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Process over product: It's more than an equation

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Process over product: It's more than an equation

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Developing number and algebra together provide opportunities for searching for patterns, conjecturing, justifying, and generalising mathematical relationships. It allows the focus to be on the process of mathematics and noticing the structure of arithmetic, rather than the product of arriving at a correct answer. Two of the big ideas in mathematics are multiplicative thinking and algebraic reasoning. By noticing the structure of multiplicative situations, students will be in a position to reason algebraically, and the process of reasoning algebraically will allow students to appreciate the value of thinking multiplicatively rather than additively.

INTRODUCTION

Improving the quality of mathematics teaching and learning is an issue of discussion around the world (Cobb & Jackson, 2011), and yet, how to support it is not as well researched as it could be (Cohen, Moffitt, & Goldin, 2007). One area where there is substantial research is around the impact that teachers have on student academic success (Chetty, Friedman, & Rockoff, 2014). How such success is accomplished has encouraged further research into how students learn mathematics (Daro, Mosher, & Corcoran, 2011).

The connections between how students learn mathematics and the way in which teachers teach are as complex as the practice of teaching (Hattie, 2015). The practice of teaching asks teachers to make informed choices on a daily basis. These choices should always be focussed on developing mathematically powerful classrooms (Schoenfeld, 2014). It is difficult to have a mathematically powerful classroom if the classroom is one that is predicated solely on finding answers as opposed to one that values noticing the process or structure so that students can reach the answer to the immediate problem, but also generalise what they have learned in order to be able to solve future problems.

Generalisation, or the noticing of structure, is fundamental to mathematical success (Mason, Graham, & Johnston-Wilder, 2005) and needs to be incorporated at all levels of the teaching and learning of mathematics. Mason et al. (2005) wrote that, when they first come to school, young children are able to generalise. While generalising is quite innate, children need to practise, strengthen, and extend this natural ability. Teachers need to ask students explicit questions about what they notice, the patterns they can see, and how they are making sense of the mathematics (Day, 2017). Whether it be learning to trust the count or developing proportional reasoning, an understanding of structure is highly desirable. For the sake of this article we will explore noticing structure in terms of developing multiplicative and algebraic reasoning.

Empson, Levi, and Carpenter (2011) suggested that children seeing mathematics as a collection of procedures, rather than a process of noticing the structure of number and operations, causes major issues later in schooling. This is, we propose, very pertinent in the areas of multiplicative thinking and algebraic reasoning. The role that multiplicative thinking plays as a foundational concept underpinning the development of further mathematical ideas has been well documented (Hurst & Hurrell, 2016; Mulligan & Watson, 1998; Siemon, Izard, Breed, & Virgona, 2006). Brown and Quinn (2006) linked an ability to think multiplicatively to success with algebraic reasoning while Jacobs, Franke, Carpenter, Levi, and Battey (2007) contended that noticing number relationships and relational thinking are key indicators of success in algebraic reasoning. Multiplicative thinking and algebraic reasoning are clearly linked and we contend that this linkage is about noticing mathematical structure and the process of generalising.

MULTIPLICATIVE THINKING

Multiplicative thinking is not easy to teach or to learn. Whereas most students enter school with informal knowledge that supports counting and early additive thinking (Sophian & Madrid, 2003) students need to re-conceptualise their understanding about number to understand multiplicative relationships (Wright, 2011). Multiplicative thinking is distinctly different from additive thinking even though it is constructed by children following on from their additive thinking processes (Clark & Kamii, 1996). We would contend that multiplicative thinking is not just about a procedural ability with multiplication, it is a deeper conceptual understanding of how multiplication acts on numbers, an understanding of the process or the structure of multiplication, and not just the product.

One of the best ways we can facilitate an exploration of structure is through implementing good tasks. Let us take for example the use of a multiplicative array to look at the structure and processes behind how we reach a product when completing a multiplication algorithm. It is difficult to explore abstract ideas such as the distributive property without having materials to manipulate. The materials, in this case the multiplicative array, promote the cultivation of the concept, and then the capacity to test any developing ideas, in order to create understanding (Hurst & Hurrell, 2016). The use of materials fits with Bruner's (1966) work on learning concepts through following a progression. Bruner's progression starts with the enactive stage (which requires concrete materials), moves to the iconic stage where pictorial representations are employed and finishes at the symbolic stage, the stage of abstraction. More contemporary researchers employ the same basic structure of learning in prosecuting the CRA (Concrete-Representation-Abstract) approach (Agrawal & Morin, 2016).

