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Abstract

We propose a natural discretisation scheme for classical projective minimal surfaces. We follow the classical geometric characterisation and classification of projective minimal surfaces and introduce at each step canonical discrete models of the associated geometric notions and objects. Thus, we introduce discrete analogues of classical Lie quadrics and their envelopes and classify discrete projective minimal surfaces according to the cardinality of the class of envelopes. This leads to discrete versions of Godeaux-Rozet, Demoulin and Tzitzéica surfaces. The latter class of surfaces requires the introduction of certain discrete line congruences which may also be employed in the classification of discrete projective minimal surfaces. The classification scheme is based on the notion of discrete surfaces which are in asymptotic correspondence. In this context, we set down a discrete analogue of a classical theorem which states that an envelope (of the Lie quadrics) of a surface is in asymptotic correspondence with the surface if and only if the surface is either projective minimal or a Q surface. Accordingly, we present a geometric definition of discrete Q surfaces and their relatives, namely discrete counterparts of classical semi-Q, complex, doubly Q and doubly complex surfaces.

Keywords: discrete differential geometry, projective minimal surface, Lie quadric, envelope

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1. Introduction

Projective geometry is arguably "The Queen of All Geometries". In 1872, this was recognised by Felix Klein whose pioneering and universally accepted Erlangen program revolutionised the way (differential) geometry was approached and treated. Thus, Klein [1] proposed that geometries should be classified in terms of groups of transformations acting on a space (of homogeneous coordinates) with projective geometry being associated with the most encompassing group, namely the group of projective collineations. Apart from placing Euclidean and affine geometry in this context, this approach gives rise to projective models of a diversity of geometries such as Lie, Möbius, Laguerre and Plücker line geometries.

Projective differential geometry was initiated by Wilczynski [2, 3, 4] representing the "American School" and later formulated in an invariant manner by the "Italian School" whose founding members were Fubini and Ĉech. Cartan is but one of a long list of distinguished geometers who were involved in the development of this subject with monographs by Fubini & Ĉech [5, 6], Lane [7], Finikov [8] and, most notably, Bol whose first two volumes [9, 10] consist of more than 700 pages. Recently, there has been a resurgence of interest in projective differential geometry. Here, we mention the monograph *Projective Differential Geometry. Old and New* by Ovsienko and Tabachnikov [11] and the *Notes on Projective Differential Geometry* by Eastwood [12]. Interestingly, as pointed out in [13], projective differential geometry also finds application in General Relativity in connection with geodesic conservation laws.

Projective geometry has proven to play a central role in both discrete differential geometry and discrete integrable systems theory [14]. Thus, it turns out that classical incidence theorems of projective geometry such as Desargues', Möbius' and Pascal's theorems lie at the heart of the integrable structure prevalent in discrete differential geometry. Their algebraic incarnations provide the origin of the integrability of the associated discrete integrable systems such as the master Hirota (dKP), Miwa (dBKP) and dCKP equations (see [15, 16] and references therein). A survey of this important subject may be found in the monograph [14]. Moreover, classical projective differential geometry has been shown to constitute a rich source of integrable geometries and associated nonlinear differential equations [17, 18].

Here, in view of the development of a canonical discrete analogue of projective differential geometry, we are concerned with the important class of

projective minimal surfaces [10]. These have a great variety of geometric and algebraic properties and are therefore custom made for the identification of a general discretisaton procedure which preserves essential geometric and algebraic features. In this connection, it should be pointed out that projective minimal surfaces have been shown to be integrable (see [17, 19] and references therein) in the sense that the underlying projective "Gauss-Mainardi-Codazzi equations" are amenable to the powerful techniques of integrable systems theory (soliton theory) [19]. It turns out that the discretisation scheme proposed here preserves integrability and even though this important aspect will be discussed in a separate publication (cf. [20]), we briefly identify and discuss a discrete analogue of the Euler-Lagrange equations for projective minimal surfaces.

Projective minimal surfaces may be characterised and classified both geometrically and algebraically. Here, we mainly focus on geometric notions and objects with a view to establishing natural discrete counterparts. Even though, by definition, projective minimal surfaces are critical points of the area functional in projective differential geometry, these may also be characterised in terms of Lie quadrics and their envelopes [10, 21]. For any (hyperbolic) surface Σ in a three-dimensional projective space $\mathbb{P}^3$, there exists a particular two-parameter family (congruence) of quadrics Q which have second-order contact with Σ. In general, this congruence of quadrics, which are known as Lie quadrics, admit four additional envelopes Ω which are termed Demoulin transforms of Σ [10, 21]. If the asymptotic lines on the surface Σ are mapped via the congruence of Lie quadrics to the asymptotic lines of at least one envelope Ω then we say that Σ and Ω are in asymptotic correspondence. In [22], we have proposed the term PMQ surface for a surface Σ which admits at least one envelope Ω (of the associated Lie quadrics) which is in asymptotic correspondence with Σ. A classical theorem [21, 23] now states that a PMQ surface is either projective minimal (PM) or a Q surface [10] (or both). A definition of the interesting but restrictive class of Q surfaces is given in the next section.

Projective minimal surfaces may be classified in terms of the number of distinct (additional) envelopes of the Lie quadrics [10, 21, 22]. If two envelopes are the same then the projective minimal surface Σ is of Godeaux-Rozet type. If there exists only one envelope (apart from Σ itself) then Σ is of Demoulin type. This classification may also be formulated in terms of certain line congruences, which has the advantage that Demoulin surfaces may be further separated into generic Demoulin surfaces and projective transforms of

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Tzitzéica surfaces. The latter have been discussed in great detail not only in the classical context of affine differential geometry but also in connection with the theory of integrable systems and integrability-preserving discretisations (see [19, 24] and references therein). On the other hand, Q surfaces are naturally defined in terms of a so-called semi-Q property which gives rise to the isolation of not only Q surfaces but also other classical classes of surfaces, namely complex, doubly Q and doubly complex surfaces [10]. The semi-Q property is defined in terms of special generators of Lie quadrics which form so-called Demoulin quadrilaterals. This is made precise in the following section.

In the following, we first briefly review the classical geometric notions and objects, and classes of surfaces mentioned above and then propose, justify and analyse in detail all of their discrete analogues. It turns out that the discrete and classical theories are remarkably close. Moreover, importantly, it may be argued that the discrete theory is more transparent and, thereby, makes the classical theory more accessible. In this connection, it is observed that it is well known that, in many instances, discrete geometries may be generated from continuous geometries by means of iterative application of transformations such as Bäcklund transformations, thereby preserving the essential features of the continuous geometries (see [14] and references therein). In [22], combinatorial and geometric properties of the afore-mentioned classical Demoulin transformation have been investigated in detail and it turns out that the Demoulin transformation applied to classical PMQ surfaces indeed generates discrete PMQ surfaces of the type proposed here.

2. Projective minimal surfaces

2.1. Algebraic classification of projective minimal surfaces

We are concerned with surfaces Σ in a three-dimensional (real) projective space $\mathbb{P}^3$ represented by $[\boldsymbol{r}] : \mathbb{R}^2 \to \mathbb{P}^3$, where $(x, y) \in \mathbb{R}^2$ are taken to be asymptotic coordinates on Σ. Since we confine ourselves to hyperbolic surfaces, the asymptotic coordinates are real. By definition of asymptotic coordinates, the second derivatives along the coordinate lines are tangent to Σ so that the vector of homogeneous coordinates $\boldsymbol{r} \in \mathbb{R}^4$ satisfies a pair of linear equations

$$\boldsymbol{r}_{xx} = p\boldsymbol{r}_y + \pi\boldsymbol{r} + \sigma\boldsymbol{r}_x, \quad \boldsymbol{r}_{yy} = q\boldsymbol{r}_x + \xi\boldsymbol{r} + \chi\boldsymbol{r}_y.$$

Then, it is well known [10, 17, 19] and may readily be verified that particular homogeneous coordinates, known as the Wilczynski lift [2, 3, 4], may be chosen such that the functions σ and χ vanish. Hence, for convenience, one may parametrise the remaining coefficients of the "projective Gauss-Weingarten equations" according to

$$\boldsymbol{r}_{xx} = p\boldsymbol{r}_y + \frac{1}{2}(V - p_y)\boldsymbol{r}, \quad \boldsymbol{r}_{yy} = q\boldsymbol{r}_x + \frac{1}{2}(W - q_x)\boldsymbol{r}$$

in terms of functions p, q, V and W. The latter are constrained by the "projective Gauss-Mainardi-Codazzi equations"

$$p_{yyy} - 2p_y W - pW_y = q_{xxx} - 2q_x V - qV_x \tag{1}$$
$$W_x = 2qp_y + pq_y \tag{2}$$
$$V_y = 2pq_x + qp_x \tag{3}$$

which may be derived from the compatibility condition $\boldsymbol{r}_{xxyy} = \boldsymbol{r}_{yyxx}$. It is noted that the Wilczynski lift is unique up to the group of transformations

$$x \rightarrow f(x), \quad y \rightarrow g(y), \quad \boldsymbol{r} \rightarrow \sqrt{f'(x)g'(y)}\,\boldsymbol{r}$$

with

$$p \rightarrow p\frac{g'(y)}{[f'(x)]^2}, \qquad q \rightarrow q\frac{f'(x)}{[g'(y)]^2} \tag{4}$$

$$V \rightarrow \frac{V + S(f)}{[f'(x)]^2}, \quad W \rightarrow \frac{W + S(g)}{[g'(y)]^2}, \tag{5}$$

where S denotes the Schwarzian derivative

$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

The quadratic form

$$pq\,dxdy$$

is a projective invariant and is known as the projective metric. Throughout the paper, we shall assume that Σ is not ruled, i.e., $pq \neq 0$.

In view of the classification of projective minimal surfaces, it turns out convenient to define functions α and β by

$$\alpha = p^2 W - pp_{yy} + \frac{p_y^2}{2}, \quad \beta = q^2 V - qq_{xx} + \frac{q_x^2}{2}$$

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so that the Gauss-Mainardi-Codazzi equations (1)-(3) adopt the form

$$\frac{\alpha_y}{p} = \frac{\beta_x}{q} \tag{6}$$

$$(\ln p)_{xy} = pq + \frac{A}{p}, \quad A_y = -p\left(\frac{\alpha}{p^2}\right)_x \tag{7}$$

$$(\ln q)_{xy} = pq + \frac{B}{q}, \quad B_x = -q\left(\frac{\beta}{q^2}\right)_y. \tag{8}$$

This is directly verified by eliminating the functions A and B.

Definition 2.1. A surface Σ in $\mathbb{P}^3$ is said to be projective minimal if it is critical for the area functional $\iint pq\,dxdy$.

Theorem 2.2 ([25]). A surface Σ in $\mathbb{P}^3$ is projective minimal if and only if

$$\frac{\alpha_y}{p} = \frac{\beta_x}{q} = 0. \tag{9}$$

There exist classical canonical classes of projective minimal surfaces as listed below. Thus, a projective minimal surface is said to be

(a) generic if $\alpha \neq 0$ and $\beta \neq 0$.

(b) of Godeaux-Rozet type if $\alpha \neq 0$, $\beta = 0$ or $\alpha = 0$, $\beta \neq 0$.

(c) of Demoulin type if $\alpha = \beta = 0$. If, in addition, $p = q$ then Σ is said to be of Tzitzéica type.

It is noted that, using a gauge transformation of the form (4), (5), one may normalise α and β to be one of -1, 1 or 0. This normalisation corresponds to canonical forms of the integrable system (6)-(9) underlying projective minimal surfaces [17].

2.2. Geometric classification of projective minimal surfaces

It turns out that the above algebraic classification of projective minimal surfaces admits a natural discrete analogue. This is the subject of a separate publication [20]. Here, our discretisation procedure is of a geometric nature based on the classical geometric classification scheme of projective minimal surfaces which involves Lie quadrics and their envelopes.

Definition 2.3. Let $[\boldsymbol{r}] : \mathbb{R}^2 \to \mathbb{P}^3$ be a parametrisation of a surface Σ in terms of asymptotic coordinates. Let $\boldsymbol{p} = \boldsymbol{r}(x, y)$ be a point on Σ and let $\boldsymbol{p}_\pm$ be two additional points on the x-asymptotic line passing through $\boldsymbol{p}$, given by $\boldsymbol{p}_\pm = \boldsymbol{r}(x \pm \epsilon, y)$. Let $l_\pm$ and l be the three lines tangent to the y-asymptotic lines at $\boldsymbol{p}_\pm$ and $\boldsymbol{p}$ respectively. These uniquely define a quadric Q_ϵ containing them as rectilinear generators. The Lie quadric at (x, y) is then the unique quadric defined by

$$Q(x, y) = \lim_{\epsilon \to 0} Q_\epsilon(x, y).$$

It is important to emphasise that the above definition of a Lie quadric may be shown to be symmetric in x and y, that is, interchanging x-asymptotic lines and y-asymptotic lines leads to the same Lie quadric Q. This is reflected in the explicit representation of the Lie quadric Q given below [10, 18]. For brevity, in the following, notationally, we do not distinguish between a Lie quadric in $\mathbb{RP}^3$ and its representation in the space of homogeneous coordinates $\mathbb{R}^4$.

Theorem 2.4. The Lie quadric Q (at a point (x, y)) admits the parametrisation

$$Q = \mathfrak{n} + \mu \mathfrak{r}^1 + \nu \mathfrak{r}^2 + \mu \nu \mathfrak{r},$$

where μ and ν parametrise the two families of generators of Q and $\{\mathfrak{r}, \mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{n}\}$ is the Wilczynski frame given by

$$\mathfrak{r} = \boldsymbol{r}, \quad \mathfrak{r}^1 = \boldsymbol{r}_x - \frac{q_x}{2q} \boldsymbol{r}, \quad \mathfrak{r}^2 = \boldsymbol{r}_y - \frac{p_y}{2p} \boldsymbol{r}$$

$$\mathfrak{n} = \boldsymbol{r}_{xy} - \frac{p_y}{2p} \boldsymbol{r}_x - \frac{q_x}{2q} \boldsymbol{r}_y + \left(\frac{p_y q_x}{4pq} - \frac{pq}{2} \right) \boldsymbol{r}.$$

It is observed that the lines $(\mathfrak{r}, \mathfrak{r}^1)$ and $(\mathfrak{r}, \mathfrak{r}^2)$ are tangent to Σ, while the line $(\mathfrak{r}, \mathfrak{n})$ is transversal to Σ and plays the role of a projective normal. It is known as the first directrix of Wilczynski.

Definition 2.5. A surface Ω parametrised by $[\boldsymbol{\omega}] : \mathbb{R}^2 \to \mathbb{P}^3$ is an envelope of the two parameter family of Lie quadrics $\{Q(x, y)\}$ associated with a surface Σ if $\boldsymbol{\omega}(x, y) \in Q(x, y)$ such that Ω touches $Q(x, y)$ at $\boldsymbol{\omega}(x, y)$.

