26}^3 \frac{(n_c^2)}{(sn^2)(tn^2)} + 0.75\bar{A}_{26}^3 \frac{1}{(tn^4)} - 2.25\bar{A}_{26}^4 \frac{1}{(tn^3)} \\
& \left. + 1.75\bar{A}_{26}^4 \frac{(n_c^2)}{(sn^2)(tn^3)} \right] \tag{F.49}
\end{aligned}$$

$$\begin{aligned}
\mathcal{CP}_{4,9} = & + 4\bar{R}_{16}^{10} + 6\bar{R}_{16}^{11} \frac{1}{tn} + \bar{W}_{56}^{10} + 1.5\bar{W}_{56}^{11} \frac{1}{tn} + \tau_0 \left[+\bar{R}_{26}^{10} + 2\bar{R}_{16}^{11} \frac{1}{tn} \right. \\
& + \bar{R}_{26}^{11} \frac{1}{tn} + 3\bar{R}_{16}^{12} \frac{1}{(tn^2)} - 0.75\bar{R}_{26}^{12} \frac{1}{(tn^2)} + 0.5\bar{W}_{56}^{11} \frac{1}{tn} + 0.75\bar{W}_{56}^{12} \frac{1}{(tn^2)} \left. \right] + \tau \left[\right. \\
& \left. + \bar{R}_{16}^{12} \frac{1}{(tn^2)} + 0.75\bar{R}_{26}^{12} \frac{1}{(tn^2)} + 0.75\bar{R}_{26}^{13} \frac{1}{(tn^3)} + 0.25\bar{W}_{56}^{12} \frac{1}{(tn^2)} \right] \tag{F.50}
\end{aligned}$$

$$\begin{aligned}
\mathcal{CP}_{4,10} = & + 2\bar{R}_{26}^{20} \frac{n_c}{sn} + 2.5\bar{R}_{26}^{21} \frac{n_c}{sntn} + \tau_0 \left[-0.5\bar{R}_{26}^{21} \frac{n_c}{sntn} - 0.75\bar{R}_{26}^{22} \frac{n_c}{sn(tn^2)} \right] + \tau \left[\right. \\
& \left. + 0.75\bar{R}_{26}^{22} \frac{n_c}{sn(tn^2)} + 0.75\bar{R}_{26}^{23} \frac{n_c}{sn(tn^3)} \right] \tag{F.51}
\end{aligned}$$

$$\mathcal{CP}_{4,11} = + \bar{A}_{16}^0 + 1.5\bar{A}_{16}^1 \frac{1}{tn} + \tau_0 \left[+0.5\bar{A}_{16}^1 \frac{1}{tn} + 0.75\bar{A}_{16}^2 \frac{1}{(tn^2)} \right] + \tau \left[+0.25\bar{A}_{16}^2 \frac{1}{(tn^2)} \right] \tag{F.52}$$

$$\begin{aligned}
\mathcal{CP}_{4,13} = & - 4\bar{A}_{16}^1 - \bar{A}_{26}^1 - 6\bar{A}_{16}^2 \frac{1}{tn} - 1.5\bar{A}_{26}^2 \frac{1}{tn} + \tau_0 \left[-2\bar{A}_{16}^2 \frac{1}{tn} \right. \\
& + 0.5\bar{A}_{26}^2 \frac{1}{tn} - 3\bar{A}_{16}^3 \frac{1}{(tn^2)} + 0.75\bar{A}_{26}^3 \frac{1}{(tn^2)} \left. \right] + \tau \left[-\bar{A}_{16}^3 \frac{1}{(tn^2)} - 0.75\bar{A}_{26}^3 \frac{1}{(tn^2)} \right. \\
& \left. - 0.75\bar{A}_{26}^4 \frac{1}{(tn^3)} \right] \tag{F.53}
\end{aligned}$$

$$\begin{aligned} \mathcal{CP}_{4,14} = & +\bar{R}_{16}^{10} + 1.5\bar{R}_{16}^{11} \frac{1}{tn} + \tau_0 \left[+0.5\bar{R}_{16}^{11} \frac{1}{tn} + 0.75\bar{R}_{16}^{12} \frac{1}{(tn^2)} \right] + \tau \left[\right. \\ & \left. + 0.25\bar{R}_{16}^{12} \frac{1}{(tn^2)} \right] \end{aligned} \quad (\text{F.54})$$

$$\mathcal{CP}_{4,18} = -\bar{A}_{16}^1 - 1.5\bar{A}_{16}^2 \frac{1}{tn} + \tau_0 \left[-0.5\bar{A}_{16}^2 \frac{1}{tn} - 0.75\bar{A}_{16}^3 \frac{1}{(tn^2)} \right] + \tau \left[-0.25\bar{A}_{16}^3 \frac{1}{(tn^2)} \right] \quad (\text{F.55})$$

$$\begin{aligned} \mathcal{CP}_{5,1} = & +\bar{A}_{26}^0 \frac{n_c}{sntn} - 2\bar{A}_{16}^1 \frac{n_c}{sn} - 0.5\bar{A}_{26}^1 \frac{n_c cs}{(sn^2)tn} + \bar{A}_{26}^1 \frac{(n_c^3)}{(sn^3)} \\ & - 3\bar{A}_{26}^1 \frac{n_c}{sn} + \bar{A}_{16}^2 \frac{n_c cs}{(sn^2)} - 0.5\bar{A}_{26}^2 \frac{(n_c^3)cs}{(sn^4)} - 0.5\bar{A}_{26}^2 \frac{n_c cs}{(sn^2)} + \tau_0 \left[\right. \\ & - 0.5\bar{A}_{26}^1 \frac{n_c}{sn(tn^2)} - \bar{A}_{16}^2 \frac{n_c}{sntn} + 0.25\bar{A}_{26}^2 \frac{n_c cs}{(sn^2)(tn^2)} - 0.5\bar{A}_{26}^2 \frac{(n_c^3)}{(sn^3)tn} \\ & + 1.5\bar{A}_{26}^2 \frac{n_c}{sntn} + 0.5\bar{A}_{16}^3 \frac{n_c cs}{(sn^2)tn} + 0.25\bar{A}_{26}^3 \frac{n_c cs}{(sn^2)tn} + 0.25\bar{A}_{26}^3 \frac{(n_c^3)cs}{(sn^4)tn} \left. \right] + \tau \left[\right. \\ & + 0.75\bar{A}_{26}^2 \frac{n_c}{sn(tn^3)} - 0.5\bar{A}_{16}^3 \frac{n_c}{sn(tn^2)} + 0.75\bar{A}_{26}^3 \frac{(n_c^3)}{(sn^3)(tn^2)} - 0.25\bar{A}_{26}^3 \frac{n_c cs}{(sn^2)(tn^3)} \\ & \left. - 1.25\bar{A}_{26}^3 \frac{n_c}{sn(tn^2)} - 0.25\bar{A}_{26}^4 \frac{(n_c^3)cs}{(sn^4)(tn^2)} - 0.25\bar{A}_{26}^4 \frac{n_c cs}{(sn^2)(tn^2)} \right] \end{aligned} \quad (\text{F.56})$$

$$\begin{aligned} \mathcal{CP}_{5,2} = & +\bar{A}_{26}^0 \frac{1}{tn} - 2\bar{A}_{16}^1 + 1.5\bar{A}_{26}^1 \frac{cs}{sntn} - 3\bar{A}_{26}^1 \frac{(n_c^2)}{(sn^2)} + \bar{A}_{26}^1 \\ & - 3\bar{A}_{16}^2 \frac{cs}{sn} + 1.5\bar{A}_{26}^2 \frac{(n_c^2)cs}{(sn^3)} - 4\bar{A}_{26}^2 \frac{(n_c^2)}{(sn^2)tn} + 1.5\bar{A}_{26}^2 \frac{cs}{sn} + \tau_0 \left[-0.5\bar{A}_{26}^1 \frac{1}{(tn^2)} \right. \\ & - \bar{A}_{16}^2 \frac{1}{tn} - 0.75\bar{A}_{26}^2 \frac{cs}{sn(tn^2)} + 1.5\bar{A}_{26}^2 \frac{(n_c^2)}{(sn^2)tn} - 0.5\bar{A}_{26}^2 \frac{1}{tn} - 1.5\bar{A}_{16}^3 \frac{cs}{sntn} \\ & \left. - 0.75\bar{A}_{26}^3 \frac{(n_c^2)cs}{(sn^3)tn} - 0.75\bar{A}_{26}^3 \frac{cs}{sntn} + 2\bar{A}_{26}^3 \frac{(n_c^2)}{(sn^2)(tn^2)} \right] + \tau \left[+0.75\bar{A}_{26}^2 \frac{1}{(tn^3)} - 0.5\bar{A}_{16}^3 \frac{1}{(tn^2)} \right. \\ & + 0.75\bar{A}_{26}^3 \frac{cs}{sn(tn^3)} - 1.25\bar{A}_{26}^3 \frac{(n_c^2)}{(sn^2)(tn^2)} + 0.75\bar{A}_{26}^3 \frac{1}{(tn^2)} + 0.75\bar{A}_{26}^4 \frac{(n_c^2)cs}{(sn^3)(tn^2)} \\ & \left. + 0.75\bar{A}_{26}^4 \frac{cs}{sn(tn^2)} - 2\bar{A}_{26}^4 \frac{(n_c^2)}{(sn^2)(tn^3)} \right] \end{aligned} \quad (\text{F.57})$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{5,3} = & -2\bar{A}_{26}^1 \frac{n_c}{sntn} - 4\bar{A}_{16}^2 \frac{n_c}{sn} + 2\bar{A}_{26}^2 \frac{n_c}{sn} - 2\bar{A}_{26}^2 \frac{(n_c^3)}{(sn^3)} + \tau_0 \left[\right. \\
& + \bar{A}_{26}^2 \frac{n_c}{sn(tn^2)} - 2\bar{A}_{16}^3 \frac{n_c}{sntn} - \bar{A}_{26}^3 \frac{n_c}{sntn} + \bar{A}_{26}^3 \frac{(n_c^3)}{(sn^3)tn} \left. \right] + \tau \left[-\bar{A}_{26}^3 \frac{n_c}{sn(tn^3)} \right. \\
& \left. + \bar{A}_{26}^4 \frac{n_c}{sn(tn^2)} - \bar{A}_{26}^4 \frac{(n_c^3)}{(sn^3)(tn^2)} \right]
\end{aligned} \tag{F.58}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{5,4} = & + \bar{R}_{26}^{10} \frac{n_c}{sntn} - 6\bar{R}_{16}^{11} \frac{n_c}{sn} + 2\bar{R}_{26}^{11} \frac{n_c}{sn} + \bar{R}_{26}^{11} \frac{(n_c^3)}{(sn^3)} \\
& - 6\bar{W}_{56}^{11} \frac{n_c}{sn} + \tau_0 \left[-6\bar{R}_{26}^{11} \frac{n_c}{sn} - \bar{R}_{26}^{11} \frac{n_c}{sn(tn^2)} + \bar{R}_{26}^{12} \frac{n_c}{sntn} - \bar{R}_{26}^{12} \frac{(n_c^3)}{(sn^3)tn} \right. \\
& \left. - 3\bar{W}_{56}^{12} \frac{n_c}{sntn} \right] + \tau \left[+\bar{R}_{26}^{12} \frac{n_c}{sn(tn^3)} - \bar{R}_{26}^{13} \frac{n_c}{sn(tn^2)} + \bar{R}_{26}^{13} \frac{(n_c^3)}{(sn^3)(tn^2)} \right]
\end{aligned} \tag{F.59}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{5,5} = & -6\bar{R}_{26}^{21} \frac{(n_c^2)}{(sn^2)} - \bar{W}_{24}^{20} \frac{1}{tn} + 6\bar{W}_{14}^{21} - \bar{W}_{24}^{21} \frac{(n_c^2)}{(sn^2)} - 2\bar{W}_{24}^{21} + \tau_0 \left[\right. \\
& + \bar{R}_{24}^{20} \frac{1}{(tn^2)} + \bar{R}_{26}^{20} \frac{1}{tn} - 6\bar{R}_{14}^{21} \frac{1}{tn} - 6\bar{R}_{16}^{21} - \bar{R}_{24}^{21} \frac{1}{(tn^3)} \\
& + \bar{R}_{24}^{21} \frac{(n_c^2)}{(sn^2)tn} + 2\bar{R}_{24}^{21} \frac{1}{tn} - \bar{R}_{26}^{21} \frac{1}{(tn^2)} + \bar{R}_{26}^{21} \frac{(n_c^2)}{(sn^2)} + 2\bar{R}_{26}^{21} \\
& - \bar{R}_{24}^{22} \frac{(n_c^2)}{(sn^2)(tn^2)} - 2\bar{R}_{24}^{22} \frac{1}{(tn^2)} + 2\bar{R}_{26}^{22} \frac{(n_c^2)}{(sn^2)tn} - 2\bar{R}_{26}^{22} \frac{1}{tn} + 6\bar{W}_{14}^{22} \frac{1}{tn} \left. \right] + \tau \left[\right. \\
& + \bar{R}_{24}^{22} \frac{1}{(tn^4)} + \bar{R}_{26}^{22} \frac{1}{(tn^3)} + \bar{R}_{24}^{23} \frac{(n_c^2)}{(sn^2)(tn^3)} + 2\bar{R}_{24}^{23} \frac{1}{(tn^3)} \\
& \left. - 2\bar{R}_{26}^{23} \frac{(n_c^2)}{(sn^2)(tn^2)} + 2\bar{R}_{26}^{23} \frac{1}{(tn^2)} \right]
\end{aligned} \tag{F.60}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{5,6} = & -10\bar{A}_{16}^1 \frac{n_c}{sn} - \bar{A}_{26}^1 \frac{n_c}{sn} + 2\bar{A}_{16}^2 \frac{n_c cs}{(sn^2)} - 0.5\bar{A}_{26}^2 \frac{n_c cs}{(sn^2)} + \tau_0 \left[\right. \\
& - 5\bar{A}_{16}^2 \frac{n_c}{sntn} + 0.5\bar{A}_{26}^2 \frac{n_c}{sntn} + \bar{A}_{16}^3 \frac{n_c cs}{(sn^2)tn} + 0.25\bar{A}_{26}^3 \frac{n_c cs}{(sn^2)tn} \left. \right] + \tau \left[\right. \\
& \left. - \bar{A}_{16}^3 \frac{n_c}{sn(tn^2)} - 0.25\bar{A}_{26}^3 \frac{n_c}{sn(tn^2)} - 0.25\bar{A}_{26}^4 \frac{n_c cs}{(sn^2)(tn^2)} \right]
\end{aligned} \tag{F.61}$$

$$\begin{aligned}
\mathcal{CP}_{5,7} = & -\bar{A}_{26}^0 \frac{1}{tn} + 2\bar{A}_{16}^1 - 3\bar{A}_{26}^1 \frac{(n_c^2)}{(sn^2)} - 1.5\bar{A}_{26}^1 \frac{cs}{sntn} - \bar{A}_{26}^1 \\
& + 3\bar{A}_{16}^2 \frac{cs}{sn} - 1.5\bar{A}_{26}^2 \frac{cs}{sn} - 1.5\bar{A}_{26}^2 \frac{(n_c^2)cs}{(sn^3)} - 2\bar{A}_{26}^2 \frac{(n_c^2)}{(sn^2)tn} + \tau_0 \left[+0.5\bar{A}_{26}^1 \frac{1}{(tn^2)} \right. \\
& + \bar{A}_{16}^2 \frac{1}{tn} + 0.75\bar{A}_{26}^2 \frac{cs}{sn(tn^2)} + 1.5\bar{A}_{26}^2 \frac{(n_c^2)}{(sn^2)tn} + 0.5\bar{A}_{26}^2 \frac{1}{tn} + 1.5\bar{A}_{16}^3 \frac{cs}{sntn} \\
& + 0.75\bar{A}_{26}^3 \frac{cs}{sntn} + \bar{A}_{26}^3 \frac{(n_c^2)}{(sn^2)(tn^2)} + 0.75\bar{A}_{26}^3 \frac{(n_c^2)cs}{(sn^3)tn} \left. \right] + \tau \left[-0.75\bar{A}_{26}^2 \frac{1}{(tn^3)} \right. \\
& + 0.5\bar{A}_{16}^3 \frac{1}{(tn^2)} - 0.75\bar{A}_{26}^3 \frac{cs}{sn(tn^3)} - 1.75\bar{A}_{26}^3 \frac{(n_c^2)}{(sn^2)(tn^2)} - 0.75\bar{A}_{26}^3 \frac{1}{(tn^2)} \\
& \left. - 0.75\bar{A}_{26}^4 \frac{cs}{sn(tn^2)} - \bar{A}_{26}^4 \frac{(n_c^2)}{(sn^2)(tn^3)} - 0.75\bar{A}_{26}^4 \frac{(n_c^2)cs}{(sn^3)(tn^2)} \right]
\end{aligned} \tag{F.62}$$

$$\begin{aligned}
\mathcal{CP}_{5,8} = & -4\bar{A}_{26}^1 \frac{n_c}{sntn} + 4\bar{A}_{16}^2 \frac{n_c}{sn} + 4\bar{A}_{26}^2 \frac{n_c}{sn} - 4\bar{A}_{26}^2 \frac{(n_c^3)}{(sn^3)} + \tau_0 \left[\right. \\
& + 2\bar{A}_{26}^2 \frac{n_c}{sn(tn^2)} + 2\bar{A}_{16}^3 \frac{n_c}{sntn} + 2\bar{A}_{26}^3 \frac{(n_c^3)}{(sn^3)tn} - 2\bar{A}_{26}^3 \frac{n_c}{sntn} \left. \right] + \tau \left[-2\bar{A}_{26}^3 \frac{n_c}{sn(tn^3)} \right. \\
& \left. + 2\bar{A}_{26}^4 \frac{n_c}{sn(tn^2)} - 2\bar{A}_{26}^4 \frac{(n_c^3)}{(sn^3)(tn^2)} \right]
\end{aligned} \tag{F.63}$$

$$\begin{aligned}
\mathcal{CP}_{5,9} = & -14\bar{R}_{16}^{11} \frac{n_c}{sn} + \bar{R}_{26}^{11} \frac{n_c}{sn} - 2\bar{W}_{56}^{11} \frac{n_c}{sn} + \tau_0 \left[-2\bar{R}_{26}^{11} \frac{n_c}{sn} \right. \\
& \left. - 4\bar{R}_{16}^{12} \frac{n_c}{sntn} - \bar{W}_{56}^{12} \frac{n_c}{sntn} \right]
\end{aligned} \tag{F.64}$$

$$\begin{aligned}
\mathcal{CP}_{5,10} = & -\bar{R}_{26}^{20} \frac{1}{tn} + 12\bar{R}_{16}^{21} - 3\bar{R}_{26}^{21} - 3\bar{R}_{26}^{21} \frac{(n_c^2)}{(sn^2)} + 6\bar{W}_{14}^{21} - \bar{W}_{24}^{21} + \tau_0 \left[\right. \\
& - 6\bar{R}_{14}^{21} \frac{1}{tn} - 6\bar{R}_{16}^{21} + \bar{R}_{24}^{21} \frac{1}{tn} + \bar{R}_{26}^{21} + 12\bar{R}_{16}^{22} \frac{1}{tn} - \bar{R}_{24}^{22} \frac{1}{(tn^2)} \\
& + \bar{R}_{26}^{22} \frac{(n_c^2)}{(sn^2)tn} - \bar{R}_{26}^{22} \frac{1}{tn} + 6\bar{W}_{14}^{22} \frac{1}{tn} \left. \right] + \tau \left[+\bar{R}_{24}^{23} \frac{1}{(tn^3)} - \bar{R}_{26}^{23} \frac{(n_c^2)}{(sn^2)(tn^2)} \right. \\
& \left. + \bar{R}_{26}^{23} \frac{1}{(tn^2)} \right]
\end{aligned} \tag{F.65}$$