A multiplicative array is a representation of objects (we have found that, at least initially, it is preferable to use tiles which can tessellate) in which the multiplier and the multiplicand are exchangeable (Figure 1). Arrays are seen as a powerful way in which to represent multiplication (Young-Loveridge & Mills, 2009). Multiplicative arrays have the potential to allow students, among other things, to visualise factors, commutativity, associativity, and distributivity (Wright 2011). In this article we will keep the focus on distributivity.

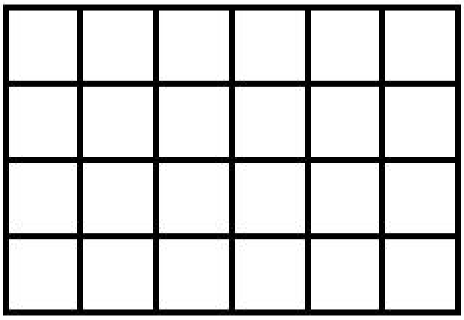


Figure 1. A multiplicative array showing 4 x 6.

Through the manipulation of a variety of arrays, the array should be established and accepted as a legitimate representation of single digit by single digit multiplication. Once this has occurred the array can then be extended into two-digit by one-digit multiplication. For example we can construct an array to represent 7 x 12 (Figure 2).

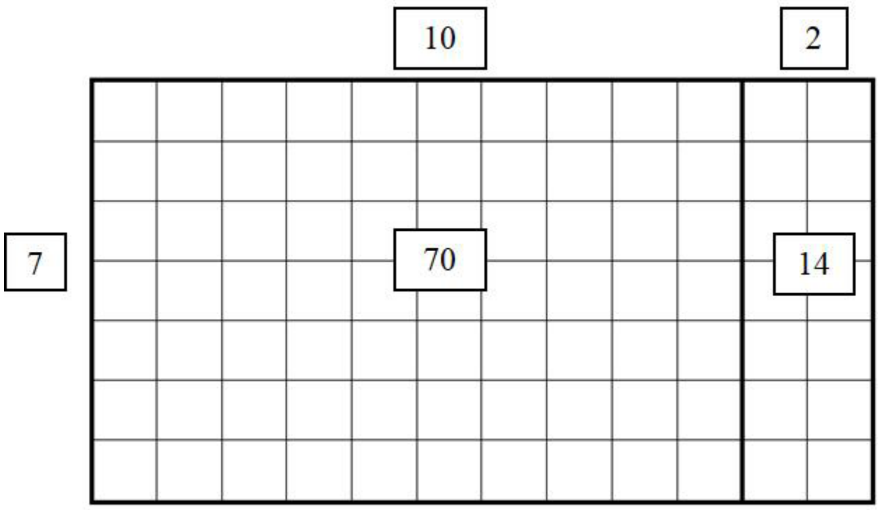


Figure 2. A partitioned 7 x 12 array.

What has been created is a visual representation of what is happening when multiplying 7 x 12, a representation which also encourages an understanding of the magnitude of numbers. Once the model has been used to develop two-by one-digit multiplication, it is then a reasonable step to move to a representation of two-by two-digit multiplication (Figure 3). This can still be done as a concrete model using tiles, but it does start to be less efficient. This loss of efficiency gives impetus for the student to move away from manipulative materials to the more efficacious, pictorial representations. One of the strengths of the pictorial representation of the array is that it illustrates a problem which is quite common when students start multiplying double digit by double digit numbers, that is not addressing all of the different parts which need to be multiplied. Students who have not used an array model may assume that 14×12 can be calculated by $10 \times 10 + 4 \times 3$. Using the array model and identifying the areas that are created, helps the students see why this is not the case.