We note that, in particular, Σ is itself an envelope of $\{Q\}$. Generically, there exist four additional envelopes as stated below [10].

Theorem 2.6. If $\alpha, \beta \geq 0$ then the Lie quadrics $\{Q\}$ possess four real additional envelopes

$$
\begin{aligned}
\boldsymbol{\omega}_{++} &= \mathfrak{n} + \hat{\mu}\mathfrak{r}^1 + \hat{\nu}\mathfrak{r}^2 + \hat{\mu}\hat{\nu}\mathfrak{r} \\
\boldsymbol{\omega}_{+-} &= \mathfrak{n} + \hat{\mu}\mathfrak{r}^1 - \hat{\nu}\mathfrak{r}^2 - \hat{\mu}\hat{\nu}\mathfrak{r} \\
\boldsymbol{\omega}_{-+} &= \mathfrak{n} - \hat{\mu}\mathfrak{r}^1 + \hat{\nu}\mathfrak{r}^2 - \hat{\mu}\hat{\nu}\mathfrak{r} \\
\boldsymbol{\omega}_{--} &= \mathfrak{n} - \hat{\mu}\mathfrak{r}^1 - \hat{\nu}\mathfrak{r}^2 + \hat{\mu}\hat{\nu}\mathfrak{r},
\end{aligned}
$$

where

$$
\hat{\mu} = \sqrt{\frac{\alpha}{2p^2}}, \quad \hat{\nu} = \sqrt{\frac{\beta}{2q^2}}.
$$

These are distinct if $\alpha, \beta \neq 0$.

Remark 2.7. The above envelopes are called the Demoulin transforms of Σ. We denote them by Σ_{++}, Σ_{+-}, Σ_{-+} and Σ_{--}. In general, by construction, these have first-order contact with the Lie quadrics, while, by definition of Lie quadrics, the surface Σ has second-order contact. It turns out that there exists a natural discrete analogue of this classical fact.

As indicated in the above theorem, the expressions for $\hat{\mu}$ and $\hat{\nu}$ imply that whether α and β vanish or not is related to the number of distinct envelopes. Accordingly, the geometric interpretation of the algebraic classification (a)-(c) is then that a projective minimal surface Σ is

(a) generic if the set of Lie quadrics $\{Q\}$ has four distinct additional envelopes.

(b) of Godeaux-Rozet type if $\{Q\}$ has two distinct additional envelopes.

(c) of Demoulin type if $\{Q\}$ has one additional envelope.

Remark 2.8. By virtue of the Gauss-Mainardi-Codazzi equation (6), Theorem 2.6 implies that a surface Σ in $\mathbb{P}^3$ is necessarily projective minimal if there exist less than four additional distinct envelopes. Specifically, if the Lie quadrics of Σ have only two additional distinct envelopes then Σ is of Godeaux-Rozet type. If the Lie quadrics of Σ have only one additional envelope then Σ is of Demoulin type.

Remark 2.9. For any fixed (x, y), the points $\boldsymbol{\omega}_{++}(x, y)$, $\boldsymbol{\omega}_{+-}(x, y)$, $\boldsymbol{\omega}_{--}(x, y)$ and $\boldsymbol{\omega}_{-+}(x, y)$ of the Demoulin transforms of Σ may be regarded as the vertices of a quadrilateral (cf. Figure 1)

$$[\boldsymbol{\omega}_{++}(x, y), \, \boldsymbol{\omega}_{+-}(x, y), \, \boldsymbol{\omega}_{--}(x, y), \boldsymbol{\omega}_{-+}(x, y)]$$

which is known as the Demoulin quadrilateral [10]. Then, the parametrisation of the envelopes set down in Theorem 2.6 shows that the extended edges $[\boldsymbol{\omega}_{++}(x, y), \, \boldsymbol{\omega}_{+-}(x, y)]$, $[\boldsymbol{\omega}_{+-}(x, y), \, \boldsymbol{\omega}_{--}(x, y)]$, $[\boldsymbol{\omega}_{--}(x, y), \, \boldsymbol{\omega}_{-+}(x, y)]$ and $[\boldsymbol{\omega}_{-+}(x, y), \, \boldsymbol{\omega}_{++}(x, y)]$ are generators of the Lie quadric $Q(x, y)$.
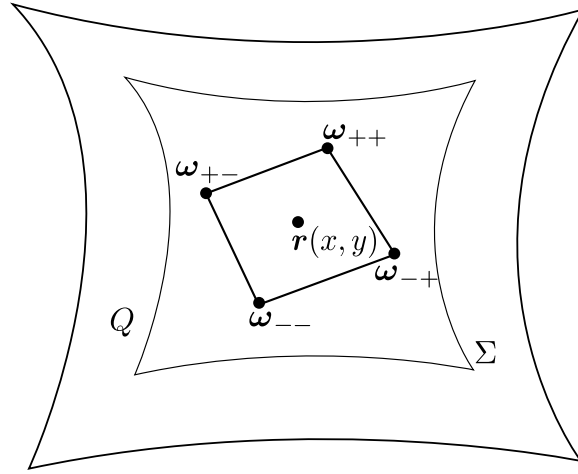

Figure 1: The Demoulin quadrilateral

Remarkably, it turns out that the Demoulin transformation acts within the class of projective minimal surfaces and, specifically, within the classes (a)-(c) [21, 23].

Theorem 2.10. Let Σ be a projective minimal surface. Then, each of its Demoulin transforms is projective minimal. Moreover, the number $n \in \{1, 2, 4\}$ of distinct Demoulin transforms of Σ is preserved by the Demoulin transformation. In particular, if Σ is of Godeaux-Rozet type then each of its Demoulin transforms is of Godeaux-Rozet type. If Σ is of Demoulin type then its transform is of Demoulin type.

The following classical theorem lies at the heart of the geometric definition and analysis of discrete projective minimal surfaces. In order to formulate

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this theorem, we first note that since any point of a surface Σ is mapped to a point of any envelope Ω via the corresponding Lie quadric, any coordinate system on the surface Σ induces a coordinate system on the envelope Ω. Hence, we say that a surface Σ and any envelope Ω are in asymptotic correspondence if the asymptotic lines on Σ are mapped to the asymptotic lines on Ω. It turns out that the notion of asymptotic correspondence gives rise to a privileged class of surfaces. Thus, the definition proposed in [22] is adopted.

Definition 2.11. A surface Σ is termed a PMQ surface if it is in asymptotic correspondence with at least one associated envelope Ω.

The above-mentioned key theorem therefore reads as follows [10, 21].

Theorem 2.12. The class of PMQ surfaces consists of projective minimal surfaces and Q surfaces.

The properties of Q surfaces and their relatives have been discussed in great detail in [10]. Some of those which are pertinent to the discrete theory developed in the following sections are now briefly mentioned. Thus, in general, for any surface Σ, any extended edge of the Demoulin quadrilateral displayed in Figure 1 such as $[\boldsymbol{\omega}_{++}(x,y), \boldsymbol{\omega}_{+-}(x,y)]$ generates a two-parameter family of lines (i.e., a line congruence) by varying the coordinates (x,y) on Σ. However, this line congruence may degenerate to a one-parameter family of lines, in which case Σ is referred to as a semi-Q surface. If this kind of degeneration is present with respect to two connected edges of the Demoulin quadrilateral such as the preceding one and $[\boldsymbol{\omega}_{++}(x,y), \boldsymbol{\omega}_{-+}(x,y)]$ then Σ is termed a Q surface with respect to the envelope Σ_{++}. In fact, the latter turns out to be a quadric generated by the two one-parameter families of extended edges. If the line congruences associated with two "opposite" edges such as $[\boldsymbol{\omega}_{++}(x,y), \boldsymbol{\omega}_{+-}(x,y)]$ and $[\boldsymbol{\omega}_{--}(x,y), \boldsymbol{\omega}_{-+}(x,y)]$ degenerate then Σ is known as a complex surface. If three or four line congruences degenerate then we refer to Σ as doubly Q or doubly complex respectively. However, it is known that doubly Q surfaces are automatically doubly complex. It turns out that the same property holds in the discrete case.

3. Lattice Lie quadrics

3.1. The frame of a discrete asymptotic net
Definition 3.1 ([14]). A $\mathbb{Z}^2$ lattice of points in $\mathbb{P}^3$ whose stars are planar is termed a discrete asymptotic net.

Let $\boldsymbol{r} : \mathbb{Z}^2 \to \mathbb{R}^4$ be a homogeneous coordinate representation of a discrete asymptotic net in $\mathbb{P}^3$. We usually suppress the arguments in $\boldsymbol{r} = \boldsymbol{r}(n_1, n_2)$ and denote an increment of n_k by a subscript k and a decrement of n_k by a subscript $\bar{k}$, that is, $\boldsymbol{r}_k$ and $\boldsymbol{r}_{\bar{k}}$ respectively, as illustrated in Figure 2. We
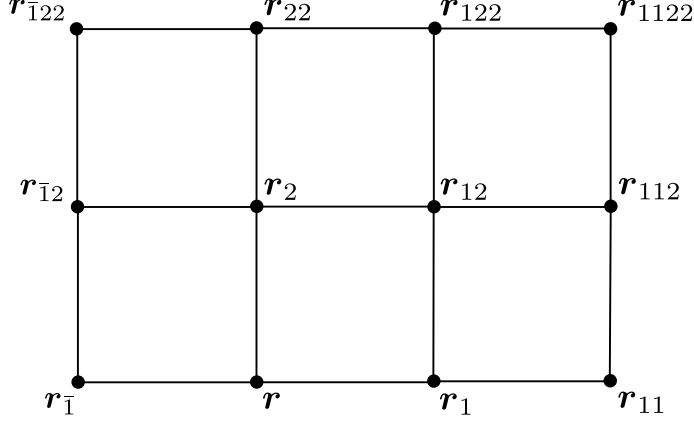

Figure 2: Labeling of a discrete asymptotic net

introduce the frame

$$V = \begin{pmatrix} \boldsymbol{r} \\ \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \boldsymbol{r}_{12} \end{pmatrix} \tag{10}$$

so that the planar star condition on the lattice gives rise to the frame equations

$$V_1 = \begin{pmatrix} \boldsymbol{r} \\ \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \boldsymbol{r}_{12} \end{pmatrix}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \alpha^0 & \alpha^1 & 0 & \alpha^3 \\ 0 & 0 & 0 & 1 \\ 0 & \beta^1 & \beta^2 & \beta^3 \end{pmatrix} \begin{pmatrix} \boldsymbol{r} \\ \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \boldsymbol{r}_{12} \end{pmatrix} = LV \tag{11}$$

$$V_2 = \begin{pmatrix} \boldsymbol{r} \\ \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \boldsymbol{r}_{12} \end{pmatrix}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \gamma^0 & 0 & \gamma^2 & \gamma^3 \\ 0 & \delta^1 & \delta^2 & \delta^3 \end{pmatrix} \begin{pmatrix} \boldsymbol{r} \\ \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \boldsymbol{r}_{12} \end{pmatrix} = MV, \tag{12}$$

where we note that in order to exclude the degenerate case of three points connected by two edges being collinear, we demand that only $\alpha^1, \beta^3, \gamma^2$ and

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δ^3 be allowed to be zero. L and M are not arbitrary but are constrained by the compatibility condition $\boldsymbol{V}_{12} = \boldsymbol{V}_{21}$, that is

$$M_1 L = L_2 M \tag{13}$$

which equates to

$$
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & \beta^1 & \beta^2 & \beta^3 \\
0 & \beta^1\gamma_1^3 + \gamma_1^0 & \gamma_1^3\beta^2 & \beta^3\gamma_1^3 + \gamma_1^2 \\
\delta_1^1\alpha^0 & \alpha^1\delta_1^1 + \beta^1\delta_1^3 & \delta_1^3\beta^2 & \alpha^3\delta_1^1 + \beta^3\delta_1^3 + \delta_1^2
\end{pmatrix}
$$
$$
=
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & \alpha_2^3\delta^1 & \delta^2\alpha_2^3 + \alpha_2^0 & \delta^3\alpha_2^3 + \alpha_2^1 \\
0 & \delta^1 & \delta^2 & \delta^3 \\
\beta_2^2\gamma^0 & \beta_2^3\delta^1 & \gamma^2\beta_2^2 + \delta^2\beta_2^3 & \gamma^3\beta_2^2 + \delta^3\beta_2^3 + \beta_2^1
\end{pmatrix}.
$$

$$\tag{14}$$

These may be regarded as the "Gauss-Mainardi-Codazzi equations" in discrete projective differential geometry.

3.2. Lattice Lie quadrics

For each quadrilateral of a discrete asymptotic net, there exists a one-parameter family of quadrics passing through the edges of that quadrilateral. Analogously to the continuous Lie quadric for projective minimal and Q surfaces, these "lattice quadrics" play a central role in defining discrete Demoulin transforms.

Definition 3.2.

(i) A quadric associated with a quadrilateral of a discrete asymptotic net is a quadric that passes through the edges of the quadrilateral. For any given quadrilateral, there exists a one parameter family of such quadrics.

(ii) Let Q and $\hat{Q}$ be two quadrics associated with neighbouring quadrilaterals of a discrete asymptotic net. Q and $\hat{Q}$ are said to satisfy the $\mathcal{C}^1$ condition if their tangent planes coincide at each point of the common edge.
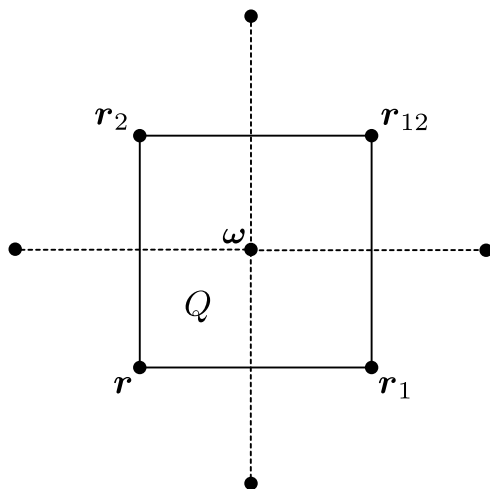
Figure 3: Combinatorial picture of a quadrilateral of a discrete asymptotic net and the star of an envelope touching an associated lattice Lie quadric

(iii) A set of quadrics $\{Q\}$ associated with the quadrilaterals of a discrete asymptotic net is termed a set of lattice Lie quadrics if the $\mathcal{C}^1$ condition holds for all neighbouring pairs of quadrics.

(iv) A discrete envelope of a set of lattice Lie quadrics $\{Q\}$ is a dual lattice of $\mathbb{Z}^2$ combinatorics such that each star touches the corresponding lattice Lie quadric (cf. Figure 3).

(v) A discrete PMQ surface is a discrete asymptotic net which admits a discrete envelope of an associated set of lattice Lie quadrics.