$$\begin{aligned} \mathcal{C}\bar{\mathcal{P}}_{5,11} = & -3\bar{A}_{16}^1 \frac{n_c}{sn} + 0.5\bar{A}_{16}^2 \frac{n_c cs}{(sn^2)} + \tau_0 \left[-1.5\bar{A}_{16}^2 \frac{n_c}{sntn} + 0.25\bar{A}_{16}^3 \frac{n_c cs}{(sn^2)tn} \right] + \tau \left[\right. \\ & \left. -0.25\bar{A}_{16}^3 \frac{n_c}{sn(tn^2)} \right] \end{aligned} \quad (\text{F.66})$$

$$\begin{aligned} \mathcal{C}\bar{\mathcal{P}}_{5,12} = & +5\bar{A}_{16}^1 - \bar{A}_{26}^1 + 7.5\bar{A}_{16}^2 \frac{cs}{sn} - 1.5\bar{A}_{26}^2 \frac{cs}{sn} + \tau_0 \left[+2.5\bar{A}_{16}^2 \frac{1}{tn} \right. \\ & \left. + 0.5\bar{A}_{26}^2 \frac{1}{tn} + 3.75\bar{A}_{16}^3 \frac{cs}{sntn} + 0.75\bar{A}_{26}^3 \frac{cs}{sntn} \right] + \tau \left[+1.25\bar{A}_{16}^3 \frac{1}{(tn^2)} - 0.75\bar{A}_{26}^3 \frac{1}{(tn^2)} \right. \\ & \left. - 0.75\bar{A}_{26}^4 \frac{cs}{sn(tn^2)} \right] \end{aligned} \quad (\text{F.67})$$

$$\mathcal{C}\bar{\mathcal{P}}_{5,13} = +18\bar{A}_{16}^2 \frac{n_c}{sn} + \tau_0 \left[+9\bar{A}_{16}^3 \frac{n_c}{sntn} \right] \quad (\text{F.68})$$

$$\mathcal{C}\bar{\mathcal{P}}_{5,14} = -3\bar{R}_{16}^{11} \frac{n_c}{sn} + \tau_0 \left[-\bar{R}_{16}^{12} \frac{n_c}{sntn} \right] \quad (\text{F.69})$$

$$\begin{aligned} \mathcal{C}\bar{\mathcal{P}}_{5,15} = & +8\bar{R}_{16}^{21} - \bar{R}_{26}^{21} + \bar{W}_{14}^{21} + \tau_0 \left[-\bar{R}_{14}^{21} \frac{1}{tn} - \bar{R}_{16}^{21} + 8\bar{R}_{16}^{22} \frac{1}{tn} \right. \\ & \left. + \bar{W}_{14}^{22} \frac{1}{tn} \right] \end{aligned} \quad (\text{F.70})$$

$$\mathcal{C}\bar{\mathcal{P}}_{5,17} = +\bar{A}_{16}^1 + 1.5\bar{A}_{16}^2 \frac{cs}{sn} + \tau_0 \left[+0.5\bar{A}_{16}^2 \frac{1}{tn} + 0.75\bar{A}_{16}^3 \frac{cs}{sntn} \right] + \tau \left[+0.25\bar{A}_{16}^3 \frac{1}{(tn^2)} \right] \quad (\text{F.71})$$

$$\mathcal{C}\bar{\mathcal{P}}_{5,18} = +4\bar{A}_{16}^2 \frac{n_c}{sn} + \tau_0 \left[+2\bar{A}_{16}^3 \frac{n_c}{sntn} \right] \quad (\text{F.72})$$

$$\mathcal{C}\bar{\mathcal{P}}_{5,20} = +\bar{R}_{16}^{21} + \tau_0 \left[+\bar{R}_{16}^{22} \frac{1}{tn} \right] \quad (\text{F.73})$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{6,1} = & -\bar{A}_{22}^0 \frac{1}{tn} + 2\bar{A}_{12}^1 - \bar{A}_{22}^1 \frac{(n_c^2)}{(sn^2)} - \bar{A}_{22}^1 - 4\bar{A}_{66}^1 \frac{(n_c^2)}{(sn^2)} \\
& + 2\bar{A}_{66}^2 \frac{(n_c^2)cs}{(sn^3)} + \tau_0 \left[+\bar{A}_{22}^1 \frac{1}{(tn^2)} + \bar{A}_{22}^2 \frac{(n_c^2)}{(sn^2)tn} + \bar{A}_{22}^2 \frac{1}{tn} \right] + \tau \left[-\bar{A}_{22}^2 \frac{1}{(tn^3)} \right. \\
& \left. - \bar{A}_{22}^3 \frac{(n_c^2)}{(sn^2)(tn^2)} - \bar{A}_{22}^3 \frac{1}{(tn^2)} - 2\bar{A}_{66}^3 \frac{(n_c^2)}{(sn^2)(tn^2)} + 0.5\bar{A}_{66}^4 \frac{(n_c^2)cs}{(sn^3)(tn^2)} \right]
\end{aligned} \tag{F.74}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{6,2} = & -\bar{A}_{22}^0 \frac{n_c}{sntn} + 2\bar{A}_{12}^1 \frac{n_c}{sn} - \bar{A}_{22}^1 \frac{(n_c^3)}{(sn^3)} - \bar{A}_{22}^1 \frac{n_c}{sn} \\
& - \bar{A}_{22}^1 \frac{n_c}{sn(tn^2)} - 4\bar{A}_{66}^1 \frac{n_c}{sn} + 2\bar{A}_{12}^2 \frac{n_c}{sntn} - \bar{A}_{22}^2 \frac{(n_c^3)}{(sn^3)tn} - \bar{A}_{22}^2 \frac{n_c}{sntn} \\
& - 6\bar{A}_{66}^2 \frac{n_c cs}{(sn^2)} + \tau_0 \left[+\bar{A}_{22}^1 \frac{n_c}{sn(tn^2)} + \bar{A}_{22}^2 \frac{(n_c^3)}{(sn^3)tn} + \bar{A}_{22}^2 \frac{n_c}{sn(tn^3)} + \bar{A}_{22}^2 \frac{n_c}{sntn} \right. \\
& \left. + \bar{A}_{22}^3 \frac{n_c}{sn(tn^2)} + \bar{A}_{22}^3 \frac{(n_c^3)}{(sn^3)(tn^2)} \right] + \tau \left[-\bar{A}_{22}^2 \frac{n_c}{sn(tn^3)} - \bar{A}_{22}^3 \frac{n_c}{sn(tn^4)} \right. \\
& \left. - \bar{A}_{22}^3 \frac{n_c}{sn(tn^2)} - \bar{A}_{22}^3 \frac{(n_c^3)}{(sn^3)(tn^2)} - 2\bar{A}_{66}^3 \frac{n_c}{sn(tn^2)} - \bar{A}_{22}^4 \frac{(n_c^3)}{(sn^3)(tn^3)} \right. \\
& \left. - \bar{A}_{22}^4 \frac{n_c}{sn(tn^3)} - 1.5\bar{A}_{66}^4 \frac{n_c cs}{(sn^2)(tn^2)} \right]
\end{aligned} \tag{F.75}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{6,3} = & -\bar{A}_{22}^0 \frac{1}{(tn^2)} + 2\bar{A}_{12}^1 \frac{1}{tn} - \bar{A}_{22}^1 \frac{1}{tn} - 2\bar{A}_{22}^1 \frac{(n_c^2)}{(sn^2)tn} \\
& + 2\bar{A}_{12}^2 \frac{(n_c^2)}{(sn^2)} - \bar{A}_{22}^2 \frac{(n_c^4)}{(sn^4)} - \bar{A}_{22}^2 \frac{(n_c^2)}{(sn^2)} - 8\bar{A}_{66}^2 \frac{(n_c^2)}{(sn^2)} + \tau_0 \left[+\bar{A}_{22}^1 \frac{1}{(tn^3)} \right. \\
& \left. + \bar{A}_{22}^2 \frac{1}{(tn^2)} + 2\bar{A}_{22}^2 \frac{(n_c^2)}{(sn^2)(tn^2)} + \bar{A}_{22}^3 \frac{(n_c^2)}{(sn^2)tn} + \bar{A}_{22}^3 \frac{(n_c^4)}{(sn^4)tn} \right] + \tau \left[\right. \\
& \left. - \bar{A}_{22}^2 \frac{1}{(tn^4)} - 2\bar{A}_{22}^3 \frac{(n_c^2)}{(sn^2)(tn^3)} - \bar{A}_{22}^3 \frac{1}{(tn^3)} - \bar{A}_{22}^4 \frac{(n_c^2)}{(sn^2)(tn^2)} \right. \\
& \left. - \bar{A}_{22}^4 \frac{(n_c^4)}{(sn^4)(tn^2)} - 2\bar{A}_{66}^4 \frac{(n_c^2)}{(sn^2)(tn^2)} \right]
\end{aligned} \tag{F.76}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{6,4} = & -6\bar{R}_{66}^{11} \frac{(n_c^2)}{(sn^2)} - \bar{W}_{25}^{10} \frac{1}{tn} + 6\bar{W}_{15}^{11} - 2\bar{W}_{25}^{11} - \bar{W}_{25}^{11} \frac{(n_c^2)}{(sn^2)} + \tau_0 \left[\right. \\
& - \bar{R}_{22}^{10} \frac{1}{tn} + 6\bar{R}_{12}^{11} - 2\bar{R}_{22}^{11} + \bar{R}_{22}^{11} \frac{1}{(tn^2)} - \bar{R}_{22}^{11} \frac{(n_c^2)}{(sn^2)} + 2\bar{R}_{22}^{12} \frac{1}{tn} \\
& + \bar{R}_{22}^{12} \frac{(n_c^2)}{(sn^2)tn} + 3\bar{R}_{66}^{12} \frac{(n_c^2)}{(sn^2)tn} + 6\bar{W}_{15}^{12} \frac{1}{tn} \left. \right] + \tau \left[-\bar{R}_{22}^{12} \frac{1}{(tn^3)} - 2\bar{R}_{22}^{13} \frac{1}{(tn^2)} \right. \\
& \left. - \bar{R}_{22}^{13} \frac{(n_c^2)}{(sn^2)(tn^2)} - 3\bar{R}_{66}^{13} \frac{(n_c^2)}{(sn^2)(tn^2)} \right]
\end{aligned} \tag{F.77}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{6,5} = & -\bar{R}_{22}^{20} \frac{n_c}{sntn} + 6\bar{R}_{12}^{21} \frac{n_c}{sn} - \bar{R}_{22}^{21} \frac{(n_c^3)}{(sn^3)} - 2\bar{R}_{22}^{21} \frac{n_c}{sn} \\
& + 6\bar{W}_{46}^{21} \frac{n_c}{sn} + \tau_0 \left[+\bar{R}_{22}^{21} \frac{n_c}{sn(tn^2)} - 6\bar{R}_{46}^{21} \frac{n_c}{sntn} - 6\bar{R}_{66}^{21} \frac{n_c}{sn} + 2\bar{R}_{22}^{22} \frac{n_c}{sntn} \right. \\
& + \bar{R}_{22}^{22} \frac{(n_c^3)}{(sn^3)tn} + 3\bar{R}_{46}^{22} \frac{n_c}{sn(tn^2)} + 3\bar{R}_{66}^{22} \frac{n_c}{sntn} + 3\bar{W}_{46}^{22} \frac{n_c}{sntn} \left. \right] + \tau \left[-\bar{R}_{22}^{22} \frac{n_c}{sn(tn^3)} \right. \\
& \left. - \bar{R}_{22}^{23} \frac{(n_c^3)}{(sn^3)(tn^2)} - 2\bar{R}_{22}^{23} \frac{n_c}{sn(tn^2)} - 3\bar{R}_{46}^{23} \frac{n_c}{sn(tn^3)} - 3\bar{R}_{66}^{23} \frac{n_c}{sn(tn^2)} \right]
\end{aligned} \tag{F.78}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{6,6} = & -\bar{A}_{12}^0 \frac{1}{tn} + 6\bar{A}_{11}^1 + 2\bar{A}_{12}^1 - \bar{A}_{12}^1 \frac{(n_c^2)}{(sn^2)} - \bar{A}_{22}^1 \\
& - 2\bar{A}_{66}^1 \frac{(n_c^2)}{(sn^2)} + \bar{A}_{66}^2 \frac{(n_c^2)cs}{(sn^3)} + \tau_0 \left[+6\bar{A}_{11}^2 \frac{1}{tn} + \bar{A}_{22}^2 \frac{1}{tn} \right] + \tau \left[-\bar{A}_{22}^3 \frac{1}{(tn^2)} \right. \\
& \left. - \bar{A}_{66}^3 \frac{(n_c^2)}{(sn^2)(tn^2)} + 0.25\bar{A}_{66}^4 \frac{(n_c^2)cs}{(sn^3)(tn^2)} \right]
\end{aligned} \tag{F.79}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{6,7} = & +4\bar{A}_{12}^1 \frac{n_c}{sn} - \bar{A}_{22}^1 \frac{n_c}{sn} + 4\bar{A}_{66}^1 \frac{n_c}{sn} + 4\bar{A}_{12}^2 \frac{n_c}{sntn} \\
& - \bar{A}_{22}^2 \frac{n_c}{sntn} + 6\bar{A}_{66}^2 \frac{n_c cs}{(sn^2)} + \tau_0 \left[+\bar{A}_{22}^2 \frac{n_c}{sntn} + \bar{A}_{22}^3 \frac{n_c}{sn(tn^2)} \right] + \tau \left[-\bar{A}_{22}^3 \frac{n_c}{sn(tn^2)} \right. \\
& \left. + 2\bar{A}_{66}^3 \frac{n_c}{sn(tn^2)} - \bar{A}_{22}^4 \frac{n_c}{sn(tn^3)} + 1.5\bar{A}_{66}^4 \frac{n_c cs}{(sn^2)(tn^2)} \right]
\end{aligned} \tag{F.80}$$

$$\begin{aligned} \mathcal{C}\bar{\mathcal{P}}_{6,8} = & +4\bar{A}_{12}^1 \frac{1}{tn} - 6\bar{A}_{12}^2 + 4\bar{A}_{12}^2 \frac{(n_c^2)}{(sn^2)} + 2\bar{A}_{22}^2 + 8\bar{A}_{66}^2 \frac{(n_c^2)}{(sn^2)} + \tau_0 \left[\right. \\ & \left. - 2\bar{A}_{22}^3 \frac{1}{tn} \right] + \tau \left[+2\bar{A}_{22}^4 \frac{1}{(tn^2)} + 2\bar{A}_{66}^4 \frac{(n_c^2)}{(sn^2)(tn^2)} \right] \end{aligned} \quad (\text{F.81})$$

$$\begin{aligned} \mathcal{C}\bar{\mathcal{P}}_{6,9} = & -\bar{R}_{12}^{10} \frac{1}{tn} + 12\bar{R}_{11}^{11} - \bar{R}_{12}^{11} \frac{(n_c^2)}{(sn^2)} - 3\bar{R}_{12}^{11} - 2\bar{R}_{66}^{11} \frac{(n_c^2)}{(sn^2)} \\ & + 6\bar{W}_{15}^{11} - \bar{W}_{25}^{11} + \tau_0 \left[+6\bar{R}_{12}^{11} - \bar{R}_{22}^{11} + 12\bar{R}_{11}^{12} \frac{1}{tn} + \bar{R}_{22}^{12} \frac{1}{tn} \right. \\ & \left. + \bar{R}_{66}^{12} \frac{(n_c^2)}{(sn^2)tn} + 6\bar{W}_{15}^{12} \frac{1}{tn} \right] + \tau \left[-\bar{R}_{22}^{13} \frac{1}{(tn^2)} - \bar{R}_{66}^{13} \frac{(n_c^2)}{(sn^2)(tn^2)} \right] \end{aligned} \quad (\text{F.82})$$

$$\begin{aligned} \mathcal{C}\bar{\mathcal{P}}_{6,10} = & +6\bar{R}_{12}^{21} \frac{n_c}{sn} - \bar{R}_{22}^{21} \frac{n_c}{sn} + 8\bar{R}_{66}^{21} \frac{n_c}{sn} + 2\bar{W}_{46}^{21} \frac{n_c}{sn} + \tau_0 \left[\right. \\ & - 2\bar{R}_{46}^{21} \frac{n_c}{sntn} - 2\bar{R}_{66}^{21} \frac{n_c}{sn} + \bar{R}_{22}^{22} \frac{n_c}{sntn} + \bar{R}_{46}^{22} \frac{n_c}{sn(tn^2)} + 5\bar{R}_{66}^{22} \frac{n_c}{sntn} \\ & \left. + \bar{W}_{46}^{22} \frac{n_c}{sntn} \right] + \tau \left[-\bar{R}_{22}^{23} \frac{n_c}{sn(tn^2)} - \bar{R}_{46}^{23} \frac{n_c}{sn(tn^3)} - \bar{R}_{66}^{23} \frac{n_c}{sn(tn^2)} \right] \end{aligned} \quad (\text{F.83})$$

$$\mathcal{C}\bar{\mathcal{P}}_{6,11} = +6\bar{A}_{11}^1 + \tau_0 \left[+6\bar{A}_{11}^2 \frac{1}{tn} \right] \quad (\text{F.84})$$

$$\begin{aligned} \mathcal{C}\bar{\mathcal{P}}_{6,12} = & +\bar{A}_{12}^1 \frac{n_c}{sn} + 2\bar{A}_{66}^1 \frac{n_c}{sn} + \bar{A}_{12}^2 \frac{n_c}{sntn} + 3\bar{A}_{66}^2 \frac{n_c cs}{(sn^2)} \\ & + \bar{A}_{66}^3 \frac{n_c}{sn(tn^2)} + 0.75\bar{A}_{66}^4 \frac{n_c cs}{(sn^2)(tn^2)} \end{aligned} \quad (\text{F.85})$$

$$\begin{aligned} \mathcal{C}\bar{\mathcal{P}}_{6,13} = & +2\bar{A}_{12}^1 \frac{1}{tn} - 12\bar{A}_{11}^2 + 2\bar{A}_{12}^2 \frac{(n_c^2)}{(sn^2)} - 3\bar{A}_{12}^2 + \bar{A}_{22}^2 \\ & + 4\bar{A}_{66}^2 \frac{(n_c^2)}{(sn^2)} + \tau_0 \left[-12\bar{A}_{11}^3 \frac{1}{tn} - \bar{A}_{22}^3 \frac{1}{tn} \right] + \tau \left[+\bar{A}_{22}^4 \frac{1}{(tn^2)} + \bar{A}_{66}^4 \frac{(n_c^2)}{(sn^2)(tn^2)} \right] \end{aligned} \quad (\text{F.86})$$

