Titre: Title:	Hybrid finite element method applied to the analysis of free vibration of spherical shell
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Date:	2013
Type:	Rapport / Report
Référence: Citation:	Menaa, M. & Lakis, A. (2013). *Hybrid finite element method applied to the analysis of free vibration of spherical shell* (Rapport technique n° EPM-RT-2013-02).

EPM–RT–2013-02

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Avril 2013

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Abstract

In this study, free vibration analysis of spherical shell is carried out. The structural model is based on a combination of thin shell theory and the classical finite element method. Free vibration equations using the hybrid finite element formulation are derived and solved numerically. The results are validated using numerical and theoretical data available in the literature. The analysis is accomplished for spherical shells of different boundary conditions and radius to thickness ratios. This proposed hybrid finite element method can be used efficiently for design and analysis of spherical shells employed in high speed aircraft structures.

1. Introduction

Shells of revolution, particularly spherical shells are one of the primary structural elements in high speed aircraft. Their applications include the propellant tank or gas-deployed skirt of space crafts. Free vibration of spherical shell has been investigated by numerous researchers experimentally and analytically.

Kalnins [1,2], studying analytically free vibrations in shallow spherical shell, selected used two auxiliary variables for the axial and circumferential displacements while considering the effect of longitudinal, transverse and rotary inertia as well as transverse shear deformation on the non-asymmetric vibration of shallow spherical shells. Navaratna [4], Webster [5], Greene et al. [7] used the classical finite element method to study the free vibration of thin spherical shell. Cohen[3] using a method of iteration like Stodola's method determined the natural frequencies and mode shapes of spherical shell method. Kraus [6] investigated the case of clamped spherical shell using a general theory which included the effects of transverse shear stress and rotational inertia. Tessler and Spiridigliozzi [8] gave frequencies of 60° clamped spherical shell and hemispherical shell for radius to thickness from 10 to 100 and their analysis was based upon shell theory. Narasimhan and Alwan [9] analyzed the axisymmetric free vibration of clamped isotropic spherical shell cap. Thick shell analysis was given by Gautham and Ganesan [10] for the analysis of a 60° clamped and simply supported spherical shells, the semi-analytical method was used to reduce the dimension of the problem. The same authors [11] investigated the analysis of a clamped isotropic hemispherical shell (ϕ_0 =90°**).** Sai Ram and Sridhar Babu [12] used the classical finite elements method to study the free vibration of composite spherical shell cap with or without a cutout. Buchanan and Rich [13] investigated the case of 60° clamped and simply supported spherical shells using classical finite elements method. Recently, Ventsel et al. [14] used a combined formulation of the boundary elements method and finite elements method to study the free vibration of an isotropic simply supported hemispherical shell with different circumferential mode numbers.

The objective of the present study is to develop a general hybrid finite element package for predicting the dynamic behavior of isotropic spherical shells with boundary conditions which can be varied as desired. The solution scheme is based on the hybrid finite element method. This method uses displacements functions derived from the shell theory instead of polynomials in classical finite element method. The element is a spherical frustum instead of the usual triangular or rectangular shell element. This developed method demonstrated precise and fast convergence with few elements. On the other hand, the present theory, because of its usage of shell classical theory for the displacement functions can easily be adapted to take the hydrodynamic effects into account. Finally, again because of the use of shell classical theory, we can obtain the high as well as the low frequencies with high accuracy.

2. Finite element formulation

In this study the structure is modeled using hybrid finite element method which is a combination of spherical shell theory and classical finite element method. In this hybrid finite element method, the displacement functions are found from exact solution of spherical shell theory rather approximated by polynomial functions done in classical finite element method. In the spherical coordinate system(R,θ,ϕ) shown in Fig. 1, five out of the six equations of equilibrium derived in reference for spherical shells under external load are written as follows :

$$
\begin{aligned}
&\frac{\partial N_\phi}{\partial \phi}+\frac{1}{\sin\phi}\frac{\partial N_{\phi\theta}}{\partial \theta}+\left(N_\phi-N_\theta\right)\cot\phi+Q_\phi=0\\
&\frac{\partial N_{\phi\theta}}{\partial \phi}+\frac{1}{\sin\phi}\frac{\partial N_\theta}{\partial \theta}+2N_{\phi\theta}\cot\phi+Q_\theta=0\\
&\frac{\partial Q_\phi}{\partial \phi}+\frac{1}{\sin\phi}\frac{\partial Q_\theta}{\partial \theta}+Q_\phi\cot\phi-(N_\phi+N_\theta)=0\\
&\frac{\partial M_\phi}{\partial \phi}+\frac{1}{\sin\phi}\frac{\partial M_{\phi\theta}}{\partial \theta}+\left(M_\phi-M_\theta\right)\cot\phi-RQ_\phi=0\\
&\frac{\partial M_{\phi\theta}}{\partial \phi}+\frac{1}{\sin\phi}\frac{\partial M_\theta}{\partial \theta}+2M_{\phi\theta}\cot\phi-RQ_\theta=0
\end{aligned}
\tag{1}
$$

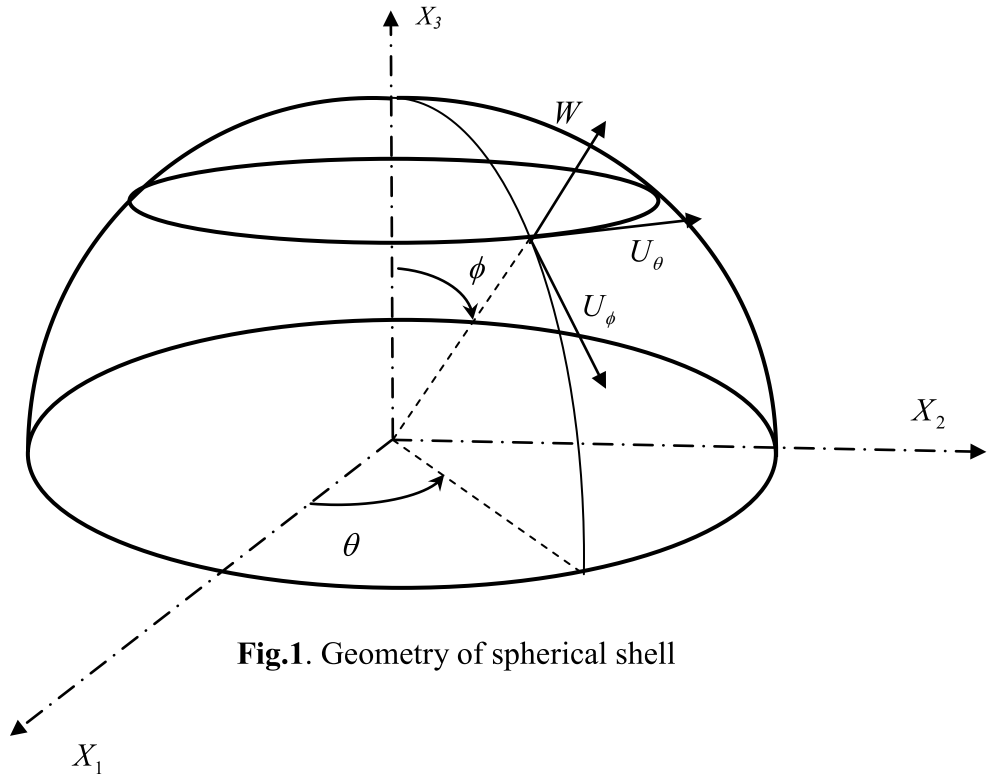

Fig.1. Geometry of spherical shell

Where N_ϕ, N_θ, $N_{\phi\theta}$ are membrane stress resultants; M_ϕ, M_θ, $M_{\phi\theta}$ the bending stress resultants and Q_ϕ, Q_θ the shear forces (Fig. 2). The sixth equation, which is an identity equation for spherical shells, is not presented here.

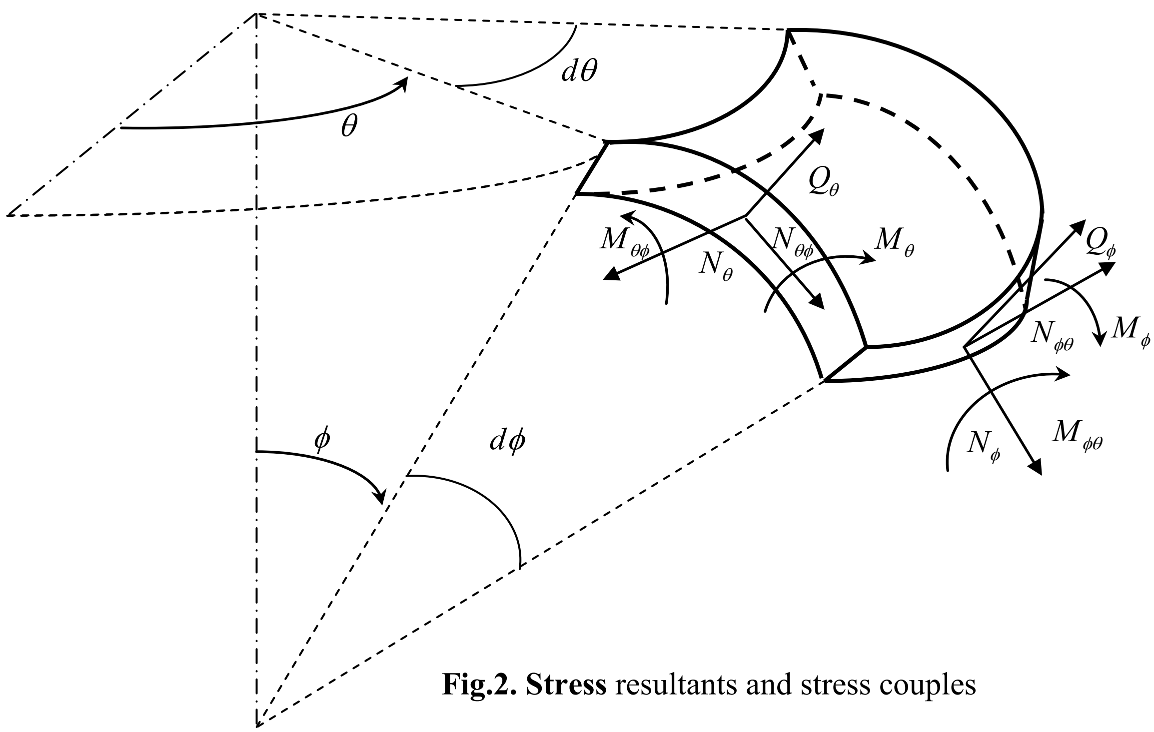

Fig.2. Stress resultants and stress couples

Strain and displacements for three displacements in axial U_ϕ, radial W and circumferential U_θ are related as follows:

$$\{\varepsilon\}=\begin{Bmatrix}\varepsilon_\phi\\ \varepsilon_\theta\\ 2\varepsilon_{\phi\theta}\\ \kappa_\phi\\ \kappa_\theta\\ 2\kappa_{\phi\theta}\end{Bmatrix}=\begin{Bmatrix}\frac{1}{R}\left(\frac{\partial U_\phi}{\partial\phi}+W\right)\\ \frac{1}{R}(\frac{1}{\sin\phi}\frac{\partial U_\theta}{\partial\theta}+U_\phi\cot\phi+W)\\ \frac{1}{R}(\frac{\partial U_\theta}{\partial\phi}+\frac{1}{\sin\phi}\frac{\partial U_\phi}{\partial\theta}-U_\theta\cot\phi)\\ \frac{1}{R^2}\left(\frac{\partial U_\phi}{\partial\phi}-\frac{\partial^2 W}{\partial\phi^2}\right)\\ \frac{1}{R^2}\left(\frac{1}{\sin\phi}\frac{\partial U_\theta}{\partial\theta}+U_\phi\cot\phi-\frac{1}{\sin^2\phi}\frac{\partial^2 W}{\partial\theta^2}-\cot\phi\frac{\partial W}{\partial\phi}\right)\\ \frac{1}{R^2}\left(\frac{\partial U_\theta}{\partial\phi}+\frac{1}{\sin\phi}\frac{\partial U_\phi}{\partial\theta}-U_\theta\cot\phi+2\frac{1}{\sin\phi}\cot\phi\frac{\partial W}{\partial\theta}-2\frac{1}{\sin\phi}\frac{\partial^2 W}{\partial\phi\partial\theta}\right)\end{Bmatrix} \quad (2)$$