We parametrise a quadric passing through the edges of the quadrilateral $[\boldsymbol{r}\ \boldsymbol{r}_1\ \boldsymbol{r}_2\ \boldsymbol{r}_{12}]$ by

$$Q(s,t) = p\boldsymbol{r}_{12} + s\boldsymbol{r}_1 + t\boldsymbol{r}_2 + st\boldsymbol{r}, \tag{15}$$

where p is fixed and labels Q among the one-parameter family of quadrics associated with the quadrilateral, and s and t are the parameters of Q which parametrise its generators.

Remark 3.3. The four (extended) edges of any quadrilateral of a discrete asymptotic net lying on the corresponding lattice Lie quadric may be regarded as a discrete analogue of second-order contact, while any planar star of a discrete envelope touching the corresponding lattice Lie quadric constitutes a discrete analogue of first-order contact.

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In order to establish the existence of lattice Lie quadrics, we first recall a key theorem established in [26] and prove it algebraically.

Theorem 3.4. Given a fixed quadric Q, the $\mathcal{C}^1$ condition uniquely determines a quadric Q_1 associated with the neighbouring quadrilateral.

Proof. We parametrise Q by (15) and Q_1 by

$$Q_1(s_1, t_1) = p_1 \boldsymbol{r}_{112} + s_1 \boldsymbol{r}_{11} + t_1 \boldsymbol{r}_{12} + s_1 t_1 \boldsymbol{r}_1 \tag{16}$$

so that, by virtue of (11), (12),

$$\begin{aligned}
Q_1 = {}& (\beta^3 p_1 + \alpha^3 s_1 + t_1) \boldsymbol{r}_{12} \\
& + (\beta^1 p_1 + \alpha^1 s_1 + s_1 t_1) \boldsymbol{r}_1 + \beta^2 p_1 \boldsymbol{r}_2 + \alpha^0 s_1 \boldsymbol{r}.
\end{aligned} \tag{17}$$

A point $\boldsymbol{X}$ on the edge common to Q and Q_1 is parametrised by $ps_1 = s$ and $t = 0$, $t_1 = \infty$. The $\mathcal{C}^1$ condition is then

$$\left| \boldsymbol{X}, \; \boldsymbol{r}_{12}, \; \left. \frac{\partial}{\partial t} Q \right|_{\boldsymbol{X}}, \; \left. \frac{\partial}{\partial \hat{t}_1} \hat{Q}_1 \right|_{\boldsymbol{X}} \right| = 0,$$

where, to obtain a finite tangent vector, we have scaled Q_1 according to $\hat{Q}_1 = \hat{t}_1 Q_1$ with $\hat{t}_1 = 1/t_1$. This yields

$$\left| s_1 \boldsymbol{r}_1, \; \boldsymbol{r}_{12}, \; \boldsymbol{r}_2 + ps_1 \boldsymbol{r}, \; p_1 \boldsymbol{r}_{112} + s_1 \boldsymbol{r}_{11} \right| = 0. \tag{18}$$

On use of the system (11), (12), this is shown to determine p_1 according to

$$p_1 = \frac{\alpha^0}{\beta^2 p}. \tag{19}$$

$\square$

Remark 3.5. We note that Theorem 3.4 makes use of the fact that Q and Q_1 share tangents plane at the points $\boldsymbol{r}_1$ and $\boldsymbol{r}_{12}$. Moreover, since (19) is independent of s, it follows that the tangent planes of Q and Q_1 coincide everywhere along the common edge, confirming the following known result.

Theorem 3.6 ([26])**.** Let Q and Q_1 be quadrics associated with two neighbouring quadrilaterals of a discrete asymptotic net. If the tangent planes of Q and Q_1 coincide at a point on their common edge other than any of the vertices connected by the common edge then the tangent planes to both quadrics coincide everywhere along the common edge.

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Remark 3.7. Let Q_2 be the quadric passing through the edges of the quadrilateral $[\boldsymbol{r}_2, \boldsymbol{r}_{12}, \boldsymbol{r}_{22}, \boldsymbol{r}_{122}]$ which satisfies the $\mathcal{C}^1$ condition with Q. Let Q_{12} and Q_{21} be the quadrics passing through the edges of the quadrilateral $[\boldsymbol{r}_{12}, \boldsymbol{r}_{112}, \boldsymbol{r}_{122}, \boldsymbol{r}_{1122}]$ which satisfies the $\mathcal{C}^1$ condition with Q_1 and Q_2 respectively (cf. Figure 4). Then, expressions similar to (19) show that the parameters p_{12} and p_{21} associated with Q_{12} and Q_{21} respectively coincide. Thus, $Q_{12} = Q_{21}$ and, hence, given one fixed quadric associated with one quadrilateral of a discrete asymptotic net, the $\mathcal{C}^1$ condition uniquely determines all other quadrics on the lattice [26]. The implication of this is summarised below.
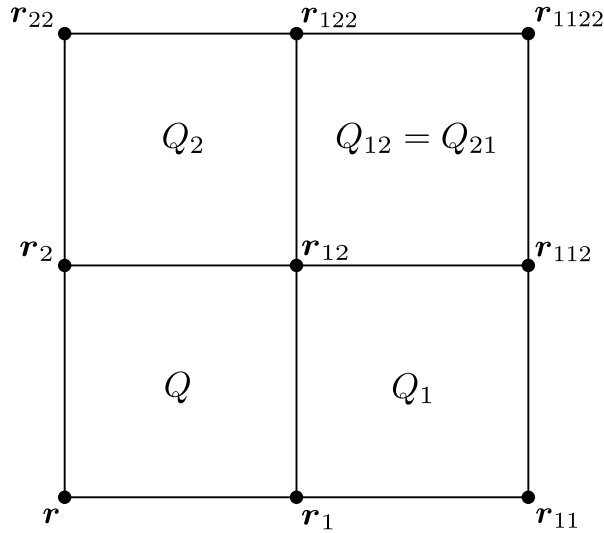
Figure 4: Quadrics on a patch of a discrete asymptotic net related by the $\mathcal{C}^1$ condition

Theorem 3.8. A set of lattice Lie quadrics is uniquely determined by a quadric associated with one quadrilateral of a discrete asymptotic net.

3.3. Generators shared by lattice Lie quadrics

In view of classifying discrete PMQ surfaces and their envelopes, we now investigate the properties of quadrics associated with the quadrilaterals of a discrete asymptotic net.

Theorem 3.9. Let Q and Q_1 be two neighbouring quadrics on a discrete asymptotic net which satisfy the $\mathcal{C}^1$ condition. Then, Q and Q_1 share

two (possibly complex or coinciding) generators (transversal to the common edge). Conversely, if there exists a generator common to Q and Q_1 then the $\mathcal{C}^1$ condition is satisfied.

Proof. At points common to Q and Q_1, we have $Q(s,t) \sim Q_1(s_1, t_1)$, which yields the equations

$$\frac{s}{p} = \frac{\beta^1 p_1 + \alpha^1 s_1 + s_1 t_1}{\beta^3 p_1 + \alpha^3 s_1 + t_1} \tag{20}$$

$$\frac{t}{p} = \frac{\beta^2 p_1}{\beta^3 p_1 + \alpha^3 s_1 + t_1} \tag{21}$$

$$\frac{st}{p} = \frac{\alpha^0 s_1}{\beta^3 p_1 + \alpha^3 s_1 + t_1}. \tag{22}$$

One solution to these equations is $t = 0$, $t_1 = \infty$ and $ps_1 = s$, which parametrises the common edge. Hence, if Q and Q_1 have a common generator then, necessarily, the corresponding parameters s and s_1 are related by $ps_1 = s$. As a result, relations (21) and (22) imply that $p_1 = \alpha^0/(\beta^2 p)$, which is precisely the $\mathcal{C}^1$ condition (19). This is geometrically evident since at the point of intersection of the common edge and generator, the generator and edge span the coinciding tangent planes of Q and Q_1 so that, as a result of Theorem 3.6, the tangent planes to Q and Q_1 coincide along their common edge. Hence, if Q and Q_1 have a shared generator then they satisfy the $\mathcal{C}^1$ condition.

Conversely, since any common generator is "labelled by $ps_1 = s$", that is, the labels s and s_1 of a generator common to Q and Q_1 are related by $ps_1 = s$, the $\mathcal{C}^1$ condition (19) implies that (21) and (22) coincide, thereby relating the parameters t and t_1 but leaving one of them arbitrary, and the remaining relation (20) reduces to the condition

$$\alpha^3 \beta^2 (s)^2 + s(\alpha^0 \beta^3 - \alpha^1 \beta^2 p) - \alpha^0 \beta^1 p = 0. \tag{23}$$

The discriminant of the latter is

$$D^1 = (\alpha^0 \beta^3 - \alpha^1 \beta^2 p)^2 + 4\alpha^0 \alpha^3 \beta^1 \beta^2 p \tag{24}$$

and determines whether the roots of (23) are real, complex conjugates or coinciding. Accordingly, the proof is complete and the following corollary holds. □

Corollary 3.10. Let Q and Q_1 be two neighbouring quadrics on a discrete asymptotic net which satisfy the $\mathcal{C}^1$ condition and

$$D^1 = (\alpha^0\beta^3 - \alpha^1\beta^2 p)^2 + 4\alpha^0\alpha^3\beta^1\beta^2 p \tag{25}$$

be the associated discriminant. If

(i) $D^1 > 0$ then Q and Q_1 share two distinct real generators.

(ii) $D^1 < 0$ then Q and Q_1 share two distinct complex conjugate generators.

(iii) $D^1 = 0$ then Q and Q_1 share two coinciding real generators.

Remark 3.11. Similarly, in the n_2 direction, Q and Q_2 share two generators labelled by $pt_2 = t$ and

$$\gamma^3\delta^1(t)^2 + t(\gamma^0\delta^3 - \gamma^2\delta^1 p) - \gamma^0\delta^2 p = 0 \tag{26}$$

with discriminant

$$D^2 = (\gamma^0\delta^3 - \gamma^2\delta^1 p)^2 + 4\gamma^0\gamma^3\delta^1\delta^2 p. \tag{27}$$

In the case $D^1 = 0$ (or $D^2 = 0$), we also have the following useful property.

Corollary 3.12. Let Q and Q_1 be two neighbouring quadrics on a discrete asymptotic net which satisfy the $\mathcal{C}^1$ condition and assume that $D^1 = 0$. Then, Q and Q_1 touch along their shared generator.

Proof. Since $D^1 = 0$, the proof of Theorem 3.9 implies that the shared generator is labelled by $ps_1 = s$, where

$$s = \frac{\alpha^1\beta^2 p - \alpha^0\beta^3}{2\alpha^3\beta^2}.$$

Let $\boldsymbol{X} = Q_1(s_1, t_1)$ be a fixed point on the shared generator which does not also lie on the common edge so that (21) leads to

$$t = \frac{2\alpha^0\beta^2 p}{\alpha^1\beta^2 p + 2\beta^2 pt_1 + \alpha^0\beta^3}.$$

Along a common generator, we then have, trivially,

$$\left| \boldsymbol{X}, \left. \frac{\partial}{\partial s}Q \right|_{\boldsymbol{X}}, \left. \frac{\partial}{\partial t}Q \right|_{\boldsymbol{X}}, \left. \frac{\partial}{\partial t_1}Q_1 \right|_{\boldsymbol{X}} \right| = 0$$

and it is easy to verify that

$$\left| \boldsymbol{X}, \left. \frac{\partial}{\partial s}Q \right|_{\boldsymbol{X}}, \left. \frac{\partial}{\partial t}Q \right|_{\boldsymbol{X}}, \left. \frac{\partial}{\partial s_1}Q_1 \right|_{\boldsymbol{X}} \right| = 0.$$

Hence, the tangent planes of Q and Q_1 coincide at $\boldsymbol{X}$. $\qquad\square$

4. Discrete projective minimal surfaces

We now intend to establish under what circumstances envelopes of sets of lattice Lie quadrics exist.

4.1. The tangency condition

Definition 4.1. Let Q and Q_1 be neighbouring quadrics on a discrete asymptotic net. Then, $\boldsymbol{\omega} \in Q$ and $\boldsymbol{\omega}_1 \in Q_1$ are said to satisfy the tangency condition if the line joining $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_1$ is tangent to both quadrics at the respective points.

Theorem 4.2. Let Q and Q_1 be two neighbouring quadrics on a 3×2 patch of a discrete asymptotic net that satisfy the $\mathcal{C}^1$ condition. Let $\boldsymbol{\omega} \in Q$ be a generic point. Then, there exists a unique point $\boldsymbol{\omega}_1 \in Q_1$ such that $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_1$ satisfy the tangency condition (cf. Figure 5).
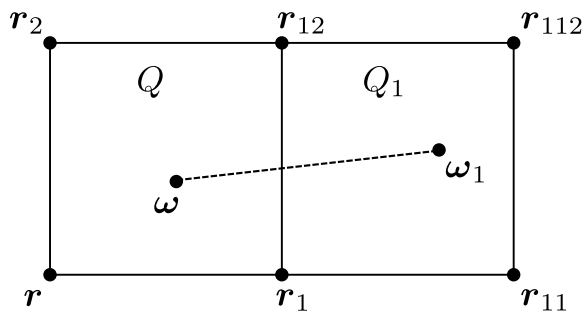
Figure 5: The tangency condition

Proof. Set $\boldsymbol{\omega} = Q(s,t)$, $\boldsymbol{\omega}_1 = Q_1(s_1, t_1)$. The tangency condition is then encapsulated in the pair

$$\left| \boldsymbol{\omega}, \; \boldsymbol{\omega}_1, \; \frac{\partial}{\partial s}Q \bigg|_{\boldsymbol{\omega}}, \; \frac{\partial}{\partial t}Q \bigg|_{\boldsymbol{\omega}} \right| = 0 \tag{28}$$

$$\left| \boldsymbol{\omega}, \; \boldsymbol{\omega}_1, \; \frac{\partial}{\partial s_1}Q_1 \bigg|_{\boldsymbol{\omega}_1}, \; \frac{\partial}{\partial t_1}Q_1 \bigg|_{\boldsymbol{\omega}_1} \right| = 0, \tag{29}$$

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which yields

$$(ps_1 - s)(\alpha^1\beta^2 pt + 2\beta^2 ptt_1 - 2\alpha^0\beta^2 p + \alpha^0\beta^3 t) = 0 \qquad (30)$$

$$\left(ps_1 - \frac{\alpha^1\beta^2 p - \alpha^0\beta^3}{2\alpha^3\beta^2}\right)\left(s - \frac{\alpha^1\beta^2 p - \alpha^0\beta^3}{2\alpha^3\beta^2}\right) - \frac{1}{(2\alpha^3\beta^2)^2}D^1 = 0. \qquad (31)$$

