Philosophia Scientiæ
Travaux d'histoire et de philosophie des sciences

23-1 | 2019
Y a-t-il encore de la place en bas ?

Axioms as Definitions: Revisiting Poincaré and Hilbert

Laura Fontanella

Electronic version
URL: http://journals.openedition.org/philosophiascientiae/1827
DOI: 10.4000/philosophiascientiae.1827
ISSN: 1775-4283

Publisher
Éditions Kimé

Printed version
Date of publication: 18 February 2019
Number of pages: 167-183
ISSN: 1281-2463

Electronic reference
Laura Fontanella, “Axioms as Definitions: Revisiting Poincaré and Hilbert”, *Philosophia Scientiæ* [Online], 23-1 | 2019, Online since 01 January 2021, connection on 31 March 2021. URL: http://journals.openedition.org/philosophiascientiae/1827 ; DOI: https://doi.org/10.4000/philosophiascientiae.1827

Axioms as Definitions: Revisiting Poincaré and Hilbert

Laura Fontanella
Université Aix-Marseille, CNRS,
Institut de Mathématiques de Marseille, Marseille (France)

Résumé : Un problème fondamental dans la réflexion sur les fondements des mathématiques consiste à déterminer ce qu'est un axiome. Cette question est spécialement importante en vue de l'étude de nouveaux axiomes pour la théorie des ensembles (tels que les axiomes de grands cardinaux) dont la légitimité est fortement controversée ; cet article s'insère dans le débat. En analysant les écrits de Poincaré et de Hilbert, nous observons que, malgré les différences profondes dans la pensée de ces deux logiciens, ils parvinrent à la même conception des axiomes de la géométrie qui ne seraient que des *définitions* déguisées. Nous généralisons cette conception des axiomes comme définitions de concepts à n'importe quel système axiomatique (en particulier aux axiomes de la théorie des ensembles).

Abstract: A fondamental problem in the discussion on the foundations of mathematics is to clarify what an axiom is. This is especially important in the light of the most recent advances in set theory where new axioms have been proposed whose legitimacy is highly controversial (for example, large cardinal axioms); this paper is a contribution to this discussion. By analysing the view of Poincaré and Hilbert on axioms, we observe that, despite the deep differences in their philosophical thinking, the two logicians came to the same conception of the axioms of geometry as *definitions* in disguise. We revisit and generalise this view by arguing that *any* axiomatic system (set theory in particular) is the definition of some concepts.

1 Introduction

Before the development of non-Euclidean geometries in the 19th century, the axioms of Euclid's Elements were regarded as absolute principles formalising

Philosophia Scientiæ, 23(1), 2019, 167–183.

the basic laws of space. The introduction of hyperbolic and elliptic geometries challenged the dominant view on the a prioricity of geometry; if the principles of geometry were *a priori*, their violation would lead to a contradiction, instead, different geometries are possible. Despite the deep differences in their philosophical views, Poincaré and Hilbert came to the same conception of the axioms of geometry: they are *definitions in disguise*; rather than asserting undeniable truths, they fix the meaning of the basic terms of geometry (point, line, etc.) that would otherwise remain undefined. The main purpose of this paper is to extend Poincaré's and Hilbert's account of geometry to any axiomatic system.

Nowadays, mathematicians no longer consider geometry as a body of indisputable principles. Pluralism in geometry is safely accepted; in fact, contemporary geometers study different geometries and this has valuable applications to physics or even cryptography and coding theory. On the other hand, it is generally believed that the axioms of theories such as set theory or arithmetic have a different status, that they should have a non-conventional, possibly intrinsic, justification. In the words of Feferman:

> When the working mathematician speaks of axioms, he or she usually means those for some particular part of mathematics such as groups, rings, vector spaces, topological spaces, Hilbert spaces, and so on. These kinds of axioms have nothing to do with self-evident propositions, nor are they arbitrary starting points. They are simply definitions of kinds of structures which have been recognized to recur in various mathematical situations. I take it that the value of these kinds of structural axioms for the organization of mathematical work is now indisputable. In contrast to the working mathematician's structural axioms, when the logician speaks of axioms, he or she means, first of all, laws of valid reasoning that are supposed to apply to all parts of mathematics, and, secondly, axioms for such fundamental concepts as number, set and function that underlie all mathematical concepts; these are properly called foundational axioms. [Feferman, Friedman *et al.* 2000, 403]

We will question Feferman distinction between structural and foundational axioms by presenting a new approach to axioms, inspired by Poincaré's and Hilbert's account of geometry, that considers any axiomatic system as the definition of some concepts. We will focus in particular on set theory and argue that this can be regarded as an axiomatic definition of the concept of "set".