Displacements U, W and V in the global cartesian coordinate system are related to displacements $U_{\phi i}$, W_i and $U_{\theta i}$ indicated in Fig 3. by:

$$\begin{Bmatrix}U\\ W\\ V\end{Bmatrix}=\begin{bmatrix}\sin\phi_i & -\cos\phi_i & 0\\ \cos\phi_i & \sin\phi_i & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}U_{\phi i}\\ W_i\\ U_{\theta i}\end{Bmatrix} \quad (3)$$

The stress vector $\{\sigma\}$ is expressed as function of strain $\{\varepsilon\}$ by

$$\{\sigma\}=[P]\{\varepsilon\} \quad (4)$$

Where $[P]$ is the elasticity matrix for an anisotropic shell given by

$$[P]=\begin{bmatrix}P_{11} & P_{12} & 0 & P_{14} & P_{15} & 0\\ P_{21} & P_{22} & 0 & P_{24} & P_{25} & 0\\ 0 & 0 & P_{33} & 0 & 0 & 0\\ P_{41} & P_{42} & 0 & P_{44} & P_{45} & 0\\ P_{51} & P_{52} & 0 & P_{54} & P_{55} & 0\\ 0 & 0 & P_{36} & 0 & 0 & P_{66}\end{bmatrix} \quad (5)$$

Upon substitution of equations (2), (4) and (5) into equations (1), a system of equilibrium equations can be obtained as a function of displacements:

$$
\begin{aligned}
&L_1\left(U_\phi, W, U_\theta, P_{ij}\right)=0 \\
&L_2\left(U_\phi, W, U_\theta, P_{ij}\right)=0 \\
&L_3\left(U_\phi, W, U_\theta, P_{ij}\right)=0
\end{aligned}
\tag{6}
$$

These three linear partial differentials operators L_1, L_2 and L_3 are given in the Appendix, and P_{ij} are elements of the elasticity matrix which, for an isotopic thin shell with thickness *h* is given by:

$$
[P]=\begin{bmatrix}
D & \nu D & 0 & 0 & 0 & 0 \\
\nu D & D & 0 & 0 & 0 & 0 \\
0 & 0 & \dfrac{(1-\nu)D}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & K & \nu K & 0 \\
0 & 0 & 0 & \nu K & K & 0 \\
0 & 0 & 0 & 0 & 0 & \dfrac{(1-\nu)K}{2}
\end{bmatrix}
\tag{7}
$$

Where $D=\dfrac{Et}{1-\nu^2}$ is the membrane stiffness and $K=\dfrac{Et^3}{12\left(1-\nu^2\right)}$ is the bending stiffness.

The element is a circumferential spherical frustum shown in Fig. 3. It has two nodal circles with four degrees of freedom; axial, radial, circumferential and rotation at each node. This element type makes it possible to use thin shell equations easily to find the exact solution of displacement functions rather than an approximation with polynomial functions as done in classical finite element method. For motions associated with the *n*th circumferential wave number we may write:

$$
\begin{Bmatrix} U_\phi(\phi,\theta) \\ W(\phi,\theta) \\ U_\theta(\phi,\theta) \end{Bmatrix}
=\begin{bmatrix} \cos n\theta & 0 & 0 \\ 0 & \cos n\theta & 0 \\ 0 & 0 & \sin n\theta \end{bmatrix}
\begin{Bmatrix} u_{\phi n}(\phi) \\ w_n(\phi) \\ u_{\theta n}(\phi) \end{Bmatrix}
=[T]\begin{Bmatrix} u_{\phi n}(\phi) \\ w_n(\phi) \\ u_{\theta n}(\phi) \end{Bmatrix}
\tag{8}
$$

The transversal displacement $w_n(\phi)$ can be expressed as:

$$
w_n(\phi)=\sum_{i=1}^{3} w_i^n \tag{9}
$$

Where

$$
w_i^n = A_i P_{\mu_i}^n(\cos\phi)+B_i Q_{\mu_i}^n(\cos\phi) \tag{10}
$$

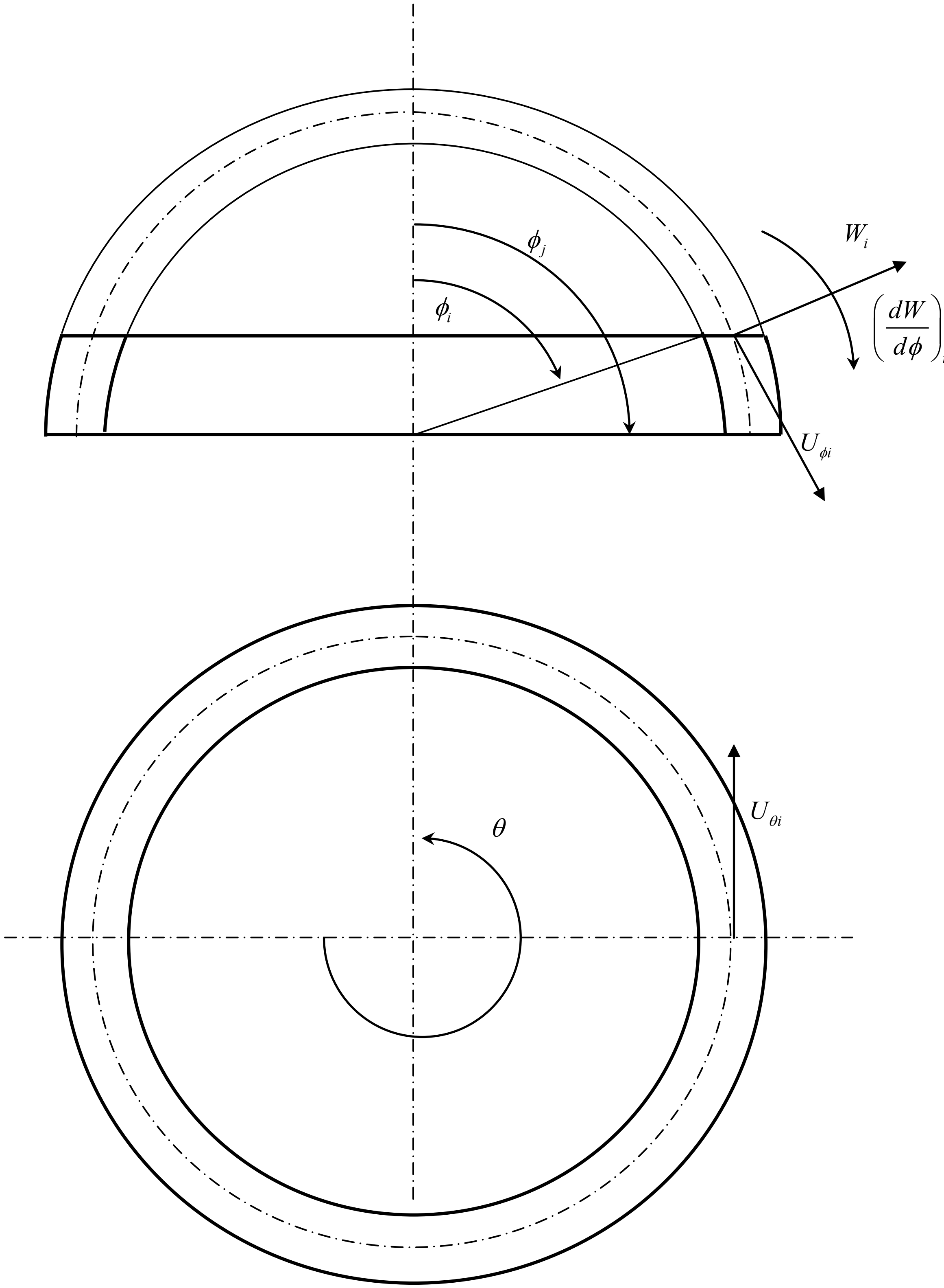

Fig. 3. Spherical frustum element

And where $P_{\mu_i}^{n}\left(\cos\phi\right), Q_{\mu_i}^{n}\left(\cos\phi\right)$ are the associated Legendre functions of the first and second kinds respectively of order n and degree μ_i.

The expression of the axial displacement $u_{\phi n}$ (ϕ) is:

$$u_{\phi n}\left(\phi\right)=\sum_{i=1}^{3}E_i\frac{dw_i^n}{d\phi}-\frac{n^2}{2\sin\phi}\psi\left(\phi\right) \tag{11}$$

Where the coefficient E_i is given by:

$$E_i=\frac{\lambda_i+k(1+\nu)-(1-\nu)}{\left(1+k\right)\left(\lambda_i-1+\nu\right)} \tag{12}$$

The auxiliary function ψ is given by the expression:

$$\psi\left(\phi\right)=A_4P_1^n\left(\cos\phi\right)+B_4Q_1^n\left(\cos\phi\right) \tag{13}$$

Finally the circumferential displacement $u_{\theta n}$ (ϕ) can be expressed as:

$$u_{\theta n}\left(\phi\right)=-n\sum_{i=1}^{3}\frac{1}{\sin\phi}E_iw_i^n+\frac{n}{2}\frac{d\psi}{d\phi} \tag{14}$$

The degree μ_i is obtained from the expression

$$\mu_i=\left(\frac{1}{4}+\lambda_i\right)^{1/2}-\frac{1}{2} \tag{15}$$

Where λ_i is one the roots of the cubic equation:

$$\lambda^3-h_1\lambda^2+h_2\lambda-h_3=0 \tag{16}$$

Where

$$\begin{aligned}h_1&=4\\h_2&=4+(1+k)(1-\nu^2)\\h_3&=2(1+k)(1-\nu^2)\end{aligned} \tag{17}$$

With $k=12\dfrac{R^2}{h^2}$

The above equation has three roots with one root is real and two other are complex conjugate roots.