$$\mathcal{C}\bar{\mathcal{P}}_{6,14} = +8\bar{R}_{11}^{11} - \bar{R}_{12}^{11} + \bar{W}_{15}^{11} + \tau_0 \left[+\bar{R}_{12}^{11} + 8\bar{R}_{11}^{12} \frac{1}{tn} + \bar{W}_{15}^{12} \frac{1}{tn} \right] \quad (\text{F.87})$$

$$\mathcal{C}\bar{\mathcal{P}}_{6,15} = +\bar{\mathbf{R}}_{12}^{21} \frac{n_c}{sn} + 2\bar{\mathbf{R}}_{66}^{21} \frac{n_c}{sn} + \tau_0 \left[+\bar{\mathbf{R}}_{66}^{22} \frac{n_c}{sntn} \right] \quad (\text{F.88})$$

$$\mathcal{C}\bar{\mathcal{P}}_{6,16} = +\bar{\mathbf{A}}_{11}^1 + \tau_0 \left[+\bar{\mathbf{A}}_{11}^2 \frac{1}{tn} \right] \quad (\text{F.89})$$

$$\mathcal{C}\bar{\mathcal{P}}_{6,18} = -8\bar{\mathbf{A}}_{11}^2 + \tau_0 \left[-8\bar{\mathbf{A}}_{11}^3 \frac{1}{tn} \right] \quad (\text{F.90})$$

$$\mathcal{C}\bar{\mathcal{P}}_{6,19} = +\bar{\mathbf{R}}_{11}^{11} + \tau_0 \left[+\bar{\mathbf{R}}_{11}^{12} \frac{1}{tn} \right] \quad (\text{F.91})$$

$$\mathcal{C}\bar{\mathcal{P}}_{6,23} = -\bar{\mathbf{A}}_{11}^2 + \tau_0 \left[-\bar{\mathbf{A}}_{11}^3 \frac{1}{tn} \right] \quad (\text{F.92})$$

$$\begin{aligned} \mathcal{C}\bar{\mathcal{P}}_{7,1} = & +2\bar{\mathbf{R}}_{16}^{10} \frac{n_c}{sn} + \bar{\mathbf{R}}_{26}^{10} \frac{n_c}{sn} - \bar{\mathbf{R}}_{16}^{11} \frac{n_c cs}{(sn^2)} - \bar{\mathbf{W}}_{56}^{10} \frac{n_c}{sn} \\ & + 0.5\bar{\mathbf{W}}_{56}^{11} \frac{n_c cs}{(sn^2)} + \tau_0 \left[-\bar{\mathbf{R}}_{26}^{10} \frac{n_c}{sn} + \bar{\mathbf{R}}_{16}^{11} \frac{n_c}{sntn} + 0.5\bar{\mathbf{R}}_{26}^{11} \frac{n_c cs}{(sn^2)} - 0.5\bar{\mathbf{R}}_{26}^{11} \frac{n_c}{sntn} \right. \\ & \left. - 0.5\bar{\mathbf{R}}_{16}^{12} \frac{n_c cs}{(sn^2)tn} - 0.25\bar{\mathbf{R}}_{26}^{12} \frac{n_c cs}{(sn^2)tn} - 0.5\bar{\mathbf{W}}_{56}^{11} \frac{n_c}{sntn} + 0.25\bar{\mathbf{W}}_{56}^{12} \frac{n_c cs}{(sn^2)tn} \right] + \tau \left[\right. \\ & \left. + 0.5\bar{\mathbf{R}}_{16}^{12} \frac{n_c}{sn(tn^2)} + 0.25\bar{\mathbf{R}}_{26}^{12} \frac{n_c}{sn(tn^2)} + 0.25\bar{\mathbf{R}}_{26}^{13} \frac{n_c cs}{(sn^2)(tn^2)} - 0.25\bar{\mathbf{W}}_{56}^{12} \frac{n_c}{sn(tn^2)} \right] \end{aligned} \quad (\text{F.93})$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{7,2} = & + 2\bar{R}_{16}^{10} + \bar{R}_{26}^{10} \frac{(n_c^2)}{(sn^2)} + 3\bar{R}_{16}^{11} \frac{cs}{sn} + \bar{R}_{26}^{11} \frac{(n_c^2)}{(sn^2)tn} - \bar{W}_{56}^{10} \\
& - 1.5\bar{W}_{56}^{11} \frac{cs}{sn} + \tau_0 \left[-\bar{R}_{26}^{10} + \bar{R}_{16}^{11} \frac{1}{tn} - \bar{R}_{26}^{11} \frac{(n_c^2)}{(sn^2)tn} - 1.5\bar{R}_{26}^{11} \frac{cs}{sn} \right. \\
& + 0.5\bar{R}_{26}^{11} \frac{1}{tn} + 1.5\bar{R}_{16}^{12} \frac{cs}{sntn} + 0.75\bar{R}_{26}^{12} \frac{cs}{sntn} - \bar{R}_{26}^{12} \frac{(n_c^2)}{(sn^2)(tn^2)} - 0.5\bar{W}_{56}^{11} \frac{1}{tn} \\
& \left. - 0.75\bar{W}_{56}^{12} \frac{cs}{sntn} \right] + \tau \left[+0.5\bar{R}_{16}^{12} \frac{1}{(tn^2)} - 0.75\bar{R}_{26}^{12} \frac{1}{(tn^2)} + \bar{R}_{26}^{12} \frac{(n_c^2)}{(sn^2)(tn^2)} \right. \\
& \left. + \bar{R}_{26}^{13} \frac{(n_c^2)}{(sn^2)(tn^3)} - 0.75\bar{R}_{26}^{13} \frac{cs}{sn(tn^2)} - 0.25\bar{W}_{56}^{12} \frac{1}{(tn^2)} \right]
\end{aligned} \tag{F.94}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{7,3} = & + \bar{R}_{26}^{10} \frac{n_c}{sntn} + 4\bar{R}_{16}^{11} \frac{n_c}{sn} + \bar{R}_{26}^{11} \frac{(n_c^3)}{(sn^3)} - 2\bar{W}_{56}^{11} \frac{n_c}{sn} + \tau_0 \left[\right. \\
& - 2\bar{R}_{26}^{11} \frac{n_c}{sn} - \bar{R}_{26}^{11} \frac{n_c}{sn(tn^2)} + 2\bar{R}_{16}^{12} \frac{n_c}{sntn} - \bar{R}_{26}^{12} \frac{(n_c^3)}{(sn^3)tn} + \bar{R}_{26}^{12} \frac{n_c}{sntn} \\
& \left. - \bar{W}_{56}^{12} \frac{n_c}{sntn} \right] + \tau \left[+\bar{R}_{26}^{12} \frac{n_c}{sn(tn^3)} + \bar{R}_{26}^{13} \frac{(n_c^3)}{(sn^3)(tn^2)} - \bar{R}_{26}^{13} \frac{n_c}{sn(tn^2)} \right]
\end{aligned} \tag{F.95}$$

$$\mathcal{C}\bar{\mathcal{P}}_{7,4} = + 3\bar{Q}_{16}^{110} \frac{n_c}{sn} \tag{F.96}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{7,5} = & + \bar{Q}_{26}^{120} \frac{(n_c^2)}{(sn^2)} - 3\bar{Y}_{14}^{210} + \bar{Z}_{45}^{120} + \tau_0 \left[+3\bar{Q}_{14}^{120} \frac{1}{tn} + 3\bar{Q}_{16}^{120} \right. \\
& - \bar{Q}_{24}^{120} \frac{1}{tn} - \bar{Q}_{26}^{120} + \bar{Q}_{24}^{121} \frac{1}{(tn^2)} + \bar{Q}_{26}^{121} \frac{1}{tn} - \bar{Q}_{26}^{121} \frac{(n_c^2)}{(sn^2)tn} \\
& - \bar{Y}_{45}^{120} \frac{1}{tn} - \bar{Y}_{56}^{120} + \bar{Y}_{24}^{210} - 3\bar{Y}_{14}^{211} \frac{1}{tn} + \bar{Z}_{45}^{121} \frac{1}{tn} \left. \right] + \tau \left[-\bar{Q}_{24}^{122} \frac{1}{(tn^3)} \right. \\
& \left. - \bar{Q}_{26}^{122} \frac{1}{(tn^2)} + \bar{Q}_{26}^{122} \frac{(n_c^2)}{(sn^2)(tn^2)} \right]
\end{aligned} \tag{F.97}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{7,6} = & + 2\bar{R}_{16}^{10} \frac{n_c}{sn} - 0.5\bar{R}_{16}^{11} \frac{n_c cs}{(sn^2)} + \tau_0 \left[+0.5\bar{R}_{16}^{11} \frac{n_c}{sntn} \right. \\
& \left. - 0.25\bar{R}_{16}^{12} \frac{n_c cs}{(sn^2)tn} \right] + \tau \left[+0.25\bar{R}_{16}^{12} \frac{n_c}{sn(tn^2)} \right]
\end{aligned} \tag{F.98}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{7,7} = & -2\bar{\mathbf{R}}_{16}^{10} - 3\bar{\mathbf{R}}_{16}^{11} \frac{cs}{sn} + \bar{\mathbf{W}}_{56}^{10} + 1.5\bar{\mathbf{W}}_{56}^{11} \frac{cs}{sn} + \tau_0 \left[+\bar{\mathbf{R}}_{26}^{10} \right. \\
& - \bar{\mathbf{R}}_{16}^{11} \frac{1}{tn} + 1.5\bar{\mathbf{R}}_{26}^{11} \frac{cs}{sn} - 0.5\bar{\mathbf{R}}_{26}^{11} \frac{1}{tn} - 1.5\bar{\mathbf{R}}_{16}^{12} \frac{cs}{sntn} - 0.75\bar{\mathbf{R}}_{26}^{12} \frac{cs}{sntn} \\
& + 0.5\bar{\mathbf{W}}_{56}^{11} \frac{1}{tn} + 0.75\bar{\mathbf{W}}_{56}^{12} \frac{cs}{sntn} \left. \right] + \tau \left[-0.5\bar{\mathbf{R}}_{16}^{12} \frac{1}{(tn^2)} + 0.75\bar{\mathbf{R}}_{26}^{12} \frac{1}{(tn^2)} + 0.75\bar{\mathbf{R}}_{26}^{13} \frac{cs}{sn(tn^2)} \right. \\
& \left. + 0.25\bar{\mathbf{W}}_{56}^{12} \frac{1}{(tn^2)} \right]
\end{aligned} \tag{F.99}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{7,8} = & -4\bar{\mathbf{R}}_{16}^{11} \frac{n_c}{sn} - \bar{\mathbf{R}}_{26}^{11} \frac{n_c}{sn} + 2\bar{\mathbf{W}}_{56}^{11} \frac{n_c}{sn} + \tau_0 \left[+2\bar{\mathbf{R}}_{26}^{11} \frac{n_c}{sn} \right. \\
& \left. - 2\bar{\mathbf{R}}_{16}^{12} \frac{n_c}{sntn} + \bar{\mathbf{W}}_{56}^{12} \frac{n_c}{sntn} \right]
\end{aligned} \tag{F.100}$$

$$\mathcal{C}\bar{\mathcal{P}}_{7,9} = +2\bar{\mathbf{Q}}_{16}^{110} \frac{n_c}{sn} \tag{F.101}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{7,10} = & -4\bar{\mathbf{Q}}_{16}^{120} + \bar{\mathbf{Y}}_{56}^{120} - \bar{\mathbf{Y}}_{14}^{210} + \tau_0 \left[+\bar{\mathbf{Q}}_{14}^{120} \frac{1}{tn} + \bar{\mathbf{Q}}_{16}^{120} + \bar{\mathbf{Q}}_{26}^{120} \right. \\
& \left. - 4\bar{\mathbf{Q}}_{16}^{121} \frac{1}{tn} + \bar{\mathbf{Y}}_{56}^{121} \frac{1}{tn} - \bar{\mathbf{Y}}_{14}^{211} \frac{1}{tn} \right]
\end{aligned} \tag{F.102}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{7,12} = & -\bar{\mathbf{R}}_{16}^{10} - 1.5\bar{\mathbf{R}}_{16}^{11} \frac{cs}{sn} + \tau_0 \left[-0.5\bar{\mathbf{R}}_{16}^{11} \frac{1}{tn} - 0.75\bar{\mathbf{R}}_{16}^{12} \frac{cs}{sntn} \right] + \tau \left[\right. \\
& \left. - 0.25\bar{\mathbf{R}}_{16}^{12} \frac{1}{(tn^2)} \right]
\end{aligned} \tag{F.103}$$

$$\mathcal{C}\bar{\mathcal{P}}_{7,13} = -3\bar{\mathbf{R}}_{16}^{11} \frac{n_c}{sn} + \tau_0 \left[-\bar{\mathbf{R}}_{16}^{12} \frac{n_c}{sntn} \right] \tag{F.104}$$

$$\mathcal{C}\bar{\mathcal{P}}_{7,15} = -\bar{\mathbf{Q}}_{16}^{120} + \tau_0 \left[-\bar{\mathbf{Q}}_{16}^{121} \frac{1}{tn} \right] \tag{F.105}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{8,1} = & -2\bar{R}_{12}^{10} + \bar{R}_{66}^{10} \frac{(n_c^2)}{(sn^2)} - 0.5\bar{R}_{66}^{11} \frac{(n_c^2)cs}{(sn^3)} + \bar{W}_{25}^{10} + \tau_0 \left[+\bar{R}_{22}^{10} \right. \\
& - \bar{R}_{22}^{11} \frac{1}{tn} - 0.5\bar{R}_{66}^{11} \frac{(n_c^2)}{(sn^2)tn} + 0.25\bar{R}_{66}^{12} \frac{(n_c^2)cs}{(sn^3)tn} \left. \right] + \tau \left[+\bar{R}_{22}^{12} \frac{1}{(tn^2)} \right. \\
& \left. + 0.75\bar{R}_{66}^{12} \frac{(n_c^2)}{(sn^2)(tn^2)} - 0.25\bar{R}_{66}^{13} \frac{(n_c^2)cs}{(sn^3)(tn^2)} \right]
\end{aligned} \tag{F.106}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{8,2} = & -2\bar{R}_{12}^{10} \frac{n_c}{sn} + \bar{R}_{66}^{10} \frac{n_c}{sn} - 2\bar{R}_{12}^{11} \frac{n_c}{sntn} + 1.5\bar{R}_{66}^{11} \frac{n_c cs}{(sn^2)} \\
& + \bar{W}_{25}^{10} \frac{n_c}{sn} + \bar{W}_{25}^{11} \frac{n_c}{sntn} + \tau_0 \left[+\bar{R}_{22}^{10} \frac{n_c}{sn} - 0.5\bar{R}_{66}^{11} \frac{n_c}{sntn} - \bar{R}_{22}^{12} \frac{n_c}{sn(tn^2)} \right. \\
& - 0.75\bar{R}_{66}^{12} \frac{n_c cs}{(sn^2)tn} \left. \right] + \tau \left[+\bar{R}_{22}^{12} \frac{n_c}{sn(tn^2)} + 0.75\bar{R}_{66}^{12} \frac{n_c}{sn(tn^2)} + \bar{R}_{22}^{13} \frac{n_c}{sn(tn^3)} \right. \\
& \left. + 0.75\bar{R}_{66}^{13} \frac{n_c cs}{(sn^2)(tn^2)} \right]
\end{aligned} \tag{F.107}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{8,3} = & -2\bar{R}_{12}^{10} \frac{1}{tn} - 2\bar{R}_{12}^{11} \frac{(n_c^2)}{(sn^2)} + 2\bar{R}_{66}^{11} \frac{(n_c^2)}{(sn^2)} + \bar{W}_{25}^{10} \frac{1}{tn} \\
& + \bar{W}_{25}^{11} \frac{(n_c^2)}{(sn^2)} + \tau_0 \left[+\bar{R}_{22}^{10} \frac{1}{tn} + \bar{R}_{22}^{11} \frac{(n_c^2)}{(sn^2)} - \bar{R}_{22}^{11} \frac{1}{(tn^2)} - \bar{R}_{22}^{12} \frac{(n_c^2)}{(sn^2)tn} \right. \\
& \left. - \bar{R}_{66}^{12} \frac{(n_c^2)}{(sn^2)tn} \right] + \tau \left[+\bar{R}_{22}^{12} \frac{1}{(tn^3)} + \bar{R}_{22}^{13} \frac{(n_c^2)}{(sn^2)(tn^2)} + \bar{R}_{66}^{13} \frac{(n_c^2)}{(sn^2)(tn^2)} \right]
\end{aligned} \tag{F.108}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{8,4} = & +\bar{Q}_{66}^{110} \frac{(n_c^2)}{(sn^2)} - 3\bar{Y}_{15}^{110} + \bar{Z}_{55}^{110} + \tau_0 \left[-3\bar{Q}_{12}^{110} + \bar{Q}_{22}^{110} \right. \\
& - \bar{Q}_{22}^{111} \frac{1}{tn} - \bar{Q}_{66}^{111} \frac{(n_c^2)}{(sn^2)tn} + 2\bar{Y}_{25}^{110} - 3\bar{Y}_{15}^{111} \frac{1}{tn} + \bar{Z}_{55}^{111} \frac{1}{tn} \left. \right] + \tau \left[\right. \\
& \left. + \bar{Q}_{22}^{112} \frac{1}{(tn^2)} + \bar{Q}_{66}^{112} \frac{(n_c^2)}{(sn^2)(tn^2)} \right]
\end{aligned} \tag{F.109}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{8,5} = & -3\bar{Q}_{12}^{120} \frac{n_c}{sn} + \bar{Y}_{25}^{120} \frac{n_c}{sn} - \bar{Y}_{46}^{210} \frac{n_c}{sn} + \tau_0 \left[+\bar{Q}_{22}^{120} \frac{n_c}{sn} \right. \\
& + \bar{Q}_{46}^{120} \frac{n_c}{sntn} + \bar{Q}_{66}^{120} \frac{n_c}{sn} - \bar{Q}_{22}^{121} \frac{n_c}{sntn} - \bar{Q}_{46}^{121} \frac{n_c}{sn(tn^2)} - \bar{Q}_{66}^{121} \frac{n_c}{sntn} \left. \right] + \tau \left[\right. \\
& \left. + \bar{Q}_{22}^{122} \frac{n_c}{sn(tn^2)} + \bar{Q}_{46}^{122} \frac{n_c}{sn(tn^3)} + \bar{Q}_{66}^{122} \frac{n_c}{sn(tn^2)} \right]
\end{aligned} \tag{F.110}$$

$$\mathcal{C}\bar{\mathcal{P}}_{8,6} = -3\bar{R}_{11}^{10} - \bar{R}_{12}^{10} + \bar{W}_{15}^{10} + \tau_0 \left[+\bar{R}_{12}^{10} - 3\bar{R}_{11}^{11} \frac{1}{tn} + \bar{W}_{15}^{11} \frac{1}{tn} \right] \quad (\text{F.111})$$