The paper is structured as follows. In section 2 we briefly explain the motivations for the present work. In sections 3 and 4, we discuss Poincaré's and Hilbert's conception of geometry, and we outline our broader conceptual approach to all axiomatic systems. In section 5 we propose a non-literal interpretation of the axioms of ZFC that justifies our view of set theory as a definition. Moreover, in this section we suggest that the very nature of the concept of "set" should not be sought in the idea of a collection of objects regarded as a totality in its own right, but rather in the possibility of performing specific operations on such collections. Finally, in section 6 we argue that our view point on set theory ties up with set theoretic pluralism.

2 Motivations for this work

Whilst axiomatic set theory was originally meant to provide a foundation for all mathematics, the development of forcing led to the discovery that many important mathematical problems cannot be solved within the classical theory of sets ZF. Several additional axioms have been introduced over the years that partially answer some of the questions that are independent from ZF: large cardinals axioms, forcing axioms, the axiom of determinacy and projective determinacy and others. This variety of strong principles, often incompatible with each other, led to debate what should be suitable criteria for new axioms for set theory. Several suggestions have been made which appeal to intrinsic or extrinsic criterias of various kind [see for instance Maddy 1988*a*,*b*], but the discussion largely depend on some preliminary questions: what foundational role shall we expect from set theory? Is there one absolute set theory? This paper is meant as a contribution to this discussion. We will argue that pluralism in set theory is the natural consequence of our approach to axioms as definitions: each theory of set is equally legitimate as a definition of the concept of "set" (see section 6).

Whether or not an axiom system is a definition in disguise is not just a linguistic matter, but changes the perspective on the role of axioms in mathematics and the goal of set theory. By arguing that any axiomatic system is the definition of some concepts, we intend to oppose to the very idea that the axioms of a theory express absolute mathematical truths. Thus, our view challenges the foundational role traditionally ascribed to set theory: we cannot expect for set theory to legitimate mathematical knowledge, although it can still aim at a fundamental role, namely providing a *conceptual* basis for mathematics by determining a concept of "set" as general as possible to embrace all suitable mathematical notions. More details about our view of the role of set theory for mathematics will be given in section 6.

We shall first clarify something that may easily lead to a misunderstanding. Every definition contains a sign or expression which had no meaning before and whose meaning is given by the definition. In mathematics it is quite common to introduce axiomatic definitions of symbols that are not included in the language of the theory considered. More precisely, suppose we have a theory T written in a language $\mathscr{L}$ and we want to define a new symbol R (a singular term, a predicate or a function) that is not in the language $\mathscr{L}$;

we can give a definition of R through sentences in the expanded language $\mathscr{L} + \{R\}$. These are called *implicit definitions*. Beth definability theorem establishes that, in first order logic, every implicit definition is equivalent to an explicit definition, namely one that depends on a formula of the original language $\mathscr{L}$ of the theory. Nevertheless, axiomatic definitions of this kind (implicit or explicit) require a background theory. In what sense then, can axiomatic systems that do not have a background theory, such as set theory, be regarded as definitions? We shall answer that in that case, the axioms fix the very meaning of the non-logical symbols of the language of the theory (i.e., the signature of the theory), such as $\in$ and $=$ in the case of set theory. These symbols are defined simultaneously through the axioms which establish their mutual relations with each other. Thus, as we will see, the axioms provide a *system of relations* between the terms so defined. To simplify the terminology, we will just say that the axioms of set theory define the concept of "set", or the axioms of arithmetic define the concept of "number", where what we actually mean is that set theory defines the symbols $\in$ and $=$, arithmetic defines 0, S, $+$ and $\times$, and so on.

3 Poincaré's account of geometry

Let us briefly discuss the evolution of geometry that led to the introduction of non-Euclidean geometries. Euclid's *Elements* is the first axiomatic presentation of a branch of mathematics; in this work, geometry is developed through five "axioms" and five "postulates". It is quite difficult to clarify the precise distinction between these two notions. For Aristotle an "axiom" is a self-evident proposition stating some general truth that is common to all sciences, while a "postulate" concerns only a specific science. It is not clear what meaning Euclid accorded to these two notions and whether or not he shared Aristotle's distinction; in any case, we can grant that in the philosophical tradition of the ancient Greeks both "axioms" and "postulates" shared the character of being *undeniable* statements. Nevertheless, one of Euclid's postulates, the fifth, was quite controversial.

The fifth postulate, also called "parallel postulate", can be stated as follows: *Given a straight line and a point outside the line, there is one and only one straight line passing through the point which is parallel to the given one.* Unlike the other principles of Euclid's axiomatisation, the legitimacy of the parallel axioms as a postulate was repeatedly questioned. While it was generally believed to be true, the common opinion seems to have been that it ought to be proved. There is evidence that Euclid himself tried to derive it from the other axioms and postulates before including it in his axiomatisation. Over the centuries, several attempts to find a demonstration of the parallel postulate (from the other axioms) were made, unsuccessfully. Then, in the 19th century, Lobachevskii, Bolyai, Gauss and Riemann considered various negations of the fifth postulate: the parallel postulate was replaced by the statements asserting the existence of *more than one* straight line (hyperbolic geometry) or *no straight line* (elliptic geometry) passing through the point and parallel to the given one. The resulting geometries were later proven to be consistent by Beltrami in 1868 (assuming the consistency of Euclid's geometry). For instance, the consistency of elliptic geometry can be proven by considering a sphere as a model, where the plane is identified with the surface of the sphere, the straight lines are the great circles, and points at each other's antipodes are taken to be equal.