The Legendre functions $P_{\mu_1}^{n}$, $P_{\mu_1}^{n-1}, Q_{\mu_1}^{n}$ and $Q_{\mu_1}^{n-1}$ are a real functions whereas $P_{\mu_i}^{n}$, $P_{\mu_i}^{n-1}, Q_{\mu_i}^{n}$ and $Q_{\mu_i}^{n-1}$ (i = 2, 3) are complex functions so we can put:

$$\begin{aligned}
P_{\mu_2}^{n} &= \operatorname{Re}(P_{\mu_2}^{n}) + i\operatorname{Im}(P_{\mu_2}^{n}) \\
P_{\mu_3}^{n} &= \operatorname{Re}(P_{\mu_2}^{n}) - i\operatorname{Im}(P_{\mu_2}^{n}) \\
Q_{\mu_2}^{n} &= \operatorname{Re}(Q_{\mu_2}^{n}) + i\operatorname{Im}(Q_{\mu_2}^{n}) \\
Q_{\mu_3}^{n} &= \operatorname{Re}(Q_{\mu_2}^{n}) - i\operatorname{Im}(Q_{\mu_2}^{n}) \\
P_{\mu_2}^{n-1} &= \operatorname{Re}(P_{\mu_2}^{n-1}) + i\operatorname{Im}(P_{\mu_2}^{n-1}) \\
P_{\mu_3}^{n-1} &= \operatorname{Re}(P_{\mu_2}^{n-1}) - i\operatorname{Im}(P_{\mu_2}^{n-1}) \\
Q_{\mu_2}^{n-1} &= \operatorname{Re}(Q_{\mu_2}^{n-1}) + i\operatorname{Im}(Q_{\mu_2}^{n-1}) \\
Q_{\mu_3}^{n-1} &= \operatorname{Re}(P_{\mu_2}^{n-1}) - i\operatorname{Im}(Q_{\mu_2}^{n-1})
\end{aligned} \tag{18}$$

Setting

$$\begin{aligned}
(n-\mu_1-1)\left(n+\mu_1\right) &= c_1 \\
(n-\mu_2-1)\left(n+\mu_2\right) &= c_2 + ic_3 \\
(n-\mu_3-1)\left(n+\mu_3\right) &= c_2 - ic_3
\end{aligned} \tag{19}$$

$$\begin{aligned}
E_1 &= e_1 \\
E_2 &= e_2 - ie_3 \\
E_3 &= e_2 + ie_3
\end{aligned} \tag{20}$$

Substituting equations (18), (19) and (20) in equations (9), (11) and (14) we have:

$$\begin{aligned}
u_{n\phi}(\phi) = {} & \left(-ne_1\cot\phi P_{\mu_1}^{n} + e_1c_1P_{\mu_1}^{n-1}\right)A_1 \\
& + \left[-ne_2\cot\phi\operatorname{Re}(P_{\mu_2}^{n}) - ne_3\cot\phi\operatorname{Im}(P_{\mu_2}^{n}) + (e_2c_2+e_3c_3)\operatorname{Re}(P_{\mu_2}^{n-1}) + (e_3c_2-e_2c_3)\operatorname{Im}(P_{\mu_2}^{n-1})\right](A_2+A_3) \\
& + \left[\ ne_3\cot\phi\operatorname{Re}(P_{\mu_2}^{n}) - ne_2\cot\phi\operatorname{Im}(P_{\mu_2}^{n}) - (e_3c_2-e_2c_3)\operatorname{Re}(P_{\mu_2}^{n-1}) + (e_2c_2+e_3c_3)\operatorname{Im}(P_{\mu_2}^{n-1})\right]i(A_2-A_3) \\
& + \left[-\frac{n^2}{2\sin\phi}P_1^{n}\right]A_4 \\
& + \left(-ne_1\cot\phi Q_{\mu_1}^{n} + e_1c_1Q_{\mu_1}^{n-1}\right)B_1 \\
& + \left[-ne_2\cot\phi\operatorname{Re}(Q_{\mu_2}^{n}) - ne_3\cot\phi\operatorname{Im}(Q_{\mu_2}^{n}) + (e_2c_2+e_3c_3)\operatorname{Re}(Q_{\mu_2}^{n-1}) + (e_3c_2-e_2c_3)\operatorname{Im}(Q_{\mu_2}^{n-1})\right](B_2+B_3) \\
& + \left[\ ne_3\cot\phi\operatorname{Re}(Q_{\mu_2}^{n}) - ne_2\cot\phi\operatorname{Im}(Q_{\mu_2}^{n}) - (e_3c_2-e_2c_3)\operatorname{Re}(Q_{\mu_2}^{n-1}) + (e_2c_2+e_3c_3)\operatorname{Im}(Q_{\mu_2}^{n-1})\right]i(B_2-B_3) \\
& + \left[-\frac{n^2}{2\sin\phi}Q_1^{n}\right]B_4
\end{aligned}$$

$$w_n(\phi) = P^n_{\mu_1} A_1 + \mathrm{Re}(P^n_{\mu_2})(A_2 + A_3) + \mathrm{Im}(P^n_{\mu_2}) i(A_2 - A_3) + Q^n_{\mu_1} B_1 + \mathrm{Re}(Q^n_{\mu_2})(B_2 + B_3) + \mathrm{Im}(Q^n_{\mu_2}) i(B_2 - B_3)$$

$$\begin{aligned}
u_{n\theta}(\phi) = & -ne_1 P^n_{\mu_1} \frac{1}{\sin\phi} A_1 \\
& - \left[ne_2 \frac{1}{\sin\phi} \mathrm{Re}(P^n_{\mu_2}) + ne_3 \frac{1}{\sin\phi} \mathrm{Im}(P^n_{\mu_2}) \right](A_2 + A_3) + \left[ne_3 \frac{1}{\sin\phi} \mathrm{Re}(P^n_{\mu_2}) - ne_2 \frac{1}{\sin\phi} \mathrm{Im}(P^n_{\mu_2}) \right] i(A_2 - A_3) \\
& + \left[-\frac{n^2}{2} \cot\phi P^n_1 + \frac{n}{2}(n-2)(n+1) P^{n-1}_1 \right] A_4 \\
& - ne_1 Q^n_{\mu_1} \frac{1}{\sin\phi} B_1 \\
& - \left[ne_2 \frac{1}{\sin\phi} \mathrm{Re}(Q^n_{\mu_2}) + ne_3 \frac{1}{\sin\phi} \mathrm{Im}(Q^n_{\mu_2}) \right](B_2 + B_3) + \left[ne_3 \frac{1}{\sin\phi} \mathrm{Re}(Q^n_{\mu_2}) - ne_2 \frac{1}{\sin\phi} \mathrm{Im}(Q^n_{\mu_2}) \right] i(B_2 - B_3) \\
& + \left[-\frac{n^2}{2} \cot\phi Q^n_1 + \frac{n}{2}(n-2)(n+1) Q^{n-1}_1 \right] B_4
\end{aligned} \tag{21}$$

In deriving the above relation we used the recursive relations:

$$\begin{aligned}
\frac{dP^n_{\mu_i}}{d\phi} &= -n \cot\phi P^n_{\mu_i} + (n - \mu_i - 1)(n + \mu_i) P^{n-1}_{\mu_i} \\
\frac{dQ^n_{\mu_i}}{d\phi} &= -n \cot\phi Q^n_{\mu_i} + (n - \mu_i - 1)(n + \mu_i) Q^{n-1}_{\mu_i}
\end{aligned} \tag{22}$$

Using matrix formulation, the displacement functions can be expressed as follows:

$$\begin{Bmatrix} U_\phi(\phi,\theta) \\ W(\phi,\theta) \\ U_\theta(\phi,\theta) \end{Bmatrix} = [T] \begin{Bmatrix} u_{\phi n}(\phi) \\ w_n(\phi) \\ u_{\theta n}(\phi) \end{Bmatrix} = [T][R]\{C\} \tag{23}$$

The vector $\{C\}$ is given by the expression:

$$\{C\}^T = \{A_1 \quad A_2 + A_3 \quad i(A_2 - A_3) \quad A_4 \quad B_1 \quad B_2 + B_3 \quad i(B_2 - B_3) \quad B_4\} \tag{24}$$

The elements of matrix $[R]$ are given in the Appendix.

In the finite element method, the vector C is eliminated in favor of displacements at elements nodes. At each finite element node, the three displacements (axial, transversal and circumferential) and the rotation are applied. The displacement of node *i* are defined by the vector:

$$\{\delta_i\} = \left\{ u^i_{\phi n} \quad w^i_n \quad \left(\frac{dw_n}{dx}\right)^i \quad u^i_{\theta n} \right\}^T \tag{25}$$

The element in Fig. 3 with two nodal lines (*i* and *j*) and eight degrees of freedom will have the following nodal displacement vector:

$$\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = \left\{ u^i_{\phi n} \quad w^i_n \quad \left(\frac{dw_n}{d\phi}\right)^i \quad u^i_{\theta n} \quad u^j_{\phi n} \quad w^j_n \quad \left(\frac{dw_n}{d\phi}\right)^j \quad u^j_{\theta n} \right\}^T = [A]\{C\} \tag{26}$$

With

$$\begin{aligned}
\frac{dw_n}{d\phi} &= \left(-n\cot\phi P^n_{\mu_1} + c_1 P^{n-1}_{\mu_1}\right)A_1 + \left[-n\cot\phi \operatorname{Re}(P^n_{\mu_2}) + c_2 \operatorname{Re}(P^{n-1}_{\mu_2}) - c_3 \operatorname{Im}(P^{n-1}_{\mu_2})\right](A_2 + A_3) \\
&+ \left[-n\cot\phi \operatorname{Im}(P^n_{\mu_2}) + c_3 \operatorname{Re}(P^{n-1}_{\mu_2}) + c_2 \operatorname{Im}(P^{n-1}_{\mu_2})\right] i(A_2 - A_3) + \left(-n\cot\phi Q^n_{\mu_1} + c_1 Q^{n-1}_{\mu_1}\right)B_1 \\
&+ \left[-n\cot\phi \operatorname{Re}(Q^n_{\mu_2}) + c_2 \operatorname{Re}(Q^{n-1}_{\mu_2}) - c_3 \operatorname{Im}(Q^{n-1}_{\mu_2})\right](B_2 + B_3) \\
&+ \left[-n\cot\phi \operatorname{Im}(Q^n_{\mu_2}) + c_3 \operatorname{Re}(Q^{n-1}_{\mu_2}) + c_2 \operatorname{Im}(Q^{n-1}_{\mu_2})\right] i(B_2 - B_3)
\end{aligned} \tag{27}$$

The terms of matrix $[A]$ are obtained from the values of matrix $[R]$ and $\frac{dw_n}{dx}$ are given in the appendix.

Now, pre-multiplying by $[A]^{-1}$ equation (26) one obtains the matrix of the constant C_i as a function of the degree of freedom:

$$\{C\} = [A]^{-1} \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{28}$$

Finally, one substitutes the vector $\{C\}$ into equation and obtains the displacement functions as follows:

$$\begin{Bmatrix} U_\phi(\phi,\theta) \\ W(\phi,\theta) \\ U_\theta(\phi,\theta) \end{Bmatrix} = [T][R][A]^{-1} \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [N] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{29}$$

The strain vector $\{\varepsilon\}$ can be determined from the displacement functions U_ϕ, U_θ, W and the deformation –displacement as:

$$\{\varepsilon\} = \begin{bmatrix} [T] & [0] \\ [0] & [T] \end{bmatrix} [Q]\{C\} = \begin{bmatrix} [T] & [0] \\ [0] & [T] \end{bmatrix} [Q][A]^{-1} \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [B] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{30}$$

Where matrix $[Q]$ is given in the appendix.

This relation can be used to find the stress vector, equation (4), in terms of the nodal degrees of freedom vector:

$$\{\sigma\} = [P][B]\begin{Bmatrix}\delta_i \\ \delta_j\end{Bmatrix} \tag{31}$$

Based on the finite element formulation, the local stiffness and mass matrices are:

$$\begin{aligned}[k]_{loc} &= \iint_A [B]^T [P][B]\,dA \\ [m]_{loc} &= \rho\, h \iint_A [N]^T [N]\,dA\end{aligned} \tag{32}$$

Where ρ the density and h is the thickness of shell.