If $ps_1 = s$ then equation (31) reduces to equation (23) and, hence, $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_1$ lie on a common generator so that $\boldsymbol{\omega}$ is non-generic. If $ps_1 \neq s$ then

$$s_1 = \frac{\alpha^1\beta^2 ps + 2\alpha^0\beta^1 p - \alpha^0\beta^3 s}{p(2\alpha^3\beta^2 s - \alpha^1\beta^2 p + \alpha^0\beta^3)} =: \mathcal{S}^1(s) \qquad (32)$$

$$t_1 = \frac{2\alpha^0\beta^2 p - \alpha^1\beta^2 pt - \alpha^0\beta^3 t}{2\beta^2 pt} =: \mathcal{T}^1(t), \qquad (33)$$

where genericity implies that $2\alpha^3\beta^2 - \alpha^1\beta^2 ps + \alpha^0\beta^3 \neq 0$. $\qquad\square$

Remark 4.3. Applying the tangency condition to a generic point $\boldsymbol{\omega}$ in the n_2 direction generates the pair of equations

$$(pt_2 - t)(\gamma^2\delta^1 ps + 2\delta^1 pss_2 - 2\gamma^0\delta^1 p + \gamma^0\delta^3 s) = 0 \qquad (34)$$

$$\left(pt_2 - \frac{\gamma^2\delta^1 p - \gamma^0\delta^3}{2\gamma^3\delta^1}\right)\left(t - \frac{\gamma^2\delta^1 p - \gamma^0\delta^3}{2\gamma^3\delta^1}\right) - \frac{1}{(2\gamma^3\delta^1)^2}D^2 = 0. \qquad (35)$$

If $pt_2 = t$ then equation (35) gives rise to equation (26) and labels the shared generators in the n_2 direction. In the generic case, (34) and (35) yield a point $\boldsymbol{\omega}_2 = Q_2(s_2, t_2)$ with

$$s_2 = \frac{2\gamma^0\delta^1 p - \gamma^2\delta^1 ps - \gamma^0\delta^3 s}{2\delta^1 ps} =: \mathcal{S}^2(s) \qquad (36)$$

$$t_2 = \frac{\gamma^2\delta^1 pt + 2\gamma^0\delta^2 p - \gamma^0\delta^3 t}{p(2\gamma^3\delta^1 t - \gamma^2\delta^1 p + \gamma^0\delta^3)} =: \mathcal{T}^2(t). \qquad (37)$$

Thus, in the generic case, the tangency condition induces 4 maps taking (s, t) to (s_1, t_1) and (s_2, t_2) which we have denoted by $\mathcal{S}^1$, $\mathcal{T}^1$ and $\mathcal{S}^2$, $\mathcal{T}^2$ respectively.

We also note that if $ps_1 \neq s$, and (31) cannot be solved for either s or s_1 then, necessarily,

$$D^1 = (\alpha^0\beta^3 - \alpha^1\beta^2 p)^2 + 4\alpha^0\alpha^3\beta^1\beta^2 p = 0 \qquad (38)$$

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and (31) becomes

$$\left(ps_1 - \frac{\alpha^1\beta^2 p - \alpha^0\beta^3}{2\alpha^3\beta^2}\right)\left(s - \frac{\alpha^1\beta^2 p - \alpha^0\beta^3}{2\alpha^3\beta^2}\right) = 0. \qquad (39)$$

This turns out to give rise to an algebraic characterisation of canonical discrete analogues of Godeaux-Rozet and Demoulin surfaces as discussed in Section 9. Here, we merely observe that if $D^1 = 0$ and $\boldsymbol{\omega}$ is a generic point then (39) implies that

$$ps_1 = s^*, \quad s^* = \frac{\alpha^1\beta^2 p - \alpha^0\beta^3}{2\alpha^3\beta^2},$$

which is a root of (23) and, hence, $\boldsymbol{\omega}_1$ lies on the shared generator of Q and Q_1. On the other hand, (39) may also be satisfied by setting $s = s^*$ and, hence, $\boldsymbol{\omega}$ lies on the shared generator.

The expressions (32) and (36) for s_1 and s_2 respectively are independent of t. Similarly, by virtue of (33) and (37), t_1 and t_2 are independent of s. Thus, we have come to the following conclusion.

Corollary 4.4. Let Σ be a discrete asymptotic net. Then, the tangency condition maps generators to generators in the sense that generic points on a generator of Q are mapped to points on a generator of Q_1 (of the same type).

4.2. Discrete projective minimal surfaces

In analogy with the continuous theory, we are interested in investigating the existence and properties of envelopes of a set of lattice Lie quadrics associated with a discrete asymptotic net Σ. Let Q be a quadric associated with a quadrilateral of Σ and choose a generic point $\boldsymbol{\omega}$ on Q. Then, by Theorem 4.2, the tangency condition uniquely determines points $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ on the neighbouring quadrics Q_1 and Q_2 respectively. Subsequent imposition of the tangency condition now generates two points $\boldsymbol{\omega}_{12}$ and $\boldsymbol{\omega}_{21}$ on Q_{12} and these are required to coincide if they are to be part of an envelope.

Theorem 4.5. Let Σ be a 3×3 patch of a discrete asymptotic net, Q, Q_1, Q_2 and Q_{12} lattice Lie quadrics of Σ as in Figure 6. Let $\boldsymbol{\omega}$ be a generic point on Q and $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_{12}, \boldsymbol{\omega}_{21}$ be points on the respective quadrics related by the tangency condition. Then, the commutativity conditions $\mathcal{S}_1^2 \circ \mathcal{S}^1 = \mathcal{S}_2^1 \circ \mathcal{S}^2$
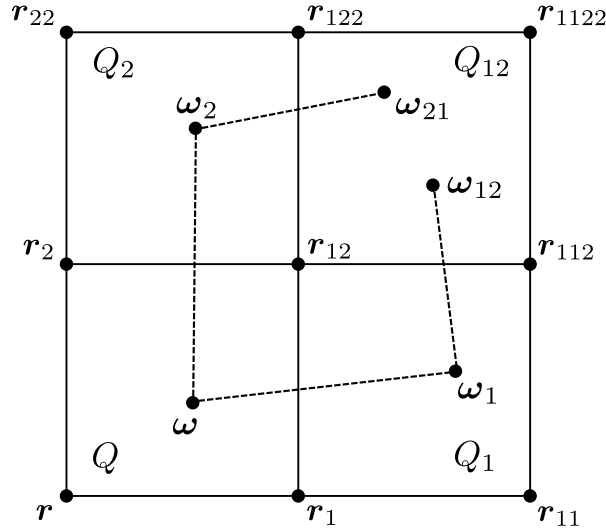
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Figure 6: (Non-)closure under the tangency condition

and $\mathcal{T}_1^2 \circ \mathcal{T}^1 = \mathcal{T}_2^1 \circ \mathcal{T}^2$ associated with the closing condition $\boldsymbol{\omega}_{21} = \boldsymbol{\omega}_{12}$ coincide and impose one scalar constraint on Σ which is independent of the choice of $\boldsymbol{\omega}$.

In order to derive a compact form of the above constraint, we require the following lemma.

Lemma 4.6. Let $\boldsymbol{V}$ be the frame of a discrete asymptotic net given by (10). Then, $\boldsymbol{V}$ admits a gauge such that

$$(1 - p^2)\frac{\beta^2 \delta^1}{\beta^1 \delta^2} = 1.$$

Proof. Under a gauge transformation

$$\boldsymbol{V} \to \boldsymbol{V}_g = \begin{pmatrix} x\boldsymbol{r} \\ x_1\boldsymbol{r}_1 \\ x_2\boldsymbol{r}_2 \\ x_{12}\boldsymbol{r}_{12} \end{pmatrix} \tag{40}$$

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the matrices L and M become

$$
L_g = \begin{pmatrix}
0 & 1 & 0 & 0 \\
\dfrac{x_{11}\alpha^0}{x} & \dfrac{x_{11}\alpha^1}{x_1} & 0 & \dfrac{x_{11}\alpha^3}{x_{12}} \\
0 & 0 & 0 & 1 \\
0 & \dfrac{x_{112}\beta^1}{x_1} & \dfrac{x_{112}\beta^2}{x_2} & \dfrac{x_{112}\beta^3}{x_{12}}
\end{pmatrix}
$$

$$
M_g = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\dfrac{x_{22}\gamma^0}{x} & 0 & \dfrac{x_{22}\gamma^2}{x_2} & \dfrac{x_{22}\gamma^3}{x_{12}} \\
0 & \dfrac{x_{122}\delta^1}{x_1} & \dfrac{x_{122}\delta^2}{x_2} & \dfrac{x_{122}\delta^3}{x_{12}}
\end{pmatrix}.
$$

The parameters of the quadric Q transform according to

$$
p_g = \left(\frac{x_1 x_2}{x x_{12}}\right) p, \quad s_g = \frac{x_2}{x} s, \quad t_g = \frac{x_1}{x} t.
$$

The expression $(1 - p_g^2)\beta_g^2 \delta_g^1/(\beta_g^1 \delta_g^2) = 1$ then becomes

$$
\left[1 - \left(\frac{x_1 x_2}{x x_{12}}\right)^2 p^2\right] \frac{\beta^2 \delta^1}{\beta^1 \delta^2} = 1 \tag{41}
$$

and "solving" this expression for the gauge function x defines the gauge. $\square$

We now present the proof of Theorem 4.5.

Proof. Set $\boldsymbol{\omega} = Q(s,t)$, $\boldsymbol{\omega}_1 = Q_1(s_1, t_1)$, $\boldsymbol{\omega}_2 = Q_2(s_2, t_2)$, where the expressions for s_1, t_1, s_2 and t_2 are given by (32), (33), (36) and (37) respectively. The two points $\boldsymbol{\omega}_{12} = Q_{12}(s_{12}, t_{12})$ and $\boldsymbol{\omega}_{21} = Q_{12}(s_{21}, t_{21})$ arise from the tangency condition associated with Q_{12} and Q_1 and Q_{12} and Q_2 respectively. The closing condition is encapsulated in the two constraints $s_{12} = s_{21}$ and $t_{12} = t_{21}$ or, equivalently, $\mathcal{S}_1^2 \circ \mathcal{S}^1(s) = \mathcal{S}_2^1 \circ \mathcal{S}^2(s)$ and $\mathcal{T}_1^2 \circ \mathcal{T}^1(t) = \mathcal{T}_2^1 \circ \mathcal{T}^2(t)$. By virtue of (32) and (36), the condition $s_{12} = s_{21}$ is given by

$$
\frac{\alpha_2^1 \beta_2^2 p_2 s_2 + 2\alpha_2^0 \beta_2^1 p_2 - \alpha_2^0 \beta_2^3 s_2}{p_2(2\alpha_2^3 \beta_2^2 s_2 - \alpha_2^1 \beta_2^2 p_2 + \alpha_2^0 \beta_2^3)} = \frac{2\gamma_1^0 \delta_1^1 p_1 - \gamma_1^2 \delta_1^1 p_1 s_1 - \gamma_1^0 \delta_1^3 s_1}{2\delta_1^1 p_1 s_1}.
$$

Remarkably, evaluation modulo (32), (36), the Gauss-Mainardi-Codazzi equations (14), the relation (19) and its counterpart $p_2 = \gamma^0/\delta^1 p$ produces a condition which is independent of s and may be best formulated as

$$\left(1 - \frac{\beta^1 \delta^2}{\beta^2 \delta^1}\right) T_2^1 = p^2 T^1,$$

where

$$T^1 = \frac{D^1}{(\alpha^0 \beta^2 p)^2} = \left(\frac{\beta^3}{\beta^2 p} - \frac{\alpha^1}{\alpha^0}\right)^2 + 4\frac{\alpha^3 \beta^1}{\alpha^0 \beta^2 p}$$

and D^1 is the discriminant (25). If we now utilise the gauge of Lemma 4.6 then the above condition reduces to

$$\Delta_2 T^1 := T_2^1 - T^1 = 0. \tag{42}$$

By symmetry, the closing condition $t_{12} = t_{21}$ leads to

$$\Delta_1 T^2 := T_1^2 - T^2 = 0, \tag{43}$$

where

$$T^2 = \frac{D^2}{(\gamma^0 \delta^1 p)^2}$$

with the discriminant D^2 given by (27). Moreover, the two constraints (42) and (43) turn out to be equivalent due to the relation

$$\frac{\beta^1}{\beta^2} \Delta_1 T^2 = \frac{\delta^2}{\delta^1} \Delta_2 T^1 \tag{44}$$

which may be extracted from the discrete Gauss-Mainardi-Codazzi equations (14). Finally, we note that the algebraic constraint (42) (or (43)) is independent of s and t, i.e., independent of the choice of $\boldsymbol{\omega}$. $\qquad\square$

Remark 4.7. Relation (44) is the discrete analogue of the Gauss-Mainardi-Codazzi equation (6) and the equivalent constraints (42), (43) may be regarded as the discrete version of the Euler-Lagrange condition (9) in the classical Theorem 2.2.

The above theorem gives rise to the following natural algebraic definition of discrete projective minimal surfaces.

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Definition 4.8. A discrete asymptotic net is a discrete projective minimal surface if $\Delta_1 T^2 = 0$ or, equivalently, $\Delta_2 T^1 = 0$, where

$$T^1 = \frac{D^1}{(\alpha^0 \beta^2 p)^2}, \quad T^2 = \frac{D^2}{(\gamma^0 \delta^1 p)^2}$$

in the admissible gauge

$$(1 - p^2)\frac{\beta^2 \delta^1}{\beta^1 \delta^2} = 1$$

of Lemma 4.6.

A detailed discussion of the discrete system (14), (42) (or (43)) underlying discrete projective minimal surfaces, including the determination of its integrability and an analogue of the algebraic classification scheme presented in Section 2, is the subject of a separate publication (cf. [20]).

5. A Cauchy problem

The algebraic discrete projective minimality condition may be exploited to state the following well-posed geometric Cauchy problem.

Theorem 5.1. A discrete projective minimal surface Σ represented by $\boldsymbol{r} : \mathbb{Z}^2 \to \mathbb{R}^4$ is uniquely determined by the Cauchy data $\{\mathscr{C}, Q_0\}$, where

$$\mathscr{C} = \{\boldsymbol{r}(\boldsymbol{n}) : \boldsymbol{n} = (0, *), (1, *), (*, 0), (*, 1)\}$$

is constrained by the planar star property and Q_0 is a fixed quadric associated with one of the quadrilaterals of $\mathscr{C}$ (Figure 7).