Poincaré's analysis of these results is quite unassailable [see Poincaré 1902]: the axioms of Euclid's geometry do not establish experimental facts, because we do not have experience of ideal straight lines and points, nor they express *a priori* knowledge for otherwise it would not be possible to violate the fifth postulate without contradiction. What is, then, the nature of these axioms? Poincaré's answer is "they are definitions in disguise".

> The axioms of geometry are therefore neither synthetic a priori intuitions nor experimental facts. They are conventions. Our choice among all possible conventions is guided by experimental facts; but it remains free and is only limited by the necessity of avoiding every contradiction [...]. In other words, the axioms of geometry (I do not speak of those of arithmetic) are only definitions in disguise. [Poincaré 1902, 75–76, translation mine]

Viewed as definitions the axioms are neither true, nor false.

> What, then, are we to think of the question: Is Euclidean geometry true? It has no meaning. One geometry cannot be more true than another; it can only be more convenient [...] because it sufficiently agrees with the properties of natural solids, those bodies which we can compare and measure by means of our senses. [Poincaré 1902, 76, translation mine]

Pluralism in geometry is the natural consequence of this approach which grants absolute freedom for the formulation of new axiomatisations, provided they do not lead to a contradiction. No geometrical system is absolute, although one can be more appropriate than others for modelling certain aspects of the physical universe. For instance, Euclidean geometry conforms to our daily experience of the distance, length, dimension of the objects around us, while elliptic geometry gives a better account for larger portions of the Earth: if we consider any three items in our apartment and measure the sum of the internal angles of the triangle formed by these items, the result of our measurement would be 180° (approximatively) in accordance with the laws of Euclidean geometry, on the contrary if we take one item in Paris, the second in Sidney and the third in Buenos Aires the resulting value would be larger than 180° as elliptic geometry predicts.

In this view, geometry is not concerned with the properties of the objects to which the geometrical system is applied, but rather with *the set of relations* that hold between the primitive terms. We shall then stress another important aspect of Poincaré's conception of geometry: the meaning of the basic terms can only be fixed through their relations with each other, thus if we take a primitive term out of the axiomatic system it would lose all meaning.

> If one wants to isolate a term and exclude its relations with other terms, nothing will remain. This term will not only become indefinable, it will become *devoid of meaning*. [Poincaré 1900, 78, translation mine]

As we will see in section 4, analogous considerations can be found in Hilbert's writings: geometry defines a system of relations between the primitive terms which are meaningless outside the axiomatic system.

Poincaré's analysis of the other branches of mathematics is surely different. In his view, arithmetic is actually synthetic a priori and its certainty is guaranteed by the intuition. Mathematical induction, he says, is a synthetic a priori principle which is "imposed upon us with such a force that we could not conceive of the contrary proposition" [Poincaré 1902, 48]. Nevertheless, we shall object that it is actually possible to consider an arithmetic where the induction principle fails. For instance, Robinson Arithmetic Q is a version of arithmetic without induction [see Robinson 1950], and one can easily exhibit a model of Q where induction fails.[1] In principle, then, we can extend the same approach for geometry to arithmetic and regard its axioms not as absolute indisputable truths but as mere definitions of certain terms, namely "zero", "successor", "sum" and "product". The very meaning of these terms changes with the axiomatisations, thus "successor" means something different in Peano Arithmetic and Robinson Arithmetics.

Concerning set theory, Poincaré was one of the most famous opponents of modern set theory for its use of impredicative concepts. We will discuss his criticisms in more detail in Section 5 and we will appeal precisely on impredicative concepts to endorse our view that the axioms of set theory should be regarded as definitions.

4 The Frege-Hilbert controversy

Hilbert's first reference to axioms as definitions appears in his *Grundlagen Der Geometry* [Hilbert 1899] where he says that his axiomatization of geometry

1. It is enough to consider the natural numbers plus two additional elements a and b, then we interpret s and $+$ as the natural successor function and addition operation on the natural numbers, but we impose $s(a) = a$, $s(b) = b$, and for every natural number n, $a + n = a$, $b + n = n$, for every element in the domain x, $x + a = b$ and $x + b = a$. This model can be easily shown to satisfy the axioms of Q, and the induction fails as $0 + a = a$ while induction would imply for every x, $0 + x = x$.

should be intended as the definition of the concepts of "point", "line" and "plane". Puzzled by this statement, Frege asks for an explanation [see Frege 1980]; the exchange that followed enlighten us about the two logicians' general views on the role of axioms in mathematics.