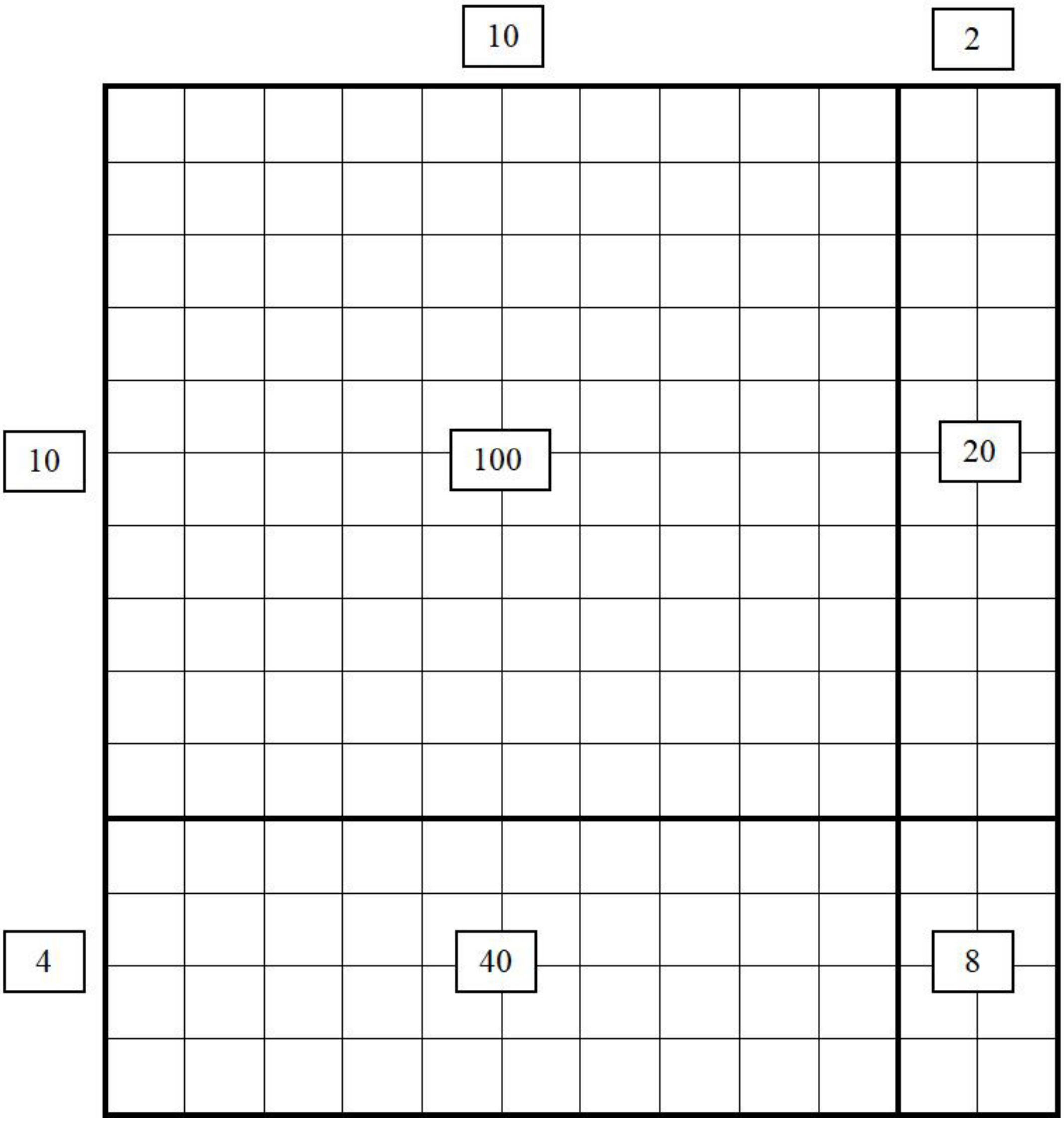
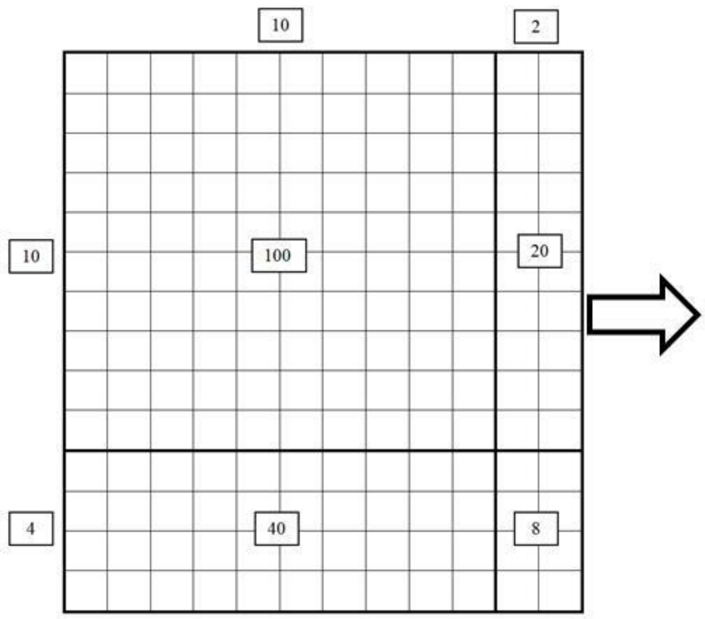