The surface element of the shell wall is $dA = R^2 \sin\phi d\phi d\theta$. After integrating over θ, the preceding equations become

$$\begin{aligned}[k]_{loc} &= \left[A^{-1}\right]^T \left(\pi R^2 \int_{\phi_i}^{\phi_j} [Q]^T [P][Q] \sin\phi d\phi\right)\left[A^{-1}\right] = \left[A^{-1}\right]^T [G]\left[A^{-1}\right] \\ [m]_{loc} &= \rho h\left[A^{-1}\right]^T \left(\pi R^2 \int_{\phi_i}^{\phi_j} [R]^T [R] \sin\phi d\phi\right)\left[A^{-1}\right] = \rho h\left[A^{-1}\right]^T [S]\left[A^{-1}\right]\end{aligned} \tag{33}$$

In the global system the element stiffness and mass matrices are

$$\begin{aligned}[k] &= [LG]^T \left[A^{-1}\right]^T [G]\left[A^{-1}\right][LG] \\ [m] &= \rho h[LG]^T \left[A^{-1}\right]^T [S]\left[A^{-1}\right][LG]\end{aligned} \tag{34}$$

Where

$$[LG] = \begin{bmatrix} \sin\phi_i & -\cos\phi_i & 0 & 0 & 0 & 0 & 0 & 0 \\ \cos\phi_i & \sin\phi_i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin\phi_j & -\cos\phi_j & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\phi_j & \sin\phi_j & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{35}$$

From these equations, one can assemble the mass and stiffness matrix for each element to obtain the mass and stiffness matrices for the whole shell: $[M]$ and $[K]$. Each elementary matrix is 8x8, therefore the final dimensions of $[M]$ and $[K]$ will be 4(N+1) where N is the number of elements of the shell.

3. Numerical results

In order to test the efficiency and the accuracy of our model, we used the theory developed in this paper to calculate the natural frequencies and modes of uniform thin elastic spherical shell, which were both non-shallow(ϕ_0 <30) and shallow, of various dimensions and under different boundary conditions. These cases have already been investigated by other authors using others methods. For purposes of comparisons among the natural frequencies obtained, it is eminently practical at this stage to introduce the non-dimensional natural frequency:

$$\Omega = \omega\, R \left(\frac{\rho}{E} \right)^{\frac{1}{2}} \tag{36}$$

Where:

ω is the natural angular frequency.

R is the radius of the reference surface.

ρ is the density.

E is the modulus of elasticity.

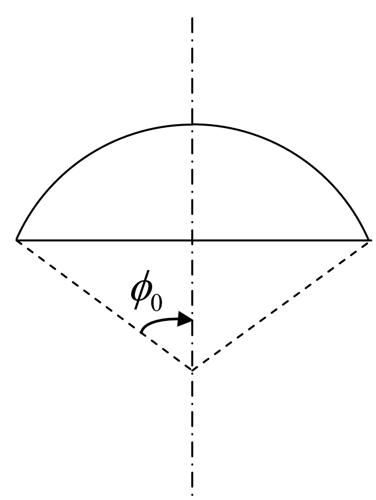

Fig.4. Definition of angle ϕ_0

3.1 <u>Case 1:</u> clamped spherical shell with ϕ_0 =10°

Narassihan and Alwar [9] investigated the case of an axisymmetric clamped spherical shell. The analysis is based on the application of the Chebyshev-Galerkin spectral method for the evaluation of free vibration frequencies and mode shapes. Sai Ram and Sreedhar Babu [12] analyzed the same case with the classical finite element method using 80 elements. Each element is an eight noded degenerated isoparametric shell element with nine degrees of freedom at each node. With our model and using 6 finite elements, the natural were computed, the results are shown in table 1.

Mode	Present	Sai Ram and Sreedhar babu[12]	Narassihan and Alwar [9]
1	1.4861	1.4577	1.4588
2	2.2498	2.2931	2.2999
3	4.4779	4.5773	4.5461

Table 1: Normalized natural frequencies for 10° clamped spherical shell with $\frac{R}{h} = 200$

3.2 <u>Case 2</u>: clamped spherical shell with ϕ_0 =30°

This case was investigated analytically by Kalnins [1] using classical theory and transverse vibration theory. With our theory, we used 8 finite elements to study the spherical shell with the results shown in table 2. The frequencies we obtained with our model are very comparable to Kalinin's values. The maxim displacement values were:

$$\frac{W_{\max}}{(U_\phi)_{\max}} = 3.61 \qquad \frac{W_{\max}}{(U_\theta)_{\max}} = 1.54$$

It was observed that at lowest natural frequency, motion of the spherical shell is mainly dominated by its radial component.

Mode	Present theory	Kalnins [1]
1	1.169	1.168
2	2.224	2.589
3	3.303	3.230
4	4.200	4.288
5	4.923	4.683

Table 2: Normalized natural frequencies for 30° clamped spherical shell with $\frac{R}{h} = 20$

3.3 <u>Case 3</u>: spherical shell ϕ_0 =60° under the three boundary conditions: clamped, simply upported and free

The free axisymmetric vibration of the spherical shell in this case was studied by Kalnins [2], Cohen [3], Navaratna [4], Webster [5], Greene et al [7], Tessler and Spiridigliozzi [8], Gautham and Ganesan [10] and Buchanan and Rich [13]. In the present investigation, the shell was investigated with 10 elements; the results are given respectively for clamped, simply supported and free shells in tables 3, 4 and 5.

The natural modes corresponding to the lowest shell natural frequencies under the two boundary conditions are illustrated in figures 5 and 6. They reveal that at the lowest natural frequency, spherical shell motion is radial.

It is easy to see that all displacements U_ϕ, W and U_θ are all zero at the top ($\phi = 0$) of the spherical shell.

Mode	Kalnins[2]	Navaratna [4]	Webster [5]	Tessler and Spiridigliozzi [8]	Gautham and Ganesan [10]	Buchanan and Rich [13]	Present theory
1	1.006	1.008	1.007	1.000	1.001	1.001	1.031
2	1.391	1.395	1.391	1.368	1.373	1.370	1.496
3	-	1.702	1.700	1.673	1.678	1.675	1.760
4	-	2.126	2.095	-	-	2.094	2.089
5	2.375	2.387	2.386	2.260	-	2.256	2.276
6	3.486	3.506	3.851	3.213	-	3.209	3.311
7	3.991	3.996	4.062	3.965	-	3.964	3.775
8	-	4.159	4.151	-	-	4.060	4.073
9	4.947	5.001	5.962	4.442	-	4.427	4.826
10	-	6.037	6.208	5.773	-	5.740	5.777

Table 3: Normalized natural frequencies for 60° clamped spherical shell with $\frac{R}{h} = 20$

Mode	Kalnins [2]	Navaratna [4]	Greene et al [7]	Cohen [3]	Gautham and Ganesan [10]	Buchanan and Rich [13]	Present Theory
1	0.962	0.963	0.974	0.959	-	0.956	0.981
2	1.334	1.338	1.338	1.325	1.315	1.308	1.412
3	-	1.653	1.652	1.646	1.639	1.612	1.646
4	2.128	2.131	2.162	-	-	2.044	2.038
5	-	2.141	-	-	-	2.059	2.115
6	3.176	3.185	-	-	-	2.965	2.934
7	3.988	3.933	-	-	-	3.837	3.871
8	-	4.159	-	-	-	4.000	4.017
9	4.575	4.601	-	-	-	4.148	4.138
10	-	6.031	-	-	-	5.608	5.773

Table 4: Normalized natural frequencies for 60° simply supported spherical shell with $\frac{R}{h}=20$

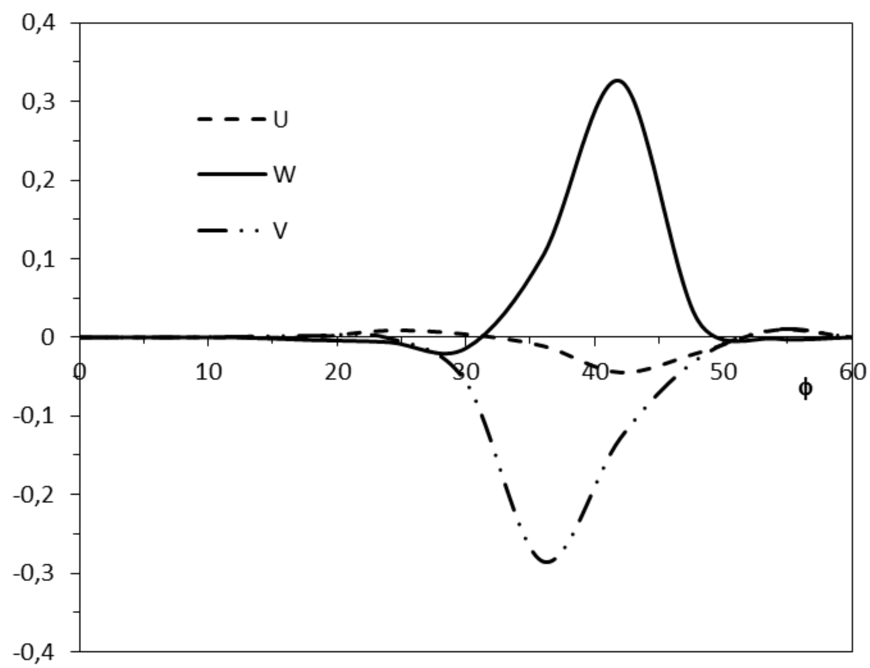

Fig.5. Displacements versus ϕ coordinate for clamped spherical shell ϕ_0 =60

Mode	Kalnins [2]	Navaratna [4]	Webster [5]	Present theory
1	0.931	0.932	0.931	0.938
2	1.088	1.094	1.089	1.062
3	1.533	1.544	1.535	1.426
4	2.348	2.363	2.360	2.425
5	2.544	2.548	2.551	2.725
6	-	2.982	2.985	2.944
7	3.497	3.519	4.023	4.264
8	-	4.971	4.950	4.973
9	4.951	4.980	5.548	5.793
10	5.230	5.543	6.224	6.605

Table 5: Normalized natural frequencies for 60° free spherical shell with $\frac{R}{h} = 20$

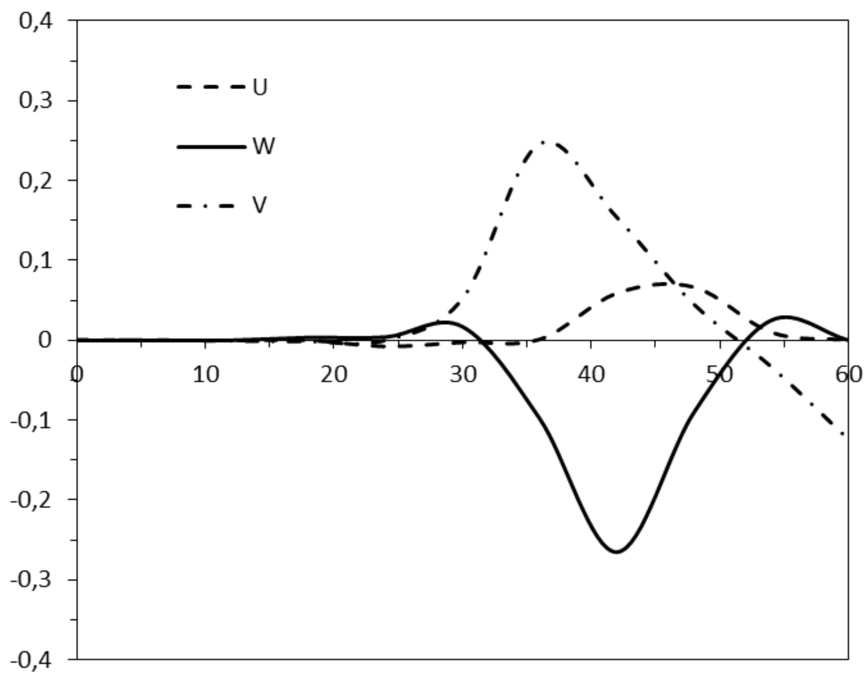

Fig.5. Displacements versus ϕ coordinate for simply supported spherical shell ϕ_0 =60

3.4 Case 4: Spherical shell with ϕ_0 =90°

Kraus [6] investigated the case of simply supported spherical shell using a general theory which included the effects of transverse shear stress and rotational inertia. For cases both with and without these effects, he determined the natural frequencies for the shell motion that was independent of θ for circumferential mode number $n = 0$. Tessler and Spiridigliozzi [8], Gautham and Ganesan [11] analyzed the case of clamped hemispherical shell. Ventsel et al. [14] studied the case of simply supported spherical shell using the boundary elements method for various circumferential mode numbers $\left(n = 0,\ n = 1, n = 2\right)$. With our model and using 12 finite elements, the natural frequencies were computed for clamped and simply supported shells. The results are shown respectively in table 7 and table 8. The maximum displacements values are:

$\left(U_{\phi}\right)_{\max} = 0.3381$ $\quad W_{\max} = 0.2317$ $\quad \left(U_{\theta}\right)_{\max} = 0.0854$

The result is that at the lowest natural frequency, the motion of spherical shell is predominately by the axial displacement.