Frege objects that axioms should be assertions, while definitions do not assert anything, but lay down something. Hilbert replies:

> In my opinion, a concept can be fixed logically only by its relations to other concepts. These relations formulated in certain statements, I call axioms, thus arriving at the view that axioms (perhaps together with propositions assigning names to concepts) are the definitions of the concepts. [Hilbert, letter to Frege 22.09.1900, translated in [Frege 1980, 51]]

Furthermore, the meaning of the terms so defined is tangled with the axioms chosen, and a different axiomatisation would change the meaning of the terms.

> [...] to try to give a definition of a point in three lines is to my mind an impossibility, for only the whole structure of axioms yields a complete definition and hence every new axiom changes the concept. [Hilbert, letter to Frege 29.12.1899, translated in [Frege 1980, 40]]

We shall rephrase this thought. Imagine we were asked to provide a precise definition for every geometrical notion. Then, for instance, we would define the notion of "triangle" as a "polygon" with three "edges" and three "vertexes". The notion of "polygon" is defined from the notion of "plane", the notion of "edge" can be defined from the notion of "line", and "vertex" is defined from "point". At this point we are supposed to define "plane", "line" and "point", but if we find a concept χ (or more concepts) that defines these notions, we will have to find a definition for χ and so on, leading to an infinite regress. Thus, as Hilbert says, we can define a concept only if we put it in relation to other concepts. We can stop this process, if we define the notions of "point", "line" and "plane" axiomatically, namely by describing their relations to one another through certain axioms.

> This is apparently where the cardinal point of the misunderstanding lies. I do not want to assume anything as known in advance; I regard my explanation in sec. 1 as the definitions of the concepts point, line, plane—if one adds again all the axioms of groups I to V as characteristic marks. If one is looking for other definitions of a "point", e.g., through paraphrase in terms of extensionless, etc., then I must indeed oppose such attempts in the most decisive way; one is looking for something one can never find, because there is nothing there and everything gets lost and becomes vague and

> tangled and degenerates into a game of hide-and-seek. [Hilbert, letter to Frege 29.12.1899, translated in [Frege 1980, 39]]

Analogous considerations can be made for the axioms of set theory. We can define every mathematical notion, including the notion of "function" and "number" from the notions of "set" and "membership" which can then be "defined" trough a series of axioms (the case of set theory will be discussed more extensively in section 5). In modern terms we say that the concept of "set" is a primitive notion as it is not defined in terms of other concepts. Nevertheless, the terminology "primitive notion" refers to something which is undefined as it is in no need for a definition, because its meaning is immediately understood; on the contrary, the view point that we want to defend here is that primitive notions are actually "defined" through axioms in the sense that the axiomatic system fixes their meaning in a formal and rigorous way.

Frege's conception of axioms represents the dominant view at the time of this correspondence with Hilbert:

> I call axioms propositions that are true but are not proven because our knowledge of them falls from a source very different from the logical source, a source which might be called spatial intuition. From the truth of the axioms it follows that they do not contradict one another. There is therefore no need for a further proof. [Hilbert, letter to Frege 27.12.1899, translated in [Frege 1980, 37]]

This evokes the delicate problem of mathematical truth. As we have seen, after claiming that geometry is a definition, Poincaré concludes that, as such, geometry is just a convention and its axioms are therefore neither true nor false. Hilbert, instead, does not intend to give up the idea that geometry is a body of truths. So in his reply to Frege, he does not deny the truth of axioms, but explains that their correctness has to be demonstrated by showing that they do not contradict each other.

> I was very much interested in your sentence: "From the truth of the axioms it follows that they do not contradict one another", because for as long as I have been thinking, writing, lecturing about these things, I have been saying the exact reverse: if the arbitrarily given axioms do not contradict one another with all their consequences, then they are true and the things defined by the axioms exist. This is for me the criterion of truth and existence. [Hilbert, letter to Frege 29.12.1899, translated in [Frege 1980, 42]]

We should point out (as C. Franks brilliantly observes in [Franks 2009]) that, despite this insistence on the necessity to prove the consistency of the axioms—not just for geometry but in general—, Hilbert was not skeptical about the correctness of mathematics, he simply had higher standard for what counts as a proof of it. While he was convinced that mathematical experience speaks for the consistency of axioms, his goal was to show that mathematics could stand on its own and prove its own consistency without appealing to extra-mathematical justifications. Now, while the consistency of geometry can be reduced to the consistency of analysis, it became clear very quickly to Hilbert himself that the *direct* consistency of analysis (i.e., not relative to another theory) would face significant difficulties. Hilbert hoped to ultimately reduce all mathematics to a unique axiomatic system and then prove its direct consistency. Today we know from Gödel's incompleteness results that Hilbert's program cannot be accomplished.