$$\begin{aligned} \mathcal{C}\bar{\mathcal{P}}_{8,7} = & -\bar{R}_{12}^{10} \frac{n_c}{sn} - \bar{R}_{66}^{10} \frac{n_c}{sn} - \bar{R}_{12}^{11} \frac{n_c}{sntn} - 1.5\bar{R}_{66}^{11} \frac{n_c cs}{(sn^2)} + \tau_0 \left[\right. \\ & \left. + 0.5\bar{R}_{66}^{11} \frac{n_c}{sntn} + 0.75\bar{R}_{66}^{12} \frac{n_c cs}{(sn^2)tn} \right] + \tau \left[-0.75\bar{R}_{66}^{12} \frac{n_c}{sn(tn^2)} - 0.75\bar{R}_{66}^{13} \frac{n_c cs}{(sn^2)(tn^2)} \right] \end{aligned} \quad (\text{F.112})$$

$$\begin{aligned} \mathcal{C}\bar{\mathcal{P}}_{8,8} = & -\bar{R}_{12}^{10} \frac{1}{tn} + 3\bar{R}_{12}^{11} - \bar{R}_{12}^{11} \frac{(n_c^2)}{(sn^2)} - 2\bar{R}_{66}^{11} \frac{(n_c^2)}{(sn^2)} - \bar{W}_{25}^{11} + \tau_0 \left[\right. \\ & \left. - \bar{R}_{22}^{11} + \bar{R}_{22}^{12} \frac{1}{tn} + \bar{R}_{66}^{12} \frac{(n_c^2)}{(sn^2)tn} \right] + \tau \left[-\bar{R}_{22}^{13} \frac{1}{(tn^2)} - \bar{R}_{66}^{13} \frac{(n_c^2)}{(sn^2)(tn^2)} \right] \end{aligned} \quad (\text{F.113})$$

$$\mathcal{C}\bar{\mathcal{P}}_{8,9} = -4\bar{Q}_{11}^{110} + \tau_0 \left[-4\bar{Q}_{11}^{111} \frac{1}{tn} \right] \quad (\text{F.114})$$

$$\mathcal{C}\bar{\mathcal{P}}_{8,10} = -\bar{Q}_{12}^{120} \frac{n_c}{sn} - \bar{Q}_{66}^{120} \frac{n_c}{sn} \quad (\text{F.115})$$

$$\mathcal{C}\bar{\mathcal{P}}_{8,11} = -\bar{R}_{11}^{10} + \tau_0 \left[-\bar{R}_{11}^{11} \frac{1}{tn} \right] \quad (\text{F.116})$$

$$\mathcal{C}\bar{\mathcal{P}}_{8,13} = +4\bar{R}_{11}^{11} + \bar{R}_{12}^{11} - \bar{W}_{15}^{11} + \tau_0 \left[-\bar{R}_{12}^{11} + 4\bar{R}_{11}^{12} \frac{1}{tn} - \bar{W}_{15}^{12} \frac{1}{tn} \right] \quad (\text{F.117})$$

$$\mathcal{C}\bar{\mathcal{P}}_{8,14} = -\bar{Q}_{11}^{110} + \tau_0 \left[-\bar{Q}_{11}^{111} \frac{1}{tn} \right] \quad (\text{F.118})$$

$$\mathcal{C}\bar{\mathcal{P}}_{8,18} = +\bar{R}_{11}^{11} + \tau_0 \left[+\bar{R}_{11}^{12} \frac{1}{tn} \right] \quad (\text{F.119})$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{9,1} = & + \bar{R}_{26}^{20} \frac{n_c}{sn} + 2\bar{R}_{66}^{20} \frac{n_c}{sn} - \bar{R}_{66}^{21} \frac{n_c cs}{(sn^2)} - \bar{W}_{46}^{20} \frac{n_c}{sn} \\
& + 0.5\bar{W}_{46}^{21} \frac{n_c cs}{(sn^2)} + \tau_0 \left[+\bar{R}_{46}^{20} \frac{n_c}{sntn} + \bar{R}_{66}^{20} \frac{n_c}{sn} - 0.5\bar{R}_{46}^{21} \frac{n_c cs}{(sn^2)tn} - 0.5\bar{R}_{46}^{21} \frac{n_c}{sn(tn^2)} \right. \\
& + 0.5\bar{R}_{66}^{21} \frac{n_c}{sntn} - 0.5\bar{R}_{66}^{21} \frac{n_c cs}{(sn^2)} + 0.25\bar{R}_{46}^{22} \frac{n_c cs}{(sn^2)(tn^2)} - 0.25\bar{R}_{66}^{22} \frac{n_c cs}{(sn^2)tn} \\
& \left. - 0.5\bar{W}_{46}^{21} \frac{n_c}{sntn} + 0.25\bar{W}_{46}^{22} \frac{n_c cs}{(sn^2)tn} \right] + \tau \left[+0.75\bar{R}_{46}^{22} \frac{n_c}{sn(tn^3)} + 1.25\bar{R}_{66}^{22} \frac{n_c}{sn(tn^2)} \right. \\
& \left. - 0.25\bar{R}_{46}^{23} \frac{n_c cs}{(sn^2)(tn^3)} - 0.25\bar{R}_{66}^{23} \frac{n_c cs}{(sn^2)(tn^2)} - 0.25\bar{W}_{46}^{22} \frac{n_c}{sn(tn^2)} \right]
\end{aligned} \tag{F.120}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{9,2} = & + \bar{R}_{26}^{20} \frac{(n_c^2)}{(sn^2)} + 2\bar{R}_{66}^{20} + \bar{R}_{26}^{21} \frac{(n_c^2)}{(sn^2)tn} + 3\bar{R}_{66}^{21} \frac{cs}{sn} - \bar{W}_{46}^{20} \\
& - 1.5\bar{W}_{46}^{21} \frac{cs}{sn} + \tau_0 \left[+\bar{R}_{46}^{20} \frac{1}{tn} + \bar{R}_{66}^{20} + 1.5\bar{R}_{46}^{21} \frac{cs}{sntn} - 0.5\bar{R}_{46}^{21} \frac{1}{(tn^2)} + 0.5\bar{R}_{66}^{21} \frac{1}{tn} \right. \\
& + 1.5\bar{R}_{66}^{21} \frac{cs}{sn} - 0.75\bar{R}_{46}^{22} \frac{cs}{sn(tn^2)} + 0.75\bar{R}_{66}^{22} \frac{cs}{sntn} - 0.5\bar{W}_{46}^{21} \frac{1}{tn} - 0.75\bar{W}_{46}^{22} \frac{cs}{sntn} \left. \right] + \tau \left[\right. \\
& \left. + 0.75\bar{R}_{46}^{22} \frac{1}{(tn^3)} + 1.25\bar{R}_{66}^{22} \frac{1}{(tn^2)} + 0.75\bar{R}_{46}^{23} \frac{cs}{sn(tn^3)} + 0.75\bar{R}_{66}^{23} \frac{cs}{sn(tn^2)} - 0.25\bar{W}_{46}^{22} \frac{1}{(tn^2)} \right]
\end{aligned} \tag{F.121}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{9,3} = & + \bar{R}_{26}^{20} \frac{n_c}{sntn} + \bar{R}_{26}^{21} \frac{(n_c^3)}{(sn^3)} + 4\bar{R}_{66}^{21} \frac{n_c}{sn} - 2\bar{W}_{46}^{21} \frac{n_c}{sn} + \tau_0 \left[\right. \\
& + 2\bar{R}_{46}^{21} \frac{n_c}{sntn} + 2\bar{R}_{66}^{21} \frac{n_c}{sn} - \bar{R}_{46}^{22} \frac{n_c}{sn(tn^2)} + \bar{R}_{66}^{22} \frac{n_c}{sntn} - \bar{W}_{46}^{22} \frac{n_c}{sntn} \left. \right] + \tau \left[\right. \\
& \left. + \bar{R}_{46}^{23} \frac{n_c}{sn(tn^3)} + \bar{R}_{66}^{23} \frac{n_c}{sn(tn^2)} \right]
\end{aligned} \tag{F.122}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{9,4} = & + 3\bar{Q}_{66}^{120} \frac{n_c}{sn} + \bar{Y}_{56}^{120} \frac{n_c}{sn} - \bar{Y}_{46}^{210} \frac{n_c}{sn} + \tau_0 \left[+\bar{Q}_{26}^{120} \frac{n_c}{sn} \right. \\
& + \bar{Q}_{46}^{120} \frac{n_c}{sntn} + \bar{Q}_{66}^{120} \frac{n_c}{sn} - \bar{Q}_{46}^{121} \frac{n_c}{sn(tn^2)} - \bar{Q}_{66}^{121} \frac{n_c}{sntn} + \bar{Y}_{56}^{121} \frac{n_c}{sntn} \left. \right] + \tau \left[\right. \\
& \left. + \bar{Q}_{46}^{122} \frac{n_c}{sn(tn^3)} + \bar{Q}_{66}^{122} \frac{n_c}{sn(tn^2)} \right]
\end{aligned} \tag{F.123}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{9,5} = & +\bar{Q}_{26}^{220} \frac{(n_c^2)}{(sn^2)} - 3\bar{Y}_{46}^{220} + \bar{Z}_{44}^{220} + \tau_0 \left[+\bar{Q}_{44}^{220} \frac{1}{(tn^2)} + 5\bar{Q}_{46}^{220} \frac{1}{tn} \right. \\
& + 4\bar{Q}_{66}^{220} - \bar{Q}_{44}^{221} \frac{1}{(tn^3)} - 2\bar{Q}_{46}^{221} \frac{1}{(tn^2)} - \bar{Q}_{66}^{221} \frac{1}{tn} - 2\bar{Y}_{44}^{220} \frac{1}{tn} - 2\bar{Y}_{46}^{220} \\
& \left. - 3\bar{Y}_{46}^{221} \frac{1}{tn} + \bar{Z}_{44}^{221} \frac{1}{tn} \right] + \tau \left[+\bar{Q}_{44}^{222} \frac{1}{(tn^4)} + 2\bar{Q}_{46}^{222} \frac{1}{(tn^3)} + \bar{Q}_{66}^{222} \frac{1}{(tn^2)} \right]
\end{aligned} \tag{F.124}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{9,6} = & +\bar{R}_{16}^{20} \frac{n_c}{sn} + \bar{R}_{66}^{20} \frac{n_c}{sn} - 0.5\bar{R}_{66}^{21} \frac{n_c cs}{(sn^2)} + \tau_0 \left[+\bar{R}_{16}^{21} \frac{n_c}{sntn} \right. \\
& \left. + 0.5\bar{R}_{66}^{21} \frac{n_c}{sntn} - 0.25\bar{R}_{66}^{22} \frac{n_c cs}{(sn^2)tn} \right] + \tau \left[+0.25\bar{R}_{66}^{22} \frac{n_c}{sn(tn^2)} \right]
\end{aligned} \tag{F.125}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{9,7} = & -2\bar{R}_{66}^{20} - 3\bar{R}_{66}^{21} \frac{cs}{sn} + \bar{W}_{46}^{20} + 1.5\bar{W}_{46}^{21} \frac{cs}{sn} + \tau_0 \left[-\bar{R}_{46}^{20} \frac{1}{tn} \right. \\
& - \bar{R}_{66}^{20} - 1.5\bar{R}_{46}^{21} \frac{cs}{sntn} + 0.5\bar{R}_{46}^{21} \frac{1}{(tn^2)} - 0.5\bar{R}_{66}^{21} \frac{1}{tn} - 1.5\bar{R}_{66}^{21} \frac{cs}{sn} \\
& + 0.75\bar{R}_{46}^{22} \frac{cs}{sn(tn^2)} - 0.75\bar{R}_{66}^{22} \frac{cs}{sntn} + 0.5\bar{W}_{46}^{21} \frac{1}{tn} + 0.75\bar{W}_{46}^{22} \frac{cs}{sntn} \left. \right] + \tau \left[-0.75\bar{R}_{46}^{22} \frac{1}{(tn^3)} \right. \\
& \left. - 1.25\bar{R}_{66}^{22} \frac{1}{(tn^2)} - 0.75\bar{R}_{46}^{23} \frac{cs}{sn(tn^3)} - 0.75\bar{R}_{66}^{23} \frac{cs}{sn(tn^2)} + 0.25\bar{W}_{46}^{22} \frac{1}{(tn^2)} \right]
\end{aligned} \tag{F.126}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{9,8} = & -\bar{R}_{26}^{21} \frac{n_c}{sn} - 4\bar{R}_{66}^{21} \frac{n_c}{sn} + 2\bar{W}_{46}^{21} \frac{n_c}{sn} + \tau_0 \left[-2\bar{R}_{46}^{21} \frac{n_c}{sntn} \right. \\
& - 2\bar{R}_{66}^{21} \frac{n_c}{sn} + \bar{R}_{46}^{22} \frac{n_c}{sn(tn^2)} - \bar{R}_{66}^{22} \frac{n_c}{sntn} + \bar{W}_{46}^{22} \frac{n_c}{sntn} \left. \right] + \tau \left[-\bar{R}_{46}^{23} \frac{n_c}{sn(tn^3)} \right. \\
& \left. - \bar{R}_{66}^{23} \frac{n_c}{sn(tn^2)} \right]
\end{aligned} \tag{F.127}$$

$$\mathcal{C}\bar{\mathcal{P}}_{9,9} = +\bar{Q}_{16}^{120} \frac{n_c}{sn} + \bar{Q}_{66}^{120} \frac{n_c}{sn} + \tau_0 \left[+\bar{Q}_{16}^{121} \frac{n_c}{sntn} \right] \tag{F.128}$$

$$\mathcal{C}\bar{\mathcal{P}}_{9,10} = -4\bar{Q}_{66}^{220} + \tau_0 \left[-4\bar{Q}_{66}^{221} \frac{1}{tn} \right] \tag{F.129}$$

$$\begin{aligned} \mathcal{C}\bar{\mathcal{P}}_{9,12} = & -\bar{R}_{66}^{20} - 1.5\bar{R}_{66}^{21}\frac{cs}{sn} + \tau_0\left[-0.5\bar{R}_{66}^{21}\frac{1}{tn} - 0.75\bar{R}_{66}^{22}\frac{cs}{sntn}\right] + \tau\left[\right. \\ & \left. - 0.25\bar{R}_{66}^{22}\frac{1}{(tn^2)} \right] \end{aligned} \quad (\text{F.130})$$

$$\mathcal{C}\bar{\mathcal{P}}_{9,13} = -\bar{R}_{16}^{21}\frac{n_c}{sn} - 2\bar{R}_{66}^{21}\frac{n_c}{sn} + \tau_0\left[-\bar{R}_{16}^{22}\frac{n_c}{sntn} - \bar{R}_{66}^{22}\frac{n_c}{sntn}\right] \quad (\text{F.131})$$

$$\mathcal{C}\bar{\mathcal{P}}_{9,15} = -\bar{Q}_{66}^{220} + \tau_0\left[-\bar{Q}_{66}^{221}\frac{1}{tn}\right] \quad (\text{F.132})$$

$$\begin{aligned} \mathcal{C}\bar{\mathcal{P}}_{10,1} = & -2\bar{R}_{26}^{20} + \bar{R}_{66}^{20}\frac{(n_c^2)}{(sn^2)} - 0.5\bar{R}_{66}^{21}\frac{(n_c^2)cs}{(sn^3)} + \bar{W}_{24}^{20} + \tau_0\left[\right. \\ & -\bar{R}_{24}^{20}\frac{1}{tn} - \bar{R}_{26}^{20} + \bar{R}_{24}^{21}\frac{1}{(tn^2)} + \bar{R}_{26}^{21}\frac{1}{tn} + 0.5\bar{R}_{66}^{21}\frac{(n_c^2)}{(sn^2)tn} \\ & \left. - 0.25\bar{R}_{66}^{22}\frac{(n_c^2)cs}{(sn^3)tn}\right] + \tau\left[-\bar{R}_{24}^{22}\frac{1}{(tn^3)} - \bar{R}_{26}^{22}\frac{1}{(tn^2)} + 0.25\bar{R}_{66}^{22}\frac{(n_c^2)}{(sn^2)(tn^2)}\right] \end{aligned} \quad (\text{F.133})$$