Proof. We note that the Cauchy data given by $\mathscr{C}$ define part of a discrete asymptotic net consisting of two strips as in Figure 7. Without loss of generality, let Q_0 be the quadric associated with the quadrilateral $[\boldsymbol{r}, \boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_{12}]$. As a result of the $\mathcal{C}^1$ condition, Q_0 uniquely determines the quadric associated with each quadrilateral along the strips. We now show that the vertex $\boldsymbol{r}_{1122}$ is uniquely determined by the projective minimality condition. The planar star condition implies that the point $\boldsymbol{r}_{1122}$ lies on the line of intersection of the planes spanned by $\{\boldsymbol{r}_{12}, \boldsymbol{r}_{11}, \boldsymbol{r}_{112}\}$ and $\{\boldsymbol{r}_{12}, \boldsymbol{r}_{22}, \boldsymbol{r}_{122}\}$ as indicated in Figure 7. For a fixed $\boldsymbol{r}_{1122}$ on this line, the quadric Q_{12} is uniquely determined by the $\mathcal{C}^1$ condition with respect to Q_1 or Q_2. Let $\boldsymbol{\omega} \in Q_0$ be a generic point. Then, the tangency condition uniquely determines points $\boldsymbol{\omega}_1 \in Q_1$,

$\boldsymbol{\omega}_2 \in Q_2$ and $\boldsymbol{\omega}_{12}, \boldsymbol{\omega}_{21} \in Q_{12}$. As shown in Theorem 4.5, the closing condition $\boldsymbol{\omega}_{12} = \boldsymbol{\omega}_{21}$ reduces to a single scalar condition and this determines the position of $\boldsymbol{r}_{1122}$ on the line of intersection. Specifically, modulo the representation of discrete asymptotic nets in terms of homogeneous coordinates, that is, modulo gauge transformations of the type (40) and their action on the matrix-valued functions L and M, specifying the quadrilaterals $[\boldsymbol{r}, \boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_{12}]$ and $[\boldsymbol{r}_1, \boldsymbol{r}_{11}, \boldsymbol{r}_{12}, \boldsymbol{r}_{112}], [\boldsymbol{r}_2, \boldsymbol{r}_{12}, \boldsymbol{r}_{22}, \boldsymbol{r}_{122}]$ modulo the planar star property is equivalent to prescribing $[\boldsymbol{r}, \boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_{12}]$ and the corresponding matrices L and M. The associated lattice Lie quadrics Q_0 and Q_1, Q_2 are uniquely determined by the prescription of the parameter p. The Gauss-Mainardi-Codazzi equations (14) supplemented by the projective minimality condition (42) or, equivalently, (43) then determine algebraically the matrices L_2 and M_1 and therefore the vertex $\boldsymbol{r}_{1122}$ via the compatible frame equations (11), (12). We can then iterate this procedure to construct simultaneously all vertices of the discrete asymptotic net, the associated family of lattice Lie quadrics and the associated envelope. $\qquad\square$

Since the Cauchy data include a fixed quadric which in turn determines a set of lattice Lie quadrics, the natural question arises as to whether a given discrete projective minimal surface admits more than one set of lattice Lie quadrics.

Theorem 5.2. A discrete projective minimal surface admits only one set of lattice Lie quadrics.

Proof. Consider the 4×3 patch of a discrete projective minimal surface in Figure 8. Let Q be a fixed member, labelled by p, of the one-parameter family of quadrics passing through the quadrilateral $[\boldsymbol{r}, \boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_{12}]$. We note that, as a result of the $\mathcal{C}^1$ condition, all other quadrics in the set of lattice Lie quadrics which contains Q are determined. Let $\boldsymbol{\omega}$ be a generic point on Q. The tangency condition then delivers points $\boldsymbol{\omega}_1 \in Q_1$, $\boldsymbol{\omega}_2 \in Q_2$ and $\boldsymbol{\omega}_{12}, \boldsymbol{\omega}_{21} \in Q_{12}$ and the closing condition $\boldsymbol{\omega}_{12} = \boldsymbol{\omega}_{21}$ determines $\boldsymbol{r}_{1122}$ uniquely. It turns out that $\boldsymbol{r}_{1122}$ is a quadratic function of p. If we now assume that $\boldsymbol{r}_{1122}(p) \sim \boldsymbol{r}_{1122}(\hat{p})$ for another quadric $\hat{Q}$ labelled by $\hat{p}$ then we obtain a quadratic equation in $\hat{p}$ which we denote by $q(\hat{p}) = 0$. Similarly, the vertex $\boldsymbol{r}_{\bar{1}22}$ generates a quadratic $\bar{q}(\hat{p}) = 0$. It is then easy to check that the only common root of q and $\bar{q}$ is p and, hence, $\hat{Q} = Q$. $\qquad\square$
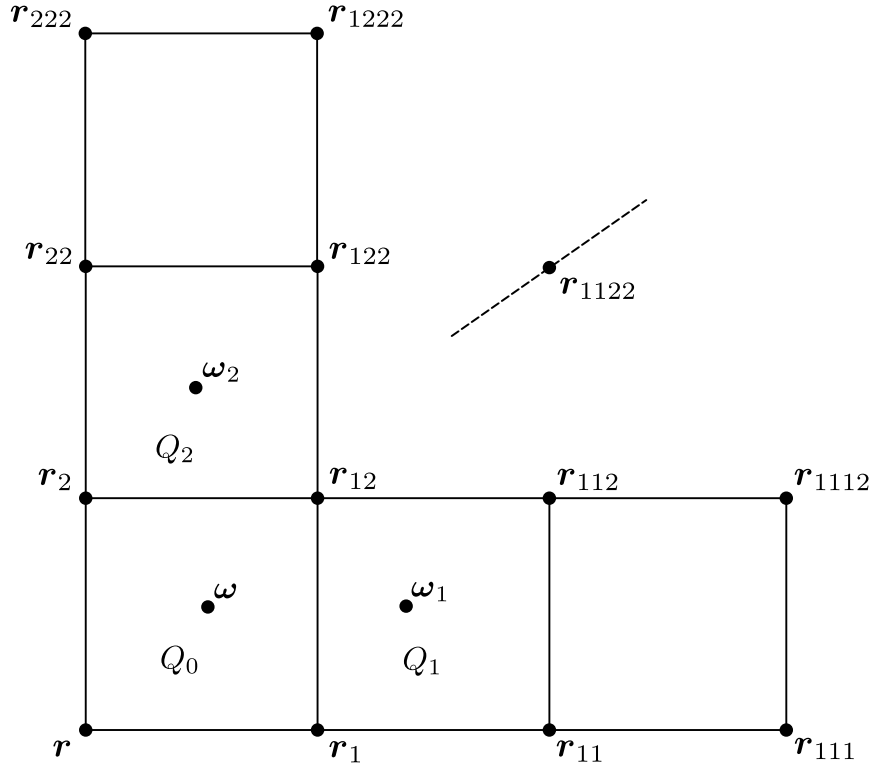
Figure 7: The evolution of Cauchy data for discrete projective minimal surfaces

6. A local classification of the envelopes of discrete PMQ surfaces

In order to produce a geometric characterisation of discrete projective minimal surfaces, it is necessary to classify all surfaces which admit envelopes, i.e., by definition, it is necessary to classify discrete PMQ surfaces. Let Σ be a discrete PMQ surface and denote by Ω an envelope of the set of associated lattice Lie quadrics of Σ. Bearing in mind the continuum limit, from now on we exclude "hybrids" of different types of surfaces by imposing a homogeneity condition on the envelope in the sense that if a property related to the envelope holds for pairs of neighbouring quadrics then it holds for all neighbouring quadrics of the same type. In the following, whenever this principle is applied, the relevant property is identified.

In the examination of different types of envelopes, the maps $\mathcal{S}^i$ and $\mathcal{T}^i$ play an essential role. Consider a 3×3 patch of Σ as displayed in Figure 9 and the associated quadrilateral of Ω. The vertices of Ω are labelled by

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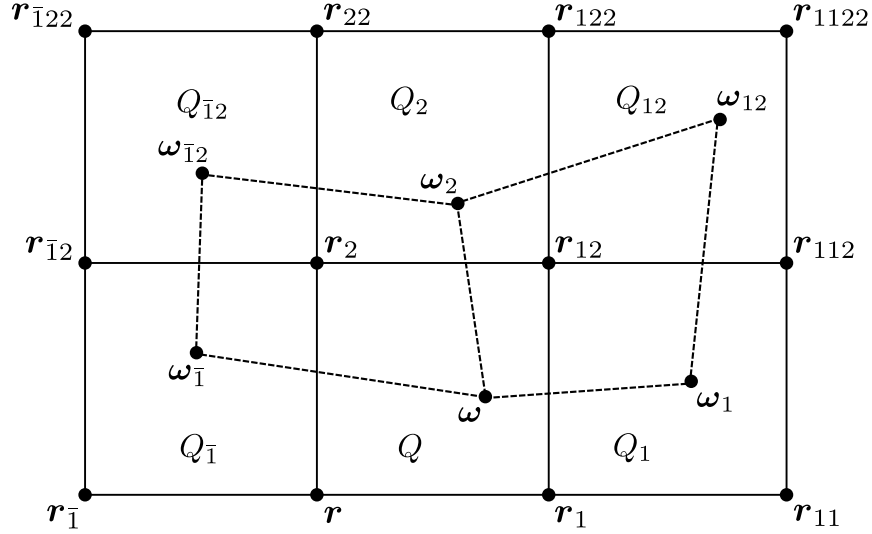
Figure 8: A 4×3 patch of a discrete projective minimal surface

$\boldsymbol{\omega} = Q(s,t)$, $\boldsymbol{\omega}_1 = Q_1(s_1, t_1)$, $\boldsymbol{\omega}_2 = Q_2(s_2, t_2)$ and $\boldsymbol{\omega}_{12} = Q_{12}(s_{12}, t_{12})$. Depending on whether or not the maps $\mathcal{S}^i$ and $\mathcal{T}^i$ are defined, whereby the homogeneity condition is taken into account, the possible cases are:

1. The generic case where all maps $\mathcal{S}^i$ and $\mathcal{T}^i$ are defined and, hence, the closing conditions are satisfied due to the existence of the discrete envelope Ω. Σ is then discrete projective minimal by Theorem 4.5. This is the case where none of the edges of the quadrilateral of Ω are shared generators of the associated pairs of quadrics.

2. The map $\mathcal{S}^2$ is defined. There are then two subcases:

   (a) $\mathcal{S}^1$ is defined. Then, since Ω is a discrete envelope, the closing condition $\mathcal{S}^2_1 \circ \mathcal{S}^1 = \mathcal{S}^1_2 \circ \mathcal{S}^2$ is satisfied and, hence, Σ is discrete projective minimal.

   (b) $\mathcal{S}^1$ is not defined. By virtue of (31), (32), this is the case when $2\alpha^3\beta^2 s - \alpha^1\beta^2 p + \alpha^0\beta^3 = 0$ and, hence,

   $$s = \frac{\alpha^1\beta^2 p - \alpha^0\beta^3}{2\alpha^3\beta^2} \qquad (45)$$

   and $D^1 = 0$. Accordingly, $\Delta_2 T^1 = 0$ and, thus, Σ is discrete projective minimal. Moreover, since $D^1 = 0$, by Corollary 3.10,
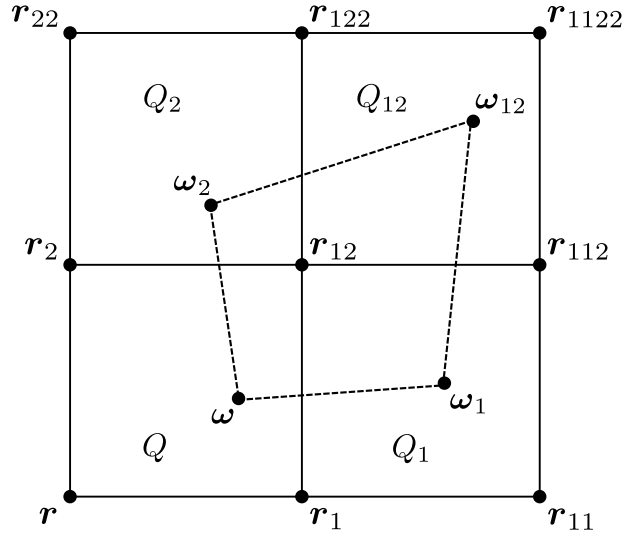
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Figure 9: A 3×3 patch of a discrete asymptotic net and an associated quadrilateral of an enevelope

neighbouring quadrics in the n_1 direction have a coinciding shared generator which is labelled by $ps_1 = s$ and (45). Thus, this case corresponds to the edges of Ω in the n_1 direction consisting of (coinciding) shared generators of neighbouring pairs of quadrics.

3. $\mathcal{S}^2$ is not defined. By (34), this is the case when $pt_2 = t$ and, hence, by Remark 4.3, the edges of Ω in the n_2 direction are shared generators. The remaining cases are then:

   (a) $\mathcal{T}^2$ is defined. There are then two additional sub-cases:

      (i) $\mathcal{T}^1$ is defined. Then similarly to 2.(a), Σ is discrete projective minimal as the projective minimality condition $\mathcal{T}_1^2 \circ \mathcal{T}^1 = \mathcal{T}_2^1 \circ \mathcal{T}^2$ is satisfied.

      (ii) $\mathcal{T}^1$ is not defined. By (30) and (33), $\mathcal{T}^1$ not being well defined corresponds to the case $ps_1 = s$ which, by the proof of Theorem 3.9, implies that the edges of Ω in the n_1 direction are shared generators of neighbouring quadrics. Thus, this case corresponds to all of the edges of the envelope Ω consisting of shared generators. This envelope always exists locally, that is, for a 3×3 patch, and is no restriction on Σ which may or may not be discrete projective minimal.

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(b) $\mathcal{T}^2$ is not defined. This case is similar to 2.(b) and leads to $D^2 = 0$. Σ is then discrete projective minimal since $\Delta_1 T^2 = 0$.

7. Geometric characterisation of discrete projective minimal surfaces and classification of discrete PMQ surfaces

As a result of the discussion of Section 6, we have the following geometric characterisation of discrete projective minimal surfaces.

Theorem 7.1. Let Σ be a discrete asymptotic net. If there exists a set of lattice Lie quadrics which admits an envelope Ω whose edges are not all shared generators of the lattice Lie quadrics then Σ is a discrete projective minimal surface. Conversely, if Σ is a discrete projective minimal surface and $D^1 \neq 0$ or $D^2 \neq 0$ then there exists an envelope Ω whose edges are not all shared generators of the lattice Lie quadrics.