For our part, by claiming that any axiomatic system is a definition, we oppose to the very idea that axioms express absolute truths. While any axiom takes the form of a sentence, it does not assert anything, it is meaningful only insofar as it contributes with the other axioms of the system to the definition of a concept. Thus, no axiom system expresses profound absolute mathematical truths, it only defines certain concepts which may be more or less suitable for modelling different things. Whatever is the source of mathematical knowledge, or whether mathematics is simply not a body of truths, is not the object of the present paper.

Hilbert further illustrates his view of the nature of the definition provided by the axioms in yet another passage of this correspondence. Frege objects that the basic concepts that are claimed to be defined by his axioms ("point", "line", "plane") are not unequivocally fixed. For instance, a "point" could be a pair of numbers, a triple, a tuple and so on. Hilbert replies:

> You say that my concepts, e.g., "point", "between" are not unequivocally fixed; e.g., "between" is understood differently on p. 20, and a point is there a pair of numbers. But it is surely obvious that every theory is only a scaffolding or schema of concepts together with their necessary relations to one another, and that the basic elements can be thought in any way one likes. If in speaking of my points I think of some system of things, e.g., the system: love, law, chimney-sweep... and then assume all my axioms as relations between these things, then my propositions, e.g., Pythagoras' theorem, are also valid for these things. [...] At the same time, the further a theory has been developed and the more finely articulated its structure, the more obvious the kind of application it has in the world of appearances and it takes a very large amount of ill will to want to apply the more subtle propositions of plane geometry or of Maxwell's theory of electricity to other appearances than the ones for which they were meant... [Hilbert, letter to Frege 29.12.1899, translated in [Frege 1980, 42]]

Thus the axioms do not point out concrete systems of things, altogether they define a *schema of concepts*. To explain this view, we may consider, as an analogy, the distinction between an individual and its shape: distinct individuals can have the same shape, since the axioms define the "shape" so to speak, not the individuals.

Both Poincaré and Hilbert refer to the axioms of geometry as fixing the *relations* between the primitive terms. In this attention for the structural aspects of the axioms, we can perhaps see in the two logicians a hint of modern structuralism. In fact, Shapiro refers to Hilbert's kind of axiomatic definitions as "structural definitions" [Shapiro 1999]. Nevertheless, one of the main concerns of *Ante Rem* structuralism is categoricity, which does not seem any of Hilbert's worries as far as geometry is concerned[2]: in fact, in the above passage, Hilbert seems to suggest that no system of things is the unique possible interpretation of the axioms. Thus, in this view, the axioms of geometry would most certainly not define an *Ante Rem* structure.

In the case of set theory, Shapiro's structuralist approach to axioms as definitions would be especially problematic, because neither ZFC, nor second-order ZFC are categorical[3] as *Ante Rem* Structuralism requires. More importantly, Shapiro replaces the requirement of consistency of a theory with the one of "satisfiability", namely the existence of a model[4], but as Shapiro himself points out, satisfiability requires a background set theory where such a model can be found, hence it cannot work as a reasonable criterion for set theory itself.

In our view, the axioms of a theory do not entail any ontological commitment to the schema of concepts so defined, the axioms only fix the conditions for a certain system of things to match the schema. We shall discuss this more precisely in the next section.

5 On the meaning of existential quantifiers

We mentioned Poincaré's skepticism about set theory due to the use of impredicative concepts. An example is given by the Powerset axiom which is formally stated as $\forall x \exists y \forall z (z \in y \iff z \subseteq x)$. This sentence, if interpreted as

---

2. On the other hand, categoricity was for Hilbert an important issue in connection with the reals.

3. Shapiro, however, remarks that second-order ZFC is quasi-categorical, namely if M and M' are two standard models of second order ZFC, then either M is isomorphic to M' or else one of them is isomorphic to an initial segment of the other. Quasi-categoricity is enough to fix certain references, for example the empty set ω, and others are unique up to isomorphism.

4. This is motivated by the observation that in second-order logic a theory can be consistent and yet not have a model. In fact, there is no completeness theorem, so if T is the conjunction of the second order axioms of Peano arithmetic and G is a standard Gödel sentence that states the consistency of T, then by the incompleteness theorem $P \wedge \neg G$ is consistent, but it has no models because every model of P is isomorphic to the natural numbers, hence G is true in all such models. Therefore, despite its consistency, $P \wedge \neg G$ fails at describing a possible structure.