Mode	Tessler and Spiridigliozzi [8]	Gautham and Ganesan [11]	Present theory
1	0.8481	0.8439	0.8327
2	1.2328	1.2317	1.1919
3	1.5902	1.5808	1.5041
4	1.9435	1.9267	1.9161

Table 7 : Normalized natural frequencies for 90° clamped spherical shell with $\frac{R}{h} = 10$

Mode	Kraus [6] $\frac{R}{h}=10$	Kraus [6] $\frac{R}{h}=50$	Ventsel et al.[14] $\frac{R}{h}=200$	Present theory $\frac{R}{h}=50$
1	0.8060	0.7548	0.7441	0.7579
2	1.2054	0.9432	0.9281	0.9034
3	1.6179	1.0152	0.9693	0.9499
4	1.9051	1.1082	-	1.1089
5	2.7205	1.2523	-	1.2759
6	2.9301	1.4576	-	1.4723
7	4.0274	1.6558	-	1.6237
8	5.5142	1.7636	-	1.7634

Table 8: Normalized natural frequencies for 90° simply supported spherical shell

4. Conclusion

The purpose of the investigation described in this paper is to determine the natural frequencies and shape modes of free vibrations of spherical shell. The modal is based on hybrid approach combining the classical finite element method and the classical shell theory. This theoretical approach is much more precise than usual finite element methods because the displacement functions are derived from exact solutions of equilibrium equations for spherical shells. The mass and stiffness matrices are determined by numerical integration.

The results obtained for spherical shells with different angles and different boundary conditions are compared with results available in the literature. Very good agreement was found. This approach resulted in a very precise element that leads to fast convergence and less numerical difficulties from the computational point of view. Because of its use of classical shell theory for the displacement functions, the presented method may easily be adapted to take fluid-structure effects into account. A paper under preparation on the effect fluid on vibrations of shells confirms this approach . For the same reason, we can obtain the high as well as low frequencies with very good accuracy.

Appendix

$$L_1(U_\phi,U_\theta,W)=\left(\frac{P_{11}}{R}+\frac{P_{41}}{R^2}\right)\left[\frac{\partial}{\partial\phi}\left(\frac{\partial U_\phi}{\partial\phi}+W\right)+\left(\frac{\partial U_\phi}{\partial\phi}+W\right)\cot(\phi)\right]$$

$$+\left(\frac{P_{12}}{R}+\frac{P_{42}}{R^2}\right)\left[\frac{\partial}{\partial\phi}\left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial\theta}+U_\phi\cot(\phi)+W\right)+\left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial\theta}+U_\phi\cot(\phi)+W\right)\cot(\phi)\right]$$

$$+\frac{1}{R}\left(\frac{P_{14}}{R}+\frac{P_{44}}{R^2}\right)\left[\frac{\partial}{\partial\phi}\left(\frac{\partial U_\phi}{\partial\phi}-\frac{\partial^2 W}{\partial\phi^2}\right)+\left(\frac{\partial U_\phi}{\partial\phi}-\frac{\partial^2 W}{\partial\phi^2}\right)\cot(\phi)\right]$$

$$+\frac{1}{R}\left(\frac{P_{15}}{R}+\frac{P_{45}}{R^2}\right)\left[\frac{\partial}{\partial\phi}\left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial\theta}+U_\phi\cot(\phi)-\frac{1}{\sin^2(\phi)}\frac{\partial^2 W}{\partial\theta^2}-\frac{\partial W}{\partial\phi}\cot(\phi)\right)+\left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial\theta}+U_\phi\cot(\phi)-\frac{1}{\sin^2(\phi)}\frac{\partial^2 W}{\partial\theta^2}-\frac{\partial W}{\partial\phi}\cot(\phi)\right)\cot(\phi)\right]$$

$$-\left(\frac{P_{21}}{R}+\frac{P_{51}}{R^2}\right)\left(\frac{\partial U_\phi}{\partial\phi}+W\right)\cot(\phi)$$

$$-\left(\frac{P_{22}}{R}+\frac{P_{52}}{R^2}\right)\left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial\theta}+U_\phi\cot(\phi)+W\right)\cot(\phi)$$

$$-\frac{1}{R}\left(\frac{P_{24}}{R}+\frac{P_{54}}{R^2}\right)\left(\frac{\partial U_\phi}{\partial\phi}-\frac{\partial^2 W}{\partial\phi^2}\right)\cot(\phi)$$

$$-\frac{1}{R}\left(\frac{P_{25}}{R}+\frac{P_{55}}{R^2}\right)\left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial\theta}+U_\phi\cot(\phi)-\frac{1}{\sin^2(\phi)}\frac{\partial^2 W}{\partial\theta^2}-\frac{\partial W}{\partial\phi}\cot(\phi)\right)\cot(\phi)$$

$$+\left(\frac{P_{33}}{R}+\frac{P_{63}}{R^2}\right)\frac{1}{\sin(\phi)}\frac{\partial}{\partial\theta}\left(\frac{\partial U_\theta}{\partial\phi}+\frac{1}{\sin(\phi)}\frac{\partial U_\varphi}{\partial\theta}-U_\theta\cot(\phi)\right)$$

$$+\frac{1}{R}\left(\frac{P_{36}}{R}+\frac{P_{66}}{R^2}\right)\frac{1}{\sin(\phi)}\frac{\partial}{\partial\theta}\left(\frac{\partial U_\theta}{\partial\phi}+\frac{1}{\sin(\phi)}\frac{\partial U_\varphi}{\partial\theta}-U_\theta\cot(\phi)+2\frac{\cot(\phi)}{\sin(\phi)}\frac{\partial W}{\partial\theta}-\frac{2}{\sin(\phi)}\frac{\partial^2 W}{\partial\phi\,\partial\theta}\right)$$

$$L_2(U_\phi,U_\theta,W)=\left(\frac{P_{21}}{R}+\frac{P_{51}}{R^2}\right)\frac{1}{\sin(\phi)}\frac{\partial}{\partial\theta}\left(\frac{\partial U_\phi}{\partial\phi}+W\right)$$

$$+\left(\frac{P_{22}}{R}+\frac{P_{52}}{R^2}\right)\frac{1}{\sin(\phi)}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial\theta}+U_\phi\cot(\phi)+W\right)$$

$$+\frac{1}{R}\left(\frac{P_{24}}{R}+\frac{P_{54}}{R^2}\right)\frac{1}{\sin(\phi)}\frac{\partial}{\partial\theta}\left(\frac{\partial U_\phi}{\partial\phi}-\frac{\partial^2 W}{\partial\phi^2}\right)$$

$$+\frac{1}{R}\left(\frac{P_{25}}{R}+\frac{P_{55}}{R^2}\right)\frac{1}{\sin(\phi)}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial\theta}+U_\phi\cot(\phi)-\frac{1}{\sin^2(\phi)}\frac{\partial^2 W}{\partial\theta^2}-\frac{\partial W}{\partial\phi}\cot(\phi)\right)$$

$$+\left(\frac{P_{33}}{R}+\frac{P_{63}}{R^2}\right)\left[\frac{\partial}{\partial\phi}\left(\frac{\partial U_\theta}{\partial\phi}+\frac{1}{\sin(\phi)}\frac{\partial U_\varphi}{\partial\theta}-U_\theta\cot(\phi)\right)+2\left(\frac{\partial U_\theta}{\partial\phi}+\frac{1}{\sin(\phi)}\frac{\partial U_\varphi}{\partial\theta}-U_\theta\cot(\phi)\right)\cot(\phi)\right]$$

$$+\frac{1}{R}\left(\frac{P_{36}}{R}+\frac{P_{66}}{R^2}\right)\left[\frac{\partial}{\partial\phi}\left(\frac{\partial U_\theta}{\partial\phi}+\frac{1}{\sin(\phi)}\frac{\partial U_\varphi}{\partial\theta}-U_\theta\cot(\phi)+2\frac{\cot(\phi)}{\sin(\phi)}\frac{\partial W}{\partial\theta}-\frac{2}{\sin(\phi)}\frac{\partial^2 W}{\partial\phi\,\partial\theta}\right)+2\left(\frac{\partial U_\theta}{\partial\phi}+\frac{1}{\sin(\phi)}\frac{\partial U_\varphi}{\partial\theta}-U_\theta\cot(\phi)+2\frac{\cot(\phi)}{\sin(\phi)}\frac{\partial W}{\partial\theta}-\frac{2}{\sin(\phi)}\frac{\partial^2 W}{\partial\phi\,\partial\theta}\right)\cot t(\phi)\right]$$

$$L_3(U_\phi, U_\theta, W) = -\left(\frac{P_{11}}{R} + \frac{P_{21}}{R}\right)\left(\frac{\partial U_\phi}{\partial \phi} + W\right)$$

$$-\left(\frac{P_{12}}{R} + \frac{P_{22}}{R}\right)\left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial \theta} + U_\phi \cot(\phi) + W\right)$$

$$-\left(\frac{P_{14}}{R^2} + \frac{P_{24}}{R^2}\right)\left(\frac{\partial U_\phi}{\partial \phi} - \frac{\partial^2 W}{\partial \phi^2}\right)$$

$$-\left(\frac{P_{15}}{R^2} + \frac{P_{25}}{R^2}\right)\left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial \theta} + U_\phi \cot(\phi) - \frac{1}{\sin^2(\phi)}\frac{\partial^2 W}{\partial \theta^2} - \frac{\partial W}{\partial \phi}\cot(\phi)\right)$$

$$+\frac{P_{41} - P_{51}}{R^2}\left(\cot(\phi)\frac{\partial}{\partial \phi}\left(\frac{\partial U_\phi}{\partial \phi} + W\right) - \left(\frac{\partial U_\phi}{\partial \phi} + W\right)\right)$$

$$+\frac{1}{R^2 \sin(\phi)}\left[P_{41}\frac{\partial}{\partial \phi}\left(\sin(\phi)\frac{\partial}{\partial \phi}\left(\frac{\partial U_\phi}{\partial \phi} + W\right)\right) + P_{51}\frac{\partial}{\partial \theta}\left(\frac{\partial U_\phi}{\partial \phi} + W\right)\right]$$

$$+\frac{P_{42} - P_{52}}{R^2}\left(\cot(\phi)\frac{\partial}{\partial \phi}\left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial \theta} + U_\phi \cot(\phi) + W\right) - \left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial \theta} + U_\phi \cot(\phi) + W\right)\right)$$

$$+\frac{1}{R^2 \sin(\phi)}\left[P_{42}\frac{\partial}{\partial \phi}\left(\sin(\phi)\frac{\partial}{\partial \phi}\left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial \theta} + U_\phi \cot(\phi) + W\right)\right) + P_{52}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial \theta} + U_\phi \cot(\phi) + W\right)\right]$$