$$\begin{aligned} \mathcal{C}\bar{\mathcal{P}}_{10,2} = & -2\bar{R}_{26}^{20}\frac{n_c}{sn} + \bar{R}_{66}^{20}\frac{n_c}{sn} - 2\bar{R}_{26}^{21}\frac{n_c}{sntn} + 1.5\bar{R}_{66}^{21}\frac{n_c cs}{(sn^2)} \\ & + \bar{W}_{24}^{20}\frac{n_c}{sn} + \bar{W}_{24}^{21}\frac{n_c}{sntn} + \tau_0\left[-\bar{R}_{24}^{20}\frac{n_c}{sntn} - \bar{R}_{26}^{20}\frac{n_c}{sn} + 0.5\bar{R}_{66}^{21}\frac{n_c}{sntn} \right. \\ & \left. + \bar{R}_{24}^{22}\frac{n_c}{sn(tn^3)} + \bar{R}_{26}^{22}\frac{n_c}{sn(tn^2)} + 0.75\bar{R}_{66}^{22}\frac{n_c cs}{(sn^2)tn}\right] + \tau\left[-\bar{R}_{24}^{22}\frac{n_c}{sn(tn^3)} \right. \\ & \left. - \bar{R}_{26}^{22}\frac{n_c}{sn(tn^2)} + 0.25\bar{R}_{66}^{22}\frac{n_c}{sn(tn^2)} - \bar{R}_{24}^{23}\frac{n_c}{sn(tn^4)} - \bar{R}_{26}^{23}\frac{n_c}{sn(tn^3)}\right] \end{aligned} \quad (\text{F.134})$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{10,3} = & -2\bar{R}_{26}^{20} \frac{1}{tn} - 2\bar{R}_{26}^{21} \frac{(n_c^2)}{(sn^2)} + 2\bar{R}_{66}^{21} \frac{(n_c^2)}{(sn^2)} + \bar{W}_{24}^{20} \frac{1}{tn} \\
& + \bar{W}_{24}^{21} \frac{(n_c^2)}{(sn^2)} + \tau_0 \left[-\bar{R}_{24}^{20} \frac{1}{(tn^2)} - \bar{R}_{26}^{20} \frac{1}{tn} - \bar{R}_{24}^{21} \frac{(n_c^2)}{(sn^2)tn} + \bar{R}_{24}^{21} \frac{1}{(tn^3)} \right. \\
& + \bar{R}_{26}^{21} \frac{1}{(tn^2)} - \bar{R}_{26}^{21} \frac{(n_c^2)}{(sn^2)} + \bar{R}_{24}^{22} \frac{(n_c^2)}{(sn^2)(tn^2)} + \bar{R}_{26}^{22} \frac{(n_c^2)}{(sn^2)tn} \\
& \left. + \bar{R}_{66}^{22} \frac{(n_c^2)}{(sn^2)tn} \right] + \tau \left[-\bar{R}_{24}^{22} \frac{1}{(tn^4)} - \bar{R}_{26}^{22} \frac{1}{(tn^3)} - \bar{R}_{24}^{23} \frac{(n_c^2)}{(sn^2)(tn^3)} \right. \\
& \left. - \bar{R}_{26}^{23} \frac{(n_c^2)}{(sn^2)(tn^2)} \right]
\end{aligned} \tag{F.135}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{10,4} = & + \bar{Q}_{66}^{120} \frac{(n_c^2)}{(sn^2)} - 3\bar{Y}_{56}^{120} + \bar{Z}_{45}^{120} + \tau_0 \left[-\bar{Q}_{24}^{120} \frac{1}{tn} - 4\bar{Q}_{26}^{120} \right. \\
& + \bar{Q}_{24}^{121} \frac{1}{(tn^2)} + \bar{Q}_{26}^{121} \frac{1}{tn} - \bar{Y}_{45}^{120} \frac{1}{tn} - \bar{Y}_{56}^{120} - 3\bar{Y}_{56}^{121} \frac{1}{tn} + \bar{Y}_{24}^{210} \\
& \left. + \bar{Z}_{45}^{121} \frac{1}{tn} \right] + \tau \left[-\bar{Q}_{24}^{122} \frac{1}{(tn^3)} - \bar{Q}_{26}^{122} \frac{1}{(tn^2)} \right]
\end{aligned} \tag{F.136}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{10,5} = & -3\bar{Q}_{26}^{220} \frac{n_c}{sn} + \bar{Y}_{24}^{220} \frac{n_c}{sn} - \bar{Y}_{46}^{220} \frac{n_c}{sn} + \tau_0 \left[-\bar{Q}_{24}^{220} \frac{n_c}{sntn} \right. \\
& - \bar{Q}_{26}^{220} \frac{n_c}{sn} + \bar{Q}_{46}^{220} \frac{n_c}{sntn} + \bar{Q}_{66}^{220} \frac{n_c}{sn} + \bar{Q}_{24}^{221} \frac{n_c}{sn(tn^2)} + \bar{Q}_{26}^{221} \frac{n_c}{sntn} \\
& \left. - \bar{Y}_{46}^{221} \frac{n_c}{sntn} \right] + \tau \left[-\bar{Q}_{24}^{222} \frac{n_c}{sn(tn^3)} - \bar{Q}_{26}^{222} \frac{n_c}{sn(tn^2)} \right]
\end{aligned} \tag{F.137}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{10,6} = & -3\bar{R}_{16}^{20} - \bar{R}_{26}^{20} + \bar{W}_{14}^{20} + \tau_0 \left[-\bar{R}_{14}^{20} \frac{1}{tn} - \bar{R}_{16}^{20} - 3\bar{R}_{16}^{21} \frac{1}{tn} \right. \\
& \left. + \bar{W}_{14}^{21} \frac{1}{tn} \right]
\end{aligned} \tag{F.138}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{10,7} = & -\bar{R}_{26}^{20} \frac{n_c}{sn} - \bar{R}_{66}^{20} \frac{n_c}{sn} - \bar{R}_{26}^{21} \frac{n_c}{sntn} - 1.5\bar{R}_{66}^{21} \frac{n_c CS}{(sn^2)} + \tau_0 \left[\right. \\
& \left. - 0.5\bar{R}_{66}^{21} \frac{n_c}{sntn} - 0.75\bar{R}_{66}^{22} \frac{n_c CS}{(sn^2)tn} \right] + \tau \left[-0.25\bar{R}_{66}^{22} \frac{n_c}{sn(tn^2)} \right]
\end{aligned} \tag{F.139}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{10,8} = & -\bar{R}_{26}^{20} \frac{1}{tn} - \bar{R}_{26}^{21} \frac{(n_c^2)}{(sn^2)} + 3\bar{R}_{26}^{21} - 2\bar{R}_{66}^{21} \frac{(n_c^2)}{(sn^2)} - \bar{W}_{24}^{21} + \tau_0 \left[\right. \\
& + \bar{R}_{24}^{21} \frac{1}{tn} + \bar{R}_{26}^{21} - \bar{R}_{24}^{22} \frac{1}{(tn^2)} - \bar{R}_{26}^{22} \frac{1}{tn} - \bar{R}_{66}^{22} \frac{(n_c^2)}{(sn^2)tn} \left. \right] + \tau \left[\right. \\
& \left. + \bar{R}_{24}^{23} \frac{1}{(tn^3)} + \bar{R}_{26}^{23} \frac{1}{(tn^2)} \right]
\end{aligned} \tag{F.140}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{10,9} = & -4\bar{Q}_{16}^{120} - \bar{Y}_{56}^{120} + \bar{Y}_{14}^{210} + \tau_0 \left[-\bar{Q}_{14}^{120} \frac{1}{tn} - \bar{Q}_{16}^{120} - \bar{Q}_{26}^{120} \right. \\
& \left. - 4\bar{Q}_{16}^{121} \frac{1}{tn} - \bar{Y}_{56}^{121} \frac{1}{tn} + \bar{Y}_{14}^{211} \frac{1}{tn} \right]
\end{aligned} \tag{F.141}$$

$$\mathcal{C}\bar{\mathcal{P}}_{10,10} = -\bar{Q}_{26}^{220} \frac{n_c}{sn} - \bar{Q}_{66}^{220} \frac{n_c}{sn} + \tau_0 \left[-\bar{Q}_{66}^{221} \frac{n_c}{sntn} \right] \tag{F.142}$$

$$\mathcal{C}\bar{\mathcal{P}}_{10,11} = -\bar{R}_{16}^{20} + \tau_0 \left[-\bar{R}_{16}^{21} \frac{1}{tn} \right] \tag{F.143}$$

$$\begin{aligned}
\mathcal{C}\bar{\mathcal{P}}_{10,13} = & +4\bar{R}_{16}^{21} + \bar{R}_{26}^{21} - \bar{W}_{14}^{21} + \tau_0 \left[+\bar{R}_{14}^{21} \frac{1}{tn} + \bar{R}_{16}^{21} + 4\bar{R}_{16}^{22} \frac{1}{tn} \right. \\
& \left. - \bar{W}_{14}^{22} \frac{1}{tn} \right]
\end{aligned} \tag{F.144}$$

$$\mathcal{C}\bar{\mathcal{P}}_{10,14} = -\bar{Q}_{16}^{120} + \tau_0 \left[-\bar{Q}_{16}^{121} \frac{1}{tn} \right] \tag{F.145}$$

$$\mathcal{C}\bar{\mathcal{P}}_{10,18} = +\bar{R}_{16}^{21} + \tau_0 \left[+\bar{R}_{16}^{22} \frac{1}{tn} \right] \tag{F.146}$$

Appendix G Properties of The Solution Basis Function

The following operators and differentiations has the closed-form result over the functions space defined by \mathbb{S} :

$$\forall r \in \mathbb{R} : \quad r = H(0, 0.5r, 0, 0) \in \mathbb{S} \tag{G.1a}$$

$$\forall r \in \mathbb{R}, \quad \forall H \in \mathbb{S}: \quad r * H = H(\beta, r * c, se, ce) \in \mathbb{S} \quad (\text{G.1b})$$

$$\forall H_1, H_2 \in \mathbb{S}: \quad H(\beta_1, c_1, se_1, ce_1) + H(\beta_2, c_2, se_2, ce_2) \in \mathbb{S} \quad (\text{G.1c})$$

$$\begin{aligned} \forall H_1, H_2 \in \mathbb{S}: \quad & H(\beta_1, c_1, se_1, ce_1) * H(\beta_2, c_2, se_2, ce_2) = \\ & H(\beta_1 \beta_2, c_1 c_2, se_1 + se_2, ce_1 + ce_2) + H(\beta_1 \beta_2^*, c_1 c_2^*, se_1 + se_2, ce_1 + ce_2) \in \mathbb{S} \end{aligned} \quad (\text{G.1d})$$

$$\forall H \in \mathbb{S}(\beta \neq 0): \quad \frac{\partial}{\partial \xi_1} H(\beta, c, se, ce) = H(\beta - 1, c * \beta / L, se, ce) \in \mathbb{S} \quad (\text{G.1e})$$

$$\begin{aligned} \forall H \in \mathbb{S}(se \neq 0, \quad ce \neq 0): \quad & \frac{\partial}{\partial \xi_2} H(\beta, c, se, ce) = \\ & H(\beta, n_c * c, se - 1, ce + 1) + H(\beta, -n_c * c, se + 1, ce - 1) \in \mathbb{S} \end{aligned} \quad (\text{G.1f})$$

Moreover, the "One" and "Zero" constants for multiplication and addition over this function space are $H(0, 1.0, 0, 0)$ and $H(0, 0, 0, 0)$ accordingly. It is also worth mentioning that multiplication of a complex number to a basis function of this type is not a member of this space ($\exists z \in \mathbb{C} \Rightarrow z * H \notin \mathbb{S}$). Using the definition of "One" it is possible to define any real constant (such as elements of $[\mathbb{C}^0]$ matrix) as a member of \mathbb{S} function space. The metrics, principle and geodesic radii of the curvature of conical shells can also be defined in form of the basis function of equation (55) :

$$\begin{aligned} H_{A_1} &\triangleq A_1(\xi_1, \xi_2) = H(0, 0.5, 0, 0), & H_{A_2} &\triangleq A_2(\xi_1, \xi_2) = H(1, 0.5L \sin(\alpha_c), 0, 0), \\ H_{1/A_1} &\triangleq \frac{1}{A_1(\xi_1, \xi_2)} = H(0, 0.5, 0, 0), & H_{1/A_2} &\triangleq \frac{1}{A_2(\xi_1, \xi_2)} = H(-1, 0.5/(L \sin(\alpha_c)), 0, 0), \\ H_{1/R_1} &\triangleq \frac{1}{R_1(\xi_1, \xi_2)} = H(0, 0, 0, 0), & H_{1/R_2} &\triangleq \frac{1}{R_2(\xi_1, \xi_2)} = H(-1, 0.5/(L \tan(\alpha_c)), 0, 0), \\ H_{1/\rho_{11}} &\triangleq \frac{1}{\rho_{11}} = H(0, 0, 0, 0), & H_{1/\rho_{22}} &\triangleq \frac{1}{\rho_{22}} = H(-1, 0.5/L, 0, 0) \end{aligned} \quad (\text{G.2})$$

Moreover by assuming:

$$\beta_R = \Re(\beta) \quad \beta_I = \Im(\beta) \quad c_R = \Re(c) \quad c_I = \Im(c) \quad y = \ln\left(\frac{\xi_1}{L}\right) \quad (\text{G.3})$$

Evaluation at a point results:

$$H(\beta, c, se, ce) \Big|_{\xi_1, \xi_2} = 2.0 \left(\frac{\xi_1}{L}\right)^{\beta_R} [c_R \cos(\beta_I y) - c_I \sin(\beta_I y)] (\sin(nc\xi_2))^{se} (\cos(nc\xi_2))^{ce} \quad (\text{G.4})$$

Finite integral over ξ_1 is given by:

$$\int_{\xi_{1,0}}^{\xi_{1,1}} H(\beta, c, 0, 0) = H(\beta + 1, cL/(\beta + 1), 0, 0) \Big|_{\xi_{1,1}} - H(\beta + 1, cL/(\beta + 1), 0, 0) \Big|_{\xi_{1,0}} \quad (\text{G.5})$$

The surface element of a truncated conical surface is defined as:

$$d\Omega = A_1 A_2 d\xi_1 d\xi_2 = 1 \times \sin(\alpha_c) \xi_1 d\xi_1 d\xi_2 = H(1, 0.5L \sin(\alpha_c)) d\xi_1 d\xi_2 \quad (\text{G.6})$$

The integration over a truncated conical shell elements defined over a domain such as $\Omega = (\xi_{1,0}.. \xi_{1,1}, 0..2\pi)$, can be obtained from:

$$\begin{aligned} \iint_{\Omega} H(\beta, c, se, ce) d\Omega &= \left(H(\beta + 2, 0.5 * cL^2 \sin(\alpha_c)/(\beta + 2), 0, 0) \Big|_{\xi_{1,0}}^{\xi_{1,1}} \right) \\ &\times \left(\int_0^{2\pi} \sin(n_c \xi_2)^{se} \cos(n_c \xi_2)^{ce} d\xi_2 \right) \end{aligned} \quad (\text{G.7})$$

Due to small set of possible values for se and ce , the second term of the right hand side of equation (G.7) can be calculated using a small look up table. For the most occurring cases of ($se = 0, ce = 2$) and ($se = 2, ce = 0$) that term is given by:

Appendix H Constitutive Matrix for Variable Thickness Conical Shells

The upper-right elements of $[\mathcal{C}\bar{\mathcal{C}}^0]$ symmetric matrix multiplied by the associated ξ_1 exponent on the left hand side, can be obtained as follows:

$$\xi_1^{-1} \mathcal{C}\bar{\mathcal{C}}_{1,1} = + \bar{A}_{11}^0 + \tau_0 \left[+ \bar{A}_{11}^1 \frac{1}{tn} \right] \quad (\text{H.1})$$

$$\xi_1^{-1} \mathcal{C}\bar{\mathcal{C}}_{1,2} = + \bar{A}_{12}^0 \quad (\text{H.2})$$

$$\xi_1^{-1} \mathcal{C}\bar{\mathcal{C}}_{1,3} = + \bar{A}_{16}^0 + \tau_0 \left[+ 0.5 \bar{A}_{16}^1 \frac{1}{tn} \right] + \tau \left[+ 0.25 \bar{A}_{16}^2 \frac{1}{(tn^2)} \right] \quad (\text{H.3})$$

$$\xi_1^{-2} \mathcal{C}\bar{\mathcal{C}}_{1,4} = + \bar{A}_{11}^1 + \tau_0 \left[+ \bar{A}_{11}^2 \frac{1}{tn} \right] \quad (\text{H.4})$$

$$\xi_1^{-2} \bar{\mathcal{C}}_{1,5} = +\bar{A}_{12}^1 \quad (\text{H.5})$$

$$\xi_1^{-2} \bar{\mathcal{C}}_{1,6} = +\bar{A}_{16}^1 + \tau_0 \left[+0.5\bar{A}_{16}^2 \frac{1}{tn} \right] \quad (\text{H.6})$$

$$\xi_1^{-2} \bar{\mathcal{C}}_{1,7} = +\bar{R}_{11}^{10} + \tau_0 \left[+\bar{R}_{11}^{11} \frac{1}{tn} \right] \quad (\text{H.7})$$

$$\xi_1^{-2} \bar{\mathcal{C}}_{1,8} = +\bar{R}_{16}^{20} + \tau_0 \left[+\bar{R}_{16}^{21} \frac{1}{tn} \right] \quad (\text{H.8})$$

$$\xi_1^{-2} \bar{\mathcal{C}}_{1,9} = +\bar{R}_{16}^{10} \quad (\text{H.9})$$

$$\xi_1^{-2} \bar{\mathcal{C}}_{1,10} = +\bar{R}_{12}^{20} \quad (\text{H.10})$$

$$\xi_1^{-1} \bar{\mathcal{C}}_{1,11} = +\bar{W}_{15}^{10} + \tau_0 \left[+\bar{R}_{12}^{10} + \bar{W}_{15}^{11} \frac{1}{tn} \right] \quad (\text{H.11})$$

$$\xi_1^{-1} \bar{\mathcal{C}}_{1,12} = +\bar{W}_{14}^{20} + \tau_0 \left[-\bar{R}_{14}^{20} \frac{1}{tn} - \bar{R}_{16}^{20} + \bar{W}_{14}^{21} \frac{1}{tn} \right] \quad (\text{H.12})$$

$$\xi_1^{-1} \bar{\mathcal{C}}_{2,2} = +\bar{A}_{22}^0 + \tau_0 \left[-\bar{A}_{22}^1 \frac{1}{tn} \right] + \tau \left[+\bar{A}_{22}^2 \frac{1}{(tn^2)} \right] \quad (\text{H.13})$$

$$\xi_1^{-1} \bar{\mathcal{C}}_{2,3} = +\bar{A}_{26}^0 + \tau_0 \left[-0.5\bar{A}_{26}^1 \frac{1}{tn} \right] + \tau \left[+0.75\bar{A}_{26}^2 \frac{1}{(tn^2)} \right] \quad (\text{H.14})$$

$$\xi_1^{-2} \bar{\mathcal{C}}_{2,4} = +\bar{A}_{12}^1 \quad (\text{H.15})$$

$$\xi_1^{-2} \bar{\mathcal{C}}_{2,5} = +\bar{A}_{22}^1 + \tau_0 \left[-\bar{A}_{22}^2 \frac{1}{tn} \right] + \tau \left[+\bar{A}_{22}^3 \frac{1}{(tn^2)} \right] \quad (\text{H.16})$$

$$\xi_1^{-2} \bar{\mathcal{C}}_{2,6} = +\bar{A}_{26}^1 + \tau_0 \left[-0.5\bar{A}_{26}^2 \frac{1}{tn} \right] + \tau \left[+0.5\bar{A}_{26}^3 \frac{1}{(tn^2)} \right] \quad (\text{H.17})$$

$$\xi_1^{-2} \bar{\mathcal{C}}_{2,7} = +\bar{R}_{12}^{10} \quad (\text{H.18})$$

$$\xi_1^{-2} \bar{\mathcal{C}}_{2,8} = +\bar{R}_{26}^{20} \quad (\text{H.19})$$

$$\xi_1^{-2} \bar{\mathcal{C}}_{2,9} = +\bar{R}_{26}^{10} + \tau_0 \left[-\bar{R}_{26}^{11} \frac{1}{tn} \right] + \tau \left[+\bar{R}_{26}^{12} \frac{1}{(tn^2)} \right] \quad (\text{H.20})$$

$$\xi_1^{-2} \bar{\mathcal{C}}_{2,10} = +\bar{R}_{22}^{20} + \tau_0 \left[-\bar{R}_{22}^{21} \frac{1}{tn} \right] + \tau \left[+\bar{R}_{22}^{22} \frac{1}{(tn^2)} \right] \quad (\text{H.21})$$

$$\xi_1^{-1} \bar{\mathcal{C}}_{2,11} = +\bar{W}_{25}^{10} + \tau_0 \left[+\bar{R}_{22}^{10} - \bar{R}_{22}^{11} \frac{1}{tn} \right] + \tau \left[+\bar{R}_{22}^{12} \frac{1}{(tn^2)} \right] \quad (\text{H.22})$$

$$\begin{aligned} \xi_1^{-1} \bar{\mathcal{C}}_{2,12} = & +\bar{W}_{24}^{20} + \tau_0 \left[-\bar{R}_{24}^{20} \frac{1}{tn} - \bar{R}_{26}^{20} + \bar{R}_{24}^{21} \frac{1}{(tn^2)} + \bar{R}_{26}^{21} \frac{1}{tn} \right] + \tau \left[\right. \\ & \left. -\bar{R}_{24}^{22} \frac{1}{(tn^3)} - \bar{R}_{26}^{22} \frac{1}{(tn^2)} \right] \quad (\text{H.23}) \end{aligned}$$

$$\xi_1^{-1} \mathcal{C}\bar{\mathcal{C}}_{3,3} = +\bar{A}_{66}^0 + 0.75\bar{A}_{66}^2 \frac{1}{(tn^2)} \quad (\text{H.24})$$

$$\xi_1^{-2} \mathcal{C}\bar{\mathcal{C}}_{3,4} = +\bar{A}_{16}^1 + \tau_0 \left[+0.5\bar{A}_{16}^2 \frac{1}{tn} \right] + \tau \left[+0.25\bar{A}_{16}^3 \frac{1}{(tn^2)} \right] \quad (\text{H.25})$$