"given a set x, the set of all subsets of x exists", involves a circularity, because for this statement to make sense, one needs to assume the very possibility of a totality of all subsets of x, precisely the Powerset of x. For this reason, if we think of this axiom as legitimating the totality of all subsets of x, we run into a circle. On the other hand, if one thinks that of statement as a way of "singling out" something which is already available as a legitimate mathematical notion, then the axiom is not problematic. In other words, for the axiom to make sense, we need to assume that in the domain of discourse over which the variables may range we already have something that plays the role of the the Powerset of x, the axiom does not literarily establish its existence or legitimate it as a valid mathematical notion, but only label it as a "set". In this sense, the most appropriate interpretation of the Powerset axiom would be "given a set x, the Powerset of x *is a set*".

In other words, the existential quantifiers in set theoretic sentences act as *filters of sets*, namely as a way of selecting sets from other collections that should not be regarded as sets. This interpretation is supported by Zermelo's terminology in his original axiomatisation of set theory:

> Set theory is concerned with a "domain" of individuals, which we shall call simply "objects" and among which are the "sets". [Zermelo 1908, 262] as translated in [Van Heijenoort 1967, 203]

Since we do not want to make any ontological commitment to the mathematical notions that are involved in set theory, we will avoid the locution "object" and we will talk of a "domain of legitimate mathematical notions" among which are the "sets". We shall not discuss the nature of the "legitimate mathematical notions" (whether they exist as part of a platonic reality immutable and independent from human thinking or they are just useful fictions) as this is irrelevant to the main thesis of the present work that any axiom system is a definition in disguise. Nor we will propose suitable criteria for establishing the legitimacy of a given mathematical notion. We will simply assume that the legitimate mathematical notions are previously available in the domain of individuals where "sets" are selected. Analogous considerations can be made for arithmetic where the existential quantifiers would act as "filters of numbers". Thus, in general, the existential quantifiers in a mathematical sentence filter out the legitimate notions that fall into the concept defined through the relevant axiom system.

Based on these considerations, we propose the following non-literal interpretation of the axioms of ZFC.

- (Extensionality) Two *sets* are equal if they contain the same *sets* as elements.
- (Pairing) Given two sets a and b, the pair $\{a, b\}$ (the collection containing exactly a and b as elements) **is a set**;
- (Separation) Given a formula $\varphi(x, \vec{p})$ with *sets* parameters $\vec{p}$, and given a *set* a, the collection of all x in a that satisfy $\varphi(x, \vec{p})$ **is a set**;

- (Union) Given a *set* a, the union of a (the collection of all *sets* that belong to some element of a) **is a set**;
- (Power Set) Given a *set* a, the collection of all subsets of a **is a set**;
- (Replacement) Given a function $f(x)$ (defined with set parameters) and a *set* a, the collection of all $f(x)$ with $x \in a$ **is a set**;
- (Foundation) A collection of *sets* that does not have an $\in$-minimal element **is not a set**;
- (Infinity) Consider all the collections *of sets* that contain the empty set and are closed by the operation $x \mapsto x \cup \{x\}$, at least one of such collections **is a set**;
- (Choice) For every family of nonempty sets *which is itself a set*, the image of the choice function **is a set**.

Under this interpretation, the axioms of ZFC are not meant as instructions for constructing mathematical objects, but rather as an axiomatic definition of the word "set".

The basic set theoretic operations such as the union, or the power set of a set are not legitimated by the axioms, instead the possibility of those operations is assumed in advance. The axioms state that the resulting collections can be considered to be "sets".

Formal existence is really a matter of what the axioms, *taken as a whole*, determine to be a "set". In fact, when we apply the existential quantifier to a certain collection, we make the collection *available for the other axioms*. In this sense, the whole theory ZF (or ZFC) defines the word "set". Now, we argue that the very nature of the concept of set should not be sought in the idea of a collection of objects regarded as a totality in its own right, but rather in the possibility of performing specific *operations* on such collections. To support this claim, let us go back to the very origin of the concept of set, that historians of mathematics date back to Cantor's "derived point-set".

> It is a well determined relation between any point in the line and a given set P to be either a limit point of it or no such point, and therefore with the point-set P the set of its limit point is conceptually co-determined; this I will denote P' and call *the first derived point-set* of P. [Cantor, as quoted in [Ferreirós 2007, 143]]

At this point, Cantor applied the operation of derivation to the derived point-set P', obtaining "the second derived point-set P'' and reiterated the process. What is crucial here is that, before Cantor, there was already talk of sets and collections of points. Therefore, Cantor's original contribution should not be sought in the concept of collections intended as totalities in their own right, but rather in the very idea that such collections were available for certain mathematical operations such as the derivation. In this spirit, we consider the concept of "set" to be related to the possibility of applying specific operations to a given collection of objects.

To further illustrate this view, we should consider for instance the Axiom of Choice, AC. The fact that AC was implicitly used in many mathematical proofs, even before Zermelo's explicit formulation, suggests that this was generally accepted as a natural principle. Thus the choice function can be regarded as a perfectly legitimate operation and the debate should not concern the legitimacy of this function, but rather the idea that the collection of objects derived from such a collection (namely the image of the choice function) can be taken to be a "set", that is whether it can be made available for the set theoretic operations definable from the other axioms.