$$+\frac{P_{44} - P_{54}}{R^3}\left(\cot(\phi)\frac{\partial}{\partial \phi}\left(\frac{\partial U_\phi}{\partial \phi} - \frac{\partial^2 W}{\partial \phi^2}\right) - \left(\frac{\partial U_\phi}{\partial \phi} - \frac{\partial^2 W}{\partial \phi^2}\right)\right)$$

$$+\frac{1}{R^3 \sin(\phi)}\left[P_{44}\frac{\partial}{\partial \phi}\left(\sin(\phi)\frac{\partial}{\partial \phi}\left(\frac{\partial U_\phi}{\partial \phi} - \frac{\partial^2 W}{\partial \phi^2}\right)\right) + P_{54}\frac{\partial}{\partial \theta}\left(\frac{\partial U_\phi}{\partial \phi} - \frac{\partial^2 W}{\partial \phi^2}\right)\right]$$

$$+\frac{P_{45} - P_{55}}{R^3}\left(\cot(\phi)\frac{\partial}{\partial \phi}\left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial \theta} + U_\phi \cot(\phi) - \frac{1}{\sin^2(\phi)}\frac{\partial^2 W}{\partial \theta^2} - \frac{\partial W}{\partial \phi}\cot(\phi)\right) - \left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial \theta} + U_\phi \cot(\phi) - \frac{1}{\sin^2(\phi)}\frac{\partial^2 W}{\partial \theta^2} - \frac{\partial W}{\partial \phi}\cot(\phi)\right)\right)$$

$$+\frac{1}{R^3 \sin(\phi)}\left[P_{45}\frac{\partial}{\partial \phi}\left(\sin(\phi)\frac{\partial}{\partial \phi}\left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial \theta} + U_\phi \cot(\phi) - \frac{1}{\sin^2(\phi)}\frac{\partial^2 W}{\partial \theta^2} - \frac{\partial W}{\partial \phi}\cot(\phi)\right)\right) + P_{55}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial \theta} + U_\phi \cot(\phi) - \frac{1}{\sin^2(\phi)}\frac{\partial^2 W}{\partial \theta^2} - \frac{\partial W}{\partial \phi}\cot(\phi)\right)\right]$$

$$+\frac{P_{63}}{R^2 \sin(\phi)}\left[\frac{\partial^2}{\partial \phi \partial \theta}\left(\frac{\partial U_\theta}{\partial \phi} + \frac{1}{\sin(\phi)}\frac{\partial U_\varphi}{\partial \theta} - U_\theta \cot(\phi)\right) + 3\cot(\phi)\left(\frac{\partial}{\partial \theta}\frac{\partial U_\theta}{\partial \phi} + \frac{1}{\sin(\phi)}\frac{\partial U_\varphi}{\partial \theta} - U_\theta \cot(\phi)\right)\right]$$

$$+\frac{P_{66}}{R^3 \sin(\phi)}\left[\frac{\partial^2}{\partial \phi \partial \theta}\left(\frac{\partial U_\theta}{\partial \phi} + \frac{1}{\sin(\phi)}\frac{\partial U_\varphi}{\partial \theta} - U_\theta \cot(\phi) + 2\frac{\cot(\phi)}{\sin(\phi)}\frac{\partial W}{\partial \theta} - \frac{2}{\sin(\phi)}\frac{\partial^2 W}{\partial \phi\, \partial \theta}\right) + 3\cot(\phi)\left(\frac{\partial U_\theta}{\partial \phi} + \frac{1}{\sin(\phi)}\frac{\partial U_\varphi}{\partial \theta} - U_\theta \cot(\phi) + 2\frac{\cot(\phi)}{\sin(\phi)}\frac{\partial W}{\partial \theta} - \frac{2}{\sin(\phi)}\frac{\partial^2 W}{\partial \phi\, \partial \theta}\right)\right]$$

$$R(1,1)=\left(-e_1 n\cot\phi P_{\mu_1}^{n}+e_1c_1P_{\mu_1}^{n-1}\right)$$

$$R(1,2)=-ne_2\cot\phi\,\mathrm{Re}(P_{\mu_2}^{n})-ne_3\cot\phi\,\mathrm{Im}(P_{\mu_2}^{n})+(e_2c_2+e_3c_3)\,\mathrm{Re}(P_{\mu_2}^{n-1})+(e_3c_2-e_2c_3)\,\mathrm{Im}(P_{\mu_2}^{n-1})$$

$$R(1,3)=ne_3\cot\phi\,\mathrm{Re}(P_{\mu_2}^{n})-ne_2\cot\phi\,\mathrm{Im}(P_{\mu_2}^{n})-(e_3c_2-e_2c_3)\,\mathrm{Re}(P_{\mu_2}^{n-1})+(e_2c_2+e_3c_3)\,\mathrm{Im}(P_{\mu_2}^{n-1})$$

$$R(1,4)=-\frac{n^2}{2\sin\phi}P_1^{n}$$

$$R(1,5)=\left(-e_1 n\cot\phi Q_{\mu_1}^{n}+e_1c_1Q_{\mu_1}^{n-1}\right)$$

$$R(1,6)=-ne_2\cot\phi\,\mathrm{Re}(Q_{\mu_2}^{n})-ne_3\cot\phi\,\mathrm{Im}(Q_{\mu_2}^{n})+(e_2c_2+e_3c_3)\,\mathrm{Re}(Q_{\mu_2}^{n-1})+(e_3c_2-e_2c_3)\,\mathrm{Im}(Q_{\mu_2}^{n-1})$$

$$R(1,7)=ne_3\cot\phi\,\mathrm{Re}(Q_{\mu_2}^{n})-ne_2\cot\phi\,\mathrm{Im}(Q_{\mu_2}^{n})-(e_3c_2-e_2c_3)\,\mathrm{Re}(Q_{\mu_2}^{n-1})+(e_2c_2+e_3c_3)\,\mathrm{Im}(Q_{\mu_2}^{n-1})$$

$$R(1,8)=-\frac{n^2}{2\sin\phi}Q_1^{n}$$

$$R(2,1)=P_{\mu_1}^{n}$$

$$R(2,2)=\mathrm{Re}(P_{\mu_2}^{n})$$

$$R\left(2,3\right)=\mathrm{Im}(P_{\mu_2}^{n})$$

$$R\left(2,4\right)=0$$

$$R(2,5)=Q_{\mu_1}^{n}$$

$$R(2,6)=\mathrm{Re}(Q_{\mu_2}^{n})$$

$$R\left(2,7\right)=\mathrm{Im}(Q_{\mu_2}^{n})$$

$$R\left(2,8\right)=0$$

$$R\left(3,1\right)=-ne_1\frac{1}{\sin\phi}P_{\mu_1}^{n}$$

$$R(3,2)=-ne_2\frac{1}{\sin\phi}\mathrm{Re}(P_{\mu_2}^{n})-ne_3\frac{1}{\sin\phi}\mathrm{Im}(P_{\mu_2}^{n})$$

$$R(3,3)=ne_3\frac{1}{\sin\phi}\mathrm{Re}(P_{\mu_2}^{n})-ne_2\frac{1}{\sin\phi}\mathrm{Im}(P_{\mu_2}^{n})$$

$$R(3,4)=-\frac{n^2}{2}\cot\phi P_1^{n}+\frac{n}{2}\left(n-2\right)\left(n+1\right)P_1^{n-1}$$

$$R\left(3,5\right)=-ne_1\frac{1}{\sin\phi}Q_{\mu_1}^{n}$$

$$R(3,6) = -ne_2 \frac{1}{\sin\phi}\mathrm{Re}(Q_{\mu_2}^n) - ne_3 \frac{1}{\sin\phi}\mathrm{Im}(Q_{\mu_2}^n)$$

$$R(3,7) = ne_3 \frac{1}{\sin\phi}\mathrm{Re}(Q_{\mu_2}^n) - ne_2 \frac{1}{\sin\phi}\mathrm{Im}(Q_{\mu_2}^n)$$

$$R(3,8) = -\frac{n^2}{2}\cot\phi Q_1^n + \frac{n}{2}\left(n-2\right)\left(n+1\right)Q_1^{n-1}$$

$A(1,j) = R(1,j)$, $A(2,j) = R(2,j)$, $A(3,j) = \frac{dw_n}{d\phi}(j)$, $A(4,j) = R(3,j)$ with $\phi = \phi_i$ $A(5,j) = R(1,j)$,

$A(6,j) = R(2,j)$; $A(7,j) = \frac{dw_n}{d\phi}(j)$, $A(8,j) = R(3,j)$ with $\phi = \phi_j$

j=1,…,8

$$Q(1,1)=\frac{1}{R}\left[e_1c_1+ne_1(\frac{1}{\sin^2\phi}+n\cot^2\phi)+1\right]P_{\mu_1}^n-\frac{e_1}{r}c_1\cot\phi P_{\mu_1}^{n-1}$$

$$Q(1,2)=\frac{1}{R}\left[\left(e_2c_2+e_3c_3\right)+ne_2(\frac{1}{\sin^2\phi}+n\cot^2\phi)+1\right]\mathrm{Re}(P_{\mu_2}^n)$$

$$+\frac{1}{R}\left[\left(e_3c_2-e_2c_3\right)+ne_3(\frac{1}{\sin^2\phi}+n\cot^2\phi)\right]\mathrm{Im}(P_{\mu_2}^n)$$

$$-\frac{1}{R}\left(e_2c_2+e_3c_3\right)\cot\phi\,\mathrm{Re}(P_{\mu_2}^{n-1})-\frac{1}{R}\left(e_3c_2-e_2c_3\right)\cot\phi\,\mathrm{Im}(P_{\mu_2}^{n-1})$$

$$Q(1,3)=-\frac{1}{R}\left[\left(e_3c_2-e_2c_3\right)+ne_3(\frac{1}{\sin^2\phi}+n\cot^2\phi)\right]\mathrm{Re}(P_{\mu_2}^n)$$

$$+\frac{1}{R}\left[\left(e_2c_2+e_3c_3\right)+ne_2(\frac{1}{\sin^2\phi}+n\cot^2\phi)+1\right]\mathrm{Im}(P_{\mu_2}^n)$$

$$+\frac{1}{R}\left(e_3c_2-e_2c_3\right)\cot\phi\,\mathrm{Re}(P_{\mu_2}^{n-1})-\frac{1}{R}\left(e_2c_2+e_3c_3\right)\cot\phi\,\mathrm{Im}(P_{\mu_2}^{n-1})$$

$$Q(1,4)=\frac{n^2}{2R}(n+1)\frac{1}{\sin\phi}\cot\phi P_1^n-\frac{n^2}{2R}(n-2)(n+1)\frac{1}{\sin\phi}P_1^{n-1}$$

$$Q(1,5)=\frac{1}{R}\left[e_1c_1+ne_1(\frac{1}{\sin^2\phi}+n\cot^2\phi)+1\right]Q_{\mu_1}^n-\frac{e_1}{r}c_1\cot\phi Q_{\mu_1}^{n-1}$$

$$Q(1,6)=\frac{1}{R}\left[\left(e_2c_2+e_3c_3\right)+ne_2(\frac{1}{\sin^2\phi}+n\cot^2\phi)+1\right]\mathrm{Re}(Q_{\mu_2}^n)$$

$$+\frac{1}{R}\left[\left(e_3c_2-e_2c_3\right)+ne_3(\frac{1}{\sin^2\phi}+n\cot^2\phi)\right]\mathrm{Im}(Q_{\mu_2}^n)$$

$$-\frac{1}{R}\left(e_2c_2+e_3c_3\right)\cot\phi\,\mathrm{Re}(Q_{\mu_2}^{n-1})-\frac{1}{R}\left(e_3c_2-e_2c_3\right)\cot\phi\,\mathrm{Im}(Q_{\mu_2}^{n-1})$$