Figure 3. A partitioned 14 x 12 array.

It would now seem appropriate to move from the pictorial representation and to introduce an algorithm. By having the array representation at hand (Figure 3), a direct comparison can be made between this representation and a more abstract representation (Figure 4). The initial use of a non-standard algorithm may be a better way to illustrate to what the abbreviated notation really refers (e.g., the six in 168 refers to 60) than beginning with the standard algorithm (Figure 5).



x	10	2
10	100	20
4	40	8

$$14 \times 12 = 100 + 40 + 20 + 8$$
$$= 168$$

Figure 4. Moving from the pictorial representation of 14 x 12 to the abstract representation.

x	10	2
10	100	$\boxed{20}$
4	$\boxed{40}$	8

$$\begin{array}{rrr} & 1 & 4 \\ \times & 1 & 2 \\ \hline & \boxed{2} & 8 \\ 1 & \boxed{4} & 0 \\ \hline 1 & 6 & 8 \\ \hline \end{array}$$

Figure 5. Illustrating the links between the non-standard and standard algorithms.

We can teach the algorithm without looking at the structure, and this would sit firmly in the area of computation, the goal being to arrive at the answer. By switching the focus about how we get to the answer, the task becomes about the structure. The answer, although still very important, is no longer the only goal, understanding how we reach the answer becomes the primary goal.

ALGEBRAIC REASONING

Kaput and Blanton (2005) defined "Algebraic reasoning [as] a process in which students generalise mathematical ideas from a particular set of instances, establish those through the discourse of argumentation and express them in increasingly formal and age appropriate ways" (p. 99). As with multiplicative thinking, in fact all mathematical thinking, truly focussing on algebraic reasoning is not simply about finding answers to problems, but about providing opportunities for discovering patterns, conjecturing, and generalising mathematical relationships (Siemon, Beswick, Brady, Clark, Faragher, & Warren, 2015). This is a process approach rather than a product approach.

Confusion about the four operations is often demonstrated by students making procedural errors, quite often by trying to use the structures that apply only to addition in other operations. For example, a common method used to add 45 + 18 is to add the tens, add the ones and then add the answers together 45 + 18 = (40 + 10) + (5 + 8) = 63. If this strategy is used for multiplication 45 x 18 = (40 x 10) + (5 x 8) provides the incorrect answer of 440. Schifter (2018) suggested that "students who make the errors … may be thinking of the correct addition strategy as the way *numbers* work, rather than how *addition* works" (p. 29).

A focus on the behaviour of the operations allows students to see that an operation is not just a procedure but also a mathematical object in its own right (Sfard, 1991). Once the operations are seen as objects with their own set of characteristics the students will be able to recognise and apply them to solve other problems. In order to do this students need to be encouraged to notice the structural properties of the operations and to explain in general terms the strategies they use when calculating answers (Schifter, 2018).