We shall, then, reconsider our approach to the thorny problem of the choice of new axioms for set theory. When discussing the axiom of choice or large cardinals axioms, we wonder whether or not a choice function *exists*, inaccessible cardinals *exist* and so on. This leaves us under the impression that the foundational goal of set theory is to detect which mathematical entities do or do not exist, are or aren't legitimate mathematical notions. But again it would be naïve to think that set theory can dictate the terms for an ontology of mathematics. In our perspective, the discussion over large cardinal axioms should not be phrased in terms of existence or non-existence of large cardinals, but rather as the problem of whether a certain "large" collection of ordinals can be made available for the operations defined by the axioms of ZFC (or ZF) without contradiction. In fact, in second-order logic one can define being "(strongly) inaccessible" as a property of *classes* then, for instance, the class of ordinals is inaccessible in this sense (provided this class is accepted as a legitimate mathematical notion). Thus the legitimacy of large cardinals axioms concerns the problem of whether or not any inaccessible collection can be taken to be a set, namely whether one can apply the other set-theoretic operations to such a collection without contradiction.

6 A multitude of concepts of set

We have outlined our view of the axioms of set theory as definitions in disguise. In this perspective, a single axiom is not an assertion inherently true or false, but *the whole system* of axioms defines the concept of "set". It follows that by adding or removing one or more axioms, we change the concept defined. So, in particular, the concept of set defined by the theory ZF is different than the one defined by the theory ZFC, or ZF+V=L, or ZF + $\exists\kappa(\kappa$ is measurable). Now, a definitional view point of axioms does not necessarily lead us to embrace a wild formalist view of mathematics considered to be a meaningless game where random theories are investigated together with their logical consequences. We cannot rule out, for instance, the possibility of an inter-subjective or innate concept of set that would make some set theories "truer" than others to the extent that they conform to the intended concept.

Nevertheless, the panorama of theories defended nowadays leaves little hope for a universal agreement on a unique concept of set. Conflicting intuitions are behind the most promising enrichments of ZF: on the one hand, we have the idea that sets must be obtained by a cumulative process, namely at each stage we throw in "the basket of sets" only those collections that are obtained from the previous stages with operations that are definable in a somehow "canonical" way—it is the case, for instance, of V=L or V=Ultimate L—, on the other hand, we have more "liberal" axioms, such as Forcing Axioms, where roughly anything that can be forced by some "nice" forcing notions is in V, in other words it is a "set".

This pluralism of concepts of sets brings us to support Feferman's thesis that the continuum hypothesis is an inherently vague question [see Feferman 2011–2012], although our arguments are somewhat different. The very meaning of the continuum changes over the theories, indeed we may even agree on what is a collection of natural numbers[5], but which of those collections are "sets" depends on the concept of "set" considered. Thus for instance, in the framework of forcing axioms, all the collections of natural numbers that can be forced by nice forcing notions are "sets", thus the continuum is quite large and, not surprisingly, CH fails (the strongest forcing axioms imply that the continuum is $\aleph_2$). On the contrary, the concept of "set" defined by the theory ZF+V=L is more restrictive, thus only few collections of natural numbers are sets in this theory, and in fact CH holds.

Now, if set theory is not a body of absolute truths, but a mere definition of some concept, it is natural to wonder in what rests its role for mathematics. The answer may be sought in the *richness* or *abstractness* of the concept defined, as all the basic mathematical notions such as groups, vector spaces, even numbers can be regarded as sets and can be made available for the set theoretic operations definable from the axioms of the theory ZF. Thus the goal of set theory does not consist in justifying the existence of mathematical notions or the truth of mathematical propositions, the aim of a theory of sets should be to define a notion of "set" as rich as possible to embrace every useful mathematical notion.

On the basis of the "richness" of the underlying concepts of sets, we may come to prefer one theory over the others. This line of thought brings us to evoke a fact that is often considered to be a natural motivation for large cardinals axioms. As pointed out by Steel,

> The language of set theory as used by the believer in V=L can certainly be translated into the language of set theory as used by the believer in measurable cardinals, via the translation

5. The continuum is the size of $\mathbb{R}$, or equivalently the size of the powerset of $\mathbb{N}$.

> $\varphi \mapsto \varphi^L$. There is no translation in the other direction. [Feferman, Friedman *et al.* 2000, 423]

In other words, the concept of set carried by V=L can be expressed in the theory of measurable cardinals, while the converse seems to be prima facie impossible. Thus, the notions of "set" underlying the theory of large cardinals is more expressive than V=L.

Nevertheless, Hamkins presented a serious challenge to this argument. He showed that:

> Even if we have very strong large cardinal axioms in our current set-theoretic universe V, there is a much larger universe V^+ in which the former universe V is a countable transitive set and the axiom of constructibility holds. [Hamkins 2014, 29]

Thus, even the axiom of constructibility is rich enough to allow us to talk about the concept of sets underlying large cardinals axioms within a model of $V = L$.