$$\xi_1^{-2} \mathcal{C}\bar{\mathcal{C}}_{3,5} = +\bar{A}_{26}^1 + \tau_0 \left[-0.5\bar{A}_{26}^2 \frac{1}{tn} \right] + \tau \left[+0.75\bar{A}_{26}^3 \frac{1}{(tn^2)} \right] \quad (\text{H.26})$$

$$\xi_1^{-2} \mathcal{C}\bar{\mathcal{C}}_{3,6} = +\bar{A}_{66}^1 + 0.5\bar{A}_{66}^3 \frac{1}{(tn^2)} \quad (\text{H.27})$$

$$\xi_1^{-2} \mathcal{C}\bar{\mathcal{C}}_{3,7} = +\bar{R}_{16}^{10} + \tau_0 \left[+0.5\bar{R}_{16}^{11} \frac{1}{tn} \right] + \tau \left[+0.25\bar{R}_{16}^{12} \frac{1}{(tn^2)} \right] \quad (\text{H.28})$$

$$\xi_1^{-2} \mathcal{C}\bar{\mathcal{C}}_{3,8} = +\bar{R}_{66}^{20} + \tau_0 \left[+0.5\bar{R}_{66}^{21} \frac{1}{tn} \right] + \tau \left[+0.25\bar{R}_{66}^{22} \frac{1}{(tn^2)} \right] \quad (\text{H.29})$$

$$\xi_1^{-2} \mathcal{C}\bar{\mathcal{C}}_{3,9} = +\bar{R}_{66}^{10} + \tau_0 \left[-0.5\bar{R}_{66}^{11} \frac{1}{tn} \right] + \tau \left[+0.75\bar{R}_{66}^{12} \frac{1}{(tn^2)} \right] \quad (\text{H.30})$$

$$\xi_1^{-2} \mathcal{C}\bar{\mathcal{C}}_{3,10} = +\bar{R}_{26}^{20} + \tau_0 \left[-0.5\bar{R}_{26}^{21} \frac{1}{tn} \right] + \tau \left[+0.75\bar{R}_{26}^{22} \frac{1}{(tn^2)} \right] \quad (\text{H.31})$$

$$\begin{aligned} \xi_1^{-1} \mathcal{C}\bar{\mathcal{C}}_{3,11} = & +\bar{W}_{56}^{10} + \tau_0 \left[+\bar{R}_{26}^{10} - 0.5\bar{R}_{26}^{11} \frac{1}{tn} + 0.5\bar{W}_{56}^{11} \frac{1}{tn} \right] + \tau \left[+0.75\bar{R}_{26}^{12} \frac{1}{(tn^2)} \right. \\ & \left. + 0.25\bar{W}_{56}^{12} \frac{1}{(tn^2)} \right] \end{aligned} \quad (\text{H.32})$$

$$\begin{aligned} \xi_1^{-1} \bar{\mathcal{C}}_{3,12} = & + \bar{W}_{46}^{20} + \tau_0 \left[-\bar{R}_{46}^{20} \frac{1}{tn} - \bar{R}_{66}^{20} + 0.5 \bar{R}_{46}^{21} \frac{1}{(tn^2)} + 0.5 \bar{R}_{66}^{21} \frac{1}{tn} \right. \\ & \left. + 0.5 \bar{W}_{46}^{21} \frac{1}{tn} \right] + \tau \left[-0.75 \bar{R}_{46}^{22} \frac{1}{(tn^3)} - 0.75 \bar{R}_{66}^{22} \frac{1}{(tn^2)} + 0.25 \bar{W}_{46}^{22} \frac{1}{(tn^2)} \right] \end{aligned} \quad (\text{H.33})$$

$$\xi_1^{-3} \bar{\mathcal{C}}_{4,4} = + \bar{A}_{11}^2 + \tau_0 \left[+ \bar{A}_{11}^3 \frac{1}{tn} \right] \quad (\text{H.34})$$

$$\xi_1^{-3} \bar{\mathcal{C}}_{4,5} = + \bar{A}_{12}^2 \quad (\text{H.35})$$

$$\xi_1^{-3} \bar{\mathcal{C}}_{4,6} = + \bar{A}_{16}^2 + \tau_0 \left[+ 0.5 \bar{A}_{16}^3 \frac{1}{tn} \right] \quad (\text{H.36})$$

$$\xi_1^{-3} \bar{\mathcal{C}}_{4,7} = + \bar{R}_{11}^{11} + \tau_0 \left[+ \bar{R}_{11}^{12} \frac{1}{tn} \right] \quad (\text{H.37})$$

$$\xi_1^{-3} \bar{\mathcal{C}}_{4,8} = + \bar{R}_{16}^{21} + \tau_0 \left[+ \bar{R}_{16}^{22} \frac{1}{tn} \right] \quad (\text{H.38})$$

$$\xi_1^{-3} \bar{\mathcal{C}}_{4,9} = + \bar{R}_{16}^{11} \quad (\text{H.39})$$

$$\xi_1^{-3} \bar{\mathcal{C}}_{4,10} = + \bar{R}_{12}^{21} \quad (\text{H.40})$$

$$\xi_1^{-2} \bar{\mathcal{C}}_{4,11} = + \bar{W}_{15}^{11} + \tau_0 \left[+ \bar{R}_{12}^{11} + \bar{W}_{15}^{12} \frac{1}{tn} \right] \quad (\text{H.41})$$

$$\xi_1^{-2} \bar{\mathcal{C}}_{4,12} = +\bar{W}_{14}^{21} + \tau_0 \left[-\bar{R}_{14}^{21} \frac{1}{tn} - \bar{R}_{16}^{21} + \bar{W}_{14}^{22} \frac{1}{tn} \right] \quad (\text{H.42})$$

$$\xi_1^{-3} \bar{\mathcal{C}}_{5,5} = +\bar{A}_{22}^2 + \tau_0 \left[-\bar{A}_{22}^3 \frac{1}{tn} \right] + \tau \left[+\bar{A}_{22}^4 \frac{1}{(tn^2)} \right] \quad (\text{H.43})$$

$$\xi_1^{-3} \bar{\mathcal{C}}_{5,6} = +\bar{A}_{26}^2 + \tau_0 \left[-0.5\bar{A}_{26}^3 \frac{1}{tn} \right] + \tau \left[+0.5\bar{A}_{26}^4 \frac{1}{(tn^2)} \right] \quad (\text{H.44})$$

$$\xi_1^{-3} \bar{\mathcal{C}}_{5,7} = +\bar{R}_{12}^{11} \quad (\text{H.45})$$

$$\xi_1^{-3} \bar{\mathcal{C}}_{5,8} = +\bar{R}_{26}^{21} \quad (\text{H.46})$$

$$\xi_1^{-3} \bar{\mathcal{C}}_{5,9} = +\bar{R}_{26}^{11} + \tau_0 \left[-\bar{R}_{26}^{12} \frac{1}{tn} \right] + \tau \left[+\bar{R}_{26}^{13} \frac{1}{(tn^2)} \right] \quad (\text{H.47})$$

$$\xi_1^{-3} \bar{\mathcal{C}}_{5,10} = +\bar{R}_{22}^{21} + \tau_0 \left[-\bar{R}_{22}^{22} \frac{1}{tn} \right] + \tau \left[+\bar{R}_{22}^{23} \frac{1}{(tn^2)} \right] \quad (\text{H.48})$$

$$\xi_1^{-2} \bar{\mathcal{C}}_{5,11} = +\bar{W}_{25}^{11} + \tau_0 \left[+\bar{R}_{22}^{11} - \bar{R}_{22}^{12} \frac{1}{tn} \right] + \tau \left[+\bar{R}_{22}^{13} \frac{1}{(tn^2)} \right] \quad (\text{H.49})$$

$$\begin{aligned} \xi_1^{-2} \bar{\mathcal{C}}_{5,12} = & +\bar{W}_{24}^{21} + \tau_0 \left[-\bar{R}_{24}^{21} \frac{1}{tn} - \bar{R}_{26}^{21} + \bar{R}_{24}^{22} \frac{1}{(tn^2)} + \bar{R}_{26}^{22} \frac{1}{tn} \right] + \tau \left[\right. \\ & \left. -\bar{R}_{24}^{23} \frac{1}{(tn^3)} - \bar{R}_{26}^{23} \frac{1}{(tn^2)} \right] \end{aligned} \quad (\text{H.50})$$

$$\xi_1^{-3} \bar{\mathcal{C}}_{6,6} = +\bar{A}_{66}^2 + 0.25\bar{A}_{66}^4 \frac{1}{(tn^2)} \quad (\text{H.51})$$

$$\xi_1^{-3} \bar{\mathcal{C}}_{6,7} = +\bar{R}_{16}^{11} + \tau_0 \left[+0.5\bar{R}_{16}^{12} \frac{1}{tn} \right] \quad (\text{H.52})$$

$$\xi_1^{-3} \bar{\mathcal{C}}_{6,8} = +\bar{R}_{66}^{21} + \tau_0 \left[+0.5\bar{R}_{66}^{22} \frac{1}{tn} \right] \quad (\text{H.53})$$

$$\xi_1^{-3} \bar{\mathcal{C}}_{6,9} = +\bar{R}_{66}^{11} + \tau_0 \left[-0.5\bar{R}_{66}^{12} \frac{1}{tn} \right] + \tau \left[+0.5\bar{R}_{66}^{13} \frac{1}{(tn^2)} \right] \quad (\text{H.54})$$

$$\xi_1^{-3} \bar{\mathcal{C}}_{6,10} = +\bar{R}_{26}^{21} + \tau_0 \left[-0.5\bar{R}_{26}^{22} \frac{1}{tn} \right] + \tau \left[+0.5\bar{R}_{26}^{23} \frac{1}{(tn^2)} \right] \quad (\text{H.55})$$

$$\xi_1^{-2} \bar{\mathcal{C}}_{6,11} = +\bar{W}_{56}^{11} + \tau_0 \left[+\bar{R}_{26}^{11} - 0.5\bar{R}_{26}^{12} \frac{1}{tn} + 0.5\bar{W}_{56}^{12} \frac{1}{tn} \right] + \tau \left[+0.5\bar{R}_{26}^{13} \frac{1}{(tn^2)} \right] \quad (\text{H.56})$$

$$\begin{aligned} \xi_1^{-2} \bar{\mathcal{C}}_{6,12} = & +\bar{W}_{46}^{21} + \tau_0 \left[-\bar{R}_{46}^{21} \frac{1}{tn} - \bar{R}_{66}^{21} + 0.5\bar{R}_{46}^{22} \frac{1}{(tn^2)} + 0.5\bar{R}_{66}^{22} \frac{1}{tn} \right. \\ & \left. + 0.5\bar{W}_{46}^{22} \frac{1}{tn} \right] + \tau \left[-0.5\bar{R}_{46}^{23} \frac{1}{(tn^3)} - 0.5\bar{R}_{66}^{23} \frac{1}{(tn^2)} \right] \end{aligned} \quad (\text{H.57})$$

$$\xi_1^{-3} \bar{\mathcal{C}}_{7,7} = +\bar{Q}_{11}^{110} + \tau_0 \left[+\bar{Q}_{11}^{111} \frac{1}{tn} \right] \quad (\text{H.58})$$

$$\xi_1^{-3} \bar{\mathcal{C}}_{7,8} = +\bar{Q}_{16}^{120} + \tau_0 \left[+\bar{Q}_{16}^{121} \frac{1}{tn} \right] \quad (\text{H.59})$$

$$\xi_1^{-3} \bar{\mathcal{C}}_{7,9} = +\bar{Q}_{16}^{110} \quad (\text{H.60})$$

$$\xi_1^{-3} \bar{\mathcal{C}}_{7,10} = +\bar{Q}_{12}^{120} \quad (\text{H.61})$$

$$\xi_1^{-2} \bar{\mathcal{C}}_{7,11} = +\bar{Y}_{15}^{110} + \tau_0 \left[+\bar{Q}_{12}^{110} + \bar{Y}_{15}^{111} \frac{1}{tn} \right] \quad (\text{H.62})$$

$$\xi_1^{-2} \bar{\mathcal{C}}_{7,12} = +\bar{Y}_{14}^{210} + \tau_0 \left[-\bar{Q}_{14}^{120} \frac{1}{tn} - \bar{Q}_{16}^{120} + \bar{Y}_{14}^{211} \frac{1}{tn} \right] \quad (\text{H.63})$$

$$\xi_1^{-3} \bar{\mathcal{C}}_{8,8} = +\bar{Q}_{66}^{220} + \tau_0 \left[+\bar{Q}_{66}^{221} \frac{1}{tn} \right] \quad (\text{H.64})$$

$$\xi_1^{-3} \bar{\mathcal{C}}_{8,9} = +\bar{Q}_{66}^{120} \quad (\text{H.65})$$

$$\xi_1^{-3} \bar{\mathcal{C}}_{8,10} = +\bar{Q}_{26}^{220} \quad (\text{H.66})$$

$$\xi_1^{-2} \bar{\mathcal{C}}_{8,11} = +\bar{Y}_{56}^{120} + \tau_0 \left[+\bar{Q}_{26}^{120} + \bar{Y}_{56}^{121} \frac{1}{tn} \right] \quad (\text{H.67})$$

$$\xi_1^{-2} \bar{\mathcal{C}}_{8,12} = +\bar{Y}_{46}^{220} + \tau_0 \left[-\bar{Q}_{46}^{220} \frac{1}{tn} - \bar{Q}_{66}^{220} + \bar{Y}_{46}^{221} \frac{1}{tn} \right] \quad (\text{H.68})$$

$$\xi_1^{-3} \bar{\mathcal{C}}_{9,9} = +\bar{Q}_{66}^{110} + \tau_0 \left[-\bar{Q}_{66}^{111} \frac{1}{tn} \right] + \tau \left[+\bar{Q}_{66}^{112} \frac{1}{(tn^2)} \right] \quad (\text{H.69})$$

$$\xi_1^{-3} \bar{\mathcal{C}}_{9,10} = +\bar{Q}_{26}^{120} + \tau_0 \left[-\bar{Q}_{26}^{121} \frac{1}{tn} \right] + \tau \left[+\bar{Q}_{26}^{122} \frac{1}{(tn^2)} \right] \quad (\text{H.70})$$

$$\xi_1^{-2} \bar{\mathcal{C}}_{9,11} = +\bar{Y}_{56}^{110} + \tau_0 \left[+\bar{Q}_{26}^{110} - \bar{Q}_{26}^{111} \frac{1}{tn} \right] + \tau \left[+\bar{Q}_{26}^{112} \frac{1}{(tn^2)} \right] \quad (\text{H.71})$$

$$\begin{aligned} \xi_1^{-2} \bar{\mathcal{C}}_{9,12} = & +\bar{Y}_{46}^{210} + \tau_0 \left[-\bar{Q}_{46}^{120} \frac{1}{tn} - \bar{Q}_{66}^{120} + \bar{Q}_{46}^{121} \frac{1}{(tn^2)} + \bar{Q}_{66}^{121} \frac{1}{tn} \right] + \tau \left[\right. \\ & \left. -\bar{Q}_{46}^{122} \frac{1}{(tn^3)} - \bar{Q}_{66}^{122} \frac{1}{(tn^2)} \right] \end{aligned} \quad (\text{H.72})$$

$$\xi_1^{-3} \bar{\mathcal{C}}_{10,10} = +\bar{Q}_{22}^{220} + \tau_0 \left[-\bar{Q}_{22}^{221} \frac{1}{tn} \right] + \tau \left[+\bar{Q}_{22}^{222} \frac{1}{(tn^2)} \right] \quad (\text{H.73})$$

$$\xi_1^{-2} \bar{\mathcal{C}}_{10,11} = +\bar{Y}_{25}^{120} + \tau_0 \left[+\bar{Q}_{22}^{120} - \bar{Q}_{22}^{121} \frac{1}{tn} \right] + \tau \left[+\bar{Q}_{22}^{122} \frac{1}{(tn^2)} \right] \quad (\text{H.74})$$

$$\begin{aligned} \xi_1^{-2} \bar{\mathcal{C}}_{10,12} = & +\bar{Y}_{24}^{220} + \tau_0 \left[-\bar{Q}_{24}^{220} \frac{1}{tn} - \bar{Q}_{26}^{220} + \bar{Q}_{24}^{221} \frac{1}{(tn^2)} + \bar{Q}_{26}^{221} \frac{1}{tn} \right] + \tau \left[\right. \\ & \left. -\bar{Q}_{24}^{222} \frac{1}{(tn^3)} - \bar{Q}_{26}^{222} \frac{1}{(tn^2)} \right] \end{aligned} \quad (\text{H.75})$$

$$\begin{aligned} \xi_1^{-1} \bar{\mathcal{C}}_{11,11} = & +\bar{Z}_{55}^{110} + \tau_0 \left[+\bar{Q}_{22}^{110} - \bar{Q}_{22}^{111} \frac{1}{tn} + 2\bar{Y}_{25}^{110} + \bar{Z}_{55}^{111} \frac{1}{tn} \right] + \tau \left[\right. \\ & \left. +\bar{Q}_{22}^{112} \frac{1}{(tn^2)} \right] \end{aligned} \quad (\text{H.76})$$

$$\begin{aligned} \xi_1^{-1} \bar{\mathcal{C}}_{11,12} = & +\bar{Z}_{45}^{120} + \tau_0 \left[-\bar{Q}_{24}^{120} \frac{1}{tn} - \bar{Q}_{26}^{120} + \bar{Q}_{24}^{121} \frac{1}{(tn^2)} + \bar{Q}_{26}^{121} \frac{1}{tn} \right. \\ & \left. -\bar{Y}_{45}^{120} \frac{1}{tn} - \bar{Y}_{56}^{120} + \bar{Y}_{24}^{210} + \bar{Z}_{45}^{121} \frac{1}{tn} \right] + \tau \left[-\bar{Q}_{24}^{122} \frac{1}{(tn^3)} - \bar{Q}_{26}^{122} \frac{1}{(tn^2)} \right] \end{aligned} \quad (\text{H.77})$$

$$\begin{aligned}
\xi_1^{-1} \bar{C} \bar{C}_{12,12} = & + \bar{Z}_{44}^{220} + \tau_0 \left[+ \bar{Q}_{44}^{220} \frac{1}{(tn^2)} + 2\bar{Q}_{46}^{220} \frac{1}{tn} + \bar{Q}_{66}^{220} - \bar{Q}_{44}^{221} \frac{1}{(tn^3)} \right. \\
& - 2\bar{Q}_{46}^{221} \frac{1}{(tn^2)} - \bar{Q}_{66}^{221} \frac{1}{tn} - 2\bar{Y}_{44}^{220} \frac{1}{tn} - 2\bar{Y}_{46}^{220} + \bar{Z}_{44}^{221} \frac{1}{tn} \left. \right] + \tau \left[+ \bar{Q}_{44}^{222} \frac{1}{(tn^4)} \right. \\
& \left. + 2\bar{Q}_{46}^{222} \frac{1}{(tn^3)} + \bar{Q}_{66}^{222} \frac{1}{(tn^2)} \right] \quad (H.78)
\end{aligned}$$