One of the important aids to noticing structure in the operations is utilising a CRA approach. Just as with multiplicative thinking, being able to visualise the ways that operations work allows students to explain how and why general strategies must work (Day, 2017). For example, if students are investigating what happens to a sum when one of the addends is increased by one, they may make a model, as in Figure 6. By noticing what changes and what stays the same in each case, students can come up with a conjecture that they can then test using diagrammatic representations, eventually arriving at a generalisation. The generalisation may be expressed in words or in symbols
$a + (b + 1) = (a + 1) + b = (a + b) + 1$.

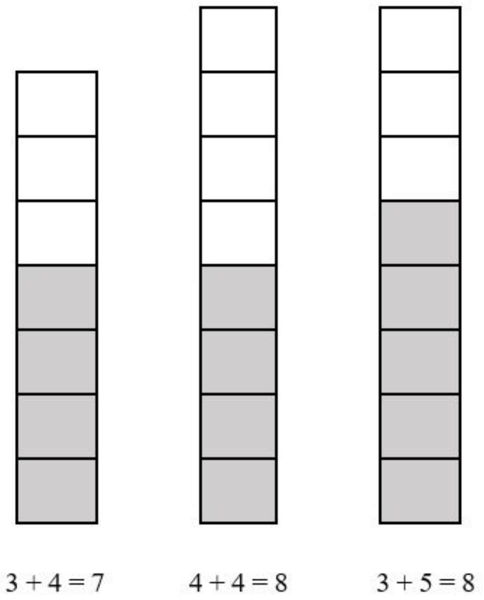


Figure 6. Representations of increasing the addend by one.

Students could then investigate what happens to the product of a number if you increase a factor by one, they may construct arrays, as in Figure 7.

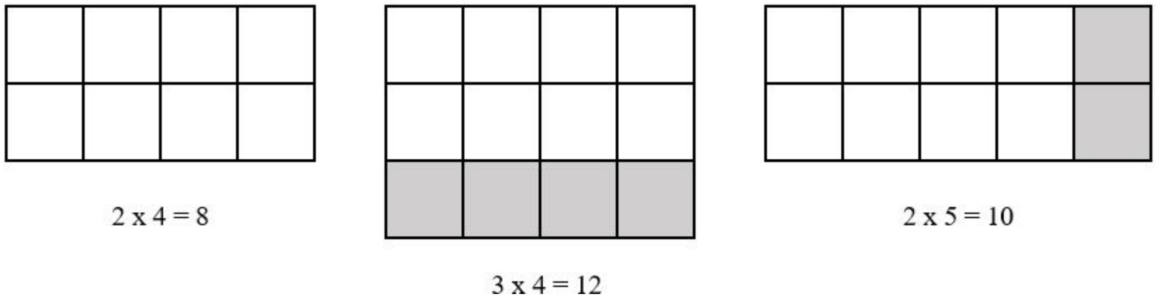


Figure 7. Arrays demonstrating increasing a factor by one.

The concrete representation allows students to physically see that increasing the number of groups (the first factor) affects the product by the number in each group (the second factor). They may have to look at several cases to recognise that this is what is happening. Students can be asked to represent the situations using other representations such as a story and a picture to illustrate their story (Schifter, 2018). Once again, by noticing what changes and what stays the same in each case, students can arrive at a conjecture to test, finally generalising the result. The generalisation may be in words or symbols $a \times b = ab$, $(a + 1)b = ab + b$ and $a(b + 1) = ab + a$, depending on the developmental stage of the students.

Importantly, students should have the opportunity to compare the differences between the different operations. In the cases mentioned above, students should have the opportunity to explain how multiplication differs from addition, otherwise the students may just see the two investigations as unrelated (Russell, Schifter, Kasman, Bastable, & Higgins, 2017; Schifter, 2018). Russell et al. (2017) created a teaching model based on five phases of investigation:

- Noticing regularity
- Articulating a claim
- Investigating through representations
- Constructing arguments
- Comparing and contrasting operations

This model can be used in a variety of situations to draw students' attention to the structure of arithmetic in order to use the structure to reason algebraically.

CONCLUSION

The argument here should not be about a choice between arithmetic fluency and an understanding of the underlying structure of the mathematics. Both can be achieved through the judicious use of quality tasks, and an attention to the processes used to reach an answer. As teachers it is important that we push the students to look at what is happening and why. Doing so helps us, and the students, to employ the Proficiency Strands as articulated in the Australian Curriculum.

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