It follows that the natural outcome of our definitional perspective is pluralism, which in contemporary set theory is represented by the "*multiverse conception*" (of which Hamkins is one of the main supporters [Hamkins 2012]). This can be described as the view that there are many distinct and equally legitimate concepts of sets, as opposed to the "*one universe view*" which in contrast asserts that there is only one absolute set concept with a corresponding absolute set-theoretic universe where every set-theoretic question has a definite answer. We should, however, add an important note: some set theorists endorse the multiverse view as an extreme form of Platonism where not just one, but many universes exist as an independent reality; our view, on the contrary, does not entail any ontological commitment to the concept defined through the axioms.

7 Conclusion

In conclusion, we have revisited Poincaré's and Hilbert's view of geometry as a definition in disguise and we have extended this approach to all axiomatic systems. Then, we have proposed an interpretation of the axioms of ZFC where the existential quantifiers are intended as filters of sets, namely as ways of singling out sets from collections that are not worth the title of "set". This naturally leads us to regard set theory as an axiomatic definition of the concept of set. Furthermore, we have argued that the very nature of this concept should not be sought in a collection of objects regarded as a totality in its own right, but rather in the idea that certain collections are available for specific operations definable from the other axioms. We have observed that the concept of set so defined changes if one adds or removes one or more axioms from the theory. This leads to a pluralism of concepts of sets varying with the theories. Finally, we have claimed that despite pluralism, set theory can still play a fundamental role for mathematics, and this is to be sought in the "richness" of the concept of set underlying the theory, which is meant to embrace all suitable mathematical notions.

Acknowledgements

I am indebted to the anonymous reviewers for their insightful comments and suggestions which greatly helped improve the quality of this paper.

Bibliography

Feferman, Solomon [2011–2012], Is the continuum hypothesis a definite mathematical problem?, `http://logic.harvard.edu/EFI_Feferman_IsCHdefinite.pdf`, for *Exploring the frontiers of incompleteness* (EFI)Project, Harvard.

Feferman, Solomon, Friedman, Harvey M., *et al.* [2000], Does mathematics need new axioms?, *Bulletin of Symbolic Logic*, 6(4), 401–446, doi: 10.2307/420965.

Ferreirós, José [2007], *Labyrinth of Thought: A History of Set Theory and its Role in Modern Mathematics*, Basel: Birkhäuser, 2nd edn., edited by E. Hiebert and E. Knobloch.

Franks, Curtis [2009], *The Autonomy of Mathematical Knowledge: Hilbert's Program Revisited*, Cambridge University Press.

Frege, Gottlob [1980], *Philosophical and Mathematical Correspondence*, Oxford: Blackwell, edited by G. Gabriel *et al.*

Hamkins, Joel David [2012], The set-theoretic multiverse, *Review of Symbolic Logic*, 5(3), 416–449, doi: 10.1017/S1755020311000359.

—— [2014], A multiverse perspective on the axiom of constructiblity, in: *Infinity and Truth*, edited by C-T Chong, Q. Feng, T. A. Slaman, & W. H. Woodin, Hackensack: World Scientific, 25–45, doi: 10.1142/9789814571043_0002.

Hilbert, David [1899], *Grundlagen der Geometrie*, Leipzig: Teubner, English translation in *Foundations of Geometry*, L. Unger (ed.), La Salle: Open Court Press, 1971.

Maddy, Penelope [1988*a*], Believing the axioms. I, *Journal of Symbolic Logic*, 53(2), 481–511, doi: 10.1017/S0022481200028425.

—— [1988*b*], Believing the axioms. II, *Journal of Symbolic Logic*, 53(3), 736–764, doi: 10.2307/2274569.

Poincaré, Henri [1900], Sur les principes de la géométrie: réponse à M. Russell, *Revue de Métaphysique et de Morale*, 8(1), 73–86.

—— [1902], *La Science et l'Hypothèse*, Paris: Flammarion, 1968, English translation in *Science and Hypothesis: The Complete Text*, M. Frappier, A. Smith, and D. J. Stump (trans.), London; New York: Bloomsbury, 2017.

Robinson, Raphael M. [1950], An essentially undecidable axiom system, in: *Proceedings of the International Congress of Mathematics*, 729–730.

Shapiro, Stewart [1999], Implicit definitions and abstraction, for *Abstraction Day 99*, University of St Andrews.

Van Heijenoort, Jean [1967], *From Frege to Gödel: A source book in mathematical logic*, Cambridge, Mass.: Harvard University Press.

Zermelo, Ernst [1908], Untersuchungen über die Grundlagen der Mengenlehre, I, *Mathematische Annalen*, 65, 261–281, English translation in [Van Heijenoort 1967, 201–215].