Appendix I Total Strain Vector Components in \mathbb{S} Function Space

The rows of linear component of $\{\mathbf{E}^\circ\}$ can be formulated as follows:

$$[\mathbf{S}_{\mathbf{E}_L^\circ}]_{1,1:K} = \{\mathbf{S}_{\mathbf{e}_{11}^\circ}\} \quad (I.1a)$$

$$[\mathbf{S}_{\mathbf{E}_L^\circ}]_{2,1:K} = \{\mathbf{S}_{\mathbf{e}_{22}^\circ}\} \quad (I.1b)$$

$$[\mathbf{S}_{\mathbf{E}_L^\circ}]_{3,1:K} = 2\{\mathbf{S}_{\mathbf{e}_{12}^\circ}\} \quad (I.1c)$$

$$[\mathbf{S}_{\mathbf{E}_L^\circ}]_{4,1:K} = \{\mathbf{S}_{\mathbf{x}_{11}^\circ}\} \quad (I.1d)$$

$$[\mathbf{S}_{\mathbf{E}_L^\circ}]_{5,1:K} = \{\mathbf{S}_{\mathbf{x}_{22}^\circ}\} \quad (I.1e)$$

$$[\mathbf{S}_{\mathbf{E}_L^\circ}]_{6,1:K} = 2\{\mathbf{S}_{\mathbf{x}_{12}^\circ}\} \quad (I.1f)$$

$$[\mathbf{S}_{\mathbf{E}_L^\circ}]_{7:8,1:K} = \{\mathbf{S}_\gamma^{\xi_1}\} \quad (I.1g)$$

$$[\mathbf{S}_{\mathbf{E}_L^\circ}]_{9:10,1:K} = \{\mathbf{S}_\gamma^{\xi_2}\} \quad (I.1h)$$

$$[\mathbf{S}_{\mathbf{E}_L^\circ}]_{11:12,1:K} = \{\mathbf{S}_\gamma\} \quad (I.1i)$$

The rows of nonlinear component of $\{\mathbf{E}^\circ\}$ can be formulated as follows:

$$[\mathbf{S}_{\mathbf{E}_L^\circ}]_{1,1:K^2} = \frac{1}{2} (\{\mathbf{S}_{\phi_1 \otimes \phi_1}\} + c_2 \{\mathbf{S}_{\phi \otimes \phi}\}) + \frac{1}{2} c_1 (\{\mathbf{S}_{\mathbf{e}_{11}^\circ \otimes \mathbf{e}_{11}^\circ}\} + \{\mathbf{S}_{\mathbf{e}_{12}^\circ \otimes \mathbf{e}_{12}^\circ}\} + 2\{\mathbf{S}_{\mathbf{e}_{12}^\circ \otimes \phi}\}) \quad (\text{I.2a})$$

$$[\mathbf{S}_{\mathbf{E}_L^\circ}]_{2,1:K^2} = \frac{1}{2} (\{\mathbf{S}_{\phi_2 \otimes \phi_2}\} + c_2 \{\mathbf{S}_{\phi \otimes \phi}\}) + \frac{1}{2} c_1 (\{\mathbf{S}_{\mathbf{e}_{22}^\circ \otimes \mathbf{e}_{22}^\circ}\} + \{\mathbf{S}_{\mathbf{e}_{12}^\circ \otimes \mathbf{e}_{12}^\circ}\} + 2\{\mathbf{S}_{\mathbf{e}_{12}^\circ \otimes \phi}\}) \quad (\text{I.2b})$$

$$[\mathbf{S}_{\mathbf{E}_L^\circ}]_{3,1:K^2} = \{\mathbf{S}_{\phi_1 \otimes \phi_2}\} + c_1 (\{\mathbf{S}_{\mathbf{e}_{11}^\circ \otimes \mathbf{e}_{12}^\circ}\} - \{\mathbf{S}_{\mathbf{e}_{11}^\circ \otimes \phi}\} + \{\mathbf{S}_{\mathbf{e}_{22}^\circ \otimes \mathbf{e}_{12}^\circ}\} + \{\mathbf{S}_{\mathbf{e}_{22}^\circ \otimes \phi}\}) \quad (\text{I.2c})$$

$$[\mathbf{S}_{\mathbf{E}_L^\circ}]_{4,12,1:K^2} = [\mathbf{H}(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})]_{9 \times K^2} \quad (\text{I.2d})$$

Appendix J Matrix Derivatives

J.1 Review of Matrix Calculus

Presenting the derivations of the following subsections requires a brief review of matrix calculus including some definitions and identities is provided here. Most of the following identities can be found in classic matrix calculus texts and articles including [41], [42] and [43].

To keep the formulations condensed the notations of this section are slightly different from the rest of the document and are mostly similar to what that has been used by Brewer in [44]. Matrices are shown by upper case bold letter (e.g. \mathbf{A}) and column vectors are presented with lower case bold letter (e.g. \mathbf{x}). The k^{th} row of a matrix such as \mathbf{A} is shown with $\mathbf{A}_{k,:}$, and the k^{th} column is shown with $\mathbf{A}_{:,k}$. The ik element of \mathbf{A} is denoted a_{ik} . The $n \times n$ unit matrix is denoted \mathbf{I}_n . The q -dimensional vector which is "1" in the in the k^{th} and zero elsewhere is called *unit vector* and is denoted:

$$\mathbf{e}_{(q)^k} \tag{J.1}$$

The dimension underscore will be dropped if the dimension can be understood from the context. The *elementary matrix* is defined as:

$$\mathbf{E}_{(p \times q) ik} \triangleq \mathbf{e}_{(p)^i} \mathbf{e}_{(q)^k}^T \tag{J.2}$$

which has the dimension $(p \times q)$ and has a single "1" element located at ik element and zero elsewhere.

The Kronecker product of $\mathbf{A}_{(p \times q)}$ and $\mathbf{B}_{(s \times t)}$ denoted by $\mathbf{A} \otimes \mathbf{B}$ is a $(ps \times qt)$ matrix defined as follows:

$$\mathbf{A} \otimes \mathbf{B} \triangleq \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1q}\mathbf{B} \\ a_{21}\mathbf{B} & \cdots & & a_{2q}\mathbf{B} \\ \vdots & \ddots & & \vdots \\ a_{p1}\mathbf{B} & a_{p2}\mathbf{B} & \cdots & a_{pq}\mathbf{B} \end{bmatrix} \tag{J.3}$$

It should be noted that:

$$\mathbf{e}_{(p)^i} \otimes \mathbf{e}_{(q)^k} = \mathbf{e}_{(pq)^{i-1)q+k}} \tag{J.4}$$

Another useful definition is the following block-unit matrix, that contains an identity matrix at its

j^{th} block and zero elsewhere and can be constructed as follows:

$$\mathbf{E} \mathbf{I}_j = \mathbf{e}_j^\top \otimes \mathbf{I}_p = \begin{bmatrix} \overbrace{\mathbf{0}_{p \times p}}^1 & \overbrace{\mathbf{0}_{p \times p}}^2 & \cdots & \overbrace{\mathbf{I}_p}^j & \overbrace{\mathbf{0}_{p \times p}}^{j+1} & \cdots & \overbrace{\mathbf{0}_{p \times p}}^q \end{bmatrix} \quad (\text{J.5})$$

It should be noted that the Kronecker product of two vectors such as $\mathbf{a}_{m \times 1}$ and $\mathbf{b}_{n \times 1}$ can be converted to classic matrix multiplication using the following operator:

$$\mathbf{a} \otimes \mathbf{b} \triangleq \underset{(m \times n)}{\text{vec}}(\mathbf{a}) \mathbf{b} = \left(\sum_{j=1}^m a_j \left(\mathbf{e}_j \otimes \mathbf{I}_n \right) \right) \mathbf{b} = \begin{bmatrix} a_1 \mathbf{I}_n \\ a_2 \mathbf{I}_n \\ \vdots \\ a_m \mathbf{I}_n \end{bmatrix}_{mn \times n} \mathbf{b} \quad (\text{J.6})$$

The vectorization operator generates a vector from a matrix such as \mathbf{A} by creating a stacking columns of matrix on top of each other from left to right:

$$\text{vec}(\mathbf{A})_{pq \times 1} \triangleq \begin{bmatrix} \mathbf{A}_{:,1} \\ \mathbf{A}_{:,2} \\ \vdots \\ \mathbf{A}_{:,q} \end{bmatrix} \quad (\text{J.7})$$

Another important matrix is the *elementary permutation matrix* that is defined as:

$$\mathbf{U}_{(p \times q)} \triangleq \sum_i^p \sum_k^q \mathbf{E}_{ik} \otimes \mathbf{E}_{ki} \quad (\text{J.8})$$

A permutation matrix such as $\mathbf{U}_{(p \times q)}$ has only a single "1" entry in each row and column. Given a row number such as r or a column number such as c for an elementary permutation matrix such as $\mathbf{U}_{(p \times q)}$ the corresponding non-zero column or row can be obtained from:

$$c_{r1} = (1 - pq) \left\lceil \frac{r}{q} \right\rceil + p(q + r - 1) \quad (\text{J.9a})$$

$$r_{1c} = (1 - pq) \left\lceil \frac{c}{p} \right\rceil + q(p + c - 1) \quad (\text{J.9b})$$

where $\lceil \cdot \rceil$ denotes the ceiling function. Following Vetter's definition for the matrix derivative, the

derivative of a matrix such as $\mathbf{A}_{(p \times q)}$ with respect to a scalar such as b is given as [43]:

$$\frac{\partial \mathbf{A}}{\partial b} = \begin{bmatrix} \frac{\partial a_{11}}{\partial b} & \frac{\partial a_{12}}{\partial b} & \dots & \frac{\partial a_{1q}}{\partial b} \\ \frac{\partial a_{21}}{\partial b} & & & \frac{\partial a_{2q}}{\partial b} \\ \vdots & \ddots & & \\ \frac{\partial a_{p1}}{\partial b} & & \dots & \frac{\partial a_{pq}}{\partial b} \end{bmatrix} \quad (\text{J.10})$$

Similarly the Vetter's definition for the derivative of a matrix such as $\mathbf{A}_{(p \times q)}$ with respect to another matrix such as $\mathbf{B}_{(s \times t)}$ is a partitioned matrix such as $\mathbf{C}_{ps \times qt}$ whose \mathbf{C}_{ik} partitions is:

$$\mathbf{C}_{ik} = \frac{\partial \mathbf{A}}{\partial b_{ik}} \quad (\text{J.11})$$

Using above definitions, some useful identities can be found in the following equations:

$$\mathbf{e}_{(p)}^T \mathbf{e}_{(p)} = \delta_{ik} \quad (\delta \text{ denote to Kronecker Delta}) \quad (\text{J.12a})$$

$$\mathbf{U} = \mathbf{U} = \mathbf{I}_p \quad (\text{J.12b})$$

$$\mathbf{U}^T = \mathbf{U} \quad (\text{J.12c})$$

$$\mathbf{U}^{-1} = \mathbf{U} \quad (\text{J.12d})$$

$$\mathbf{U} = \mathbf{U}^T = \mathbf{U}^{-1} \quad (\text{J.12e})$$

$$(\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}) \quad (\text{J.12f})$$

$$(\mathbf{A} + \mathbf{H}) \otimes (\mathbf{B} + \mathbf{R}) = \mathbf{A} \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{R} + \mathbf{H} \otimes \mathbf{B} + \mathbf{H} \otimes \mathbf{R} \quad (\text{J.12g})$$

$$(\mathbf{A} \otimes \mathbf{B})^T = \mathbf{A}^T \otimes \mathbf{B}^T \quad (\text{J.12h})$$

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{D} \otimes \mathbf{G}) = (\mathbf{AD} \otimes \mathbf{BG}) \quad (\text{if dimensions allow the operation}) \quad (\text{J.12i})$$

$$(\mathbf{B} \otimes \mathbf{A}) = \mathbf{U} \underset{(s \times p)}{(\mathbf{A} \otimes \mathbf{B})} \mathbf{U} \underset{(q \times t)}{\quad} \quad (\text{J.12j})$$

$$c(\mathbf{A} \otimes \mathbf{B}) = (c\mathbf{A}) \otimes \mathbf{B} = \mathbf{A} \otimes (c\mathbf{B}) \quad (c \in \mathbb{C}) \quad (\text{J.12k})$$

$$(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1} \text{ (if dimensions allow the operation)} \quad (\text{J.12l})$$

$$\text{vec}(\mathbf{A} + \mathbf{H}) = \text{vec}(\mathbf{A}) + \text{vec}(\mathbf{H}) \quad (\text{J.12m})$$

$$\text{vec}(\mathbf{A}^\top) = \mathbf{U} \underset{(p \times q)}{\cdot} \text{vec}(\mathbf{A}) \quad (\text{J.12n})$$

$$\text{vec}(\mathbf{AD}) = (\mathbf{I}_s \otimes \mathbf{A}) \text{vec}(\mathbf{D}) = (\mathbf{D}^\top \otimes \mathbf{I}_p) \text{vec}(\mathbf{A}) = (\mathbf{D}^\top \otimes \mathbf{A}) \text{vec}(\mathbf{I}_q) \quad (\text{J.12o})$$

$$\text{vec}(\mathbf{AD}) = \sum_{k=1}^q (\mathbf{D}^\top)_{:,k} \otimes \mathbf{A}_{:,k} \quad (\text{J.12p})$$

$$\frac{\partial(\mathbf{AF})}{\mathbf{B}} = \frac{\partial(\mathbf{A})}{\mathbf{B}} (\mathbf{I}_t \otimes \mathbf{F}) + (\mathbf{I}_s \otimes \mathbf{A}) \frac{\partial(\mathbf{F})}{\mathbf{B}} \quad (\text{J.12q})$$

$$\frac{\partial(\mathbf{A} \otimes \mathbf{C})}{\mathbf{B}} = \frac{\partial(\mathbf{A})}{\mathbf{B}} \otimes \mathbf{C} + \left(\mathbf{I}_s \otimes \underset{(p \times r)}{\mathbf{U}} \right) \left(\frac{\partial \mathbf{C}}{\mathbf{B}} \otimes \mathbf{A} \right) \left(\mathbf{I}_t \otimes \underset{(l \times q)}{\mathbf{U}} \right) \quad (\text{J.12r})$$

$$\frac{\partial \mathbf{y}}{\mathbf{y}} = \text{vec}(\mathbf{I}_q) \quad (\text{J.12s})$$

$$\frac{\partial \mathbf{y}^\top}{\mathbf{y}} = \mathbf{I}_q \quad (\text{J.12t})$$

Dimensions of matrices and vectors in (J.12) are: $\mathbf{A}_{p \times q}$, $\mathbf{B}_{s \times t}$, $\mathbf{C}_{r \times l}$, $\mathbf{D}_{q \times s}$, $\mathbf{F}_{q \times u}$, $\mathbf{G}_{t \times u}$, $\mathbf{H}_{p \times q}$ and $\mathbf{y}_{q \times 1}$.

J.2 Linear-Linear Component

The linear-linear component of energies in the equations of motion has the following shape:

$$\Phi_{11} = \mathbf{x}^\top \mathbf{Y} \mathbf{x} \quad (\text{J.13})$$

where the dimensions are: $\mathbf{x}_{1 \times K}$ and $\mathbf{x}_{1 \times K}^\top$ and $\mathbf{Y}_{K \times K}$. Using identities of (J.12) The derivative of Φ_{11} can be expanded:

$$\begin{aligned} \frac{\partial \Phi_{11}}{\partial \mathbf{x}} &= \frac{\partial \mathbf{x}^\top}{\partial \mathbf{x}} (\mathbf{I}_1 \otimes \mathbf{Y} \mathbf{x}) + (\mathbf{I}_K \otimes \mathbf{x}^\top) \left(\frac{\partial (\mathbf{Y} \mathbf{x})}{\partial \mathbf{x}} \right) = \mathbf{I}_K \mathbf{Y} \mathbf{x} + (\mathbf{I}_K \otimes \mathbf{x}^\top) \text{vec}(\mathbf{Y}) \\ &= \mathbf{Y} \mathbf{x} + (\mathbf{Y}^\top \otimes \mathbf{I}_1) \text{vec}(\mathbf{x}^\top) = \mathbf{Y} \mathbf{x} + \mathbf{Y}^\top \mathbf{x} \end{aligned} \quad (\text{J.14})$$

It should be noted that if \mathbf{Y} be a symmetric matrix, then $\mathbf{Y} = \mathbf{Y}^\top$; hence (J.14) becomes:

$$\frac{\partial \Phi_{11}}{\partial \mathbf{x}} = 2\mathbf{Y} \mathbf{x} \quad (\text{J.15})$$

which is the familiar representation of linear mass and stiffness matrices.

J.3 Linear-Nonlinear Component

The nonlinear-linear component of energy has the following form:

$$\Phi_{12} = \mathbf{x}^\top \mathbf{Y} (\mathbf{x} \otimes \mathbf{x}) \quad (\text{J.16})$$

where the dimensions are: $\mathbf{x}_{1 \times K}^\top$, $(\mathbf{x} \otimes \mathbf{x})_{K^2 \times 1}$ and $\mathbf{Y}_{K \times K^2}$. At the first step we vectorize Φ_{12} to the following series:

$$\Phi_{12} = \sum_{i=1}^{K^2} \Phi_{12,i} \triangleq \sum_{i=1}^{K^2} \overbrace{(\mathbf{x}^\top \mathbf{Y}_{:,i})}^{\triangleq f_{1i}} \overbrace{\left(\mathbf{e}_{(K^2)}^\top (\mathbf{x} \otimes \mathbf{x}) \right)}^{\triangleq f_{2i}} \quad (\text{J.17})$$

It should be noted that both f_{1i} and f_{2i} have scalar type, therefore both inner products are commutative or:

$$\left(\mathbf{x}^\top \mathbf{Y}_{:,i} \right) \left(\mathbf{e}_{(K^2)}^\top (\mathbf{x} \otimes \mathbf{x}) \right) = \left((\mathbf{Y}_{:,i})^\top \mathbf{x} \right) \left((\mathbf{x} \otimes \mathbf{x})^\top \mathbf{e}_{(K^2)} \right) = (\mathbf{Y}_{:,i})^\top \left(\mathbf{x} (\mathbf{x} \otimes \mathbf{x})^\top \mathbf{e}_{(K^2)} \right) \quad (\text{J.18})$$

The second term on the right hand side of equation (J.18) is a vector by size of $K^2 \times 1$, therefore employing equation (J.12o) yields:

$$\begin{aligned} \left(\mathbf{x}(\mathbf{x} \otimes \mathbf{x})^\top \mathbf{e}_i \right)_{(K^2)} &= \text{vec} \left(\mathbf{x}(\mathbf{x} \otimes \mathbf{x})^\top \mathbf{e}_i \right)_{(K^2)} = \left(\mathbf{e}_i^\top \otimes \mathbf{I}_K \right) \text{vec}(\mathbf{x}(\mathbf{x} \otimes \mathbf{x})^\top) \\ &= \left(\mathbf{e}_i^\top \otimes \mathbf{I}_K \right) (\mathbf{x} \otimes (\mathbf{x} \otimes \mathbf{x})) = \underline{\left(\mathbf{e}_i^\top \otimes \mathbf{I}_K \right) \mathbf{x}^{\otimes 3}} \end{aligned} \quad (\text{J.19})$$

Subsequently:

$$\Phi_{12,i} = \left(\mathbf{Y}_{:,i} \right)^\top \left(\mathbf{e}_i^\top \otimes \mathbf{I}_K \right) \mathbf{x}^{\otimes 3} = \left[\mathbf{Y}_{i,:}^\top \left(\mathbf{e}_i^\top \otimes \mathbf{I}_K \right) \right] \mathbf{x}^{\otimes 3} \quad (\text{J.20})$$