Proof. By the discussion of Section 6, if Ω does not entirely consist of shared generators then Σ is discrete projective minimal. Conversely, suppose that Σ is discrete projective minimal. In the general case $D^1 \neq 0$ and $D^2 \neq 0$, any choice of a generic point $\boldsymbol{\omega} \in Q$ gives rise to an envelope Ω since the maps $\mathcal{S}^i$ and $\mathcal{T}^i$ in (32), (33), (36) and (37) are defined and compatible by virtue of the algebraic minimality condition. Due to the genericity of $\boldsymbol{\omega}$, the edges of Ω are not shared generators. If, for instance, $D^1 = 0$ but $D^2 \neq 0$ then the map $\mathcal{S}^1$ simplifies to (cf. (32))

$$\mathcal{S}^1(s) = s_1 = \frac{\alpha^1 \beta^2 p - \alpha^0 \beta^3}{2\alpha^3 \beta^2 p} \tag{46}$$

(which is, in fact, independent of s), provided that

$$s \neq \frac{\alpha^1 \beta^2 p - \alpha^0 \beta^3}{2\alpha^3 \beta^2} \tag{47}$$

and the maps $\mathcal{S}^i$ and $\mathcal{T}^i$ are still compatible. Even if the condition (47) is violated then we may regard (46) as a definition of the map $\mathcal{S}^1$ (with the associated tangency condition (31) being satisfied) and the maps $\mathcal{S}^i$ and $\mathcal{T}^i$ remain compatible since the constraint (47) does not enter the compatibility condition. In this case, both $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_1$ lie on the generator shared by Q and Q_1 but regardless of whether (47) holds or not, since the (initial) parameter t is arbitrary, we may choose it in such a manner that the edges of the corresponding envelope Ω in n_2 direction are not shared generators. □

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Remark 7.2. If we refer to envelopes which do not consist entirely of shared generators as being generic, then the above theorem implies that, in general, discrete projective minimal surfaces admit a two-parameter family of generic envelopes.

While Theorem 7.1 characterises all discrete projective minimal surfaces, there also exist surfaces with an associated set of lattice Lie quadrics which admit envelopes but may not be discrete projective minimal.

Definition 7.3. A discrete Q surface is a discrete asymptotic net which admits an envelope Ω whose (extended) edges are shared generators of the associated lattice Lie quadrics so that the coordinate polygons of Ω are straight lines.

Figure 10: A discrete Q surface and the associated envelope

Remark 7.4. It follows from the definition of a Q surface that each straight coordinate polygon of the given envelope is a generator common to all quadrics

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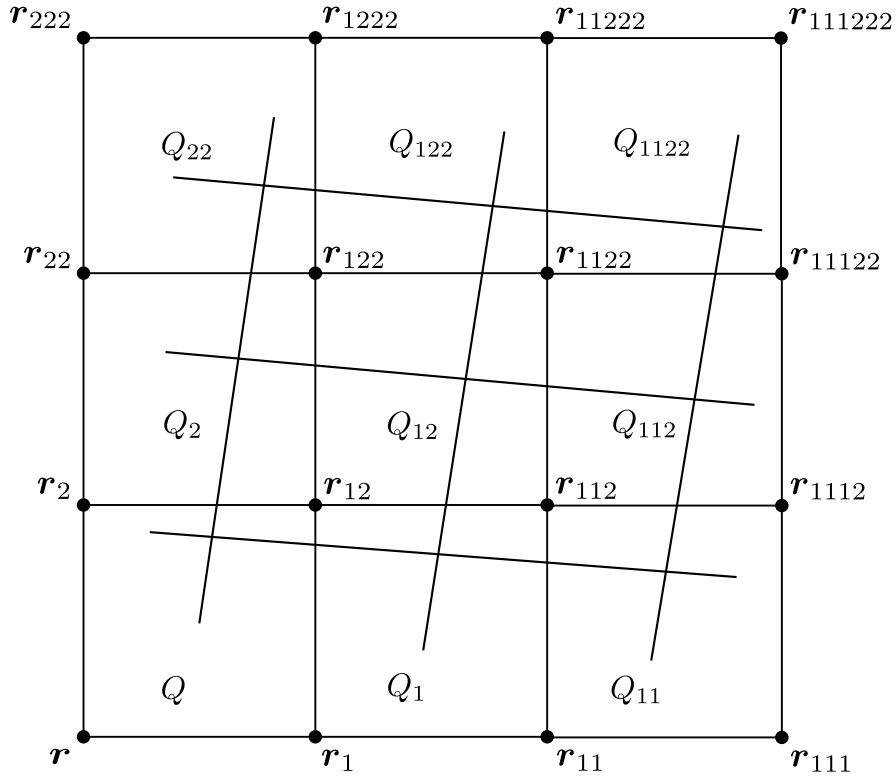

along a strip of the Q surface. In analogy with the continuous theory, this envelope is a discrete quadric as it consists of two 1-parameter (discrete) families of lines. In fact, it is evident that their exists a unique (continuous) quadric which passes through those lines. If a discrete Q surface admits an envelope which is not entirely made up of shared generators then, by Theorem 7.1, it is discrete projective minimal. Thus, the classes of discrete projective minimal and discrete Q surfaces are, *a priori*, not disjoint.

In order to establish in what sense the class of discrete PMQ surfaces consist only of discrete projective minimal and discrete Q surfaces, we make use of the fact that if there exists an envelope whose edges along a strip are shared generators then these shared generators are identical. Hence, by the homogeneity condition, if two quadrics Q and Q_1 have a (coinciding) shared generator which forms part of an envelope then all quadrics along that strip are assumed to have the same (coinciding) generator. Then, as a result of Theorem 7.1, Definition 7.3 and the homogeneity condition, we can draw the following conclusion.

Corollary 7.5. The class of discrete PMQ surfaces consists of discrete projective minimal and discrete Q surfaces.

8. Discrete semi-Q, complex, doubly Q and doubly complex surfaces

We now investigate in more detail the geometric nature of Q surfaces and classes intimately related to Q surfaces. The elementary building block of the notion of a Q surface is the notion of a semi-Q surface.

Definition 8.1. A discrete asymptotic net is said to be a discrete semi-Q surface if it admits a set of lattice Lie quadrics such that quadrics associated with each strip of a given type share a generator, as in the top left of Figure 11. We call the direction (n_1 or n_2) along which quadrics share a generator the semi-Q direction.

We now examine surfaces which are semi-Q in more than one way.

8.1. Discrete complex, doubly Q and doubly complex surfaces

Definition 8.2. As illustrated in Figure 11, a discrete asymptotic net with an associated set of lattice Lie quadrics is said to be a discrete

(i) complex surface if all quadrics on each strip of a given type share two (possibly coinciding) generators, that is, if it is doubly semi-Q in one direction.

(ii) doubly Q surface if it is a discrete complex surface with respect to one direction and a discrete semi-Q surface with respect to the other direction, that is, if it is doubly semi-Q in one direction and semi-Q in the other.

(iii) doubly complex surface if it is a discrete complex surface in both directions, that is, if it is doubly semi-Q in both directions.

Remark 8.3. By definition, discrete Q surfaces are discrete asymptotic nets which are semi-Q in both directions.

As in the continuous theory, it turns out that not all of these classes of surfaces are distinct. Firstly, we need the following well-known fact.

Lemma 8.4. Let $\hat{Q}$ be a quadric, L a line. Then,

(i) if L intersects $\hat{Q}$ at three points then it is a generator of $\hat{Q}$,

(ii) if L intersects $\hat{Q}$ at a point and touches $\hat{Q}$ at a different point then it is a generator of $\hat{Q}$.

Theorem 8.5. A discrete doubly Q surface is doubly complex.

Proof. Without loss of generality, we assume that the surface is discrete semi-Q in the n_2 direction and discrete complex in the n_1 direction. Let $\hat{Q}$ be the unique quadric passing through three neighbouring shared generators W, W_1 and W_{11} as in Figure 12. Denote by U and V the generators common to the quadrics of the n_1 strip containing the quadric Q as in Figure 12. Since U and V and their shifts in the n_2 direction intersect W, W_1 and W_{11}, by Lemma 8.4, U, V and their n_2 shifts are all generators of $\hat{Q}$. Now, consider the three quadrics Q, Q_2, and $Q_{\bar{2}}$ in an n_2 strip as in Figure 12. We treat the cases of coinciding and non-coinciding generators in the n_1 direction separately.

(i) Suppose firstly that $U \neq V$. We know by Theorem 3.9 that Q and Q_2 have a second shared generator G (which may or may not coincide with W). G intersects U_2, V_2, U (and V) and, hence, by Lemma 8.4,
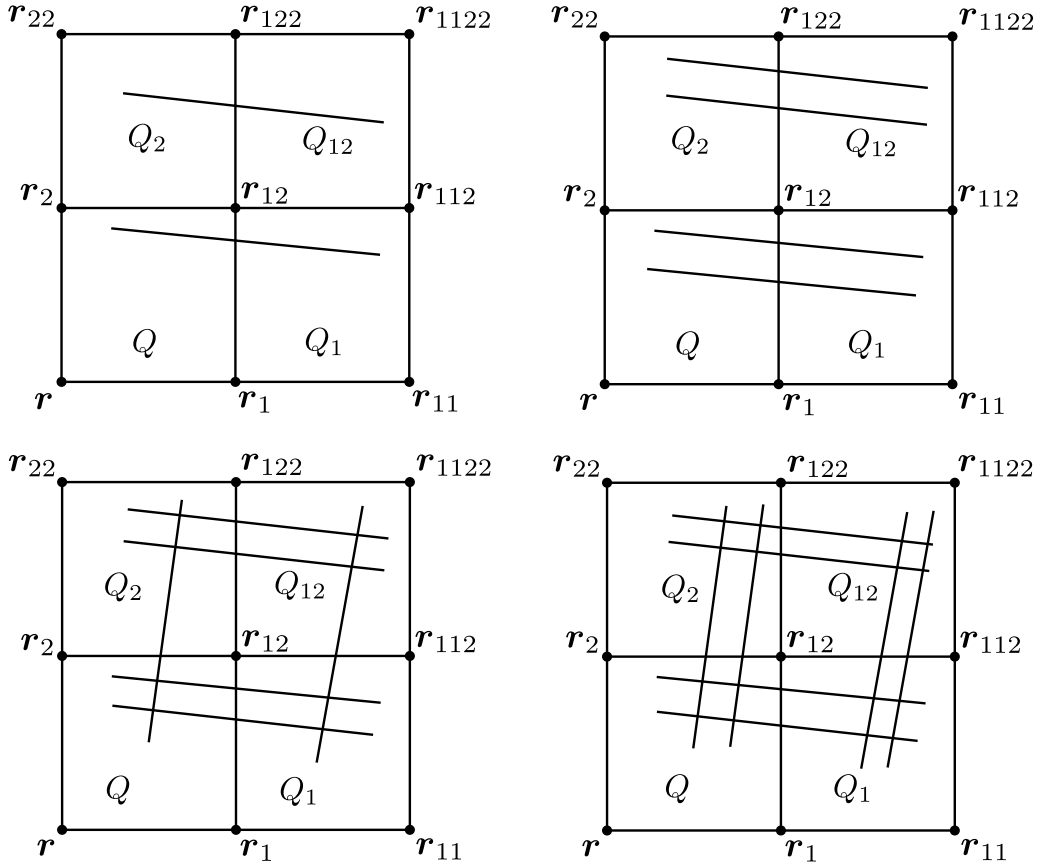
Figure 11: Discrete semi-Q (t-l), complex (t-r), doubly Q (b-l) and doubly complex (b-r) surfaces

G is a generator of $\hat{Q}$. Thus, by extension, G also intersects $U_{\bar{2}}$ and $V_{\bar{2}}$. Moreover, since the edge $[\boldsymbol{r}, \boldsymbol{r}_1]$ is a generator of $Q_{\bar{2}}$ and since G intersects this edge as well and, thus, intersects three generators of $Q_{\bar{2}}$, by Lemma 8.4, G is also a generator of $Q_{\bar{2}}$.

(ii) Suppose now that $U = V$. Denote by $\boldsymbol{H}$ the point of intersection of U and W, by $\boldsymbol{H}_1$ the point of intersection of U and W_1 etc. Then, since the tangent plane of Q at $\boldsymbol{H}$ is spanned by U and W, Q and $\hat{Q}$ touch at $\boldsymbol{H}$. Similarly, $\hat{Q}$ and Q_1 touch at $\boldsymbol{H}_1$ and $\hat{Q}$ and Q_{11} touch at H_{11}. Moreover, by Corollary 3.12, the tangent planes of Q at $\boldsymbol{H}_1$ and $\boldsymbol{H}_{11}$ coincide with those of Q_1 and Q_{11} respectively. Consequently, the tangent planes of $\hat{Q}$ and Q coincide at the three points $\boldsymbol{H}$, $\boldsymbol{H}_1$ and

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$\boldsymbol{H}_{11}$ and, therefore, $\hat{Q}$ touches Q along U. Then, since G is a generator of Q and Q_2, it intersects U and U_2 at points $\boldsymbol{K}$ and $\boldsymbol{K}_2$, say, and, hence, touches $\hat{Q}$ at $\boldsymbol{K}$ and $\boldsymbol{K}_2$. Thus, by Lemma 8.4, G is a generator of $\hat{Q}$. By extension, G intersects $Q_{\bar{2}}$ at a point on the edge $[\boldsymbol{r}, \boldsymbol{r}_1]$ and touches $Q_{\bar{2}}$ at $\boldsymbol{K}_{\bar{2}}$ (being the intersection of G and $U_{\bar{2}}$) since $\hat{Q}$ touches $Q_{\bar{2}}$ along $U_{\bar{2}}$. Thus, G is a generator of $Q_{\bar{2}}$. $\qquad\square$

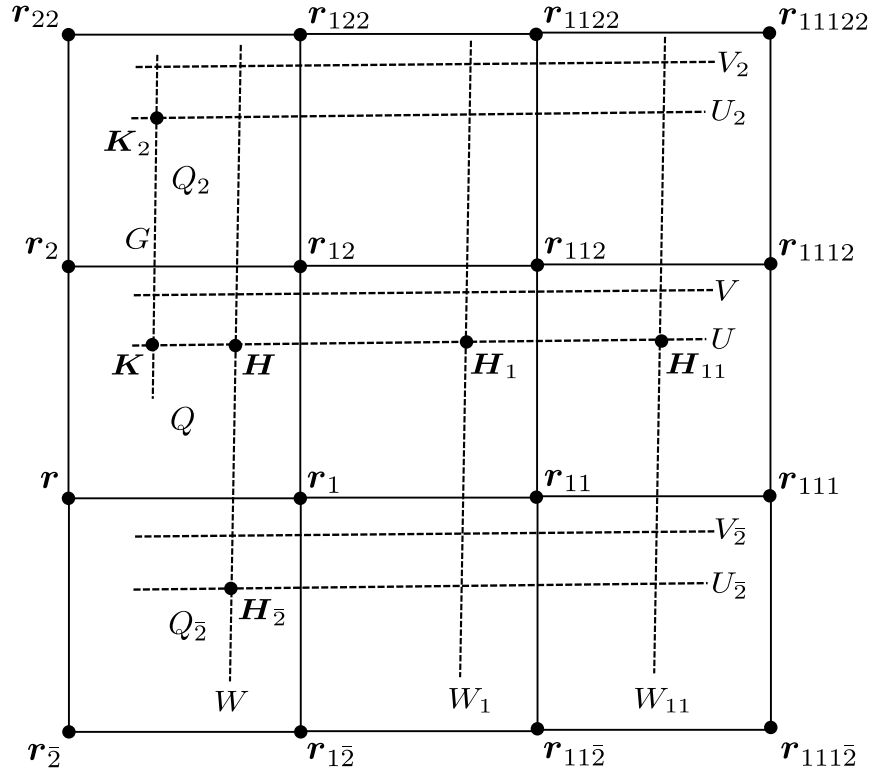Figure 12: A 4×4 patch of a discrete doubly Q surface

9. Discrete Godeaux-Rozet, Demoulin and Tzitzéica surfaces

We now turn to an investigation of special types of discrete projective minimal surfaces and define discrete analogues of Godeaux-Rozet, Demoulin and Tzitzéica surfaces.