$$Q(1,7)=-\frac{1}{R}\left[\left(e_3c_2-e_2c_3\right)+ne_3(\frac{1}{\sin^2\phi}+n\cot^2\phi)\right]\mathrm{Re}(Q_{\mu_2}^n)$$

$$+\frac{1}{R}\left[\left(e_2c_2+e_3c_3\right)+ne_2(\frac{1}{\sin^2\phi}+n\cot^2\phi)+1\right]\mathrm{Im}(Q_{\mu_2}^n)$$

$$+\frac{1}{R}\left(e_3c_2-e_2c_3\right)\cot\phi\,\mathrm{Re}(Q_{\mu_2}^{n-1})-\frac{1}{R}\left(e_2c_2+e_3c_3\right)\cot\phi\,\mathrm{Im}(Q_{\mu_2}^{n-1})$$

$$Q(1,8)=\frac{n^2}{2R}(n+1)\frac{1}{\sin\phi}\cot\phi Q_1^n-\frac{n^2}{2R}(n-2)(n+1)\frac{1}{\sin\phi}Q_1^{n-1}$$

$$Q(2,1) = \frac{1}{R}\left[1 - ne_1(n\frac{1}{\sin^2\phi} + \cot^2\phi)\right]P_{\mu_1}^n + \frac{e_1}{r}c_1\cot\phi P_{\mu_1}^{n-1}$$

$$Q(2,2) = \frac{1}{R}\left[1 - ne_2(n\frac{1}{\sin^2\phi} + \cot^2\phi)\right]\mathrm{Re}(P_{\mu_2}^n) - \frac{ne_3}{R}(n\frac{1}{\sin^2\phi} + \cot^2\phi)\,\mathrm{Im}(P_{\mu_2}^n)$$
$$+\frac{1}{R}(e_2c_2 + e_3c_3)\cot\phi\,\mathrm{Re}(P_{\mu_2}^{n-1}) + \frac{1}{R}(e_3c_2 - e_2c_3)\cot\phi\,\mathrm{Im}(P_{\mu_2}^{n-1})$$

$$Q(2,3) = \frac{ne_3}{R}(n\frac{1}{\sin^2\phi} + \cot^2\phi)\,\mathrm{Re}(P_{\mu_2}^n) + \frac{1}{R}\left[1 - ne_2(n\frac{1}{\sin^2\phi} + \cot^2\phi)\right]\mathrm{Im}(P_{\mu_2}^n)$$
$$-\frac{1}{R}(e_3c_2 - e_2c_3)\cot\phi\,\mathrm{Re}(P_{\mu_2}^{n-1}) + \frac{1}{R}(e_2c_2 + e_3c_3)\cot\phi\,\mathrm{Im}(P_{\mu_2}^{n-1})$$

$$Q(2,4) = -\frac{n^2}{2R}(n+1)\frac{1}{\sin\phi}\cot\phi P_1^n + \frac{n^2}{2R}(n-2)(n+1)\frac{1}{\sin\phi}P_1^{n-1}$$

$$Q(2,5) = \frac{1}{R}\left[1 - ne_1(\frac{1}{\sin^2\phi} + n\cot^2\phi)\right]Q_{\mu_1}^n + \frac{e_1}{r}c_1\cot\phi Q_{\mu_1}^{n-1}$$

$$Q(2,6) = \frac{1}{R}\left[1 - ne_2(n\frac{1}{\sin^2\phi} + \cot^2\phi)\right]\mathrm{Re}(Q_{\mu_2}^n) - \frac{ne_3}{R}(n\frac{1}{\sin^2\phi} + \cot^2\phi)\,\mathrm{Im}(Q_{\mu_2}^n)$$
$$+\frac{1}{R}(e_2c_2 + e_3c_3)\cot\phi\,\mathrm{Re}(Q_{\mu_2}^{n-1}) + \frac{1}{R}(e_3c_2 - e_2c_3)\cot\phi\,\mathrm{Im}(Q_{\mu_2}^{n-1})$$

$$Q(2,7) = \frac{ne_3}{R}(n\frac{1}{\sin^2\phi} + \cot^2\phi)\,\mathrm{Re}(Q_{\mu_2}^n) + \frac{1}{R}\left[1 - ne_2(n\frac{1}{\sin^2\phi} + \cot^2\phi)\right]\mathrm{Im}(Q_{\mu_2}^n)$$
$$-\frac{1}{R}(e_3c_2 - e_2c_3)\cot\phi\,\mathrm{Re}(Q_{\mu_2}^{n-1}) + \frac{1}{R}(e_2c_2 + e_3c_3)\cot\phi\,\mathrm{Im}(Q_{\mu_2}^{n-1})$$

$$Q(2,8) = -\frac{n^2}{2R}(n+1)\frac{1}{\sin\phi}\cot\phi Q_1^n + \frac{n^2}{2R}(n-2)(n+1)\frac{1}{\sin\phi}Q_1^{n-1}$$

$$Q(3,1) = \frac{2n}{R} e_1 (n+1) \frac{1}{\sin\phi} \cot\phi P_{\mu_1}^n - \frac{2n}{R} e_1 c_1 \frac{1}{\sin\phi} P_{\mu_1}^{n-1}$$

$$Q(3,2) = \frac{2n}{R} e_2 (n+1) \frac{1}{\sin\phi} \cot\phi \operatorname{Re}(P_{\mu_2}^n) + \frac{2n}{R} e_3 (n+1) \frac{1}{\sin\phi} \cot\phi \operatorname{Im}(P_{\mu_2}^n)$$
$$- \frac{2n}{R} (e_2 c_2 + e_3 c_3) \frac{1}{\sin\phi} \operatorname{Re}(P_{\mu_2}^{n-1}) - \frac{2n}{R} (e_3 c_2 - e_2 c_3) \frac{1}{\sin\phi} \operatorname{Im}(P_{\mu_2}^{n-1})$$

$$Q(3,3) = -\frac{2n}{R} e_3 (n+1) \frac{1}{\sin\phi} \cot\phi \operatorname{Re}(P_{\mu_2}^n) + \frac{2n}{R} e_2 (n+1) \frac{1}{\sin\phi} \cot\phi \operatorname{Im}(P_{\mu_2}^n)$$
$$+ \frac{2n}{R} (e_3 c_2 - e_2 c_3) \frac{1}{\sin\phi} \operatorname{Re}(P_{\mu_2}^{n-1}) - \frac{2n}{R} (e_2 c_2 + e_3 c_3) \frac{1}{\sin\phi} \operatorname{Im}(P_{\mu_2}^{n-1})$$

$$Q(3,4) = \frac{n}{2R}(n+1)\left[(n-2) + n\left(\frac{1}{\sin^2\phi} + \cot^2\phi\right)\right] P_1^n - \frac{n}{R}(n-2)(n+1)\cot\phi P_1^{n-1}$$

$$Q(3,5) = \frac{2n}{R} e_1 (n+1) \frac{1}{\sin\phi} \cot\phi Q_{\mu_1}^n - \frac{2n}{R} c_1 \frac{1}{\sin\phi} Q_{\mu_1}^{n-1}$$

$$Q(3,6) = \frac{2n}{R} e_2 (n+1) \frac{1}{\sin\phi} \cot\phi \operatorname{Re}(Q_{\mu_2}^n) + \frac{2n}{R} e_3 (n+1) \frac{1}{\sin\phi} \cot\phi \operatorname{Im}(Q_{\mu_2}^n)$$
$$- \frac{2n}{R} (e_2 c_2 + e_3 c_3) \frac{1}{\sin\phi} \operatorname{Re}(Q_{\mu_2}^{n-1}) - \frac{2n}{R} (e_3 c_2 - e_2 c_3) \frac{1}{\sin\phi} \operatorname{Im}(Q_{\mu_2}^{n-1})$$

$$Q(3,7) = -\frac{2n}{R} e_3 (n+1) \frac{1}{\sin\phi} \cot\phi \operatorname{Re}(Q_{\mu_2}^n) + \frac{2n}{R} e_2 (n+1) \frac{1}{\sin\phi} \cot\phi \operatorname{Im}(Q_{\mu_2}^n)$$
$$+ \frac{2n}{R} (e_3 c_2 - e_2 c_3) \frac{1}{\sin\phi} \operatorname{Re}(Q_{\mu_2}^{n-1}) - \frac{2n}{R} (e_2 c_2 + e_3 c_3) \frac{1}{\sin\phi} \operatorname{Im}(Q_{\mu_2}^{n-1})$$

$$Q(3,8) = \frac{n}{2R}(n+1)\left[(n-2) + n\left(\frac{1}{\sin^2\phi} + \cot^2\phi\right)\right] Q_1^n - \frac{n}{R}(n-2)(n+1)\cot\phi Q_1^{n-1}$$

$$Q(4,1)=\frac{1}{R^2}\left[\left(e_1-1\right)c_1+n\left(e_1-1\right)\left(\frac{1}{\sin^2\phi}+n\cot^2\phi\right)\right]P_{\mu_1}^{n}-\frac{e_1-1}{R^2}c_1\cot\phi P_{\mu_1}^{n-1}$$

$$\begin{aligned}Q(4,2)=&\frac{1}{R^2}\left[\left(e_2-1\right)c_2+e_3c_3+n\left(e_2-1\right)\left(\frac{1}{\sin^2\phi}+n\cot^2\phi\right)\right]\mathrm{Re}\left(P_{\mu_2}^{n}\right)\\&+\frac{1}{R^2}\left[e_3c_2-\left(e_2-1\right)c_3+ne_3\left(\frac{1}{\sin^2\phi}+n\cot^2\phi\right)\right]\mathrm{Im}\left(P_{\mu_2}^{n}\right)\\&-\frac{1}{R^2}\left[\left(e_2-1\right)c_2+e_3c_3\right]\cot\phi\,\mathrm{Re}\left(P_{\mu_2}^{n-1}\right)-\frac{1}{R^2}\left[e_3c_2-\left(e_2-1\right)c_3\right]\cot\phi\,\mathrm{Im}\left(P_{\mu_2}^{n-1}\right)\end{aligned}$$

$$\begin{aligned}Q(4,3)=&-\frac{1}{R^2}\left[e_3c_2-\left(e_2-1\right)c_3+ne_3\left(\frac{1}{\sin^2\phi}+n\cot^2\phi\right)\right]\mathrm{Re}\left(P_{\mu_2}^{n}\right)\\&+\frac{1}{R^2}\left[\left(e_2-1\right)c_2+e_3c_3+n\left(e_2-1\right)\left(\frac{1}{\sin^2\phi}+n\cot^2\phi\right)\right]\mathrm{Im}\left(P_{\mu_2}^{n}\right)\\&+\frac{1}{R^2}\left[e_3c_2-\left(e_2-1\right)c_3\right]\cot\phi\,\mathrm{Re}\left(P_{\mu_2}^{n-1}\right)-\frac{1}{R^2}\left[\left(e_2-1\right)c_2+e_3c_3\right]\cot\phi\,\mathrm{Im}\left(P_{\mu_2}^{n-1}\right)\end{aligned}$$

$$Q(4,4)=\frac{n^2}{2R^2}\left(n+1\right)\frac{1}{\sin\phi}\cot\phi P_1^n-\frac{n^2}{2R^2}(n-2)(n+1)\frac{1}{\sin\phi}P_1^{n-1}$$

$$Q(4,5)=\frac{1}{R^2}\left[\left(e_1-1\right)c_1+n\left(e_1-1\right)\left(\frac{1}{\sin^2\phi}+n\cot^2\phi\right)\right]Q_{\mu_1}^{n}-\frac{e_1-1}{R^2}c_1\cot\phi Q_{\mu_1}^{n-1}$$