Recalling equation (J.5), the bracketed term on the left hand side of equation (J.20) can be simplified as:

$$\tilde{\mathbf{y}}_{12}^\top = \sum_{i=1}^{K^2} \left[\mathbf{Y}_{i,:}^\top \left(\mathbf{e}_i^\top \otimes \mathbf{I}_K \right) \right] = \sum_{i=1}^{K^2} \mathbf{Y}_{i,:}^\top \mathbf{E} \mathbf{I}_i = (\text{vec}(\mathbf{Y}))^\top \quad (\text{J.21})$$

Therefore:

$$\Phi_{12} = \mathbf{x}^\top \mathbf{Y} (\mathbf{x} \otimes \mathbf{x}) = \tilde{\mathbf{y}}_{12}^\top \mathbf{x}^{\otimes 3} \quad (\text{J.22})$$

Using equation (J.12r), for the derivative of $\mathbf{x}^{\otimes 2}$ with respect to \mathbf{x} we have:

$$\begin{aligned} \frac{\partial \mathbf{x}^{\otimes 2}}{\partial \mathbf{x}} &= \frac{\partial \mathbf{x} \otimes \mathbf{x}}{\partial \mathbf{x}} = \frac{\partial \mathbf{x}}{\partial \mathbf{x}} \otimes \mathbf{x} + \left(\mathbf{I}_K \otimes \mathbf{U} \right) \left(\frac{\partial \mathbf{x}}{\partial \mathbf{x}} \otimes \mathbf{x} \right) \left(\mathbf{I}_1 \otimes \mathbf{U} \right) \\ &= \text{vec}(\mathbf{I}_K) \otimes \mathbf{x} + \left(\mathbf{I}_K \otimes \mathbf{U} \right) (\text{vec}(\mathbf{I}_K) \otimes \mathbf{x}) \\ &= \left(\mathbf{I}_{K^3} + \mathbf{I}_K \otimes \mathbf{U} \right) (\text{vec}(\mathbf{I}_K) \otimes \mathbf{x}) \\ &= \left(\mathbf{I}_{K^3} + \mathbf{I}_K \otimes \mathbf{U} \right) (\text{vec}(\mathbf{I}_K \otimes \mathbf{x})) \end{aligned} \quad (\text{J.23})$$

Introducing the following alias:

$$\mathbf{W3} = \mathbf{I}_{K^3} + \mathbf{I}_K \otimes \mathbf{U} \quad (\text{J.24})$$

and using equation (J.12r), for the derivative of $\mathbf{x}^{\otimes 3}$ with respect to \mathbf{x} we have:

$$\begin{aligned} \frac{\partial \mathbf{x}^{\otimes 3}}{\partial \mathbf{x}} &= \frac{\partial (\mathbf{x} \otimes \mathbf{x}^{\otimes 2})}{\partial \mathbf{x}} = \frac{\partial \mathbf{x}}{\partial \mathbf{x}} \otimes \mathbf{x}^{\otimes 2} + \left(\mathbf{I}_K \otimes \mathbf{U} \right)_{(K \times K^2)} \left(\frac{\partial \mathbf{x}^{\otimes 2}}{\partial \mathbf{x}} \otimes \mathbf{x} \right) \left(\mathbf{I}_1 \otimes \mathbf{U} \right)_{(1 \times 1)} \\ &= \text{vec}(\mathbf{I}_K) \otimes \mathbf{x}^{\otimes 2} + \left(\mathbf{I}_K \otimes \mathbf{U} \right)_{(K \times K^2)} \left((\mathbf{W} \mathbf{3} \text{vec}(\mathbf{I}_K \otimes \mathbf{x})) \otimes \mathbf{x} \right) \\ &= \text{vec}I_{(K^2 \times K^2)}(\text{vec}(\mathbf{I}_K)) \mathbf{x}^{\otimes 2} + \left(\mathbf{I}_K \otimes \mathbf{U} \right)_{(K \times K^2)} \left((\mathbf{W} \mathbf{3} \text{vec}(\mathbf{I}_K \otimes \mathbf{x})) \otimes \mathbf{x} \right) \end{aligned} \quad (\text{J.25})$$

Recalling (J.12p) and the structure of \mathbf{I}_K that contains only K non-zero entries yields:

$$\begin{aligned} \mathbf{W} \mathbf{3}(\text{vec}(\mathbf{I}_K \otimes \mathbf{x})) &= \text{vec}(\mathbf{W} \mathbf{3}(\text{vec}(\mathbf{I}_K \otimes \mathbf{x}))) = \sum_{c=1}^{K^3} (\mathbf{I}_K \otimes \mathbf{x})_{:,c}^T \otimes \mathbf{W} \mathbf{3}_{:,c} \\ &= \sum_{m=1}^K \sum_{j=1}^K x_j \mathbf{W} \mathbf{3}_{:,j+(m-1)(K+1)K} \end{aligned} \quad (\text{J.26})$$

Introducing $m' \triangleq j + (m-1)(K+1)K$ as an alias and substituting equation (J.26) in (J.54) and considering the fact that x_j is a scalar results:

$$\mathbf{W} \mathbf{3}(\text{vec}(\mathbf{I}_K \otimes \mathbf{x})) \otimes \mathbf{x} = \sum_{m=1}^K \sum_{j=1}^K (\mathbf{W} \mathbf{3}_{:,m'}) \otimes (x_j \mathbf{x}) \quad (\text{J.27})$$

On the other hand:

$$(x_j \mathbf{x}) = \mathbf{E} \mathbf{I}_{(K \times K^2)} \mathbf{x}^{\otimes 2} \quad (\text{J.28})$$

Kronecker product of equation (J.28) can be converted to matrix multiplication using equation (J.6):

$$\sum_{m=1}^K \sum_{j=1}^K (\mathbf{W} \mathbf{3}_{:,m'}) \otimes (x_j \mathbf{x}) = \left(\sum_{m=1}^K \sum_{j=1}^K \text{vec}I_{(K^3 \times K)}(\mathbf{W} \mathbf{3}_{:,m'}) \mathbf{E} \mathbf{I}_{(K \times K^2)} \right) \mathbf{x}^{\otimes 2} \quad (\text{J.29})$$

To simplify equation (J.27) further, first recalling (J.24) we decompose $\mathbf{W} \mathbf{3}_{:,m'}$ back to its components again. Both components are permutation matrix type so at each column they have only a single "non-zero" entry or they are unit vector type.

For the first component, it should be noted that $(\mathbf{I}_{K^3})_{:,m'} = \mathbf{e}_{(m')K^3}$. Therefore the summation within

Defining the following alias:

$$\mathbf{W4} = \begin{pmatrix} \mathbf{I}_K \otimes \mathbf{U} \\ \text{\scriptsize $(K \times K^2)$} \end{pmatrix} \quad (\text{J.39})$$

Subsequently using equations (J.31),(J.34) and (J.38) yields:

$$\frac{\partial \mathbf{x}^{\otimes 3}}{\partial \mathbf{x}} \triangleq \mathbf{V}_{12} \mathbf{x}^{\otimes 2} = (\mathbf{V14} + \mathbf{W4}(\mathbf{U14} + \mathbf{V14})) \mathbf{x}^{\otimes 2} \quad (\text{J.40})$$

Returning to equation (J.22) and using (J.12q):

$$\begin{aligned} \frac{\partial \Phi_{12}}{\partial \mathbf{x}} &= \frac{\partial (\tilde{\mathbf{y}}_{12}^\top \mathbf{x}^{\otimes 3})}{\partial \mathbf{x}} = \frac{\partial \tilde{\mathbf{y}}_{12}^\top}{\partial \mathbf{x}} (\mathbf{I}_1 \otimes \mathbf{x}^{\otimes 3}) + (\mathbf{I}_K \otimes \tilde{\mathbf{y}}_{12}^\top) \left(\frac{\partial \mathbf{x}^{\otimes 3}}{\partial \mathbf{x}} \right) \\ &= 0 + (\mathbf{I}_K \otimes \tilde{\mathbf{y}}_{12}^\top) (\mathbf{V}_{12}) \mathbf{x}^{\otimes 2} \end{aligned} \quad (\text{J.41})$$

Defining the following alias:

$$\tilde{\mathbf{Z}}_{12} \triangleq \mathbf{I}_K \otimes \tilde{\mathbf{y}}_{12}^\top = \begin{bmatrix} \tilde{\mathbf{y}}_{12}^\top & \mathbf{0}_{(1 \times K^3)} & \mathbf{0}_{(1 \times (K^4 - 2K^3))} \\ \mathbf{0}_{(1 \times K^3)} & \tilde{\mathbf{y}}_{12}^\top & \mathbf{0}_{(1 \times (K^4 - 2K^3))} \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{(1 \times (K^4 - 2K^3))} & \mathbf{0}_{(1 \times K^4)} & \tilde{\mathbf{y}}_{12}^\top \end{bmatrix} \quad (\text{J.42a})$$

The differentiation of equation (J.59) can be written as:

$$\frac{\partial \Phi_{12}}{\partial \mathbf{x}} \triangleq \tilde{\mathbf{K}}_{12} \mathbf{x}^{\otimes 2} = (\tilde{\mathbf{Z}}_{12} \mathbf{V}_{12}) \mathbf{x}^{\otimes 2} \quad (\text{J.43})$$

J.4 Nonlinear-Linear Component

The nonlinear-linear component of energy has the following form:

$$\Phi_{21} = (\mathbf{x} \otimes \mathbf{x})^\top \mathbf{Y} \mathbf{x} \quad (\text{J.44})$$

where the dimensions are: $(\mathbf{x} \otimes \mathbf{x})_{1 \times K^2}^\top$, $\mathbf{x}_{K \times 1}$ and $\mathbf{Y}_{K^2 \times K}$. Recalling that since this a scalar and inner products of participating vectors are commutative yields:

$$\Phi_{21} = (\mathbf{x} \otimes \mathbf{x})^\top \mathbf{Y} \mathbf{x} = (\mathbf{Y} \mathbf{x})^\top (\mathbf{x} \otimes \mathbf{x}) = \mathbf{x}^\top \mathbf{Y}^\top (\mathbf{x} \otimes \mathbf{x}) \quad (\text{J.45})$$

Equation (J.45) has the same form as equation (J.16). Therefore by introducing $\mathbf{Y}\mathbf{T} = \mathbf{Y}^\top$ the derivative can be written as:

$$\frac{\partial \Phi_{21}}{\partial \mathbf{x}} \triangleq \tilde{\mathbf{K}}_{21} \mathbf{x}^{\otimes 2} = (\tilde{\mathbf{Z}}_{21} \mathbf{V}_{12}) \mathbf{x}^{\otimes 3} \quad (\text{J.46})$$

where $\tilde{\mathbf{Z}}_{21}$ can be obtained by substituting \mathbf{Y}_{21}^\top in place of \mathbf{Y} in equations (J.21) and (J.42a).

J.5 Nonlinear-Nonlinear Component

The nonlinear-linear component of energy has the following form:

$$\Phi_{22} = (\mathbf{x} \otimes \mathbf{x})^\top \mathbf{Y} (\mathbf{x} \otimes \mathbf{x}) \quad (\text{J.47})$$

where the dimensions are: $(\mathbf{x} \otimes \mathbf{x})_{1 \times K^2}^\top$, $(\mathbf{x} \otimes \mathbf{x})_{K^2 \times 1}$ and $\mathbf{Y}_{K^2 \times K^2}$. At the first step we vectorize Φ_{22} to the following series:

$$\Phi_{22} = \sum_{i=1}^{K^2} \Phi_{22,i} \triangleq \sum_{i=1}^{K^2} \overbrace{((\mathbf{x} \otimes \mathbf{x})^\top \mathbf{Y}_{:,i})}^{\triangleq f_{1i}} \overbrace{\left(\mathbf{e}_{(K^2)}^\top (\mathbf{x} \otimes \mathbf{x}) \right)}^{\triangleq f_{2i}} \quad (\text{J.48})$$

It should be noted that both f_{1i} and f_{2i} have scalar type, therefore both inner products are commutative or:

$$\left((\mathbf{x} \otimes \mathbf{x})^\top \mathbf{Y}_{:,i} \right) \left(\mathbf{e}_{(K^2)}^\top (\mathbf{x} \otimes \mathbf{x}) \right) = \left((\mathbf{Y}_{:,i})^\top (\mathbf{x} \otimes \mathbf{x}) \right) \left((\mathbf{x} \otimes \mathbf{x})^\top \mathbf{e}_{(K^2)} \right) = (\mathbf{Y}_{:,i})^\top \left((\mathbf{x} \otimes \mathbf{x}) (\mathbf{x} \otimes \mathbf{x})^\top \mathbf{e}_{(K^2)} \right) \quad (\text{J.49})$$

The second term on the right hand side of equation (J.49) is a vector by size of $K^2 \times 1$, therefore employing equation (J.12o) yields:

$$\begin{aligned} \left((\mathbf{x} \otimes \mathbf{x}) (\mathbf{x} \otimes \mathbf{x})^\top \mathbf{e}_{(K^2)} \right) &= \text{vec} \left((\mathbf{x} \otimes \mathbf{x}) (\mathbf{x} \otimes \mathbf{x})^\top \mathbf{e}_{(K^2)} \right) = \left(\mathbf{e}_{(K^2)}^\top \otimes \mathbf{I}_{K^2} \right) \text{vec} \left((\mathbf{x} \otimes \mathbf{x}) (\mathbf{x} \otimes \mathbf{x})^\top \right) \\ &= \left(\mathbf{e}_{(K^2)}^\top \otimes \mathbf{I}_{K^2} \right) \left((\mathbf{x} \otimes \mathbf{x}) \otimes (\mathbf{x} \otimes \mathbf{x}) \right) = \underline{\left(\mathbf{e}_{(K^2)}^\top \otimes \mathbf{I}_{K^2} \right) \mathbf{x}^{\otimes 4}} \end{aligned} \quad (\text{J.50})$$

Subsequently:

$$\Phi_{22,i} = (\mathbf{Y}_{:,i})^\top \left(\mathbf{e}_{(K^2)}^\top \otimes \mathbf{I}_{K^2} \right) \mathbf{x}^{\otimes 4} = \left[\mathbf{Y}_{i,:}^\top \left(\mathbf{e}_{(K^2)}^\top \otimes \mathbf{I}_{K^2} \right) \right] \mathbf{x}^{\otimes 4} \quad (\text{J.51})$$

Recalling equation (J.5), the bracketed term on the left hand side of equation (J.51) can be simplified as:

$$\hat{y}_{22}^\top = \sum_{i=1}^{K^2} \left[\mathbf{Y}_{i,:}^\top \left(\mathbf{e}_{(K^2)}^\top \otimes \mathbf{I}_{K^2} \right) \right] = \sum_{i=1}^{K^2} \mathbf{Y}_{i,:}^\top \mathbf{E} \mathbf{I}_i = (\text{vec}(\mathbf{Y}))^\top \quad (\text{J.52})$$

Introducing the following alias:

$$\mathbf{W5} = \mathbf{I}_{K^5} + \mathbf{I}_K \otimes_{(K^2 \times K^2)} \mathbf{U} \quad (\text{J.53})$$

and using equation (J.12r), for $\mathbf{x}^{\otimes 4}$ we have:

$$\frac{\partial \mathbf{x}^{\otimes 4}}{\partial \mathbf{x}} = \mathbf{W5} \left[(\mathbf{W3}(\text{vec}(\mathbf{I}_K) \otimes \mathbf{x})) \otimes \mathbf{x}^{\otimes 2} \right] = \mathbf{W5} \left[(\mathbf{W3}(\text{vec}(\mathbf{I}_K \otimes \mathbf{x}))) \otimes \mathbf{x}^{\otimes 2} \right] \quad (\text{J.54})$$

Taking similar approach to what presented in section J.3, yields:

$$\begin{aligned} \left[(\mathbf{W3}(\text{vec}(\mathbf{I}_K) \otimes \mathbf{x})) \otimes \mathbf{x}^{\otimes 2} \right] &= \left[(\mathbf{W3}(\text{vec}(\mathbf{I}_K \otimes \mathbf{x}))) \otimes \mathbf{x}^{\otimes 2} \right] = \sum_{m=1}^K \sum_{j=1}^K (\mathbf{W3}_{:,m'}) \otimes (x_j \mathbf{x}^{\otimes 2}) \\ &= \left(\sum_{m=1}^K \sum_{j=1}^K \text{vec} \mathbf{I}_{(K^3 \times K^2)} (\mathbf{W3}_{:,m'}) \mathbf{E} \mathbf{I}_j \right) \mathbf{x}^{\otimes 3} \end{aligned} \quad (\text{J.55})$$

Again decomposing $\mathbf{W3}$ back to its components and after some mathematical manipulations and considering the location of "1" entries in each column of \mathbf{I}_{K^3} for the first component we have:

$$\mathbf{V} \mathbf{I5}_{(K^5 \times K^3)} \triangleq \sum_{m=1}^K \sum_{j=1}^K \text{vec} \mathbf{I}_{(K^3 \times K^2)} \left((\mathbf{I}_{K^3})_{:,m'} \right) \mathbf{E} \mathbf{I}_j = \begin{matrix} & & & & 1 & \dots & & K \\ & & & & 1 & & & \mathbf{I}_{K^2} & \mathbf{0}_{K^2} & \dots & \mathbf{0}_{K^2} \\ & & & & 2 & & & \mathbf{0}_{K^2} & \mathbf{I}_{K^2} & \dots & \mathbf{0}_{K^2} \\ & & & & \vdots & & & \vdots & \vdots & \ddots & \vdots \\ & & & & K & & & \mathbf{0}_{K^2} & \mathbf{0}_{K^2} & \dots & \mathbf{I}_{K^2} \\ & & & & K+1 & & & \mathbf{0}_{K^2} & \mathbf{0}_{K^2} & \dots & \mathbf{0}_{K^2} \\ & & & & \vdots & & & \vdots & \vdots & \ddots & \vdots \\ & & & & K(K+1)+1 & & & \mathbf{I}_{K^2} & \mathbf{0}_{K^2} & \dots & \mathbf{0}_{K^2} \\ & & & & K(K+1)+2 & & & \mathbf{0}_{K^2} & \mathbf{I}_{K^2} & \dots & \mathbf{0}_{K^2} \\ & & & & \vdots & & & \vdots & \vdots & \ddots & \vdots \\ & & & & K(K+1)+K & & & \mathbf{0}_{K^2} & \mathbf{0}_{K^2} & \dots & \mathbf{I}_{K^2} \\ & & & & \vdots & & & \vdots & \vdots & \ddots & \vdots \\ & & & & K^3 & & & \mathbf{0}_{K^2} & \mathbf{0}_{K^2} & \dots & \mathbf{I}_{K^2} \end{matrix} \quad (\text{J.56})$$

The differentiation of equation (J.59) can be written as:

$$\frac{\partial \Phi_{22}}{\partial \mathbf{x}} = \tilde{\mathbf{K}}_{22} \mathbf{x}^{\otimes 3} \triangleq (\tilde{\mathbf{Z}}_{22} \mathbf{V}_{22}) \mathbf{x}^{\otimes 3} \quad (\text{J.61})$$