9.1. Discrete Godeaux-Rozet and Demoulin surfaces

In view of the classification of Section 6, we propose the following algebraic definition.

Definition 9.1. A discrete asymptotic net Σ is a

(i) discrete Godeaux-Rozet surface if $D^1 = 0$ or $D^2 = 0$.

(ii) discrete Demoulin surface if $D^1 = D^2 = 0$.

Let Σ be a discrete PMQ surface and Ω an envelope corresponding to a set of lattice Lie quadrics of Σ. We may associate with each vertex $\boldsymbol{\omega}$ of Ω four lines passing through $\boldsymbol{\omega}$ and the four vertices of the corresponding quadrilateral of Σ. Taking one of these lines and its shifts along the lattice generates a discrete line congruence. Hence, there exist four such discrete line congruences $\mathscr{L}$, $\mathscr{L}^1$, $\mathscr{L}^2$ and $\mathscr{L}^{12}$ as displayed in Figure 13. A priori, generically, there exists a two-parameter family of each line congruence generated by moving the point $\boldsymbol{\omega}$ around on a fixed lattice Lie quadric.
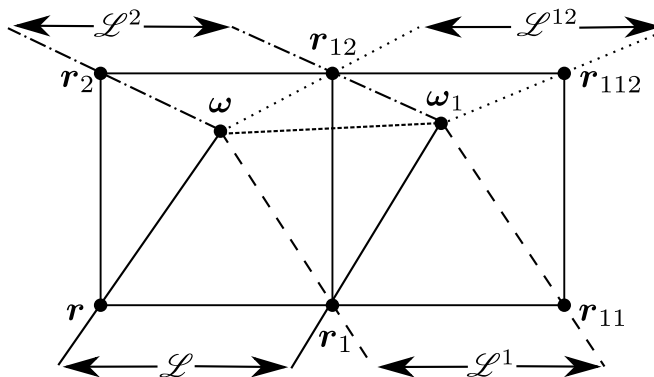

Figure 13: Four types of line congruences

In order to produce a geometric characterisation of discrete Godeaux-Rozet and Demoulin surfaces in terms of these line congruences, it is convenient to introduce the following definition.

Definition 9.2. A discrete line congruence $\mathscr{L}^* = \{l(n_1, n_2)\}$ is said to have the intersection property in the direction n_1 (or n_2) if neighbouring lines in the n_1 (or n_2) direction intersect, that is, $l \cap l_1 \neq \emptyset$ (or $l \cap l_2 \neq \emptyset$).

Theorem 9.3. Let Σ be a discrete projective minimal surface with an associated discrete envelope Ω such that the edges of Ω in the n_1 direction are not shared generators of the lattice Lie quadrics of Σ. Then, $D^1 = 0$ if and only if there exists a line congruence which possesses the intersection property in the n_1 direction. Moreover, if $\mathscr{L}$ (or $\mathscr{L}^1$) admits such an intersection property then so does $\mathscr{L}^2$ (or $\mathscr{L}^{12}$) and vice versa. These statements apply, *mutatis mutandis*, if one considers the direction n_2.

Proof. Consider the patch of Σ in Figure 14. Since the edges of Ω in the n_1 direction are not shared generators, given two vertices of Ω, $\boldsymbol{\omega} \in Q$ and $\boldsymbol{\omega}_1 \in Q_1$, it follows that either $\boldsymbol{\omega}$ or $\boldsymbol{\omega}_1$ does not lie on a generator common to Q and Q_1. Without loss of generality, we assume that $\boldsymbol{\omega}$ does not lie on a shared generator. Then, $\boldsymbol{\omega}_1 = Q_1(s_1, t_1)$, where s_1 and t_1 are given by (32) and (33). We now consider the line congruence $\mathscr{L}$ (defined by the line through $\boldsymbol{\omega}$ and $\boldsymbol{r}$ and its shifts) and assume that the lines $l = [\boldsymbol{\omega}, \boldsymbol{r}]$ and $l_1 = [\boldsymbol{\omega}_1, \boldsymbol{r}_1]$ intersect. Their point of intersection, $\boldsymbol{I}^1$, is then determined by $f\boldsymbol{r} + g\boldsymbol{r}_1 + h\boldsymbol{\omega} + \boldsymbol{\omega}_1 = 0$, that is,

$$
\begin{aligned}
\boldsymbol{r}(f + hst + \alpha^0 s_1) &+ \boldsymbol{r}_1(g + hs + \beta^1 p_1 + \alpha^1 s_1 + s_1 t_1) \\
&+ \boldsymbol{r}_2(h + \beta^2 p_1) + \boldsymbol{r}_{12}(hp + \beta^3 p_1 + \alpha^3 s_1 + t_1) = 0.
\end{aligned}
\tag{48}
$$

Equating each component equal to zero then yields four conditions which may be formulated as:

$$\boldsymbol{r}: \qquad f = \alpha^0 \left(\frac{s}{p} - s_1 \right)$$

$$\boldsymbol{r}_1: \qquad g = \frac{\alpha^0 \beta^1}{\beta^2 p} - \frac{\alpha^0 s}{pt} + \alpha^1 s_1 + s_1 t_1$$

$$\boldsymbol{r}_2: \qquad h = -\frac{\alpha^0}{pt}$$

$$\boldsymbol{r}_{12}: \qquad (\alpha^0 \beta^3 - \alpha^1 \beta^2 p)^2 + 4\alpha^0 \alpha^3 \beta^1 \beta^2 p = 0, \quad \text{i.e.,} \quad D^1 = 0.$$

Conversely, if $D^1 = 0$ then choosing f, g and h as above implies that (48) holds so that $\boldsymbol{I}^1$ exists. By symmetry, the above arguments also hold for the line congruence $\mathscr{L}^2$. $\qquad \square$

As a result of Theorem 9.3, it follows that discrete Godeaux-Rozet surfaces are essentially characterised by the intersection property of their line congruences.
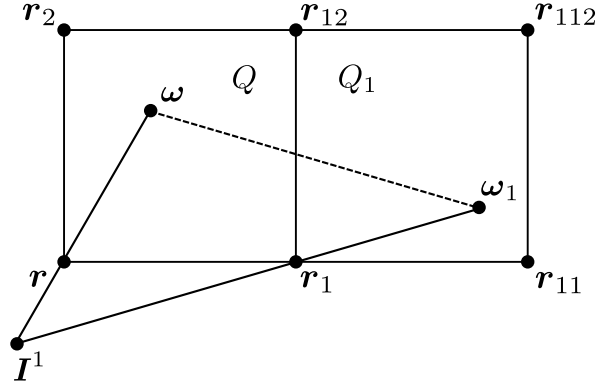
Figure 14: Intersection of lines belonging to the line congruence $\mathscr{L}$

Corollary 9.4. Under the assumption of Theorem 9.3, a discrete projective minimal surface is a discrete Godeaux-Rozet surface if and only if there exists a line congruence associated with any envelope which possesses the intersection property in the n_1 or n_2 direction.

Remark 9.5. In the proof of Theorem 9.3, since $\boldsymbol{\omega}$ does not lie on a shared generator, $D^1 = 0$ implies by virtue of (31) that

$$s_1 = \frac{\alpha^1 \beta^2 p - \alpha^0 \beta^3}{2\alpha^3 \beta^2 p}$$

and, hence, by Remark 4.3, $\boldsymbol{\omega}_1$ is on the shared generator of Q and Q_1. $\boldsymbol{I}^1$ is then given by

$$\boldsymbol{I}^1 = \boldsymbol{\omega} + \frac{f}{h}\boldsymbol{r} = p\boldsymbol{r}_{12} + s\boldsymbol{r}_1 + t\boldsymbol{r}_2 + ps_1 t\boldsymbol{r}. \tag{49}$$

On the other hand, if $\boldsymbol{\omega}_1$ does not lie on the generator common to Q and Q_1 then $D^1 = 0$ implies that $\boldsymbol{\omega}$ lies on the shared generator. In this case, the line congruences which have the intersection property are $\mathscr{L}^1$ and $\mathscr{L}^{12}$. Accordingly, by virtue of the homogeneity assumption, if $D^1 = 0$ and $\boldsymbol{\omega}$ does not lie on the generator common to Q and Q_1 then it must lie on the generator common to $Q_{\bar{1}}$ and Q. Thus, for a discrete Godeaux-Rozet surface, the set of generic envelopes (cf. Remark 7.2) consists of two one-parameter families. It turns out that these two families coincide in the following sense.

Theorem 9.6. Let Ω with vertices labelled by $\boldsymbol{\omega}$ be an envelope associated with a set of lattice Lie quadrics $\{Q\}$ of a discrete Godeaux-Rozet surface for

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which the line congruence $\mathscr{L}$ has the intersection property in the n_1 direction and for which any vertex $\boldsymbol{\omega}$ does not lie on a generator shared by the lattice Lie quadrics Q and Q_1. The discrete asymptotic net $\tilde{\Omega}$ defined by $\tilde{\boldsymbol{\omega}} = \boldsymbol{\omega}_1$ then constitutes another envelope for which $\tilde{\mathscr{L}}^1$ has the intersection property in n_1 direction and $\tilde{\mathscr{L}}^1 \ni \tilde{l} = l_1 \in \mathscr{L}$ (cf. Figure 15).

Proof. By Remark 9.5, if $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_1$ are two vertices of the envelope Ω associated with quadrics Q and Q_1 respectively, then $\boldsymbol{\omega}_1$ must lie on the shared generator of Q and Q_1 since, by assumption, $\boldsymbol{\omega}$ does not. Hence, $\boldsymbol{\omega}_1$ may be regarded as a point $\tilde{\boldsymbol{\omega}}$ of the quadric Q. Since the quadrics Q and Q_1 touch along the common generator, the tangent planes of Q and Q_1 at $\tilde{\boldsymbol{\omega}} = \boldsymbol{\omega}_1$ coincide. Hence, the envelope Ω, having the intersection property with respect to $\mathscr{L}$, may also be interpreted as an envelope $\tilde{\Omega}$, having the intersection property with respect to the congruence $\tilde{\mathscr{L}}^1$ consisting of the lines $\tilde{l} = [\tilde{\boldsymbol{\omega}}, \boldsymbol{r}_1] = [\boldsymbol{\omega}_1, \boldsymbol{r}_1] = l_1 \in \mathscr{L}$. $\qquad\square$
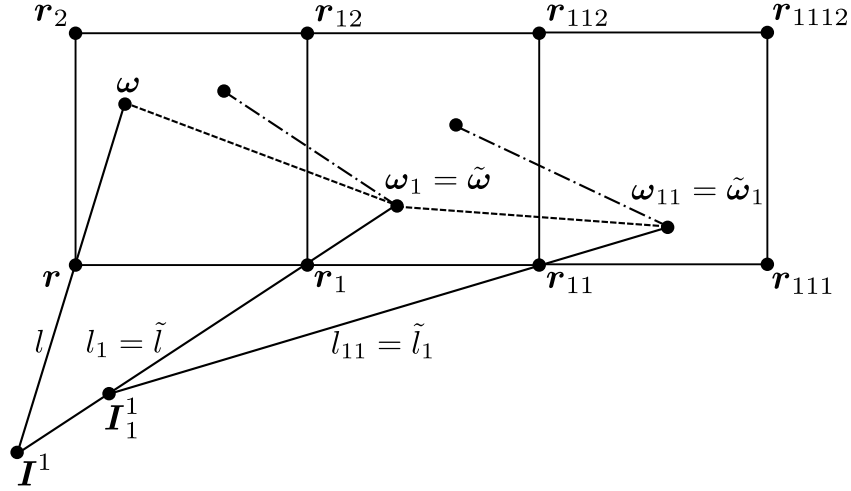

Figure 15: Coinciding envelopes of discrete Godeaux-Rozet surfaces – shared generators are represented by dashed-dotted lines

Remark 9.7. We note that, for a discrete Godeaux-Rozet surface corresponding to $D_1 = 0$, if $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_1$ are vertices of a generic envelope such that $\boldsymbol{\omega} = Q(s,t)$ does not lie on the generator common to Q and Q_1 then $\boldsymbol{\omega}_1$ as a point on Q is represented by $\boldsymbol{\omega}_1 = Q(\tilde{s}, \tilde{t})$, where $\tilde{s}$ is given by (45) and,

remarkably, $\tilde{t} = t$. Specifically, evaluation of $\boldsymbol{\omega}_1 = Q_1(s_1, t_1)$ with s_1 and t_1 given by (32) and (33) yields

$$\boldsymbol{\omega}_1 \sim p\boldsymbol{r}_{12} + \tilde{s}\boldsymbol{r}_1 + t\boldsymbol{r}_2 + \tilde{s}t\boldsymbol{r}$$

so that, indeed, $\boldsymbol{\omega}_1$ is the point of intersection of the generator common to Q and Q_1 and the generator of Q of the other type, passing through $\boldsymbol{\omega}$ and labelled by t.

The above discussion shows that we can also characterise discrete Demoulin surfaces in terms of line congruences. In particular, since a discrete Demoulin surface is a Godeaux-Rozet surface with respect to both the n_1 and n_2 directions, as a result of Theorem 9.3 and the fact that we may always choose $\boldsymbol{\omega} \in Q$ such that it does not lie on a generator common to Q and Q_1 or Q and Q_2, the following holds true.

Corollary 9.8. Under the assumptions of Theorem 9.3, a discrete projective minimal surface Σ is a discrete Demoulin surface if and only if there exists a line congruence associated with any envelope which has the intersection property in both directions (as indicated in Figure 16 for the line congruence $\mathscr{L}$).