$$\begin{aligned}Q(4,6)=&\frac{1}{R^2}\left[\left(e_2-1\right)c_2+e_3c_3+n\left(e_2-1\right)\left(\frac{1}{\sin^2\phi}+n\cot^2\phi\right)\right]\mathrm{Re}\left(Q_{\mu_2}^{n}\right)\\&+\frac{1}{R^2}\left[e_3c_2-\left(e_2-1\right)c_3+ne_3\left(\frac{1}{\sin^2\phi}+n\cot^2\phi\right)\right]\mathrm{Im}\left(Q_{\mu_2}^{n}\right)\\&-\frac{1}{R^2}\left[\left(e_2-1\right)c_2+e_3c_3\right]\cot\phi\,\mathrm{Re}\left(Q_{\mu_2}^{n-1}\right)-\frac{1}{R^2}\left[e_3c_2-\left(e_2-1\right)c_3\right]\cot\phi\,\mathrm{Im}\left(Q_{\mu_2}^{n-1}\right)\end{aligned}$$

$$\begin{aligned}Q(4,7)=&-\frac{1}{R^2}\left[e_3c_2-\left(e_2-1\right)c_3+ne_3\left(\frac{1}{\sin^2\phi}+n\cot^2\phi\right)\right]\mathrm{Re}\left(Q_{\mu_2}^{n}\right)\\&+\frac{1}{R^2}\left[\left(e_2-1\right)c_2+e_3c_3+n\left(e_2-1\right)\left(\frac{1}{\sin^2\phi}+n\cot^2\phi\right)\right]\mathrm{Im}\left(Q_{\mu_2}^{n}\right)\\&+\frac{1}{R^2}\left[e_3c_2-\left(e_2-1\right)c_3\right]\cot\phi\,\mathrm{Re}\left(Q_{\mu_2}^{n-1}\right)-\frac{1}{R^2}\left[\left(e_2-1\right)c_2+e_3c_3\right]\cot\phi\,\mathrm{Im}\left(Q_{\mu_2}^{n-1}\right)\end{aligned}$$

$$Q(4,8)=\frac{n^2}{2R^2}\left(n+1\right)\frac{1}{\sin\phi}\cot\phi Q_1^n-\frac{n^2}{2R^2}(n-2)(n+1)\frac{1}{\sin\phi}Q_1^{n-1}$$

$$Q(5,1)=\frac{n}{R^2}\left(1-e_1\right)\left(\cot^2\phi+n\frac{1}{\sin^2\phi}\right)P_{\mu_1}^n+\frac{\left(e_1-1\right)}{R^2}c_1\cot\phi P_{\mu_1}^{n-1}$$

$$Q(5,2)=\frac{n}{R^2}\left(1-e_2\right)\left(\cot^2\phi+n\frac{1}{\sin^2\phi}\right)\mathrm{Re}\left(P_{\mu_2}^n\right)-\frac{ne_3}{R^2}\left(\cot^2\phi+n\frac{1}{\sin^2\phi}\right)\mathrm{Im}\left(P_{\mu_2}^n\right)$$
$$+\frac{1}{R^2}\left[\left(e_2-1\right)c_2+e_3c_3\right]\cot\phi\,\mathrm{Re}\left(P_{\mu_2}^{n-1}\right)+\frac{1}{R^2}\left[e_3c_2-\left(e_2-1\right)c_3\right]\cot\phi\,\mathrm{Im}\left(P_{\mu_2}^{n-1}\right)$$

$$Q(5,3)=\frac{ne_3}{R^2}\left(\cot^2\phi+n\frac{1}{\sin^2\phi}\right)\mathrm{Re}\left(P_{\mu_2}^n\right)+\frac{n}{R^2}\left(1-e_2\right)\left(\cot^2\phi+n\frac{1}{\sin^2\phi}\right)\mathrm{Im}\left(P_{\mu_2}^n\right)$$
$$-\frac{1}{R^2}\left[e_3c_2-\left(e_2-1\right)c_3\right]\cot\phi\,\mathrm{Re}\left(P_{\mu_2}^{n-1}\right)+\frac{1}{R^2}\left[\left(e_2-1\right)c_2+e_3c_3\right]\cot\phi\,\mathrm{Im}\left(P_{\mu_2}^{n-1}\right)$$

$$Q(5,4)=-\frac{n}{2R^2}\left(n+1\right)\frac{1}{\sin\phi}\cot\phi P_1^n+\frac{n}{2R^2}\left(n-2\right)\left(n+1\right)\frac{1}{\sin\phi}P_1^{n-1}$$

$$Q(5,5)=\frac{n}{R^2}\left(1-e_1\right)\left(\cot^2\phi+n\frac{1}{\sin^2\phi}\right)Q_{\mu_1}^n+\frac{\left(e_1-1\right)}{R^2}c_1\cot\phi Q_{\mu_1}^{n-1}$$

$$Q(5,6)=\frac{n}{R^2}\left(1-e_2\right)\left(\cot^2\phi+n\frac{1}{\sin^2\phi}\right)\mathrm{Re}\left(Q_{\mu_2}^n\right)-\frac{ne_3}{R^2}\left(\cot^2\phi+n\frac{1}{\sin^2\phi}\right)\mathrm{Im}\left(Q_{\mu_2}^n\right)$$
$$+\frac{1}{R^2}\left[\left(e_2-1\right)c_2+e_3c_3\right]\cot\phi\,\mathrm{Re}\left(Q_{\mu_2}^{n-1}\right)+\frac{1}{R^2}\left[e_3c_2-\left(e_2-1\right)c_3\right]\cot\phi\,\mathrm{Im}\left(Q_{\mu_2}^{n-1}\right)$$

$$Q(5,7)=\frac{ne_3}{R^2}\left(\cot^2\phi+n\frac{1}{\sin^2\phi}\right)\mathrm{Re}\left(Q_{\mu_2}^n\right)+\frac{n}{R^2}\left(1-e_2\right)\left(\cot^2\phi+n\frac{1}{\sin^2\phi}\right)\mathrm{Im}\left(Q_{\mu_2}^n\right)$$
$$-\frac{1}{R^2}\left[e_3c_2-\left(e_2-1\right)c_3\right]\cot\phi\,\mathrm{Re}\left(Q_{\mu_2}^{n-1}\right)+\frac{1}{R^2}\left[\left(e_2-1\right)c_2+e_3c_3\right]\cot\phi\,\mathrm{Im}\left(Q_{\mu_2}^{n-1}\right)$$

$$Q(5,8)=-\frac{n}{2R^2}\left(n+1\right)\frac{1}{\sin\phi}\cot\phi Q_1^n+\frac{n}{2R^2}\left(n-2\right)\left(n+1\right)\frac{1}{\sin\phi}Q_1^{n-1}$$

$$Q(6,1)=\frac{2n}{R^2}(n+1)(e_1-1)\frac{1}{\sin\phi}\cot\phi P_{\mu_1}^n+\frac{2n}{R^2}(1-e_1)c_1\frac{1}{\sin\phi}P_{\mu_1}^{n-1}$$

$$Q(6,2)=\frac{2n(n+1)}{R^2}(e_2-1)\frac{1}{\sin\phi}\cot\phi\operatorname{Re}\left(P_{\mu_2}^n\right)+\frac{2n(n+1)}{R^2}e_3\frac{1}{\sin\phi}\cot\phi\operatorname{Im}\left(P_{\mu_2}^n\right)$$
$$-\frac{2n}{R^2}\left[(e_2-1)c_2+e_3c_3\right]\frac{1}{\sin\phi}\operatorname{Re}\left(P_{\mu_2}^{n-1}\right)-\frac{2n}{R^2}\left[e_3c_2-(e_2-1)c_3\right]\frac{1}{\sin\phi}\operatorname{Im}\left(P_{\mu_2}^{n-1}\right)$$

$$Q(6,3)=-\frac{2n(n+1)}{R^2}e_3\frac{1}{\sin\phi}\cot\phi\operatorname{Re}\left(P_{\mu_2}^n\right)+\frac{2n(n+1)}{R^2}(e_2-1)\frac{1}{\sin\phi}\cot\phi\operatorname{Im}\left(P_{\mu_2}^n\right)$$
$$+\frac{2n}{R^2}\left[e_3c_2-(e_2-1)c_3\right]\frac{1}{\sin\phi}\operatorname{Re}\left(P_{\mu_2}^{n-1}\right)-\frac{2n}{R^2}\left[(e_2-1)c_2+e_3c_3\right]\frac{1}{\sin\phi}\operatorname{Im}\left(P_{\mu_2}^{n-1}\right)$$

$$Q(6,4)=\frac{n}{2R^2}(n+1)\left[(n-2)+n\left(\frac{1}{\sin^2\phi}+\cot^2\phi\right)\right]P_1^n-\frac{n}{R^2}(n-2)(n+1)\cot\phi P_1^{n-1}$$

$$Q(6,5)=\frac{2n}{R^2}(n+1)(e_1-1)\frac{1}{\sin\phi}\cot\phi Q_{\mu_1}^n+\frac{2n}{R^2}(1-e_1)c_1\frac{1}{\sin\phi}Q_{\mu_1}^{n-1}$$

$$Q(6,6)=\frac{2n(n+1)}{R^2}(e_2-1)\frac{1}{\sin\phi}\cot\phi\operatorname{Re}\left(Q_{\mu_2}^n\right)+\frac{2n(n+1)}{R^2}e_3\frac{1}{\sin\phi}\cot\phi\operatorname{Im}\left(Q_{\mu_2}^n\right)$$
$$-\frac{2n}{R^2}\left[(e_2-1)c_2+e_3c_3\right]\frac{1}{\sin\phi}\operatorname{Re}\left(Q_{\mu_2}^{n-1}\right)-\frac{2n}{R^2}\left[e_3c_2-(e_2-1)c_3\right]\frac{1}{\sin\phi}\operatorname{Im}\left(Q_{\mu_2}^{n-1}\right)$$

$$Q(6,7)=-\frac{2n(n+1)}{R^2}e_3\frac{1}{\sin\phi}\cot\phi\operatorname{Re}\left(Q_{\mu_2}^n\right)+\frac{2n(n+1)}{R^2}(e_2-1)\frac{1}{\sin\phi}\cot\phi\operatorname{Im}\left(Q_{\mu_2}^n\right)$$
$$+\frac{2n}{R^2}\left[e_3c_2-(e_2-1)c_3\right]\frac{1}{\sin\phi}\operatorname{Re}\left(Q_{\mu_2}^{n-1}\right)-\frac{2n}{R^2}\left[(e_2-1)c_2+e_3c_3\right]\frac{1}{\sin\phi}\operatorname{Im}\left(Q_{\mu_2}^{n-1}\right)$$

$$Q(6,8)=\frac{n}{2R^2}(n+1)\left[(n-2)+n\left(\frac{1}{\sin^2\phi}+\cot^2\phi\right)\right]Q_1^n-\frac{n}{R^2}(n-2)(n+1)\cot\phi Q_1^{n-1}$$

In deriving the above relation we used the recursive relations:

$$\frac{d^2P_\mu^n}{d\phi^2}=\left[(n-\mu-1)(n+\mu)+n\left(\frac{1}{\sin^2\phi}+n\cot^2\phi\right)\right]P_\mu^n-\cot\phi(n-\mu-1)(n+\mu)P_\mu^{n-1}$$
$$\frac{d^2Q_\mu^n}{d\phi^2}=\left[(n-\mu-1)(n+\mu)+n\left(\frac{1}{\sin^2\phi}+n\cot^2\phi\right)\right]Q_\mu^n-\cot\phi(n-\mu-1)(n+\mu)Q_\mu^{n-1}$$

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