Remark 9.9. Remark 9.5 implies that for a discrete Demoulin surface and a given generic envelope Ω, each vertex of Ω lies on a shared generator with one of the two neighbouring quadrics in the n_1 direction and on a shared generator with one of the two neighbouring quadrics in the n_2 direction. Thus, a discrete Demoulin surface has four generic envelopes, the four vertices of which associated with a quadric Q constitute the points of intersection $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ of the four generators of Q shared with the neighbouring four quadrics as depicted in Figure 17. By Remark 9.7, the point $\boldsymbol{A}_1$ is the intersection of the generator common to Q and Q_1 and the generator common to Q and $Q_{\bar{2}}$. Hence, it follows that $\boldsymbol{A}_1 = \boldsymbol{D}$. Similarly, $\boldsymbol{B}_1 = \boldsymbol{C}$ and for reasons of symmetry, $\boldsymbol{A}_2 = \boldsymbol{B}$ and $\boldsymbol{D}_2 = \boldsymbol{C}$. We conclude that the four envelopes of a discrete Demoulin surface coincide in the sense of Theorem 9.6 with $\boldsymbol{A}_{12} = \boldsymbol{C}$.

In summary, the following statement may be made.

Corollary 9.10. Discrete projective minimal surfaces and their subclasses may be characterised in terms of the number of envelopes of the associated set of lattice Lie quadrics.
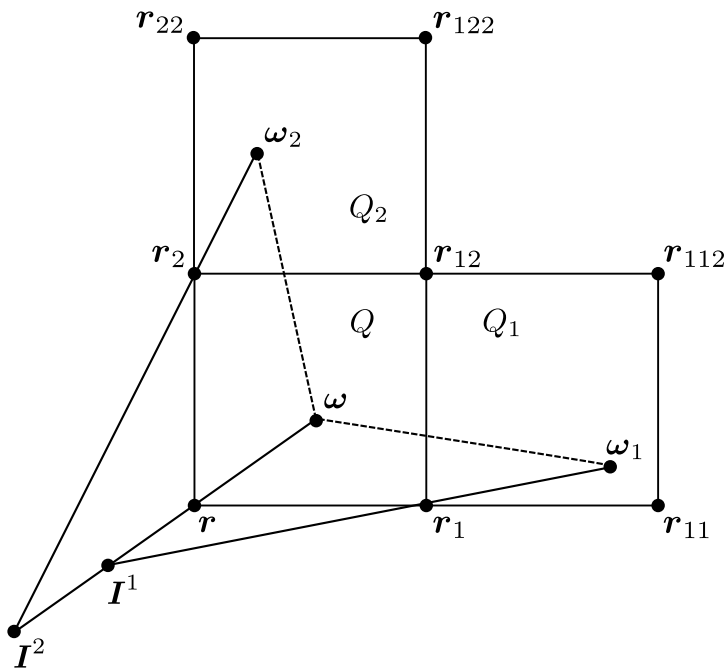
Figure 16: Intersection of three members of the line congruence $\mathscr{L}$ which has the intersection property in both directions.

  (i) In general, a discrete projective minimal surface admits a two-parameter family of generic envelopes.

 (ii) In general, a discrete Godeaux-Rozet surface admits two one-parameter families of generic envelopes. The two families coincide modulo a relabelling of vertices.

(iii) In general, a discrete Demoulin surface admits four generic envelopes which coincide modulo a relabelling of their vertices.

9.2. Discrete Tzitzéica surfaces

We have shown that discrete Demoulin surfaces admit four envelopes which coincide modulo a relabelling of the vertices and that the four associated line congruences which possess the intersection property in both directions consist of the same lines. In the following, we therefore focus without loss of generality on the envelope of a discrete Demoulin surface for which $\mathscr{L}$ possesses the intersection property as depicted in Figure 16. As in
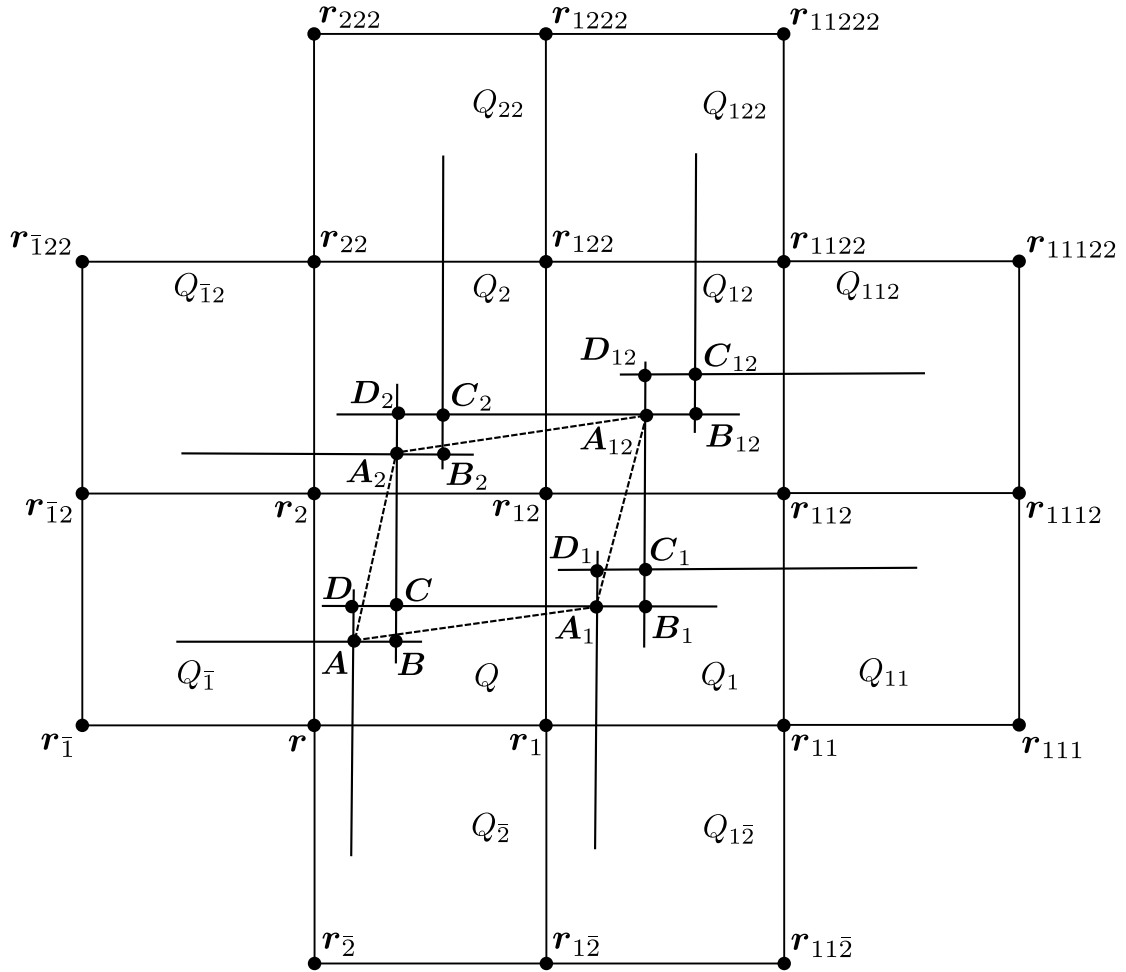
Figure 17: The vertices of the four envelopes of a discrete Demoulin surface regarded as the points of intersection of shared generators.

the preceding, we denote by $\boldsymbol{I}^1$ the point of intersection of a line $l \in \mathcal{L}$ and the neighbouring line l_1 and, similarly, the point of intersection of the line l and the neighbouring line l_2 is labelled by $\boldsymbol{I}^2$. It is natural to examine the case when the points $\boldsymbol{I}^1$ and $\boldsymbol{I}^2$ coincide. Since $\boldsymbol{I}^1$ and $\boldsymbol{I}^2$ lie on l, this corresponds to one additional condition on the discrete Gauss-Mainardi-Codazzi equations. Comparison of the parametrisation (49) of $\boldsymbol{I}^1$ and its counterpart for $\boldsymbol{I}^2$, namely

$$\boldsymbol{I}^2 = p\boldsymbol{r}_{12} + s\boldsymbol{r}_1 + t\boldsymbol{r}_2 + pt_2 s\boldsymbol{r},$$

shows that this condition is given by

$$s_1 t = t_2 s. \tag{50}$$

Likewise, there exists only one condition for the points $\boldsymbol{I}^1$ and $\boldsymbol{I}^1_1$ to coincide since both $\boldsymbol{I}^1$ and $\boldsymbol{I}^1_1$ lie on the line l_1. Specifically, evaluation of $\boldsymbol{I}^1_1$ modulo the frame equations (11) leads to

$$\begin{aligned}
\boldsymbol{I}^1_1 &= p_1\boldsymbol{r}_{112} + s_1\boldsymbol{r}_{11} + t_1\boldsymbol{r}_{12} + p_1 s_{11} t_1 \boldsymbol{r}_1 \\
&= \left(\frac{\alpha^0 \beta^3}{p\beta^2} + \alpha^3 s_1 + t_1 \right) \boldsymbol{r}_{12} + \left(\frac{\alpha^0 \beta^1}{p\beta^2} + \alpha^1 s_1 + \frac{\alpha^0 s_{11} t_1}{p\beta^2} \right) \boldsymbol{r}_1 \\
&\quad + \frac{\alpha^0}{p}\boldsymbol{r}_2 + \alpha^0 s_1 \boldsymbol{r}
\end{aligned} \tag{51}$$

which needs to be compared with

$$\boldsymbol{I}^1 \sim \frac{\alpha^0}{t}\boldsymbol{r}_{12} + \frac{\alpha^0}{p}st\boldsymbol{r}_1 + \frac{\alpha^0}{p}\boldsymbol{r}_2 + \alpha^0 s_1 \boldsymbol{r}. \tag{52}$$

The coefficients of $\boldsymbol{r}$ and $\boldsymbol{r}_2$ in the expansions (51) and (52) match and the $\boldsymbol{r}_{12}$-components are likewise identical modulo the expressions for s_1 and t_1 given by (32) and (33) respectively since $D^1 = 0$. Hence, as expected, matching the coefficients of $\boldsymbol{r}_1$ produces the only condition which may be formulated as

$$s_{11} t_1 = t_{12} s_1.$$

Comparison with (50) shows that this encodes the coincidence of the points $\boldsymbol{I}^1_1$ and $\boldsymbol{I}^2_1$. Hence, for reasons of symmetry, we have established the following theorem.

Theorem 9.11. Let Σ be a discrete Demoulin surface and $\boldsymbol{I}^1$ and $\boldsymbol{I}^2$ be the points of intersection of neighbouring lines of the line congruence $\mathscr{L}$. If the points $\boldsymbol{I}^1$ and $\boldsymbol{I}^2$ coincide for each quadrilateral then $\boldsymbol{I}^1 = \boldsymbol{I}^2$ does not depend on the quadrilateral so that all lines of the line congruence $\mathscr{L}$ meet in a point.

In light of the above theorem, we define a special subclass of discrete Demoulin surfaces which may be regarded as discrete analogues of (projective transforms of) Tzitzéica surfaces (see, e.g., [19] and references therein).

Definition 9.12. A discrete projective minimal surface Σ is said to be a projective transform of a discrete Tzitzéica surface if there exists an envelope and an associated line congruence such that all lines of the line congruence meet in a point.

Remarkably, the particular discrete projective minimal surfaces Σ defined above constitute projective transforms of the integrable discrete Tzitzéica surfaces proposed in [24] in an entirely different context. Indeed, we first note that the condition of coinciding points of intersection $\boldsymbol{I}^1$ and $\boldsymbol{I}^2$, namely $s_1 t = t_2 s$, guarantees the existence of a potential φ related to s and t by

$$\varphi_1 = -t\varphi, \quad \varphi_2 = -s\varphi. \tag{53}$$

This potential φ turns out to be a scalar solution of the discrete asymptotic net conditions (cf. (11))

$$\boldsymbol{r}_{11} = \alpha^0 \boldsymbol{r} + \alpha^1 \boldsymbol{r}_1 + \alpha^3 \boldsymbol{r}_{12}, \quad \boldsymbol{r}_{22} = \gamma^0 \boldsymbol{r} + \gamma^2 \boldsymbol{r}_2 + \gamma^3 \boldsymbol{r}_{12}. \tag{54}$$

For instance, substitution of φ into $(54)_1$ and evaluation by means of (53) yield

$$t_1 t = \alpha^0 - \alpha^1 t + \alpha^3 s_1 t, \tag{55}$$

which is indeed satisfied by virtue of the expressions for t_1 and s_1 obtained from (33) and (31) respectively. Accordingly, in the affine gauge

$$\hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{\varphi}, \tag{56}$$

the discrete asymptotic net conditions adopt the form

$$
\begin{aligned}
\hat{\boldsymbol{r}}_{11} - \hat{\boldsymbol{r}}_1 &= \hat{\alpha}^1 (\hat{\boldsymbol{r}}_1 - \hat{\boldsymbol{r}}) + \hat{\alpha}^3 (\hat{\boldsymbol{r}}_{12} - \hat{\boldsymbol{r}}_1) \\
\hat{\boldsymbol{r}}_{22} - \hat{\boldsymbol{r}}_2 &= \hat{\gamma}^2 (\hat{\boldsymbol{r}}_2 - \hat{\boldsymbol{r}}) + \hat{\gamma}^3 (\hat{\boldsymbol{r}}_{12} - \hat{\boldsymbol{r}}_2).
\end{aligned}
\tag{57}
$$

On the other hand, the constancy of the point of intersection $I^1 = I^2$ may be formulated as

$$I^1 \sim \boldsymbol{e}_4, \qquad \boldsymbol{e}_4 = (0\ 0\ 0\ 1)^T \tag{58}$$

modulo an appropriate projective transformation so that the parametrisation (49) gives rise to

$$p\frac{\varphi_{12}\varphi}{\varphi_1\varphi_2}(\hat{\boldsymbol{r}}_{12} + \hat{\boldsymbol{r}}) - (\hat{\boldsymbol{r}}_1 + \hat{\boldsymbol{r}}_2) \sim \boldsymbol{e}_4. \tag{59}$$

If we now interpret the first three components $\boldsymbol{r}^a$ of $\hat{\boldsymbol{r}}$ as the position vector of a representation Σ^a in centro-affine geometry of the discrete surface Σ then, by virtue of (57), Σ^a constitutes a discrete asymptotic net which is constrained by

$$\boldsymbol{r}_{12}^a + \boldsymbol{r}^a = h(\boldsymbol{r}_1^a + \boldsymbol{r}_2^a), \qquad h = \frac{\varphi_1\varphi_2}{p\varphi_{12}\varphi}. \tag{60}$$

The latter may be interpreted geometrically if we define a discrete affine normal $\boldsymbol{N}^a$ associated with a quadrilateral $[\boldsymbol{r}^a, \boldsymbol{r}_1^a, \boldsymbol{r}_2^a, \boldsymbol{r}_{12}^a]$ to be the line connecting the midpoints of the diagonals of the quadrilateral. Thus, the constraint (60) shows that all affine normals $\boldsymbol{N}^a$ meet at a point (namely the origin of the coordinate system). This is precisely the property on which the definition of the discrete affine spheres proposed in [24] is based.

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